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Quasipartition and planar quasipartition algebras

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Abstract. The quasi-partition algebras were introduced by Daugherty and the first author in 2014, as centralizers of the symmetric group. Here we provide a more general construction using idempotents which allows us to define the half quasi-partition algebra. Our construction allows us to describe the planar analogues of these quasi-partition algebras. In this case the planar subalgebras are centralizer algebras of the quantum group $U_q(\mathfrak{sl}_2)$ and of dimensions equal to Motzkin and Riordan numbers. We use a Bratteli-like diagram to describe how the representation theories of these algebras are related.

Keywords: diagram algebras, representation theory, Motzkin and Riordan numbers

1 Introduction

The partition algebra was defined independently in the work of Martin and his coauthors [10, 11] and Jones [9] in the early 1990s as a natural generalization of centralizer algebras such as the Brauer and Temperley-Lieb algebras. It is of interest for combinatorial representation theory because it provides a dual approach to resolving some of the open combinatorial problems related to the representation theory of the symmetric group. The partition algebras are related to the Kronecker coefficients [3] and to the restriction and plethysm coefficients [12].

For integers k and n with $n \geq 2k$ and $V_n = \mathbb{C}^n$, then $V_n^{\otimes k}$ is an S_n module with the diagonal action of a permutation on the k tensors. The centralizer of this action is isomorphic to the partition algebra $\mathsf{P}_k(n)$. It is possible to understand the tensor products of permutation modules as sequences of restriction and induction [4] of the trivial S_n-module, $\mathbb{S}^{(n)}$, since we have $V_n^{\otimes k} \cong (\mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{Res}_{S_{n-1}}^{S_n})^k \mathbb{S}^{(n)}$. Following [8], we denote the half-partition algebras as $\mathsf{P}_{k+\frac{1}{2}}(n)$. These algebras lie in between two partition algebras and are isomorphic to centralizers of the symmetric group S_{n-1} acting on $\mathrm{Res}_{S_{n-1}}^{S_n} V_n^{\otimes k}$. This defines a structure of embeddings and inclusions as

$$P_0(n) \hookrightarrow \mathsf{P}_{\frac{1}{2}}(n) \subseteq \mathsf{P}_1(n) \hookrightarrow \mathsf{P}_{\frac{3}{2}}(n) \subseteq \mathsf{P}_2(n) \hookrightarrow \cdots$$

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that makes it possible to construct the irreducible representations using what is known as the "basic construction" (see Section 4 of [8]).

The quasi-partition algebra was introduced by Daugherty and the first author [5] by considering the centralizer of the symmetric groups when they act instead on $(\mathbb{S}^{(n-1,1)})^{\otimes k}$ where $\mathbb{S}^{(n-1,1)}$ is the simple S_n-module indexed by $(n-1,1)$. It is well known that $V_n \cong \mathbb{S}^{(n-1,1)} \oplus \mathbb{S}^{(n)}$. Let proj denote the projection that maps V_n to $\mathbb{S}^{(n-1,1)}$. We have that

$$(\mathbb{S}^{(n-1,1)})^{\otimes k} \cong (\mathrm{proj} \circ \mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{Res}_{S_{n-1}}^{S_n})^k \mathbb{S}^{(n)} .$$

This decomposition into three operations gives rise to three families of quasi-partition algebras. Notably there is a half quasi-partition algebra that is the centralizer of the symmetric group S_{n-1} when it acts on $\mathrm{Res}_{S_{n-1}}^{S_n}(\mathbb{S}^{(n-1,1)})^{\otimes k}$ and another algebra that is the centralizer when S_n acts diagonally on $\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n}(\mathbb{S}^{(n-1,1)})^{\otimes k} \cong (\mathbb{S}^{(n-1,1)})^{\otimes k} \otimes V_n$.

This paper develops the quasi-partition algebras both as centralizer algebras (Theorem 3.9) and as projections of the partition algebra multiplied on the left and right by an idempotent (Equation (3.4)). The main results are the construction of a tower of quasi-partition algebras (Subsection 3.3) and an explicit description of bases of the simple modules of $\mathsf{QP}_k(n)$ (Section 3.1). The tower of algebras is used to relate the dimensions of the irreducibles of these families using the inclusions and projections (Theorem 3.10).

A motivation for introducing these algebras is to gain a better understanding of the representation theory of the symmetric group. An important insight from the reference [3] is that reduced Kronecker coefficients arise as multiplicities in the restriction and induction of simple partition algebra modules. In analogy, it can be shown that the coefficients occurring in the restriction/induction of simple quasi-partition algebras are also the reduced Kronecker coefficients. This is because there is a see-saw pair which relates these coefficients:

$$\begin{tikzcd}
\mathsf{QP}_{k+\ell}(n) \arrow[dr, dash] & & S_n \times S_n \arrow[dl, dash] \\
& \left(\mathbb{S}^{(n-1,1)}\right)^{\otimes k+\ell} & \\
\mathsf{QP}_k(n) \otimes \mathsf{QP}_\ell(n) \arrow[uu, hook] \arrow[ur, dash] & & S_n \arrow[uu, hook] \arrow[ul, dash]
\end{tikzcd}$$

This relationship implies that the reduced Kronecker coefficients, which are multiplicities of the restriction of an $S_n \times S_n$ module to S_n, are also the multiplicities of a simple $\mathsf{QP}_k(n) \otimes \mathsf{QP}_\ell(n)$ module in the restriction of a simple $\mathsf{QP}_{k+\ell}(n)$ module.

We conclude by describing the planar quasi-partition and half quasi-partition algebras. These algebras are isomorphic to centralizer algebras of the quantum group $U_q(\mathfrak{sl}_2)$ and have dimensions which are given by the Motzkin and Riordan numbers.

2 Preliminaries

The partition algebra was originally defined by Martin in [11]. All the results in this section are due to Martin and his collaborators, see [10] and references therein. For a nice survey on the partition algebra see [8].

For $k \in \mathbb{Z}_{>0}$, $x \in \mathbb{C}$, we let $\mathsf{P}_k(x)$ denote the complex vector space with bases given by all set partitions of $[k] \cup [\overline{k}] := \{1, 2, \ldots, k, \overline{1}, \overline{2}, \ldots, \overline{k}\}$. A part of a set partition is called a *block*. For a given block B, the set $B \cap [\overline{k}]$ denotes the subset of all barred elements of B are referred to as the *bottom* of B and the set $B \cap [k]$ denotes the subset of all unbarred elements of B and is referred to as the *top* of B. Notice that for a given set partition d on $2k$ elements, then $d \cap [k]$ and $d \cap [\overline{k}]$ are set partitions on k elements. We will let $\widehat{\mathsf{P}}_k$ denote the set of all set partitions of $\{1, 2, \ldots, k, \overline{1}, \overline{2}, \ldots, \overline{k}\}$.

Blocks with a single element will be referred to as *singletons*. Blocks containing at least one element from $[k]$ and one element from $[\overline{k}]$ will be called *propagating blocks*; all other blocks will be called *non-propagating blocks*.

For example,

$$d = \{\{1, 2, 4, \overline{2}, \overline{5}\}, \{3\}, \{5, 6, 7, \overline{3}, \overline{4}, \overline{6}, \overline{7}\}, \{8, \overline{8}\}, \{\overline{1}\}\},$$

is a set partition (for $k = 8$) with 5 blocks. The block $B = \{1, 2, 4, \overline{2}, \overline{5}\}$ is propagating. The block $\{3\}$ is a singleton.

A set partition in $\widehat{\mathsf{P}}_k$ can be represented by a *partition diagram* consisting of a frame with k distinguished points on the top and bottom boundaries, which we call vertices. We number the top vertices from left to right by $1, 2, \ldots, k$ and the bottom vertices similarly by $\overline{1}, \overline{2}, \ldots, \overline{k}$. We create a graph with connected components corresponding to the blocks of the set partition such that there is a path of edges between two vertices if they belong to the same block. A partition diagram is an equivalence class of graphs, where the equivalence is given by having the same connected components. In displaying the diagrams, we often omit the numbering on the vertices in the interest of keeping the diagrams less cluttered.

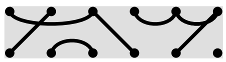
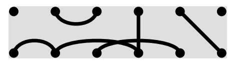
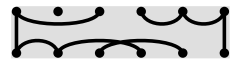
We will use the word *diagram* to refer to any element of $\widehat{\mathsf{P}}_k$ or equivalently its partition diagram. Examples of set partitions represented as diagrams are given in Example 2.1.

We define an internal product, $d_1 \cdot d_2$, of two diagrams d_1 and d_2 using the concatenation of d_1 above d_2, where we identify the bottom vertices of d_1 with the top vertices of d_2. If there are m connected components consisting only of middle vertices, then

$$d_1 \cdot d_2 = x^m d_3$$

where d_3 is the diagram with the middle vertices components removed.

Example 2.1. Consider the set partitions $d_1 = \{\{1, 3, \overline{4}\}, \{2, \overline{1}\}, \{4, 5, 6, \overline{5}\}, \{\overline{2}, \overline{3}\}, \{\overline{6}\}\}$ and $d_2 = \{\{1\}, \{2, 3\}, \{4, \overline{1}, \overline{2}, \overline{4}\}, \{5, \overline{6}\}, \{6\}, \{\overline{3}, \overline{5}\}\}$ in $\mathsf{P}_6(x)$. Which have the diagram representation given below. When we stack d_1 on top of d_2, there are two components containing only middle vertices, hence the coefficient x^2 in the product.

$d_1 =$  and $d_2 =$  then $d_1 d_2 = x^2$ .

Extending this by linearity defines a multiplication on $\mathsf{P}_k(x)$. With this product, $\mathsf{P}_k(x)$ becomes an associative algebra with unit of dimension $B(2k)$, the *Bell number* which enumerates the number of set partitions of a set with $2k$ elements.

A diagram is *planar* if the blocks of the diagram can be drawn so they do not intersect (to be clear, the blocks are not permitted to leave the bounding box). The span of the planar diagrams of size k is a subalgebra of the partition algebra $\mathsf{P}_k(x)$ which we denote $\mathsf{PP}_k(x)$. This subalgebra is of dimension equal to the Catalan number C_{2k}.

Both $\mathsf{P}_{k+1}(x)$ and $\mathsf{PP}_{k+1}(x)$ have a subalgebra spanned by the diagrams with $k+1$ and $\overline{k+1}$ are in the same block. These subalgebras will be denoted $\mathsf{P}_{k+\frac{1}{2}}(x)$ and $\mathsf{PP}_{k+\frac{1}{2}}(x)$ and have dimensions equal to $B(2k+1)$ and C_{2k+1} respectively. The planar partition algebras $\mathsf{PP}_k(x)$ and $\mathsf{PP}_{k+\frac{1}{2}}(x)$ are known to be isomorphic to the Temperley-Lieb algebras $\mathsf{TL}_{2k}(\sqrt{x})$ and $\mathsf{TL}_{2k+1}(\sqrt{x})$ respectively.

Given diagrams $d_1 \in \widehat{\mathsf{P}}_{k_1}$ and $d_2 \in \widehat{\mathsf{P}}_{k_2}$, we denote by $d_1 \otimes d_2$ the diagram in $\widehat{\mathsf{P}}_{k_1+k_2}$ obtained by placing d_2 to the right of d_1. Alternatively, in terms of set partition notation

$$d_1 \otimes d_2 = d_1 \cup \{\{b + k_1 : b \in B\} : B \in d_2\}\ .$$

This external product is extended linearly to a product of elements from $\mathsf{P}_{k_1}(x)$ and $\mathsf{P}_{k_2}(x)$ with the result being an element in $\mathsf{P}_{k_1+k_2}(x)$.

Let $\mathbf{1} := \{\{1, \overline{1}\}\}$ and $\mathsf{p} := \{\{1\}, \{\overline{1}\}\}$ denote special elements of $\hat{\mathsf{P}}_1$. For a fixed k, we denote the identity element of $\mathsf{P}_k(x)$ by $\mathbf{1}^{\otimes k}$ and the elements $\mathsf{p}_j := \mathbf{1}^{\otimes j-1} \otimes \mathsf{p} \otimes \mathbf{1}^{\otimes k-j} \in \hat{\mathsf{P}}_k$ for $1 \leq j \leq k$. For a complete presentation of the partition algebra see Theorem 1.11 in [8].

Let $V_n = \mathbb{C}^n$, the symmetric group acts on V_n via the permutation matrices

$$\sigma \cdot v_i = v_{\sigma(i)}, \quad \text{for } \sigma \in S_n.$$

Thus, S_n acts diagonally on a basis of simple tensors of $V_n^{\otimes k}$,

$$\sigma \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)}\ .$$

There is an action of $\mathsf{P}_r(x)$ on an element of $V_n^{\otimes k}$ which we do not explicitly use and so we do not state it here. Using this action, we have that for $n \geq 2k$, $k \in \mathbb{Z}_{\geq 0}$, $\mathsf{P}_k(n) \cong \mathrm{End}_{S_n}(V_n^{\otimes k})$ and for $n \geq 2k+1$ and $k \in \mathbb{Z}_{\geq 0}$, $\mathsf{P}_{k+\frac{1}{2}}(n) \cong \mathrm{End}_{S_{n-1}}(\mathrm{Res}^{S_n}_{S_{n-1}} V_n^{\otimes k})$. For details and proofs see [4, 8, 9].

The planar partition algebras (through the isomorphism with the Temperley-Lieb algebra) have a similar interpretation as centralizer of the quantized universal enveloping algebra when acting on the $2k$-fold tensor of the defining representation $V(1)^{\otimes 2k}$ [6].

3 Quasi-partition algebras

For $k \in \mathbb{Z}_{>0}$, the quasi-partition algebra $\mathsf{QP}_k(n)$ was introduced in [5] as the centralizer algebra $\mathrm{End}_{S_n}((\mathbb{S}^{(n-1,1)})^{\otimes k})$, where $\mathbb{S}^{(n-1,1)}$ is the irreducible representation of the symmetric group, S_n. In this section, we give a more general definition and introduce the half quasi-partition algebras, $\mathsf{QP}_{k+\frac{1}{2}}(x)$.

Let $k \in \mathbb{Z}_{\geq 0}$ and let J be any subset of $[k] = \{1, 2, \ldots, k\}$, we set $\mathsf{p}_\emptyset := \mathbf{1}^{\otimes k}$ and $\mathsf{p}_J := \prod_{j \in J} \mathsf{p}_j$. We define $\pi := \mathbf{1} - \frac{1}{x}\mathsf{p}$ and using tensor notation we define an idempotent $\pi^{\otimes k}$ in $\mathsf{P}_k(x)$ as follows

$$\pi^{\otimes k} := \left(\mathbf{1}^{\otimes k} - \frac{1}{x}\mathsf{p}_1\right)\left(\mathbf{1}^{\otimes k} - \frac{1}{x}\mathsf{p}_2\right)\cdots\left(\mathbf{1}^{\otimes k} - \frac{1}{x}\mathsf{p}_k\right) = \sum_{J \subseteq [k]} \frac{1}{(-x)^{|J|}}\mathsf{p}_J \,. \tag{3.1}$$

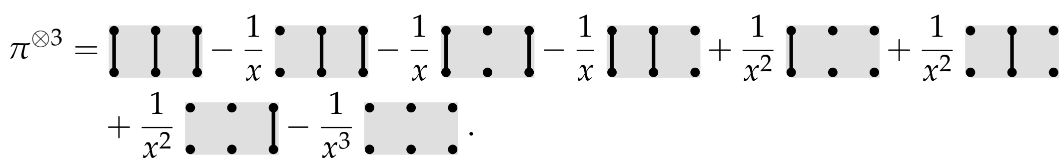
The corresponding idempotent in $\mathsf{P}_{k+\frac{1}{2}}(x) \subseteq \mathsf{P}_{k+1}(x)$ will be denoted by $\pi_{k+1}^{\otimes k} := \pi^{\otimes k} \otimes \mathbf{1}$ to indicate that it is contained in the larger algebra. For $k \in \mathbb{Z}_{\geq 0}$ and any diagram $d \in \widehat{\mathsf{P}}_k$, we define

$$\overline{d} = \pi^{\otimes k} d \pi^{\otimes k}.$$

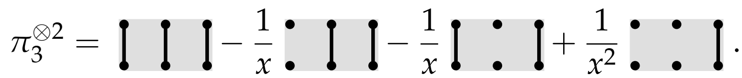
And similarly, for $d \in \widehat{\mathsf{P}}_{k+\frac{1}{2}}$, we define $\overline{d} = \pi_{k+1}^{\otimes k} d \pi_{k+1}^{\otimes k}$. For integers $k \geq 0$, and $d \in \mathsf{P}_{k+1}(x)$, we define

$$\tilde{d} = \pi_{k+1}^{\otimes k} d \pi_{k+1}^{\otimes k} \,.$$

Example 3.1. The idempotent in $\mathsf{P}_3(x)$ is $\pi^{\otimes 3} = \mathbf{1}^{\otimes 3} - \frac{1}{x}\mathsf{p}_1 - \frac{1}{x}\mathsf{p}_2 - \frac{1}{x}\mathsf{p}_3 + \frac{1}{x^2}\mathsf{p}_1\mathsf{p}_2 + \frac{1}{x^2}\mathsf{p}_1\mathsf{p}_3 + \frac{1}{x^2}\mathsf{p}_2\mathsf{p}_3 - \frac{1}{x^3}\mathsf{p}_1\mathsf{p}_2\mathsf{p}_3$. This element expressed using diagrams is



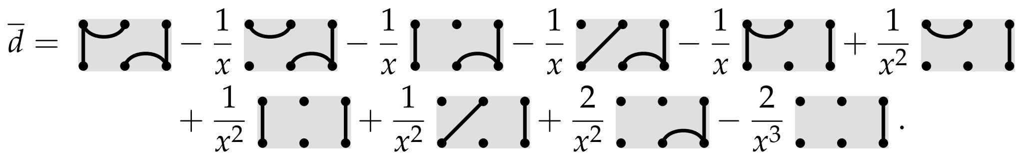
The idempotent in $\mathsf{P}_{2+\frac{1}{2}}(x)$ is $\pi_3^{\otimes 2} = \mathbf{1}^{\otimes 3} - \frac{1}{x}\mathsf{p}_1 - \frac{1}{x}\mathsf{p}_2 + \frac{1}{x^2}\mathsf{p}_1\mathsf{p}_2$ and this expression in terms of diagrams is



Lemma 3.2. *For $r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ if $d \in \widehat{\mathsf{P}}_r$ is a diagram with one or more singletons, then $\overline{d} = 0$.*

Let d be a diagram without singletons, we note that $\overline{d}$ is equal to a sum of elements d plus other terms with at least one singleton.

Example 3.3. For $d = \{\{1,2,\overline{1}\},\{3,\overline{2},\overline{3}\}\}$ a diagram in $\mathsf{P}_{2+\frac{1}{2}}(x)$, we compute directly:



For $r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we set $\widehat{\mathsf{D}}_r = \{d : d \in \widehat{\mathsf{P}}_r \text{ without singletons}\}$, and define

$$\mathsf{QP}_r(x) = \mathbb{C}(x)\text{-Span}\{\overline{d} \,|\, d \in \widehat{\mathsf{D}}_r\}. \tag{3.2}$$

If r is an integer, we call $\mathsf{QP}_r(x)$ the *quasi-partition algebra* and if r is half an integer, the *half quasi-partition algebra*.

Now consider the subalgebra of $\mathsf{P}_{k+1}(x)$,

$$\widetilde{\mathsf{QP}}_{k+1}(x) = \mathbb{C}(x)\text{-Span}\{\tilde{d} \,|\, d \in \mathsf{P}_{k+1}(x)\}\ .$$

We note that the basis of $\widetilde{\mathsf{QP}}_{k+1}(x)$ is:

$$\{\tilde{d} : d \in \widehat{\mathsf{P}}_{k+1} \text{ has no singletons in } [k] \cup [\overline{k}]\}\ . \tag{3.3}$$

The index set are the diagrams which have no singletons in the first k positions but that may have singletons in the last position.

Hence, the first step is the natural inclusion of $\mathsf{QP}_{k+\frac{1}{2}}(x)$ in $\widetilde{\mathsf{QP}}_{k+1}(x)$. It should be made clear that $\widetilde{\mathsf{QP}}_{k+1}(x)$ is larger than both $\mathsf{QP}_{k+\frac{1}{2}}(x)$ and $\mathsf{QP}_{k+1}(x)$ since, for instance, $\pi_{k+1}^{\otimes k}\mathsf{p}_{k+1} \in \widetilde{\mathsf{QP}}_{k+1}(x)$ but it is not an element of either $\mathsf{QP}_{k+\frac{1}{2}}(x)$ or $\mathsf{QP}_{k+1}(x)$.

Thus far we have introduced algebras in our tower so that for each $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \mathsf{QP}_k(x) &= \pi^{\otimes k}\mathsf{P}_k(x)\pi^{\otimes k}\ , \\ \mathsf{QP}_{k+\frac{1}{2}}(x) &= \pi_{k+1}^{\otimes k}\mathsf{P}_{k+\frac{1}{2}}(x)\pi_{k+1}^{\otimes k}\ , \\ \widetilde{\mathsf{QP}}_{k+1}(x) &= \pi_{k+1}^{\otimes k}\mathsf{P}_{k+1}(x)\pi_{k+1}^{\otimes k}\ . \end{aligned} \tag{3.4}$$

The second step is to explain how they are related. There is a projection from $\widetilde{\mathsf{QP}}_{k+1}(x)$ to $\mathsf{QP}_{k+1}(x)$ which, for each $d \in \widehat{\mathsf{P}}_{k+1}$, $\tilde{d} \in \widetilde{\mathsf{QP}}_{k+1}(x)$ is sent to $\overline{d} = (\mathbf{1}^{\otimes k+1} - \frac{1}{x}\mathsf{p}_{k+1})\tilde{d}(\mathbf{1}^{\otimes k+1} - \frac{1}{x}\mathsf{p}_{k+1}) \in \mathsf{QP}_{k+1}(x)$.

Therefore we have the following chain of inclusions and projections:

$$\mathsf{QP}_0(x) \hookrightarrow \mathsf{QP}_{\frac{1}{2}}(x) \subseteq \widetilde{\mathsf{QP}}_1(x) \twoheadrightarrow \mathsf{QP}_1(x) \hookrightarrow \mathsf{QP}_{1+\frac{1}{2}}(x) \subseteq \widetilde{\mathsf{QP}}_2(x) \twoheadrightarrow \mathsf{QP}_2(x) \hookrightarrow \cdots\ . \tag{3.5}$$

The dimensions of these algebras are determined by counting the elements in Equations (3.2) and (3.3). The dimension of $\mathsf{QP}_k(x)$, $\dim(\mathsf{QP}_k(x))$, is equal to the number of set partitions of $[k] \cup [\overline{k}]$ without blocks of size one and is equal to (see [5])

$$\dim(\mathsf{QP}_k(x)) = \sum_{j=1}^{2k}(-1)^{j-1}B(2k-j) + 1$$

and are every other term in [15] sequence A000296, while $\dim(\mathsf{QP}_{k+\frac{1}{2}}(x)) = B(2k)$ and is every other term in [15] sequence A000110. The sequence of dimensions of $\widetilde{\mathsf{QP}}_{k+1}(x)$ is given by [15] sequence A207978. Using the standard counting technique of inclusion-exclusion we deduce that

$$\dim(\widetilde{\mathsf{QP}}_{k+1}(x)) = \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} B(2k+2-s) \ .$$

Example 3.4. The sequence of dimensions of the algebras for $0 \leq k \leq 6$ is given in the table below.

k	0	1	2	3	4	5	6
$\dim(\mathsf{QP}_k(x))$	1	1	4	41	715	17722	580317
$\dim(\mathsf{QP}_{k+\frac{1}{2}}(x))$	1	2	15	203	4140	115975	4213597
$\dim(\widetilde{\mathsf{QP}}_{k+1}(x))$	2	7	67	1080	25287	794545	31858034

3.1 Representations of quasi-partition algebras

For this section, let $x = n \in \mathbb{Z}_{\geq 0}$ with $n \geq 2k$.

Recall that a block B is called *propagating* if it contains at least one element from each $[k]$ and $[\overline{k}]$. Define $V(k, m)$ to be the vector space spanned by the diagrams corresponding to set partitions of $[k] \cup [\overline{k}]$ with $\overline{m+1}, \ldots, \overline{k}$ in singleton blocks, and all other $\overline{j}$ are in propagating blocks where $\overline{j}$ is the only barred element in its block. We call these *(k, m)-diagrams*. For a diagram $d \in \hat{\mathsf{P}}_k$, let $p(d)$ denote the number of propagating blocks. A (k, m)-diagram is called *(k, m)-standard* if its propagating blocks $B_{\overline{1}}, \ldots, B_{\overline{m}}$ satisfy $\max(B_{\overline{j-1}} \cap [k]) < \max(B_{\overline{j}} \cap [k])$ for all $1 \leq j \leq m$.

For $0 \leq m \leq k$ and $\nu \vdash m$, a basis of the simple $\mathsf{P}_k(x)$ module $\Delta_k(\nu)$ is defined by

$$\mathcal{B}_k(\nu) = \{ d \otimes T \,|\, d \text{ is a } (k, m)\text{-standard and } T \text{ is a standard tableau of shape } \nu \}. \quad (3.6)$$

A diagram $d \in \mathsf{P}_k(x)$ acts on a basis element $d' \otimes T$ of $\Delta_k(\nu)$ by left multiplication,

$$d \cdot d' \otimes T = \begin{cases} dd' \otimes T & \text{if } p(dd') = m \\ 0 & \text{otherwise} \end{cases}, \quad (3.7)$$

in the case that $p(dd') = m$, we may factor $dd' = x^a d_1 \tau$ where d_1 is a (k, m)-standard diagram and $\tau \in S_m$. Hence, $dd' \otimes T = x^a d_1 \otimes \tau \cdot T$, where τ acts on T by permuting the entries of the tableau and $\tau \cdot T$ might not be standard, but can be written as a linear combination of standard tableaux using the Garnir straightening algorithm for Specht modules (see for instance [14]).

Following [7] the elements of $\mathcal{B}_k(\nu)$ can be combined into a single object that is represented by a set valued tableau.

Definition 3.5. For $k \in \mathbb{Z}_{\geq 0}$, $n \geq 2k$ and $0 \leq i \leq k$, let λ be a partition of n, a *$[k]$-set valued tableau T of shape λ* satisfies the following conditions:

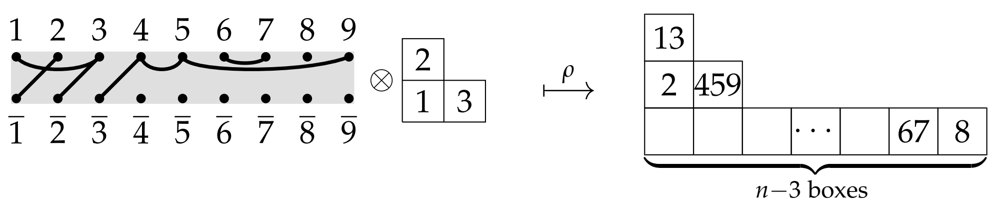
1. The sets filling the boxes of the Young diagram of λ form a set partition α of $[k]$, the sets in α are called blocks.

2. Every box in rows $\lambda_2, \dots, \lambda_\ell$ is filled with a block in α.

3. Boxes at the end of the first row of λ could contain blocks of α and, because of the condition that $n \geq 2k$, there are at least k empty boxes preceding the boxes containing sets.

Let $\mathcal{T}_k(\lambda)$ denote the set of all $[k]$-set valued tableaux of shape λ.

Example 3.6. Correspondence between a basis element $d \otimes T \in \Delta_9((2,1))$ and a $[9]$-set valued tableau of shape $(n-3,2,1)$.



We define $\mathcal{Q}_k(\nu)$ to be the set of nonzero $\pi^{\otimes k} d \otimes T$ (that is, d has no singletons in the top row), for $d \otimes T \in \mathcal{B}_k(\nu)$. Define QP_k^ν to be the $\mathbb{C}(x)$-Span of the elements in $\mathcal{Q}_k(\nu)$ for every $\nu \vdash m$ and $0 \leq m \leq k$.

Theorem 3.7. *Let $k \in \mathbb{Z}_{\geq 0}$, the set $\{\mathsf{QP}_k^\nu \,|\, \nu \vdash m \text{ where } 0 \leq m \leq k\}$ forms a complete set of mutually non-isomorphic simple modules for $\mathsf{QP}_k(x)$.*

Example 3.8. For $k = 2$ there are four simple modules of $\mathsf{QP}_2(n)$ and all are of dimension one and we display them using the correspondence with set valued tableaux:

$$\mathsf{QP}_2^{\emptyset} = \mathbb{C}\text{-Span}\left\{ \begin{array}{|c|c|c|c|c|}\hline \ & \ & \cdots & \ & 12 \\ \hline \end{array} - \frac{1}{n} \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \cdots & \ & 1 & 2 \\ \hline \end{array} \right\}$$

$$\mathsf{QP}_2^{(1)} = \mathbb{C}\text{-Span}\left\{ \begin{array}{|c|c|c|c|}\hline 12 \\ \hline \ & \ & \cdots & \ \\ \hline \end{array} - \frac{1}{n} \begin{array}{|c|c|c|c|}\hline 1 \\ \hline \ & \ & \cdots & 2 \\ \hline \end{array} - \frac{1}{n} \begin{array}{|c|c|c|c|}\hline 2 \\ \hline \ & \ & \cdots & 1 \\ \hline \end{array} \right\}$$

$$\mathsf{QP}_2^{(2)} = \mathbb{C}\text{-Span}\left\{ \begin{array}{|c|c|c|c|c|}\hline 1 & 2 \\ \hline \ & \ & \cdots & \ & \ \\ \hline \end{array} \right\} \qquad \mathsf{QP}_2^{(1,1)} = \mathbb{C}\text{-Span}\left\{ \begin{array}{|c|c|c|c|}\hline 2 \\ \hline 1 \\ \hline \ & \ & \cdots & \ \\ \hline \end{array} \right\}.$$

In [13] we give a similar description of the simple modules of the half partition algebras. From that construction, it is possible to give a similar description of the simple modules of $\mathsf{QP}_{k+\frac{1}{2}}(x)$. However we will see in the next section that $\mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathsf{P}_k(n-1)$.

3.2 Quasi-partition algebras as centralizers

Let $V_n = \mathbb{C}\text{-Span}\{v_1, v_2, \ldots, v_n\}$, then it is well known that $\mathbb{S}_{S_n}^{(n-1,1)} \cong \mathbb{C}\text{-Span}\{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n\}$ and $\mathbb{S}_{S_n}^{(n)} \cong \mathbb{C}\text{-Span}\{v_1 + v_2 + \cdots + v_n\}$ and that $V_n \cong \mathbb{S}_{S_n}^{(n-1,1)} \oplus \mathbb{S}_{S_n}^{(n)}$ as an S_n-module.

We refer the reader to [1, 2, 4] for the action of the elements $\mathsf{P}_k(n)$ when it acts on $V_n^{\otimes k}$. This action realizes the partition algebra as a centralizer algebra $\mathsf{P}_k(n) \cong \mathrm{End}_{S_n}(V_n^{\otimes k})$. In this section we state the corresponding realizations of the quasi-partition algebras as centralizer algebras.

Theorem 3.9. *For $n, k \in \mathbb{Z}_{>0}$, if $n \geq 2k$, then*

$$\mathsf{QP}_k(n) \cong \mathrm{End}_{S_n}\left(\left(\mathbb{S}_{S_n}^{(n-1,1)}\right)^{\otimes k}\right), \qquad \mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathrm{End}_{S_{n-1}}\left(\mathrm{Res}_{S_{n-1}}^{S_n}\left(\mathbb{S}_{S_n}^{(n-1,1)}\right)^{\otimes k}\right),$$

$$\textit{and} \qquad \widetilde{\mathsf{QP}}_{k+1}(n) \cong \mathrm{End}_{S_n}\left(\left(\mathbb{S}_{S_n}^{(n-1,1)}\right)^{\otimes k} \otimes V_n\right).$$

Since we have that

$$\mathrm{Res}_{S_{n-1}}^{S_n}\mathbb{S}_{S_n}^{(n-1,1)} \cong \mathbb{S}_{S_{n-1}}^{(n-2,1)} \oplus \mathbb{S}_{S_{n-1}}^{(n-1)} \cong V_{n-1}$$

it follows that $\mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathsf{P}_k(n-1)$.

3.3 Dimensions of irreducible modules and a Bratteli diagram

The interpretation of the quasi-partition algebras as centralizer algebras allows us to relate the dimensions of the irreducibles in the following recursive formulae.

Theorem 3.10. *Let $n \geq 2k+1$, then for $\mu \vdash n-1$ such that $|\overline{\mu}| < k$, then*

$$\dim\left(\mathsf{QP}^{\mu}_{k+\frac{1}{2}}(n)\right) = \sum_{\lambda \leftarrow \mu} \dim(\mathsf{QP}^{\lambda}_k(n)), \quad \dim(\widetilde{\mathsf{QP}}^{\lambda}_k(n)) = \sum_{\mu \rightarrow \lambda} \dim\left(\mathsf{QP}^{\mu}_{k+\frac{1}{2}}(n)\right), \quad (3.8)$$

$$\dim(\mathsf{QP}^{\lambda}_k(n)) = \dim(\widetilde{\mathsf{QP}}^{\lambda}_k(n)) - \dim(\mathsf{QP}^{\lambda}_{k-1}(n)). \quad (3.9)$$

Each row of the diagram on the left in Figure 1 displays partitions $\overline{\lambda}$ where λ is in the index set of the irreducible representations of the chain algebras from Theorem 3.10. The irreducible representations of $\mathsf{QP}_k(n)$ are displayed in red, $\mathsf{QP}_{k+\frac{1}{2}}(n)$ are displayed in blue, $\widetilde{\mathsf{QP}}_{k+1}(n)$ are displayed in green.

Let $\lambda \rightarrow \mu$ represent the relation that λ is obtained from μ by removing a cell. The relations between the irreducibles in the rows of the diagram are summarized as follows:

- (Equation (3.8)) Between the $\mathsf{QP}_k(n)$ and $\mathsf{QP}_{k+\frac{1}{2}}(n)$ rows there is an edge from $\overline{\lambda}$ to $\overline{\mu}$ if $\overline{\mu} = \overline{\lambda}$ or $\overline{\mu} \to \overline{\lambda}$ (alternatively, if $\mu \to \lambda$).
- (Equation (3.8)) Between the $\mathsf{QP}_{k+\frac{1}{2}}(n)$ and $\widetilde{\mathsf{QP}}_{k+1}(n)$ rows there is an edge from $\overline{\mu}$ to $\overline{\lambda}$ if $\overline{\lambda} = \overline{\mu}$ or $\overline{\lambda} \leftarrow \overline{\mu}$ (alternatively, these two conditions may be stated as 'if $\lambda \leftarrow \mu$').
- (Equation (3.9)) Between the $\widetilde{\mathsf{QP}}_{k+1}(n)$ and $\mathsf{QP}_k(n)$ rows there is an edge from $\overline{\lambda}$ to $\overline{\lambda}$ but the dimension of the irreducible $\overline{\lambda}$ is equal to the dimension of the irreducible $\overline{\lambda}$ minus the dimension of the irreducible $\overline{\lambda}$ at $k-1$.

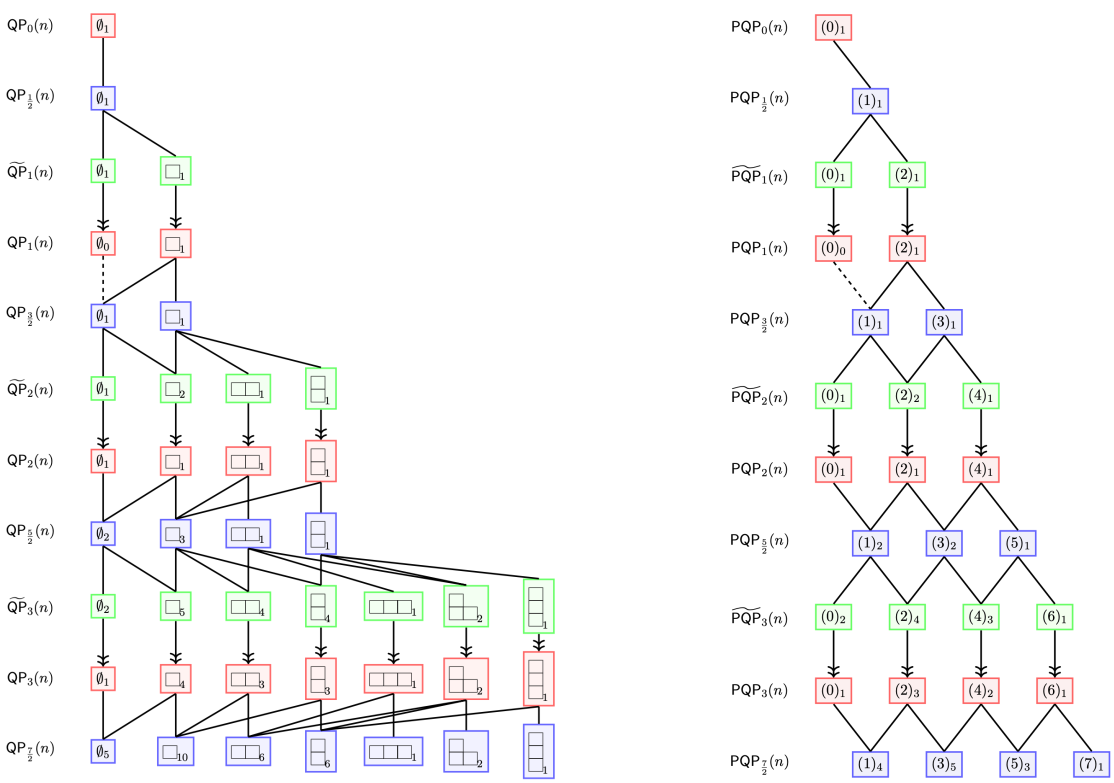


Figure 1: On the left, a Bratteli like-diagram showing the relations of the tower of algebras in Equation (3.5); and on the right the corresponding diagram for the planar counterpart. The subscripts within the colored boxes indicate the dimensions of the irreducibles.

The diagram is similar to a Bratteli diagram except that, because of the projection operation from $\widetilde{\mathsf{QP}}_{k+1}(n)$ to $\mathsf{QP}_{k+1}(n)$, the dimension is no longer the number of paths in the diagram and is instead something slightly more complex.

4 Planar quasi-partition algebras

We now proceed to develop a similar construction to the quasi-partition algebra by considering a subalgebra of the planar partition algebra. Due to space considerations and that we have provided details on the quasi-partition algebras already, our presentation of these algebras here will be briefer, but analogues of the results for the quasi-partition algebras in this setting can be shown using similar methods.

We define three subalgebras of $\mathsf{PP}_r(x)$ for $r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ by

$$\begin{aligned} \mathsf{PQP}_k(x) &= \pi^{\otimes k}\mathsf{PP}_k(x)\pi^{\otimes k}\,, \\ \mathsf{PQP}_{k+\frac{1}{2}}(x) &= \pi^{\otimes k}_{k+1}\mathsf{PP}_{k+\frac{1}{2}}(x)\pi^{\otimes k}_{k+1}\,, \\ \widetilde{\mathsf{PQP}}_{k+1}(x) &= \pi^{\otimes k}_{k+1}\mathsf{PP}_{k+1}(x)\pi^{\otimes k}_{k+1}\,. \end{aligned} \tag{4.1}$$

Example 4.1. The sequence of dimensions of $\mathsf{PQP}_k(x)$, $\mathsf{PQP}_{k+\frac{1}{2}}(x)$ and $\widetilde{\mathsf{PQP}}_{k+1}(x)$ for $0 \leq k \leq 13$ is

k	0	1	2	3	4	5	6
$\dim(\mathsf{PQP}_k(x))$	1	1	3	15	91	603	4213
$\dim(\mathsf{PQP}_{k+\frac{1}{2}}(x))$	1	2	9	51	323	2188	15511
$\dim(\widetilde{\mathsf{PQP}}_{k+1}(x))$	2	6	30	178	1158	7986	57346

The first row of this table is given by [15] sequence A099251 and is every other term of the Riordan numbers (A005043). The second row of this table is given by [15] sequence A026945 which is every other term of the Motzkin numbers (A001006). The third row of this table is every other term in the [15] sequence A005554 which are a sum of two successive Motzkin numbers.

The planar partition algebra $\mathsf{PP}_k(x)$ is isomorphic to the Temperley-Lieb algebra $TL_{2k}(\sqrt{x})$. For $q \in \mathbb{C}$, let $U_q(\mathfrak{sl}_2)$ denote the quantum group of the Lie algebra $\mathfrak{sl}_2$ and recall that its simple modules are classically denoted by $V(i)$, where i is a nonnegative integer. For example, $V(0)$ is the trivial representation, $V(1) \cong \mathbb{C}^2$ and $V(2)$ is the adjoint representation. It is well known that $TL_k \cong \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V(1)^{\otimes k})$ see [6] for more details. Using well known tensor rules, we have that $\mathbb{V} := V(1)^{\otimes 2} \cong V(0) \oplus V(2)$.

Theorem 4.2. *Let r be a nonzero integer and $0 \neq q \in \mathbb{C}$ is not a root of unity, and set $\mathbb{V} = V(0) \oplus V(2)$, then we have the following*

$$PQP_r((q+q^{-1})^2) \cong End_{U_q(\mathfrak{sl}_2)}(V(2)^{\otimes r}),$$

$$PQP_{r+\frac{1}{2}}((q+q^{-1})^2) \cong End_{U_q(\mathfrak{sl}_2)}(V(2)^{\otimes r} \otimes V(1)),$$

and

$$\widetilde{PQP}_{r+1}((q+q^{-1})^2) \cong End_{U_q(\mathfrak{sl}_2)}(V(2)^{\otimes r} \otimes \mathbb{V}).$$

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