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A CALCULUS OF FIGURATE NUMBERS AND FINITE DIFFERENCES 14

$$N_{s,r}(n) = \frac{1}{(s-1)!} S_{s,r}(n) + O\left(\frac{n^{s-1}}{x^{r-1} \log x}\right) + O(n^{s-2}e^{3x}).$$

The choice of x is still at our disposal. It is advantageous to take $x = \frac{1}{4} \log n$; with this choice of x we obtain

$$N_{s,r}(n) = \frac{n^{s-1}}{(s-1)!} \mathfrak{S}_{s,r}(n) + O\left(\frac{n^{s-1}}{\log^{r-1} n \log \log n}\right).$$

This is our first assertion and the second now follows immediately by Lemma 3.

5. The conjugate problem. In conclusion it may be of some interest to mention the conjugate problem of the problem discussed above. This consists in the determination of an asymptotic formula for the number $F(x) = F(x; r; k_1, \dots, k_s)$ of positive integers $n \leq x$ which are such that the highest common factor of any r of the integers $n+k_1, \dots, n+k_r$ is 1. An account of this problem (in a more general form) has previously appeared in this MONTHLY†.

† L. Mirsky, On coprime values taken by given polynomials, this Monthly, vol. 55, 1948, pp. 88-89.

A CALCULUS OF FIGURATE NUMBERS AND FINITE DIFFERENCES*

E. T. FRANKEL, Pittsburgh, Pennsylvania

- 1. Introduction. The purpose of this article is to derive an operational calculus in the field of finite differences, which is based on the properties of generalized figurate numbers. The procedures necessarily yield old as well as new results, but the emphasis will be on methods and results which are believed to be new.
- 2. Figurate Numbers are here generalized to include figurate numbers of negative order as well as the traditional series of positive order [1]. The more general numbers are derived by using 1, 0, 0, · · · as a generating series, and operating on it by repeated summation for figurate numbers of positive order and by repeated inverse summation for figurate numbers of negative order.

The generating series $1, 0, 0, \cdots$ is defined as the series of figurate numbers of zero order. In our notation, superscripts will represent orders of figurate numbers, positive or negative, and subscripts 0, 1, 2, · · · will represent the successive terms. Thus, the (r+1)th term in the series of figurate numbers of the nth order will be represented by F_r^n . The operation of repeated summation is, of course, the process of successive cumulative addition which connects the 1950 A CALCULUS

Tolumus, or tion, Thus, regardieships are true:

- (1)
- (2)
- (3)

Table 1 presents: to 7, and terms rang

	F_r^{-1}	F
0	1	1
1	-7	-5
2	21	15
3	-35	-20
4	. 35	15
5	-21	-6
6	7	1
7	-1	0

In standard com

$$(4) F_r^n = \binom{n+r}{r}$$

$$(5) \quad F_r^{-n} := \begin{pmatrix} -n \\ -n \end{pmatrix}$$

Conversely,

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^{*} Based on a presentation by the author at the Mathematics Colloquium of the University of Pittsburgh, April 24, 1947, through the courtesy of Professor J. S. Taylor, Head of the Department.

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$$\frac{1}{\log x} + O(n^{z-2}e^{3x}).$$

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$$\frac{n^{n-1}}{n \log \log n}$$
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include figurate numbers of positive order [1]. The more · as a generating series, and ite numbers of positive order umbers of negative order.

the series of figurate numbers represent orders of figurate $1, 2, \cdots$ will represent the e series of figurate numbers ration of repeated summation addition which connects the

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adjacent columns, or rows, of Pascal's triangle. Inverse summation, as the term implies, is the operation which reverses or "undoes" the operation of summation. Thus, regardless of the sign of the superscript, the following basic relationships are true:

(1)
$$F_r^{n+1} = F_0^n + F_1^n + F_2^n + \dots + F_r^n$$
(2)
$$F_r^{n+1} = F_r^n + F_{r-1}^{n+1}, \quad r > 0$$

(2)
$$F_r^{n+1} = F_r^n + F_{r-1}^{n+1}, \qquad r > 0$$

(3)
$$F_0^m = F_0^{-n} = 1.$$

Table 1 presents a short table of figurate numbers of orders ranging from -7to 7, and terms ranging from 0 to 7.

TABLE 1

7	Negative Order						Positive Order									
	F_r^{-7}	F_r^{-6}	F_r^{-5}	F_{r}^{-4}	F_r^{-3}	F_{τ}^{-2}	F_r^{-1}	F_r^0	$F_r^{\scriptscriptstyle m I}$	F_r^2	F_{i}	F_r^4	$F_r^{\mathbf{b}}$	F_r^g	F_{p}^{j}	
0	1 - 7	1	1	1	1	1	1	1	1	1	1	. 1	1	1	1	
2	21	-6 15	-5 10	~ −4	-3 3	-2 1	$-1 \\ 0$	0	1	2.	3	4	5	6	7 /- 2	417
3	-35	20	-10	-4	-1	0	0	0	1	3 4	6 10	10 20	15 35	21 56	28 /	
4 5	35 21	15 6	. 5	1	0	0	0	0	1	5	15	35	70	126	210	
6	7	1	-1	0	0	0	0	0	1	6	21 28	56	126	252	462	
7	-1	0	0	0	0	0	0	0	1	8	36	84 120	210 330	462 792	924 1716	

In standard combinatorial notation,

(4)
$$F_r^n = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \binom{r+1+n-2}{n-1} = F_{n-1}^{r+1}$$

(5)
$$F_r^{-n} = {\binom{-n+r-1}{r}} = {\binom{-(n-r+1)}{r}} = {(-1)}^r {\binom{n}{r}}$$

(6)
$$= (-1)^r \binom{n}{n-r} = (-1)^n F_{n-r}^{-n}.$$

Conversely,

(7)
$$\binom{a}{b} = F_b^{a-b+1} = (-1)^b F_b^{-a}$$

and

(8)
$${\binom{-a}{b}} = F_b^{-a-b+1} = F_b^{-(a+b-1)} = (-1)^b {\binom{a+b-1}{b}}.$$

In the notation of the calculus of finite differences $\Delta u_x = u_{x+1} - u_x$. Since, from (2)

(9)
$$F_{r+1}^{n-1} = F_{r+1}^{n} - F_{r}^{n},$$

$$\Delta F_r^n = F_{r+1}^{n-1}.$$

In particular,

$$\Delta F_0^n = F_1^n - F_0^n = F_1^{n-1}$$
$$\Delta^2 F_0^n = \Delta F_1^{n-1} = F_2^{n-2}$$

and, in general,

$$\Delta^t F_0^n = F_t^{n-t}.$$

Thus, the leading term and leading differences of the series of figurate numbers F_r^n , $(r=0, 1, 2, \cdots)$, are the figurate numbers F_0^n , F_1^{n-1} , F_2^{n-2} , etc., which, as in Table 1, form a downward diagonal to the left, starting with F_0^n . It may be observed that the numbers above such a diagonal do not appear at all when a series of figurate numbers is successively differenced by the repeated operation $\Delta u_x = u_{x+1} - u_x$. Apparently the operation of inverse summation, which we shall represent by the symbol S^{-1} (corresponding to S for summation), is a more general type of difference operation since it produces the numbers above the leading difference diagonal in addition to the leading differences themselves. This suggests the desirability of studying similar patterns derived from other series.

3. Summation and Inverse Summation of Series. When generalized figurate numbers are arranged as in Table 1 any two adjacent columns bear mutually inverse relationships to each other. The series at the right is the summation of the series at the left. The series at the left is the inverse summation of the series at the right. In effect, any one of the columns may be regarded as a generating series from which all of the others are derived.

Analogously, the successive equidistant values of a function u_x , $(x=0, 1, 2, \dots, r)$, may be treated as a generating series and operated upon by repeated summation and by repeated inverse summation. If u_r represents the general term of the generating series, Su_r the general term of the summation series, and $S^{-1}u_r$ the general term of the inverse summation series, the following relationships are true:

$$(12) S^0 u_r = u_r$$

(13)
$$Su_r = u_0 + u_1 + u_2 + \cdots + u_r = \sum_{t=0}^r u_t$$

(14)
$$S^{2}u_{r} = Su_{0} + Su_{1} + Su_{2} + \cdots + Su_{r} = \sum_{i=0}^{r} Su_{i}, \quad etc.$$

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$$S^{-1}u_r = u_r - \frac{1}{2}$$

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\-alogous to (11)

(9)

When r = 0, no operation we. Therefore, analo, (20)

Table 2 exhibits bas mation series and the

x	Inverse S
0 .	To To
1	5
* * *	
7	

mutually inverse operation Δ and its inverse

4. Parallel Leadin pose u, is a rational ir of equidistant terms universe summation. T is, it lists parallel lea

x	5-(0-1)4
0	tro
1	di
2	4,
n-1	d=-1
	d.
n+1	0
	0

fferences $\Delta u_x = u_{x+1} - u_x$. Since,

es of the series of figurate numbers F_0^n , F_1^{n-1} , F_2^{n-2} , etc., which, left, starting with F_0^n . It may be all do not appear at all when a need by the repeated operation erse summation, which we shall be S for summation), is a more oduces the numbers above the lead differences themselves. ar powers derived from other

ries. When generalized figurate djacent columns bear mutually the right is the summation of the inverse summation of the mns may be regarded as a genrived.

les of a function u_x , $(x=0, 1, and operated upon by repeated if <math>u_r$ represents the general term of the summation series, and a series, the following relation-

$$\sum_{t=0}^{r} u_t$$

$$5u_r = \sum_{t=0}^{r} Su_t, \quad etc.$$

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(15)
$$S^{-1}u_r = u_r - u_{r-1}, \qquad r > 0$$

(16)
$$S^{-2}u_r = S^{-1}u_r - S^{-1}u_{r-1}, \qquad r > 0, \quad etc.$$

In particular,

$$S^{-1}u_1 = u_1 - u_0 = \Delta u_0$$

(18)
$$S^{-2}u_2 = S^{-1}u_2 - S^{-1}u_1 = \Delta S^{-1}u_1 = \Delta^2 u_0.$$

Analogous to (11)

$$\Delta^t S^n u_0 = S^{n-t} u_t.$$

When r = 0, no operation of summation or of inverse summation is performed on u_0 . Therefore, analogous to (3),

$$S^{m}u_{0} = S^{-n}u_{0} = u_{0}.$$

Table 2 exhibits basic relationships between the generating series, the summation series and the inverse summation series. It should be noted that the

TABLE 2

x	Inverse Summation $S^{-1}u_x$	Generating Series S^0u_x	Summation Su _z
0	<i>u</i> ₀	u_0	uo
1	$u_1 - u_0$	u_1	$u_0 + u_1$
	• • •		
r	$u_r - u_{r-1}$	u_r	$u_0+u_1+\cdots+u_r$

mutually inverse operations S and S^{-1} are commutative. By contrast, the operation Δ and its inverse Δ^{-1} (or Σ) are not commutative [2].

4. Parallel Leading Differences and Intersecting Leading Differences. Suppose u_x is a rational integral function of the *n*th degree in x, and that the series of equidistant terms u_x , $(x=0, 1, \dots, n+1)$, has been operated on by repeated inverse summation. Table 3 summarizes certain results of such operations; that is, it lists parallel leading differences and intersecting leading differences of a

Table 3

x	$S^{-(n+1)}u_x$	$S^{-n}u_{\pm}$	$S^{-(n-1)}u_x$	 $S^{-2}u_x$	$S^{-1}u_x$	S^0x_2
0	u_0	u_0	u_0	 и0	u_0	uo
1	d_1				Δu_0	24 1
2	d_2			$\Delta^2 u_0$	-	24 2
* *, *						
n-1	d_{n-1}		$\Delta^{n-1}u_0$			u_{n-1}
n	d_n	$\Delta^n u_0$				u_n
n+1	0	$\Delta^n u_0$				u_{n+1}
	0					****

rational integral function of the nth degree in x. In column $S^{-n}u_x$ the nth differences, starting with $\Delta^n u_0$, are constant. In the successive columns starting with $S^{-(n+1)}u_x$ the (n+1)th and higher orders of differences vanish. In general, the terms d_1, d_2, \cdots, d_n in column $S^{-(n+1)}u_x$ are not zero. We designate them the parallel leading differences of the generating series, in order to distinguish them from $\Delta u_0, \Delta^2 u_0, \cdots, \Delta^n u_0$ which we call the intersecting leading differences. In this way, both sets of differences may be derived by repeated inverse summation of the (n+1)th order.

The rth parallel leading difference is

(21)
$$d_r = S^{-(n+1)}u_r.$$

The rth intersecting leading difference is

$$\Delta^r u_0 = S^{-r} u_r.$$

5. Criss-Cross Multiplication of Series. Having given two series a_x and b_x ($x=0, 1, \dots, r$), a third series $c_x=a_x*b_x$ can be obtained by a process of criss-cross multiplication shown in Table 4. The symbol of criss-cross multiplication

TABLE 4

x	a_z	b_x	$c_x = a_x * b_x$
0	<i>a</i> ₀	<i>b</i> ₀	$c_0 = a_0 b_0$ $c_1 = a_0 b_1 + a_1 b_0$
1	a_1	b_1	$c_1 = a_0 b_1 + a_1 b_0$
• • •	• • •		• • •
r	a_{r}	b_r	$c = a_r * b_r = \sum_{t=0}^r a_t b_{r-t}$

(*) is somewhat suggestive of the operation which it represents. The symbol is taken from the theory of the Laplace Transformation where it has an analogous significance in connection with certain continuous functions [3]. The operation of criss-cross multiplication obviously conforms to the commutative law; that is, $a_r * b_r = b_r * a_r$. It is easily seen that the operation of criss-cross multiplication conforms to the distributive law; that is, $a_r * (b_r + c_r) = a_r * b_r + a_r * c_r$. We shall now demonstrate that it also conforms to the associate law.

Proof:

$$(23) (a_r * b_r) * c_r = (a_0 * b_0)c_r + (a_1 * b_1)c_{r-1} + \cdots + (a_{r-1} * b_{r-1})c_1 + (a_r * b_r)c_0.$$

Multiplying out and regrouping the terms, we obtain

$$(24) (a_r * b_r) * c_r = a_0(b_r * c_r) + a_1(b_{r-1} * c_{r-1}) + \cdots + a_r(b_0 * c_0)$$

$$= a_r * (b_r * c_r).$$

(26)
$$(a_r * b_r) * c_r = b_0(a_r * c_r) + b_1(a_{r-1} * c_{r-1}) + \cdots + b_r(a_0 * c_0)$$

$$= b_r * (a_r * c_r).$$

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6. Theorems on

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In column $S^{-n}u_x$ the *n*th differecessive columns starting with erences vanish. In general, the t zero. We designate them the es, in order to distinguish them ersecting leading differences. In d by repeated inverse summa-

ving given two series a_x and b_x , e obtained by a process of crissabol of criss-cross multiplication

$$c_x = a_x * b_x$$

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$\cdots$$

$$c = a_r * b_r = \sum_{t=0}^r a_t b_{r-t}$$

thich it represents. The symbol is mation where it has an analogous rous functions [3]. The operation is to the commutative law; that ation of criss-cross multiplication $(b_r+c_r)=a_r*b_r+a_r*c_r$. We shall associate law.

$$(a_0*c_0)$$

6. Theorems on Criss-Cross Multiplication, Figurate Numbers, and Summation of Series.

THEOREM 1. Repeated summation of a series n times is equivalent to criss-cross multiplication of the series by the nth order of figurate numbers. That is,

(28)
$$S^{n}u_{r} = u_{r}*F_{r}^{n}$$

$$= u_{0}F_{r}^{n} + u_{1}F_{r-1}^{n} + \cdots + u_{r-1}F_{1}^{n} + u_{r}F_{0}^{n}$$

(30)
$$= u_0 \binom{n+r-1}{r} + u_1 \binom{n+r-2}{r-1} + \cdots + u_{r-1} \binom{n}{1} + u_r \binom{n-1}{0}$$

(31)
$$= u_0 \binom{n+r-1}{n-1} + u_1 \binom{n+r-2}{n-1} + \cdots + u_{r-1} \binom{n}{n-1} + u_r \binom{n-1}{n-1}.$$

An equivalent result has been given by Sheppard [4] with a reference to Elderton [5].

THEOREM 2. Repeated inverse summation of a series n times is equivalent to criss-cross multiplication of the series by the (-n)th order of figurate numbers. That is,

(32)
$$S^{-n}u_r = u_r * F_r^{-n}$$

(33) $= u_0 F_r^{-n} + u_1 F_{r-1}^{-n} + \cdots + u_{r-1} F_1^{-n} + u_r F_0^{-n}$

(34)
$$= u_0(-1)^r \binom{n}{r} + u_1(-1)^{r-1} \binom{n}{r-1} + \cdots + u_r \binom{n}{0}$$

(35)
$$= u_r - \binom{n}{1} u_{r-1} + \binom{n}{2} u_{r-2} - \cdots + (-1)^r \binom{n}{r} u_0.$$

An equivalent result has been given by Dwyer [6].

THEOREM 3. Criss-cross multiplication of figurate numbers of order r (positive or negative) by figurate numbers of order s (positive or negative) results in figurate numbers of order r+s.

(36) That is,
$$F_m^r * F_m^s = F_m^{r+s}$$
(37) Corollary:
$$F_m^r * F_m^s * F_m^{-t} \text{ etc.} = F_m^{r+s-t} \text{ etc.}$$

THEOREM 4. If $u_x(x=0, 1, 2, \dots, n)$ is a rational integral function of the nth degree in x, the rth parallel leading difference is

(38)
$$d_r = S^{-(n+1)} u_r = u_r * F_r^{-(n+1)}$$

and the rth intersecting leading difference is

(39)
$$\Delta^{r} u_{0} = S^{-r} u_{r} = u_{r} * F_{r}^{-r} = d_{r} * F_{r}^{n-r+1}.$$

THEOREM 5. Let ux and vx represent the successive terms of two series for $x=0, 1, 2, etc., and let u_{\tau} * v_{\tau}$ represent the general term of their criss-cross product. If the first series is operated on by repeated summation of the mth order, and the second series by repeated inverse summation of the nth order, then the general term of the criss-cross product of the two new series is

$$(40) S^{m} u_{r} * S^{-n} v_{r} = (F^{m}_{r} * u_{r}) * (F^{-n}_{r} * v_{r}) = F^{m-n}_{r} * (u_{r} * v_{r}) = S^{m-n} (u_{r} * v_{r}).$$

Theorem 6. If in the preceding formula n = m, it becomes

(41)
$$S^{m}u_{r}*S^{-m}v_{r} = F_{r}^{0}*(u_{r}*v_{r}) = u_{r}*v_{r}.$$

This last result is equivalent to Dwyer's Successive Cumulation Theorem [6], which he applied to the development of new techniques for the computation of moments of statistical frequency distributions.

Proofs of theorems 1 to 6 present no unusual features, and are omitted here.

7. Applications. To derive a general formula for the (r+1)th term in the kth order of repeated summation of the series $u_x(x=0, 1, \dots, r)$, a rational integral function of the nth degree in x:

In Table 3, any column operated upon by repeated summation and inverse summation will produce the entire field of numbers. Therefore, any column operated upon by criss-cross multiplication with figurate numbers will produce the entire field. If we choose the column which contains the parallel leading differences of the original series, that is, if we apply criss-cross multiplication by figurate numbers to the series u_0 , d_1 , d_2 , \cdots , d_n , then any term in any column can be represented by an expression of not more than n+1 terms involving the parallel leading differences and figurate numbers.

Applying the foregoing principles, the desired term is S^ku_r , and

(42)
$$S^k u_r = u_r * F_r^k = (d_r * F_r^{n+1}) * F_r^k = d_r * F_r^{k+n+1}$$

(42)
$$S^{k}u_{r} = u_{r}*F_{r}^{k} = (d_{r}*F_{r}^{n+1})*F_{r}^{k} = d_{r}*F_{r}^{k+n+1}$$

(43) $= u_{0}F_{r}^{k+n+1} + d_{1}F_{r-1}^{k+n+1} + \cdots + d_{n}F_{r-n}^{k+n+1}$

(44)
$$= u_0 \binom{k+n+r}{r} + d_1 \binom{k+n+r-1}{r-1} + \cdots + d_n \binom{k+r}{r-n}$$

(45)
$$= u_0 \binom{k+n+r}{k+n} + d_1 \binom{k+n+r-1}{k+n} + \cdots + d_n \binom{k+r}{k+n}.$$

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$$S^{m-n}(u_r*v_r).$$

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immation and inverse fore, any column operabers will produce the parallel leading differoss multiplication by y term in any column -1 terms involving the

 u_r , and

$$\cdots + d_n \binom{k+r}{r-n}$$

$$\cdots + d_n \binom{k+r}{k+n}.$$

The parallel leading differences d_1 , d_2 , etc. which are required in applying the above formulas may be obtained either by repeated inverse summation or by critic order -(n+1), that is, by $F^{-(n+1)}$. It may be observed that the combinatorial expressions represent figurate numbers of constant order (k+n+1). Such expressions are simpler to evaluate numerically than the binomial coefficients of constant order which appear in corresponding formulas in terms of the intersecting leading differences Δ , Δ^2 , Δ^3 , etc.

8. Numerical Examples. A few numerical examples will illustrate the application of the summation formula described in the preceding section.

Example (a): Derive an expression for the sum of the cubes of the first r odd numbers. Here, $u_x = (2x+1)^3$, the expression is of the third degree, and the parallel leading differences may be obtained by criss-cross multiplication by figurate numbers of the fourth negative order, that is, by F_x^{-4} , as follows:

x	$u_x = (2x+1)^3$	F_x^{-4}	$d_x = u_x * F_x^{-4}$
0	1	1	1
. 1	27	-4	23
2	125	6	23
3	343	-4	1

In this case, k=1; n=3; $u_0=1$; $d_1=23$; $d_2=23$; $d_3=1$; and the required summation ends with the term u_{r-1} . Substituting,

$$Su_{r-1} = {r+3 \choose 4} + 23{r+2 \choose 4} + 23{r+1 \choose 4} + {r \choose 4}$$

$$= \frac{(r+3)(r+2)(r+1)r}{24} + \frac{23(r+2)(r+1)r(r-1)}{24} + \frac{23(r+1)r(r-1)(r-2)}{24} + \frac{r(r-1)(r-2)(r-3)}{24}.$$

When this is multiplied out, it reduces to

$$1^3 + 3^3 + \cdots + (2r-1)^3 = r^2(2r^2-1).$$

Example (b): Find polynomial expressions for the rth term and the sum of r terms of the series -6, -3, 10, 39, 90, 169, \cdots .

Since, in this case, inspection does not reveal the degree of the general term, the parallel differences are formed by repeated inverse summation, as follows:

From the above difference table, since terminal zeros appear in column $S^{-4}u_x$, we assume that the desired polynomial is of the third degree. The term desired is u_{r-1} , since that is the rth term of the series u_0 , u_1 , u_2 , \cdots , u_{r-1} . Substituting in formula (45), k=0; n=3; $u_0=-6$; $d_1=21$; $d_2=-14$; $d_3=5$; and for the rth term of the series

$$S^{0}u_{r-1} = -6\binom{r+2}{3} + 21\binom{r+1}{3} - 14\binom{r}{3} + 5\binom{r-1}{3}$$
$$= r^{3} - r^{2} - r - 5.$$

Similarly, for the sum of the first r terms,

$$Su_{r-1} = -6\binom{r+3}{4} + 21\binom{r+2}{4} - 14\binom{r+1}{4} + 5\binom{r}{4}$$
$$= r(3r^3 + 2r^2 - 9r - 68)/12,$$

9. Vandermonde's Theorem. Using the notation $r^{(k)} = r(r-1) \cdot \cdot \cdot \cdot (r-k+1)$, the identity known as Vandermonde's Theorem or the factorial binomial theorem [7] may be written

(46)
$$(r+s)^{(k)} = r^{(k)} + {k \choose 1} r^{(k-1)} s + {k \choose 2} r^{(k-2)} s^{(2)} + \cdots$$

which is analogous in form to the binomial expansion

(47)
$$(r+s)^{k} = r^{k} + {k \choose 1} r^{k-1} s + {k \choose 2} r^{k-2} s^{2} + \cdots .$$

The factorial binomial theorem may be proved starting with Theorem 3, as follows:

(48)
$$F_{k}^{-r-s} = F_{k}^{-r} * F_{k}^{-s}$$

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$$(49) \quad (-1)^k \binom{r}{}$$

Eliminating

$$(50) \qquad \binom{r+s}{k}$$

Multiplying

(52)
$$(r+s)^{(k)}$$

10. A Proble urate numbers of ary, 1946, Vol... Prove that

The expression

$$(-1)^r \binom{r}{r}$$

The expression

Therefore, the o

$$(-1)^{m}$$

$S^{-3}u_x$	$S^{-4}u_{-}=d_{-}$
-6	-6
15	21
1	-14
6	5
. 6	0
6	0

eros appear in column $S^{-1}u_x$, and degree. The term desired u_1, u_2, \dots, u_{r-1} . Substituting u_1, u_2, \dots, u_{r-1} substituting u_1, u_2, \dots, u_{r-1} and for the rth

$$\binom{r}{3} + 5 \binom{r-1}{3}$$

$$\binom{r+1}{4} + 5\binom{r}{4}$$

 $r^{(k)} = r(r-1) \cdot \cdot \cdot (r-k+1),$ 1 or the factorial binomial

$$r^{(k-2)}s^{(2)} + \cdots$$

-

ion

$$-2s^2 + \cdots$$

tarting with Theorem 3, as

$$(49) \quad (-1)^{k} {r + s \choose r} = \overline{F_{k}}^{r} \overline{F_{0}}^{s} + \overline{F_{k-1}} \overline{F_{1}}^{s} + \dots + \overline{F_{0}}^{r} \overline{F_{k}}^{s}$$

$$= (-1)^{k} {r \choose k} {s \choose 0} + (-1)^{k-1} {r \choose k-1} (-1)^{1} {s \choose 1}$$

$$+ \dots + {r \choose 0} (-1)^{k} {s \choose k}.$$

Eliminating $(-1)^k$ from both sides of the equation,

(50)
$$\binom{r+s}{k} = \binom{r}{k} \binom{s}{0} + \binom{r}{k-1} \binom{s}{1} + \dots + \binom{r}{0} \binom{s}{k}, \text{ or }$$

(51)
$$\frac{(r+s)^{(k)}}{k!} = \frac{r^{(k)}s^{(0)}}{k!0!} + \frac{r^{(k-1)}s^{(1)}}{(k-1)!1!} + \cdots + \frac{r^{(0)}s^{(k)}}{0!k!} .$$

Multiplying both sides by k! we obtain

$$(52) \quad (r+s)^{(k)} = r^{(k)}s^{(0)} + \binom{k}{1}r^{(k-1)}s^{(1)} + \binom{k}{2}r^{(k-2)}s^{(2)} + \cdots + r^{(0)}s^{(k)}.$$

10. A Problem in Binomial Coefficients. The properties of generalized figurate numbers can be applied to solve Problem 4189 in the Monthly, February, 1946, Vol. 53, No. 2, page 103. Prove that

$$\sum_{r=0}^{m} (-1)^r \binom{m+k+a}{r+k+a} \binom{r+k}{k} = \binom{m+a-1}{m}, \qquad a \ge 0.$$

The expression

$$(-1)^{r} \binom{m+k+a}{r+k+a} = (-1)^{r} \binom{m+k+a}{m-r} = (-1)^{m} F_{m-r}^{-(m+k+a)}.$$

The expression

$$\binom{r+k}{k} = \binom{r+\frac{5}{4}k}{r} = F_r^{k+1}.$$

Therefore, the original expression is equivalent to

$$(-1)^{m} \cdot \sum_{r=0}^{m} F_{m-r}^{-(m+k+a)} \cdot F_{r}^{k+1} = (-1)^{m} F_{m}^{-(m+k+a)} * F_{m}^{k+1}$$
$$= (-1)^{m} F_{m}^{-(m+a-1)} = {m+a-1 \choose m}.$$

The result is independent of the restriction $a \ge 0$. The above proof may be compared with the proof in the Monthly, June–July, 1947, page 347.

11. Summation of Product Series. Theorem 6 can be applied to find the sum of a product series such as u_rv_0 , $u_{r-1}v_1$, \cdots , u_0v_r , where v_x is a polynomial of degree n. The sum of the series is evidently

(53)
$$u_r * v_r = u_r v_0 + u_{r-1} v_1 + \cdots + u_0 v_r.$$

Introducing figurate numbers of the (n+1)th order, positive and negative,

(54)
$$u_r * v_r = (F_r^{n+1} * u_r) * (F_r^{-(n+1)} * v_r)$$

(55)
$$= (F_r^{n+1} * u_r) * d_r.$$

Changing back from criss-cross multiplication by figurate numbers to repeated summation,

(56)
$$u_r * v_r = S^{n+1} u_r * d_r,$$

When the above is expanded, since the function v_x is of the *n*th degree, the terms involving d_{n+1} , d_{n+2} , etc., will vanish. Therefore, the desired sum

(57)
$$u_r * v_r = d_0 S^{n+1} u_r + d_1 S^{n+1} u_{r-1} + \cdots + d_n S^{n+1} u_{r-n},$$

where

$$d_t = F_t^{-(n+1)} * v_t,$$
 $(t = 0, 1, \dots, n).$

This result is equivalent to one given by Dwyer [6] for the evaluation of $\sum_{0}^{k} u_{a+x}v_{a+x}$.

12. Parallel Leading Differences of Powers of Integers. Although Dwyer used another type of notation, the differencing operation which he applied is identical with the operation of inverse summation as defined in this article. Carrying out his objective of simplifying the computation of statistical moments, Dwyer derived general expressions and recurrence formulas for what, in our terminology, are the parallel leading differences of the powers of integers, which are analogous to the "differences of zero" of actuarial literature. For the powers r^n , $(r = 0, 1, 2, \cdots)$, since the function is of the nth degree, Theorem 4 is applicable, and we have

(58)
$$d_r = S^{-(n+1)} r^n = F_r^{-(n+1)} * r^n.$$

An equivalent formula, in another connection has been given by Wall [8]. Values of d_1, d_2, \cdots are given in Table 5 from n=1 to n=11.

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7	n = 1	n=2 n
1	1	1
2		1
3		-
4		
5		
6		
7		
8		
9		
10		
11		
Totals	1 1!	- 2 2!

These values of for powers of in terms of factoria a weighted avera

 Max Sasuly, titled "Figurate Nun torical references.

2. Charles Jord New York, 1947, pa 2a. J. F. Steffer

3. Gustav Doet tions, New York, 19

4. W. F. Shepp. London Mathematic

5. W. P. Elder 6. P. S. Dwyer.

matical Statistics, V

7. L. M. Milne

8. H. S. Wall, Monthly, February above proof may be come, page 347.

be applied to find the sum here v_x is a polynomial of

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positive and negative,

rate numbers to repeated

th degree, the terms

 $d_n S^{n+1} u_{r-n}$

$$(t=0, 1, \cdots, n).$$

6] for the evaluation of

ttegers. Although Dwyer tion which he applied is is defined in this article. Itation of statistical morence formulas for what, of the powers of integers, tuarial literature. For the in the degree, Theorem 4

been given by Wall [8]. 1 to n=11.

TABLE 5 n = 10n = 11.1 152637~ Totals 9! 10! 11! 1! 2! 3! 4! 5! 6! 8!

These values of d may be substituted in Formula 45 to derive expressions for powers of integers and their sums in terms of figurate numbers or in terms of factorial products. From a statistical point of view, r^n turns out to be a weighted average of factorial products, in which the d's are the weights. Thus,

(59)
$$r^{n} = \frac{d_{1}r^{(n)} + d_{2}(r+1)^{(n)} + \cdots + d_{n}(r+n-1)^{(n)}}{d_{1} + d_{2} + \cdots + d_{n}}.$$

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8. H. S. Wall, "On the Expansion of an Integral of Stieltjes," The American Mathematical Monthly, February, 1932, page 98.