Optional Survey

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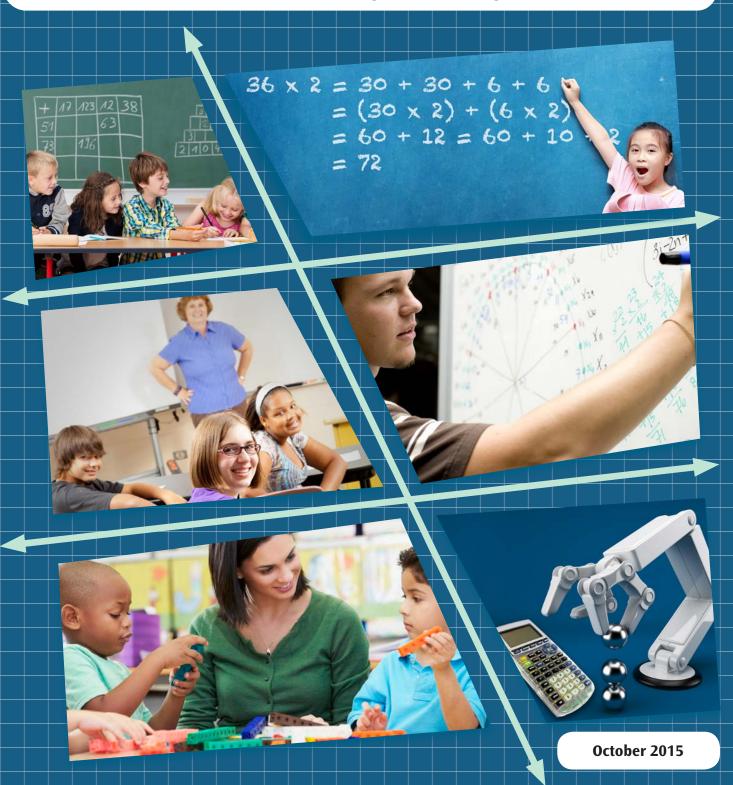**https://www.surveymonkey.com/r/CC_Doc_Use_Survey**

Executive Summary

Mathematics Framework

for California Public Schools

Kindergarten Through Grade Twelve

$$36 \times 2 = 30 + 30 + 6 + 6$$
$$= (30 \times 2) + (6 \times 2)$$
$$= 60 + 12 = 60 + 10 + 2$$
$$= 72$$

October 2015

What's in the *Mathematics Framework* for me?

▶ **Classroom teachers and other educators of all grade levels** will find explanations of the three major principles underlying mathematics instruction: focus, coherence, and rigor. They will also find descriptions and examples of the mathematics standards, including the Standards for Mathematical Practice, and guidance on instruction and learning aligned with the mathematics standards.

▶ **Coaches/mentors and professional learning providers** will find information about the vision for focused, coherent, and rigorous mathematics instruction and learning. In addition, they will find chapters on instructional strategies and implementing high-quality mathematics instruction, including discussion of effective professional learning.

▶ **Site and district administrators** will find information about the vision for focused, coherent, and rigorous mathematics instruction and learning, the systemic supports necessary to ensure all students succeed in mathematics, and guidance on meeting the increased language demands of the mathematics standards. They will also find criteria for evaluating instructional materials at the local level.

▶ **University faculty in teacher preparation programs** will find information about the standards and the vision for focused, coherent, and rigorous mathematics instruction and learning that prospective teachers and in-service teachers are expected to address.

▶ **Parents and communities** will find grade-level and course-level expectations and examples of students' mathematical work, as well as explanations of the Standards for Mathematical Practice.

▶ **Curriculum developers** will find expectations for instructional materials and models of appropriate instructional approaches and assessment practices.

The complete *Mathematics Framework* is available online at
http://www.cde.ca.gov/ci/ma/cf/

Acknowledgments

What do teachers and administrators need to know about the new mathematics standards? How will the standards impact teaching and learning?

The *Mathematics Framework for California Public Schools* (*Mathematics Framework*) answers these questions and more. This executive summary highlights essential information and guidance in the *Mathematics Framework* and is intended to introduce the reader to the wealth of information and support for teachers, administrators, and parents/guardians it provides.

The purpose of the *Mathematics Framework* is to support implementation of California's standards for mathematics. Grade-level and course-level chapters provide examples of what standards-based instruction and learning look like; other chapters focus on universal access, instructional strategies, assessment, and supporting high-quality instruction.

Understanding the California Common Core State Standards for Mathematics

The new California Common Core State Standards for Mathematics (CA CCSSM) define what students should understand and be able to do in the study of mathematics. The standards are designed to prepare students for college, careers, and civic life—developing mathematically competent individuals who can use mathematics in their personal lives, at work, and as a means for comprehending and influencing the world in which they will live after they graduate from high school.

The standards are based on three major principles: **focus**, **coherence**, and **rigor**, which are the foundations for effective curricula and instruction. These principles are the basis of how students acquire conceptual understanding, procedural skill and fluency, and the ability to apply mathematics to solve problems.

Focus is necessary so that students have sufficient time to think about, practice, and integrate concepts that are new at the grade. Teachers encourage rich classroom discussion and interaction to support students' broader mathematical understanding.

Coherence arises from mathematical connections. The standards are based on a progression of learning and are designed to help administrators and teachers connect learning within and across grades.

> ### Major Principles of the California Common Core State Standards for Mathematics
>
> **FOCUS** ▶ Place strong emphasis where the standards focus.
>
> **COHERENCE** ▶ Think across grades, and link to major topics in each grade.
>
> **RIGOR** ▶ In major topics, pursue with equal intensity:
> - conceptual understanding;
> - procedural skills and fluency;
> - application.

Rigor requires that conceptual understanding, procedural skill and fluency, and application be approached with equal intensity. The word understand in the standards sets clear expectations for conceptual understanding. The standards are also explicit where fluency is expected. Teachers help students make steady progress toward procedural skill and computational fluency. Teachers also support students' ability to access concepts from a number of different perspectives and to apply mathematics to solve real-world problems.

The CA CCSSM include **Standards for Mathematical Practice** (the same at each grade level) and **Standards for Mathematical Content** (different at each grade). These two types of standards support instruction with an equal focus on developing students' ability to engage in the practice standards and on developing conceptual understanding of and procedural fluency in the content standards. The standards for each grade and course level are included in the *Mathematics Framework*, and sample problems and explanations in each grade-level and course-level chapter identify connections between the two types of standards.

The **Standards for Mathematical Practice (MP)** describe attributes of mathematically proficient students who reason mathematically and apply mathematics to solve real-world problems. The *Mathematics Framework* offers grade- and course-level classroom explanations and examples of how teachers might support student learning of these standards. For example, teachers can use questioning strategies to help students make sense of problems and explain their thinking. Teachers might ask, "What do you notice?" (**MP.1**), "What is the relationship of the quantities?" (**MP.2**), or "What math drawing or diagram could you make and label to represent the problem?" (**MP.4**)

Standards for Mathematical Practice (MP)

MP.1	Make sense of problems and persevere in solving them.
MP.2	Reason abstractly and quantitatively.
MP.3	Construct viable arguments and critique the reasoning of others.
MP.4	Model with mathematics.
MP.5	Use appropriate tools strategically.
MP.6	Attend to precision.
MP.7	Look for and make use of structure.
MP.8	Look for and express regularity in repeated reasoning.

The **Standards for Mathematical Content** were built on progressions of topics across a number of grade levels and are organized by domains within each grade level for kindergarten through grade eight. The standards for higher mathematics were first developed by conceptual categories and then organized into courses. The following table illustrates how the Mathematical Domains and Conceptual Categories are distributed across the K–12 mathematical content standards.

Grade	K	1	2	3	4	5	6	7	8	Higher Mathematics Conceptual Categories	
K-8 Domains	Counting and Cardinality (CC)						Ratios and Proportional Relationships (RP)	Functions (F)	Functions (F)	Functions (F)	**Modeling (★)**
	Operations and Algebraic Thinking (OA)						Expression and Equations (EE)			Algebra (A)	
	Number and Operations in Base Ten (NBT)						The Number System (NS)			Number and Quantity (N)	
				Number and Operations - Fractions (NF)							
	Measurement and Data (MD)						Statistics and Probability (SP)			Statistics and Probability (SP)	
	Geometry (G)						Geometry (G)			Geometry (G)	

Kindergarten Through Grade Eight (K–8) Chapters

The *Mathematics Framework* includes grade-level chapters to help teachers understand the standards, focus instruction on the major work (or emphases) at the grade level, and connect learning within and across grades. Examples of standards-based problems help teachers understand the math content called for in the standards and highlight various ways students might approach, visualize, think about, and solve problems. Each chapter includes examples of how to engage students in activities that connect the Standards for Mathematical Content with the Standards for Mathematical Practice and contains a full-page sample classroom activity, "Connecting to the Standards for Mathematical Practice." The content standards are included at the end of each grade-level chapter and also embedded throughout the chapter narrative for easy reference. The section titled "Essential Learning for the Next Grade" reminds teachers of the important knowledge, skills, and understanding that students will need to succeed in future grades. (See the grade-level chapters.)

A Note Regarding Transitional Kindergarten

The *Framework* includes guidance for teachers and administrators regarding instruction in transitional kindergarten (TK). Unlike kindergarten, TK does not have grade-specific content standards. To facilitate district-level discussions on a modified mathematics curriculum for TK, the *Framework* includes tables that highlight connections between particular California Preschool Learning Foundations and the kindergarten standards from the CA CCSSM.

Higher Mathematics Chapters

The *Mathematics Framework* includes higher mathematics chapters organized into courses based on two "Pathways" (Traditional Pathway and Integrated Pathway) that each represent three years of instruction. The Traditional Pathway is more aligned with traditional higher mathematics courses, and the Integrated Pathway is a new set of courses that blends standards from all conceptual categories in an integrated way. The *Mathematics Framework* identifies the standards in each course and provides teacher-friendly narratives that describe important nuances of each standard and new learning expectations, such as the use of transformations to prove geometric theorems in Geometry. Statistics and Probability standards are included in each course, building on students' work with formal statistics that begins in grade six. Mathematical Modeling is also featured prominently in each higher mathematics course to help teachers understand how this conceptual category overlaps with all the others. (See Appendix B: Mathematical Modeling for more detail.)

Each course is described in its own chapter in the *Mathematics Framework* and includes an overview of the course followed by a detailed description of the mathematics content standards that are included in the course. Throughout, there are teacher-friendly examples that illustrate the mathematical ideas and connect the mathematical practice standards to the content standards. Standards that might be new to many secondary teachers are explained more fully than standards that have appeared in the curriculum prior to the adoption of the CA CCSSM.

Students who successfully complete the three higher mathematics courses may be interested in taking a "Fourth Year" course described in the *Mathematics Framework*, Statistics and Probability or Precalculus, or another higher mathematics course. Decisions regarding which mathematics course students take, including acceleration decisions, and which courses a school offers should be made thoughtfully. To assist teachers and districts with these crucial decisions, the *Mathematics Framework* includes two appendix documents: Appendix D: Course Placement and Sequences; and Appendix F: Higher Mathematics Pathways Standards Chart.

Kindergarten

What does the *Mathematics Framework* say about kindergarten instruction and learning?

Kindergarten students develop an understanding of the relationship between numbers, quantities, and counting. Instructional time should focus on two critical areas: (1) representing and comparing whole numbers, initially with sets of objects; and (2) describing shapes and space. More learning time in kindergarten should be devoted to numbers than to other topics. Kindergarten students also work toward fluency with addition and subtraction of whole numbers within 5.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table K-1 Kindergarten Cluster-Level Emphases in the *Mathematics Framework* highlights the content emphases in the grade.

Kindergarten students are introduced to addition and subtraction with small numbers. Number pairs that total 10 are foundational for students' ability to work fluently within base-ten numbers and operations. Students also learn the teen numbers are composed of 10 ones and some more ones—an important foundational concept that sets the stage for place-value understanding and mental calculations.

The grade-level chapters in the *Mathematics Framework* include the standards and also provide explanations and sample problems to help teachers understand the standards and teach the content. The following examples from the *Mathematics Framework* focus on just two of the Kindergarten standards.

What do the standards say?

Standard K.OA.4
"For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation."
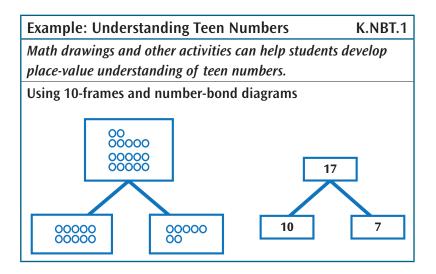
Examples: Tools and Strategies for Making a Ten	K.0A.4
A student places 3 objects on a 10-frame and then determines how many more are needed to "make a ten." Students may use electronic versions of 10-frames to develop this skill (MP.5)	

A student snaps 10 cubes together to make a pretend train.

- The student breaks the train into two parts. He or she identifies how many cubes are in each part and records the associated equation (10 = _____ + _____).

- The student breaks the train into two parts. He or she counts how many cubes are in one part and determines how many are in the other part without directly counting that part. Then the student records the associated equation (if the counted part has 4 cubes, the equation would be 10 = 4 + _____).

- The student covers up part of the train, without counting the covered part. He or she counts the cubes that are showing and determines how many are covered up. Then the student records the associated equation (if the counted part has 7 cubes, the equation would be 10 = 7 + _____) [**MP.8**].

- The student tosses 10 two-color counters on the table and records how many of each color are facing up (**MP.8**).

Standard K.NBT.1

"Compose and decompose numbers from 11 to 19 into 10 ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of 10 ones and one, two, three, four, five, six, seven, eight, or nine ones."

Example: Understanding Teen Numbers K.NBT.1
Math drawings and other activities can help students develop place-value understanding of teen numbers.
Using 10-frames and number-bond diagrams

What support does the *Mathematics Framework* provide?

Teachers help students lay a foundation for understanding the base-ten system by drawing special attention to the number 10. These examples from the *Mathematics Framework* provide teachers with visual representations students might use to "find the number that makes 10" (K.OA.4) or decompose a number into "10 ones and some further ones" (K.NBT.1). Students can also record the decomposition with an equation, such as 17 = 10 + 7.

Students' understanding of numbers progresses from the concrete to the abstract. Problems that encourage students to use visual models as tools for learning also support connections between the Standards for Mathematical Content (K.OA.4 and K.NBT.1) and the Standards for Mathematical Practice, "Use appropriate tools strategically (MP.5)" and "Look for and make use of structure (MP.7)." To reinforce student understanding teachers might ask, "Why was it helpful to use the 10-frame when you worked with number pairs that make 10?" or "What pattern do you notice in the 'teen' numbers?" (See Table K-2 Standards for Mathematical Practice—Explanations and Examples for Kindergarten.)

Grade One

What does the *Mathematics Framework* say about grade one instruction and learning?

In grade one, students develop the concept of place value by viewing 10 ones as a unit called a *ten*, an essential concept in the base-ten number system. In grade one, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole-number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of and composing and decomposing geometric shapes. Students also work toward fluency in addition and subtraction with whole numbers within 10.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 1-1 Grade One Cluster-Level Emphases in the *Mathematics Framework* highlights the content emphases in the grade.

Students in first grade represent word problems (e.g., using objects, drawings, and equations) and relate strategies to a written method to solve various types of addition and subtraction problems. (See Tables 1-4 and 1-5 Grade-One Addition and Subtraction Problem Types.)

The grade-level chapters in the *Mathematics Framework* include the standards and also provide explanations and sample problems to help teachers understand the standards and teach the content. The following examples from the *Mathematics Framework* focus on just one of the Grade One standards.

What does the standard say?

Standard 1.OA.1

"Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem."

Focus, Coherence, and Rigor
Comparing numbers and groups in kindergarten will progress to comparing addition and subtraction situations in grade one. For example, "Which is more?" or "Which is less?" will progress to "How many more?" or "How many less?"

Examples: Strategies for Solving "Compare" Problems	1.OA.1

Abel has 9 balls. Susan has 3 balls. How many more balls does Abel have than Susan?

Students use objects to represent the two sets of balls and compare them.

Teachers may also ask the related question, "How many fewer balls does Susan have than Abel?"

9 Balls

3 Balls	?? Balls

Students also use comparison bars. Rather than representing the actual objects with malnipulatives or drawings, they use the numbers in the problem to represent the quantities.

OOOOOOOOO

OOO	???

9 Balls

3 Balls	? Balls

What support does the *Mathematics Framework* provide?

First grade students add and subtract within 20 to solve various types of word problems. "Compare" problems are introduced in first grade. These examples from the *Mathematics Framework* provide teachers with visual representations students might use to understand and solve compare problems. Comparison bars can help students move away from representing all objects in a problem to representing objects solely with numbers. Although most adults know to solve "compare" problems with subtraction, students often represent these problems as unknown-addend problems (e.g., $3 + ? = 9$). Understanding subtraction as an unknown-addend addition problem (e.g., $9 - 3 = ?$ can be written as $3 + ? = 9$ and thought of as "How much more do I need to add to 3 to get 9?") is an essential understanding students will need in middle school to extend arithmetic to negative rational numbers.

Problems that encourage students to discuss and explain their thinking support connections between the Standards for Mathematical Content (1.OA.1) and the Standards for Mathematical Practice, "Reason abstractly and quantitatively (MP.2)" and "Look for and make use of structure (MP.7)." Students make use of structure when they work subtraction as an unknown-addend problem. To reinforce student understanding, teachers might ask students, "How do you know?" or "What do you notice when____?" (See Table 1-2 Standards for Mathematical Practice—Explanations and Examples for Grade One.)

Grade Two

What does the *Mathematics Framework* say about grade two instruction and learning?

In grade two, students continue to build upon their mathematical foundation as they focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes. Students also work toward fluency with addition and subtraction within 20 using mental strategies and within 100 using strategies based on place value, properties of operations, and the relationship between addition and subtraction. They know from memory all sums of two one-digit numbers.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 2-1 Grade-Two Cluster-Level Emphases in the *Mathematics Framework* highlights the content emphases in the grade.
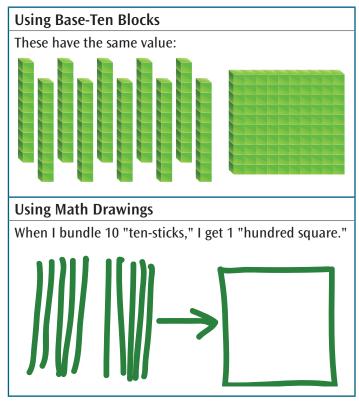
Place-value understanding is central to multi-digit computations. In grade two, students learn three-digit numbers represent amounts of hundreds, tens, and ones, and they apply this understanding to develop and use methods to add and subtract within 1000.

The grade-level chapters in the *Mathematics Framework* include the standards and also provide explanations and sample problems to help teachers understand the standards and teach the content. The following examples from the *Mathematics Framework* focus on just one of the Grade Two standards.

What does the standard say?

Standard 2NBT.7

"Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds."

Using Base-Ten Blocks

These have the same value:

Using Math Drawings

When I bundle 10 "ten-sticks," I get 1 "hundred square."

Adapted from Fuson and Beckmann 2013 and UA Progressions Documents 2012b.

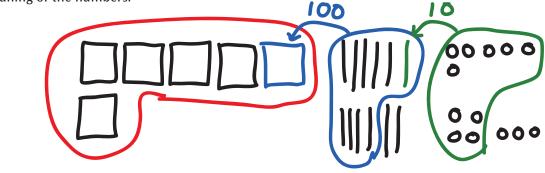**What support does the *Mathematics Framework* provide?**

These examples from the *Mathematics Framework* provide teachers with visual representations students might use to relate addition strategies to written methods. Students initially work with math drawings or manipulatives alongside written methods to help them connect each step of the operation to the value of each digit within a number. Ultimately students understand why algorithms work and will exclusively use written methods. These foundations and understandings will also prepare students for concepts, skills, and problem solving with multiplication and division, which are introduced in grade three.

Problems that encourage students to explain their thinking also support connections between the Standards for Mathematical Content (2.NBT.7) and the Standards for Mathematical Practice, "Make sense of problems and persevere in solving them (MP.1)" and "Construct viable arguments and critique the reasoning of others (MP.3)." To reinforce student understanding, teachers might ask, "What strategy or math drawing could you use to represent the quantities?" or "Explain how you solved the problem?" (See Table 2-2 Standards for Mathematical Practice—Explanations and Examples for Grade Two.)

Example: Addition Method Supported with Math Drawing   2.NBT.7

Addition Method 2: In this written addition method, digits representing newly composed units are placed below the addends from which they were derived, to the right to indicate that they are represented as a larger, newly composed unit. The addition proceeds right to left. The advantage to placing the composed units as shown is that it is clearer where they came from—e.g., the 1 and 3 that came from the sum of the ones-place digits (6 + 7) are close to each other. This eliminates confusion that can arise from traditional methods involving "carrying," which tends to separate the two digits that came from 13 and obscure the meaning of the numbers.

$$
\begin{array}{r}
4\ \ 5\ \ 6 \\
+\ 1\ \ 6\ \ 7 \\
\hline
\end{array}
\qquad
\begin{array}{r}
4\ \ 5\ \ 6 \\
+\ 1\ \ 6\ \ 7 \\
1 \\
\hline
3
\end{array}
\qquad
\begin{array}{r}
4\ \ 5\ \ 6 \\
+\ 1\ \ 6\ \ 7 \\
1\ \ 1 \\
\hline
2\ \ 3
\end{array}
\qquad
\begin{array}{r}
4\ \ 5\ \ 6 \\
+\ 1\ \ 6\ \ 7 \\
1\ \ 1 \\
\hline
6\ \ 2\ \ 3
\end{array}
$$

Add the ones, 6 + 7, and record these as 13, with 3 in the ones place and a 1 underneath the tens column.	Add the tens, 5 + 6 + 1, and record these 12 tens with 2 in the tens place and 1 under the hundreds column.	Add the hundreds, 4 + 1 + 1, and record these 6 hundreds in the hundreds column.

Grade Three

What does the *Mathematics Framework* say about grade three instruction and learning?

In grade three, students continue to build upon their mathematical foundation as they focus on four critical areas: (1) developing understanding of multiplication and division, as well as strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with a numerator of 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes. Students also work toward fluency with addition and subtraction within 1000 and multiplication and division within 100. By the end of grade three, students know all products of two one-digit numbers from memory.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 3-1 Grade Three Cluster-Level Emphases in the *Mathematics Framework* highlights the content emphases in the grade.

In grade three, students are formally introduced to fractions as numbers, thus broadening their understanding of the number system. Students use the number line as a tool to compare fractions and recognize equivalent fractions.

The grade-level chapters in the *Mathematics Framework* include the standards and also provide explanations and sample problems to help teachers understand the standards and teach the content. The following examples from the *Mathematics Framework* focus on just one of the Grade Three standards.

What does the standard say?

Standard 3.NF.3a
"Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line."

Focus, Coherence, and Rigor

When working with fractions, teachers should emphasize two main ideas:

- Specifying the whole
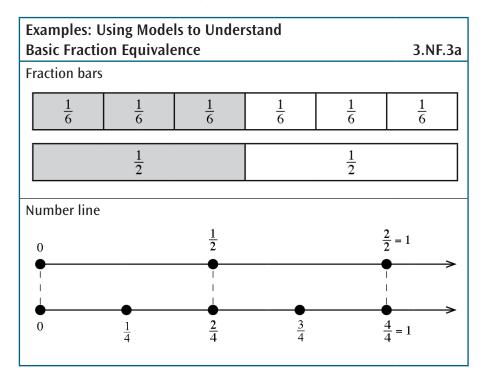- Explaining what is meant by "equal parts"

Student understanding of fractions hinges on understanding these ideas.

What support does the *Mathematics Framework* provide?

These examples from the *Mathematics Framework* provide teachers with visual representations students might use to understand and compare fractions. The number line reinforces the analogy between fractions and whole numbers. A goal is for students to see unit fractions as the basic building blocks of all fractions, in the same sense that the number 1 is the basic building block of whole numbers. Students develop an understanding of the size of fractions by locating fractions on a number line. Given two fractions—thus two points on the number line—students understand the fraction to the left is smaller and the fraction to the right is larger. Students might use fraction bars, with the same whole divided into different numbers of pieces to help them understand equivalent fractions. Conceptual work with fractions in grade three will help students add and subtract fractions with like denominators and multiply fractions by whole numbers in grade four.

Problems that encourage students to explain their thinking and to use fraction models also support connections between the Standards for Mathematical Content (3.NF.3a) and the Standards for Mathematical Practice, "Model with mathematics (MP.4)" and "Look for and express regularity in repeated reasoning (MP.8)." To reinforce student understanding, teachers might ask, "Why was it helpful to use a number line to represent the quantities?" or "What predictions or generalizations can this pattern support?" (See Table 3-2 Standards for Mathematical Practice—Explanations and Examples for Grade Three.)

Examples: Using Models to Understand Basic Fraction Equivalence	3.NF.3a

Fraction bars

| $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

| $\frac{1}{2}$ | $\frac{1}{2}$ |

Number line

Grade Four

What does the *Mathematics Framework* say about grade four instruction and learning?

In grade four, students continue to build a strong foundation for higher mathematics and should focus on three critical areas: (1) developing understanding and fluency with multi-digit multiplication and developing understanding of dividing to find quotients involving multi-digit dividends; (2) developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; and (3) understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry. Students also work toward fluency in addition and subtraction within 1,000,000 using the standard algorithm.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 4-1 Grade Four Cluster-Level Emphases in the *Mathematics Framework* highlights the content emphases in the grade.

In grade four, students extend multiplication and division to include whole numbers greater than 100. Standards 4.NBT.5–6 call for students to use visual representations, such as area and array models, to explain these operations.

The grade-level chapters in the *Mathematics Framework* include the standards and also provide explanations and sample problems to help teachers understand the standards and teach the content. The following example from the *Mathematics Framework* focuses on just one of the Grade Four standards.

What does the standard say?

Standard 4.NBT.5
"Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models."

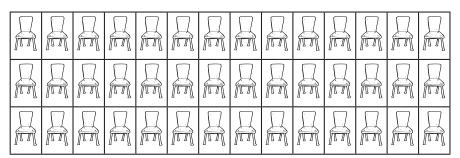**What support does the *Mathematics Framework* provide?**

This example from the *Mathematics Framework* includes visual representations and strategies students might use to solve the multiplication problem. It describes how students build on prior learning and shows connections between the Standards for Mathematical Content (4.NBT.5) and the Standards for Mathematical Practice, "Reason abstractly and quantitatively (MP.2)" and "Use appropriate tools strategically (MP.5)." To reinforce student understanding, teachers might ask, "How do you know?" or "What is the relationship of the quantities and your drawing?" (See Table 4-2 Standards for Mathematical Practice—Explanations and Examples for Grade Four.) By reasoning about the connections between math drawings and written numerical work, students can see multiplication and division algorithms as summaries of their reasoning about quantities.

Example: Area Models and Strategies for Multi-Digit Multiplication with a Single-Digit Multiplier 4.NBT.5
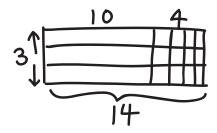
Chairs are being set up for a small play. There should be 3 rows of chairs and 14 chairs in each row. How many chairs will be needed?

Solution: As in grade three, when students first made the connection between array models and the area model, students might start by drawing a sketch of the situation. They can then be reminded to see the chairs as if surrounded by unit squares and hence a model of a rectangular region. With base-ten blocks or math drawings (**MP.2, MP.5**), students represent the problem and see it broken down into $3 \times (10 + 4)$.

Making a sketch like the one above becomes cumbersome, so students move toward representing such drawings more abstractly, with rectangles, as shown to the right. This builds on the work begun in grade three. Such diagrams help children see the distributive property: "3×14 can be written as $3 \times (10 + 4)$, and I can do the multiplications separately and add the results: $3 \times (10 + 4) = 3 \times 10 + 3 \times 4$. The answer is $30 + 12 = 42$, or 42 chairs."

Grade Five

What does the *Mathematics Framework* say about grade five instruction and learning?

In grade five, students continue to build a strong foundation for higher mathematics and should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to two-digit divisors, integrating decimal fractions into the place-value system, developing understanding of operations with decimals to hundredths, and developing fluency with whole-number and decimal operations; and (3) developing understanding of volume. Students also fluently multiply multi-digit whole numbers using the standard algorithm.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 5-1 Grade Five Cluster-Level Emphases in the *Mathematics Framework* highlights the content emphases in the grade.

In grade five, students apply and extend previous understandings of multiplication and division to multiply and divide fractions. By encouraging students to use fraction models to build an understanding of fraction operations, teachers help students lay a foundation for the algorithms that will follow.

The grade-level chapters in the *Mathematics Framework* include the standards and also provide explanations and sample problems to help teachers understand the standards and teach the content. The following examples from the *Mathematics Framework* focus on just one of the Grade Five standards.

What does the standard say?

Standard 5.NF.4b
"Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas."
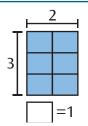

What support does the *Mathematics Framework* provide?

The examples from the *Mathematics Framework* connect to prior learning and provide teachers with visual representations and strategies students might use to understand how to multiply fractions. Students explain their thinking using area models, where the factors in a multiplication problem represent the side lengths of the rectangle and the product represents the area.

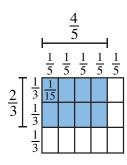| Examples of the Reasoning Called for in Standard 5.NF.4b	
Prior to grade five, students worked with examples of finding products as finding areas. In general, the factors in a multiplication problem represent the lengths of a rectangle and the product represents the area.	**Student:** "By counting the side lengths of this rectangle and the number of square units, I see that 2×3 = 6."
When students move to examples such as $2 \times \frac{2}{3}$, they recognize that one side of a rectangle is less than a unit length (in this case, some sides have lengths that are mixed numbers). The idea of the picture is the same, but finding the area of the rectangle can be a little more challenging and requires reasoning about unit areas and the number of parts into which the unit areas are being divided.	**Student:** "I made a rectangle with sides of 2 units and $\frac{2}{3}$ of a unit. I can see that the 2-unit squares in the pictures are each divided into 3 equal parts (representing $\frac{1}{3}$), with two shaded in each unit square (4 total). That means that the total area of the shaded rectangle is $\frac{4}{3}$ square units."
Finally, when students move to examples such as $\frac{2}{3} \times \frac{4}{5}$, they see that the division of the side lengths into fractional parts creates a division of the unit area into fractional parts as well. Students will discover that the fractional parts of the unit area are related to the denominators of the original fractions. At right, a 1×1 square is divided into thirds in one direction and fifths in another. This results in the unit square itself being divided into fifteenths. This reasoning shows why $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$.	**Student:** "I created a unit square and divided it into fifths in one direction and thirds in the other. This allows me to shade a rectangle of dimensions $\frac{2}{3}$ and $\frac{4}{5}$. I noticed that 15 of the new little rectangles make up the entire unit square, so they must be fifteenths $\left(\frac{1}{15}\right)$. Altogether, I had 2×4 of those fifteenths. So my answer is $\frac{8}{15}$."

Prior to grade five, students reasoned about multiplication using area models with whole numbers. In grade five, students represent more challenging examples with fractions and discover the fractional parts of the unit area are related to the denominators of the original fractions. Problems that encourage students to explain their thinking and use math drawings also support connections between the Standards for Mathematical Content (5.NF.4b) and the Standards for Mathematical Practice, "Make sense of a problem (MP.1)" and "Model with mathematics (MP.4)." To reinforce student understanding, teachers might ask, "Does this make sense?" or "What are some ways to represent the quantities?" (See Table 5-2 Standards for Mathematical Practice—Explanations and Examples for Grade Five.)

Grade Six

What does the *Mathematics Framework* say about grade six instruction and learning?

Grade six is an especially important year for bridging the concrete concepts of arithmetic and the abstract thinking of algebra. Students complete developing their skills with operations on rational numbers and delve into the multiplicative thinking demanded by proportional reasoning. Instructional time should focus on four critical areas: (1) connecting ratio, rate, and percentage to whole number multiplication and division and using concepts of ratio and rate to solve problems; (2) completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers; (3) writing, interpreting, and using expressions and equations; and (4) developing understanding of statistical thinking. Students also work toward fluency with multi-digit division and multi-digit decimal operations.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 6-1 Grade Six Cluster-Level Emphases highlights the content emphases in the grade.

Students' prior understanding of, and skill with, multiplication, division, and fractions contribute to their study of ratios, proportional relationships, including percentage, and unit rates in grade six.

The grade-level chapters in the *Mathematics Framework* include the standards and also provide explanations and sample problems to help teachers understand the standards and teach the content. The following example from the *Mathematics Framework* focuses on just one of the Grade Six standards.

What does the standard say?

Standard 6.RP.3
"Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations."

What support does the *Mathematics Framework* provide?

This example from the *Mathematics Framework* illustrates how simple ratio reasoning with pictures can be translated to a table or double number line diagram. Students recognize patterns in the table or on a diagram that reflect the relationship between the quantities a and b in the ratio $a:b$. They use this pattern recognition as a basis for seeing that if the number a is multiplied by a number k, then so is the number b. This example helps illustrate the intent of the standard and shows connections between the Standards for Mathematical Content (6.RP.3) and Standards for Mathematical Practice, "Reason abstractly and quantitatively (MP.2)" and "Use appropriate tools strategically (MP.5)."

In these examples, the tools are diagrams and tables, and students begin to reason abstractly by moving from pictures to the language and notation of ratios. (See Table 6.2 Standards for Mathematical Practice—Explanations and Examples for Grade Six.)

Example: Representing Ratios in Different Ways   6.RP.3a

A juice recipe calls for 5 cups of grape juice for every 2 cups of peach juice. How many cups of grape juice are needed for a batch that uses 8 cups of peach juice?

Using Ratio Reasoning: "For every 2 cups of peach juice, there are 5 cups of grape juice, so I can draw groups of the mixture to figure out how much grape juice I would need." [In the illustrations below, 🍇 represents 1 cup of grape juice and 🍑 represents 1 cup of peach juice.]
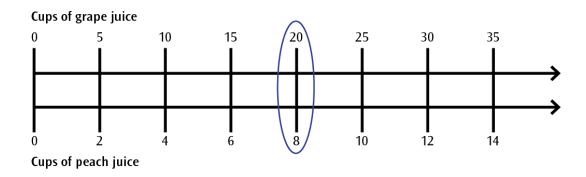
"It's easy to see that when you have $4 \times 2 = 8$ cups of peach juice, you need $4 \times 5 = 20$ cups of grape juice."

Using a Table: "I can set up a table. That way it's easy to see that every time I add 2 more cups of peach juice, I need to add 5 cups of grape juice."

+5

Cups of Grape Juice	Cups of Peach Juice
5	2
10	4
15	6
20	8
25	10

+2

Using a Double Number Line Diagram: "I set up a double number line, with cups of grape juice on the top and cups of peach juice on the bottom. When I count up to 8 cups of peach juice, I see that this brings me to 20 cups of grape juice."

Cups of grape juice

| 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 |

| 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |

Cups of peach juice

Grade Seven

What does the *Mathematics Framework* say about grade seven instruction and learning?

In grade seven, instructional time should focus on four critical areas: (1) developing understanding of and applying proportional relationships, including percentages; (2) developing understanding of operations with rational numbers (i.e., positive and negative numbers) and working with expressions and linear equations; (3) solving problems that involve scale drawings and informal geometric constructions and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and (4) drawing inferences about populations based on samples. Students also work toward fluently solving equations, such as $7x + 2.5 = 13$ and $\frac{1}{2}(x + 5) = 34$.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 7-1 Grade Seven Cluster-Level Emphases in the *Mathematics Framework* highlights the content emphases in the grade.

In grade seven, students develop more fluency with algebraic expressions and understand that different ways to write expressions can reveal different things, as described in Standard 7.EE.2.

What does the standard say?

Standard 7.EE.2
"Understand that rewriting an expression in different forms in a problem context can shed light on the problem and

Example: Working with Expressions	7.EE.2

2. While Chris was driving a Canadian car, he figured out a way to mentally convert the outside temperature that the car displayed in degrees Celsius to degrees Fahrenheit. This was his method: "I took the temperature it showed and doubled it. Then I subtracted one-tenth of that doubled amount. Finally, I added 32 to get the Fahrenheit temperature." The standard expression for finding the temperature in degrees Fahrenheit when the Celsius reading is known is $\frac{9}{5}C + 32$, where C is the temperature in degrees Celsius. Was Chris's method correct?

Solution: If C is the temperature in degrees Celsius, then the first step in Chris's calculation was to find $2C$. Then, he subtracted one-tenth of that quantity, which yielded $\frac{1}{10}(2C)$. Finally, he added 32. The resulting expression was $2C - \frac{1}{10}(2C) + 32$. This could be rewritten as $2C - \frac{1}{5}C + 32$. Combining the first two terms, we got $2C - \frac{1}{5}C + 32 = \left(2 - \frac{1}{5}\right)C + 32 = \left(\frac{10}{5} - \frac{1}{5}\right)C + 32 = \frac{9}{5}C + 32$. Chris's calculation was correct.

how the quantities in it are related. *For example*, $a + 0.05a = 1.05a$ *means that "increase by 5%" is the same as "multiply by 1.05."*

What support does the *Mathematics Framework* provide?

This example from the *Mathematics Framework* illustrates how different forms of the same expression can be useful. When doing calculations mentally, $2C - \frac{1}{10}(2C) + 32$ can be easier to work with than $\frac{9}{5}C + 32$, and the expressions each represent the same quantity. This example helps illustrate the intent of the standard and shows connections between the Standards for Mathematical Content (7.EE.2) and Standards for Mathematical Practice, "Reason abstractly and quantitatively (MP.2)" and "Look for and make use of structure (MP.7)." Students contextualize to understand the meanings of the variables in an expression. In the example, students see that $\frac{9}{5}C$ can be represented by $2C - \frac{2}{10}C$. (See Table 7.2 Standards for Mathematical Practice—Explanations and Examples for Grade Seven.)

Grade Eight

What does the *Mathematics Framework* say about grade eight instruction and learning?

In grade eight, instructional time should focus on three critical areas: (1) formulating and reasoning about expressions and equations, including modeling the relationship between two quantities (e.g., absences and math scores) with a linear equation, and solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence and understanding and applying the Pythagorean Theorem. Students also work towards fluency with solving sets of two equations with two unknowns.

To support the instructional focus at the grade level, some standards and clusters of standards require a greater instructional emphasis than others. Table 8-1 Grade Eight Cluster-Level Emphasis highlights the content emphases in the grade.

In grade eight, students use functions to model relationships between quantities (Standards 8.F.4–5). They begin to explore the concept of a function, in the linear context, and apply their understanding to real situations (Standard 8.SP.3).

What does the standard say?

Standard 8.SP.3

"Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept."

What support does the *Mathematics Framework* provide?

The example from the *Mathematics Framework* illustrates how linear functions can be applied to real-life situations This example helps illustrate the intent of the standard and shows connections between the Standards for Mathematical Content (8.SP.3) and Standards for Mathematical Practice, "Model with mathematics (MP.4)" and "Look for and make use of structure (MP.7)." Students engage in mathematical modeling by representing a real-world situation with a linear function; they make use of structure when they approximate a line of best fit and interpret the slope as a rate of change. (See Table 8.2 Standards for Mathematical Practice—Explanations and Examples for Grade Eight.)

Example: Finding a Linear Model for a Data Set   8.SP.3

Given data from students' math scores and absences, make a scatter plot. Informally fit a line to the graph, and determine an approximate linear fundtion that models the data. What would you expect to be the score of a student with 4 absences?

Solution:

Absences	3	5	1	1	3	6	5	3	0	7	8
Math Scores	65	50	95	85	80	34	70	56	100	24	45
Absences	2	9	0	6	6	2	0	5	7	9	1
Math Scores	71	30	95	55	42	90	92	60	50	10	80

Students can use graphing software to find a line of best fit. Such a line might be $y = -8x + 95$. They interpret this equation as defining a function that gives the approximate score of a student based on the number of absences they have. We would expect a student with 4 absences to have a score of approximately $y = -8(4) + 95 = 63$.

Math Scores

Higher Mathematics: Traditional Pathway

What does the *Mathematics Framework* say about the Traditional Pathway?

The goal of the Traditional Pathway is to present students with a rigorous and challenging curriculum that prepares them for 21st century careers and college. The Traditional Pathway organizes standards from the six higher mathematics domains into a sequence of courses that more closely aligns to the traditional sequence of Algebra I, Geometry, and Algebra II. Like the courses in the Integrated Pathway, Algebra I, Geometry, and Algebra II prepare students for more advanced mathematics courses.

What's new in the Traditional Pathway?

Algebra I: One of the important changes in the CA CCSSM, beginning in Grade 8, is a more prominent focus on the concept of a *function* and student understanding of the notation and skills accompanying this idea, including viewing sequences as functions. The topics of Algebra I can be summarized as offering a survey of linear, exponential, and quadratic functions and their properties. To most teachers, the inclusion of exponential functions will present a new teaching challenge. Their inclusion presents an opportunity to contrast them with linear functions: as linear functions grow by an equal additive change over

Building Functions	F-BF

Build a function that models a relationship between two quantities.
[For F-BF.1–2, linear, exponential, and quadratic]

1. Write a function that describes a relationship between two quantities. ★

   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★

   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Example: Exponential Growth	F-BF.1–2

When a quantity grows with time by a multiplicative factor greater than 1, it is said the quantity grows exponentially. Hence, if an initial population of bacteria, P_0, doubles each day, then after t days, the new population is given by $P(t) = P_0 2^t$. This expression can be generalized to include different growth rates, r, as in $P(t) = P_0 r^t$. A more specific example illustrates the type of problem that students may face after they have worked with basic exponential functions:

On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If the algae continue to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. Based on the current rate at which the algae are growing, this will happen on June 30.

Possible Questions to Ask:

a. When will the lake be covered halfway?

b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae, as a function of time (in days) that passes since the algae were introduced into the lake.

equal intervals, exponential functions grow by equal multiplicative change over equal intervals. The Number and Quantity standards of Algebra I focus on exponents and rational and irrational numbers; the Algebra standards focus on solving linear and quadratic equations and inequalities; the Function standards focus on properties of linear, exponential, and quadratic functions and compare properties of all three; and the Statistics and Probability standards focus on scatter plots and informal lines of best fit. Finally, the Modeling standards tie everything together by allowing students opportunities to apply these ideas to the real world.
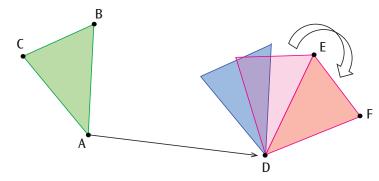
Geometry: Continuing with the more traditional organization, the Geometry course is focused solely on geometric concepts although now with connections to probability and statistics and modeling. A significant departure from a traditional geometry course is the use of transformational geometry, introduced in grade eight, as a means for justifying relationships and proving theorems. For example, new to many teachers are the more precise definitions of *congruent* and *similar*: Two shapes are *congruent* if there is a sequence of rigid motions that carries one onto the other; two shapes are *similar* if there is a sequence of rigid motions and a dilation that carries one onto the other. These definitions and the geometric transformations allow for the justification of triangle congruence relationships and similarity relationships, which are then used to justify further theorems and results. The Geometry course culminates in the study of right triangle trigonometry, simple conic sections, and proofs of the relationships between parallel and perpendicular lines and the accompanying relationships between their slopes.

Example: Reasoning About Congruence	**G-CO.7**

Standard **G-CO.7** explicitly states that students show that two triangles are congruent *if and only if* corresponding pairs of sides and corresponding pairs of angles are congruent (**MP.3**). The depth of reasoning here is fairly substantial, as students must be able to show, using rigid motions, that congruent triangles have congruent corresponding parts and that, conversely, if the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions that takes one triangle to the other. The second statement may be more difficult to justify than the first for most students, so a justification is presented here. Suppose there are two triangles $\triangle ABC$ and $\triangle DEF$ such that the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$ results in pairs of sides and pairs of angles being congruent. If one triangle were drawn on a fixed piece of paper and the other drawn on a separate transparency, then a student could illustrate a translation, T, that takes point A to point D. A simple rotation R about point A, if necessary, takes point B to point E, which is certain to occur because $\overline{AB} \cong \overline{DE}$ and rotations preserve lengths. A final step that may be needed is a reflection S about the side AB, to take point C to point F. It is important to note why the image of point C is actually F. Since $\angle A$ is reflected about line $\overline{AB}$, its measure is preserved. Therefore, the image of side $\overline{AC}$ at least lies on line $\overline{DF}$, since $\angle A \cong \angle D$. But since $\overline{AC} \cong \overline{DF}$, it must be the case that the image of point C coincides with F. The previous discussion amounts to the fact that the sequence of rigid motions, T, followed by R, followed by S, maps $\triangle ABC$ exactly onto $\triangle DEF$. Therefore, if it is known that the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions carrying one onto the other; that is, they are congruent. The informal proof presented here should be accessible to students in the Geometry course; see figure below.

Illustration of the Reasoning That Congruent Corresponding Parts Imply Triangle Congruence. Point A is translated to D, the resulting image of $\triangle ABC$ is rotated so as to place B onto E, and the image is then reflected along line segment DE to match point C to F.

Algebra II: In Algebra II, students study more general polynomials. They experiment with right triangle trigonometry and extend trigonometric functions to all real numbers. There is more emphasis on modeling as students use their understanding of functions to practice modeling real-life situations that may not fit simple relationships like linear, exponential, or quadratic.

Example: Modeling Daylight Hours   F-TF.5
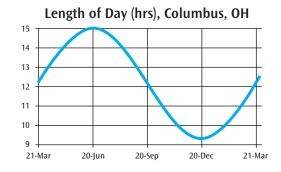
By looking at data for length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), $C = \dfrac{2\pi}{365}$, and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83\sin\left(\dfrac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t, the day of the year from March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.

Students can investigate many other trigonometric modeling situations, such as simple predator–prey models, sound waves, and noise-cancellation models.

Source: UA Progressions Documents 2013c, 19.

Length of Day (hrs), Columbus, OH

What support does the *Mathematics Framework* provide?

The *Mathematics Framework* clarifies the expectations of the standards in each course, indicates the importance of each standard for student success, and provides examples of the types of problems and accompanying instruction that students should expect.
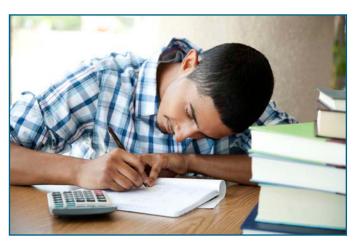
Higher Mathematics: Integrated Pathway

What does the *Mathematics Framework* say about the Integrated Pathway?

The goal of the Integrated Pathway courses is to present students with a rigorous and challenging curriculum that prepares them for 21st century careers and college. The primary way the Integrated Pathway differs from the Traditional Pathway is that in each of the three courses it presents common mathematics topics from all six conceptual categories of Common Core higher mathematics: Number and Quantity, Algebra, Functions, Geometry, Modeling and Statistics, and Probability. Like the courses in the Traditional Pathway, Mathematics I, II, and III prepare students for more advanced mathematics courses.

What's new in the Integrated Pathway?

Mathematics I: One of the important changes in the CA CCSSM, beginning in Grade 8, is a more prominent focus on the concept of a *function* and student understanding of the notation and skills accompanying this idea. The topics of Mathematics I formalize and extend students' understanding of linear functions—functions of the form $f(x) = mx + b$—and their properties. The Number and Quantity standards of Mathematics I focus on rates; the Algebra standards focus on solving linear equations and systems of linear equations; the Function standards focus on properties of linear functions and contrast those with exponential functions; the Geometry standards focus on geometric relationships between graphs of linear functions (e.g., parallel and perpendicular lines and rectilinear shapes); the Statistics and Probability standards focus on scatter plots and informal lines of best fit. Finally, the Modeling standards tie everything together, allowing students opportunities to apply these ideas in the real world.

Reasoning with Equations and Inequalities A-REI

Solve systems of equations. [Linear systems]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

When solving systems of equations, students also make frequent use of substitution—for example, when solving the system $2x - 9y = 5$ and $y = \frac{1}{3}x + 1$, the expression $\frac{1}{3}x + 1$ can be substituted for y in the first equation to obtain $2x - 9\left(\frac{1}{3}x + 1\right) = 5$. Students also solve such systems approximately, by using graphs and tables of values (**A-REI.5–6**).

Example	A-REI.6

Solving simple systems of equations. To get started with understanding how to solve systems of equations by linear combinations, students can be encouraged to interpret a system in terms of real-world quantities, at least in some cases. For instance, suppose one wanted to solve this system:

$$3x + y = 40$$
$$4x + 2y = 58$$

Now consider the following scenario: Suppose 3 CDs and a magazine cost $40, while 4 CDs and 2 magazines cost $58.

- What happens to the price when you add 1 CD and 1 magazine to your purchase?
- What is the price if you decided to buy only 2 CDs and no magazine?

Answering these questions amounts to realizing that since $(3x + y) + (x + y) = 40 + 18$, we must have that $x + y = 18$. Therefore, $(3x + y) + (-1)(x + y) = 40 + (-1)18$, which implies that $2x = 22$, or 1 CD costs $11. The value of y can now be found using either of the original equations: $y = 7$.

Mathematics II: Continuing with the development of an understanding of functions, in Mathematics II students work with quadratic functions as functions that have different growth patterns than either linear or exponential functions. They further their facility with equations and expressions by translating between the various forms of a quadratic and solving quadratic equations, ultimately leading to the quadratic formula. Simple trigonometric functions and conic sections like circles are introduced in the Geometry standards. The Statistics and Probability standards introduce more probability and use area models to support these concepts.

Students in Mathematics II extend their work with exponents to include quadratic functions and equations. To extend their understanding of these quadratic expressions and the functions defined by such expressions, students investigate properties of quadratics and their graphs in the Functions domain. It may be best to present the solving of quadratic equations in the context of functions. For instance, if the equation $h(t) = -16t^2 + 50t + 150$ defines the height of a projectile launched with an initial velocity of 50 ft/s from a height of 150 ft, then asking at which time t the object hits the ground is asking for which t is found at $h(t) = 0$. That is, students now need to solve the equation $-16t^2 + 50t + 150 = 0$ and require new methods for doing so. Students have investigated how to "undo" linear and simple exponential functions in Mathematics I; now they do so for quadratic functions and discover that the process is more complex.

Example	A-REI.4a

When solving quadratic equations of the form $(x-p)^2 = q$, students rely on the understanding that they can take square roots of both sides of the equation to obtain the following:

$$\sqrt{(x-p)^2} = \sqrt{q} \qquad (1)$$

In the case that $\sqrt{q}$ is a real number, this equation can be solved for x. A common mistake is to quickly introduce the $\pm$ symbol here, without understanding where the symbol comes from. Doing so without care often leads students to think that $\sqrt{9} = \pm 3$, for example.

Note that the quantity $\sqrt{a^2}$ is simply a when $a \geq 0$ (as in $\sqrt{5^2} = \sqrt{25} = 5$), while $\sqrt{a^2}$ is equal to $-a$ (the opposite of a) when $a < 0$ (as in $\sqrt{(-4)^2} = \sqrt{16} = 4$). But this means that $\sqrt{a^2} = |a|$. Applying this to equation (1) yields $|x-p| = \sqrt{q}$. Solving this simple absolute value equation yields that $x - p = \sqrt{q}$ or $-(x-p) = \sqrt{q}$. This results in the two solutions $x = p + \sqrt{q}, p - \sqrt{q}$.

Mathematics III: In Mathematics III, students study more general polynomials. They experiment with right triangle trigonometry and extend trigonometric functions to all real numbers. There is a heavier focus on modeling here as students use their understanding of functions to practice modeling real-life situations that may not fit simple relationships like linear, exponential, or quadratic.

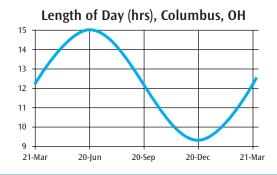**Example: Modeling Daylight Hours**   F-TF.5

By looking at data for length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), $C = \dfrac{2\pi}{365}$, and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83\sin\left(\dfrac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t, the day of the year from March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.

Students can investigate many other trigonometric modeling situations, such as simple predator–prey models, sound waves, and noise-cancellation models.

Source: UA Progressions Documents 2013c, 19.

Length of Day (hrs), Columbus, OH

Chapters on a Variety of Instructional Topics

The *Mathematics Framework* provides additional support for teachers and other educators in chapters focused on a variety of instructional topics. Beginning with a chapter on access to mathematics for all of California's diverse students, these chapters offer guidance for teachers and school and district administrators. Rich in detail, the chapters are crucial tools for the implementation of the CA CCSSM. Very brief summaries of the content of each chapter are provided in this section. The summaries are presented in the order the chapters appear in the *Mathematics Framework*.

The *Mathematics Framework* also includes a number of documents in the appendix that support mathematics instruction and learning. Two of the six appendices are summarized in this section—one on financial literacy and one on mathematical modeling.

Universal Access

What does the *Mathematics Framework* say about Universal Access?

> The ultimate goal of mathematics programs in California is to ensure universal access to high-quality curriculum and instruction so that all students are college and career ready.

The CA CCSSM provide an historic opportunity to improve access to rigorous mathematics for all students. The standards call for students to be active participants in their learning, not only through problem solving but also discussing, listening, explaining, demonstrating, reading, writing, representing, and presenting. These increased language demands may pose challenges for all students and even greater challenges for English learners and students who are reading or writing below grade level.

What support does the *Mathematics Framework* provide for Universal Access?

The *Mathematics Framework* recognizes that the increased language and cognitive demands of mathematics learning require different approaches to instructional support. The "Universal Access" chapter offers guidance on planning for universal access, helping students meet the new language demands of the mathematics standards, and strategies to support English learners. The principles of universal design for learning (UDL) and multi-tiered systems of support (MTSS) are highlighted to support equitable access and opportunity for all students to achieve the CA CCSSM. The chapter also provides insights into common student errors and includes sections on planning for instruction for students with disabilities, at-risk students, and advanced learners.

Instructional Strategies

What does the *Mathematics Framework* say about Instructional Strategies?

The *Mathematics Framework* clarifies the expectations for instruction to achieve student success with the CA CCSSM. According to the *Mathematics Framework*, "The three major principles on which the CA CCSSM are based are focus, coherence, and rigor." The *Mathematics Framework* emphasizes that teachers, curriculum specialists, and other instructional leaders should focus on a balance between these three principles

> **Standards for Mathematical Content**
>
> The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:
>
> - Focus—Instruction is focused on grade-level standards.
> - Coherence—Instruction should be attentive to learning across grades and to linking major topics within grades.
> - Rigor—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

when developing best practices for mathematics instruction. Instruction that connects the Standards for Mathematical Practice and the Standards for Mathematical Content integrates all three principles.

What support does the *Mathematics Framework* provide for incorporating new instructional strategies?

The *Mathematics Framework* recognizes that teaching styles vary for a number of reasons. And while the *Mathematics Framework* does not prescribe any particular teaching style over another, it does emphasize that teaching for focus, coherence, and rigor will often consist of a blend of three general teaching models: Explicit Instruction, Interactive Instruction, and Implicit Instruction. Many teachers will recognize that they already make use of all three teaching models naturally. To achieve the focus, coherence, and rigor mentioned above, the type of instructional model used by the teacher will depend upon the learning needs of students and the mathematical content that is being presented. For example, explicit instruction models best support practice to mastery of skills, while implicit models link information to students' prior learning to help them develop deeper conceptual understanding and problem solving abilities.

The *Mathematics Framework* contains several general examples of each type of instruction, as well as specific grade-level examples at the end of the chapter. The *Mathematics Framework* also includes a brief discussion of student discourse and its importance in the mathematics classroom.

Supporting High-Quality Common Core Mathematics Instruction

What does the *Mathematics Framework* say about support for instruction?

The planning and implementation of effective and efficient mathematics instruction that meets the needs of every student requires broad support. The *Mathematics Framework* sets forth suggestions for how both administrators and the community can support high-quality instruction for all California students. In particular, the *Mathematics Framework* emphasizes the importance of long-range, collaborative, and administrator-supported professional learning of teachers. Professional learning that addresses both teacher content knowledge and effective and appropriate pedagogical strategies for the classroom will support student understanding of the CA CCSSM.

What is the role of administrators?

The *Mathematics Framework* recognizes that two of the many responsibilities of school site administrators are supporting and evaluating teachers. It presents several forms of professional learning for teachers and notes the importance of administrator participation in and support of such opportunities, as well as administrators' own responsibility to understand the new CA CCSSM. Evaluation of teachers should be based on the new vision of classrooms set forth in the Mathematical Practice Standards— that of all students reasoning, discussing, and questioning, as well as listening and practicing.

Mathematical Practice (MP) Standards
The Mathematical Practice (MP) Standards represent a shift toward students "doing mathematics" in the classroom, and teachers must not only understand the practices of the discipline, but how these practices can occur in school mathematics. To support this change, professional learning should accomplish the following results: • Engage teachers in the posing and solving of problems, requiring teachers to make sense out of problems and learn to persevere in solving them (MP.1). • Encourage teachers to explain their reasoning, make conjectures, and critique each other's reasoning in a safe environment (MP.3). • Allow teachers to learn which tools are appropriate for the mathematics at hand and gather experience with the use of those tools in the classroom (MP.5).

What are other supports that ensure high-quality instruction?

The *Mathematics Framework* discusses the large network of entities that contributes to student success in schools, including teachers, administrators, school board members, professional learning providers, parents and the community, and the higher education community. Each plays a role, and all must collaborate and focus their efforts on discovering and implementing best practices for CA CCSSM instruction.

Technology in the Teaching of Mathematics

What does the *Mathematics Framework* say about using technology in mathematics?

Education technology can help support access to and understanding of the standards-based academic curriculum. The use of technology is directly integrated into the CA CCSSM and integral to preparing students for college and 21st century careers. Educational technology can facilitate simple computation and the visualization of mathematics situations and relationships, allowing students to better comprehend mathematical concepts in practice.

> When used effectively, educational technology can enhance student understanding of mathematical concepts, bolster student engagement, and strengthen problem-solving skills.
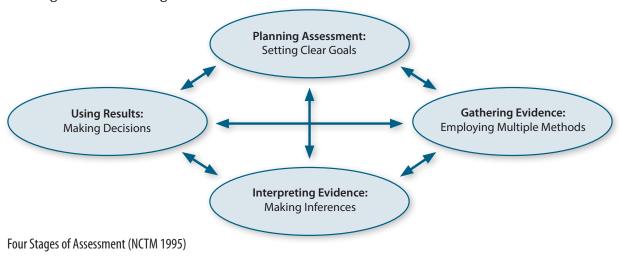

What support does the* Mathematics Framework *provide for using technology in mathematics?

The CA CCSSM make specific reference to using technology tools in a number of cases, especially in the middle grades and high school. The "Technology in the Teaching of Mathematics" chapter of the *Mathematics Framework* provides examples of classroom use of technology for elementary, middle, and high school standards that facilitate students' exploration and deeper understanding of mathematical concepts. It offers guidance for teachers in the use of assistive technology to help students with disabilities gain access to the full mathematics curriculum and perform tasks that might otherwise be difficult or impossible. The chapter also guides teachers on incorporating education technology into instruction to provide a challenging and interesting educational environment for all students.

Assessment

What does the *Mathematics Framework* say about assessment?

According to the *Mathematics Framework*, "Assessment provides students with frequent feedback on their performance, teachers with diagnostic tools for gauging students' depth of understanding, parents with information about their children's performance in the context of program goals, and administrators with a means for measuring student achievement." Assessment tools should have a clear purpose for instruction, for example, to measure a student's understanding of a particular concept or to assess a student's facility with a given algorithm. Assessment may happen on the individual level as well as the classroom or even schoolwide level. Moreover, the *Mathematics Framework* recognizes that there are multiple forms of assessment that may be used for different purposes. The important message is that assessment should have a clear purpose of supporting and enhancing student learning.

Planning Assessment:
Setting Clear Goals

Using Results:
Making Decisions

Gathering Evidence:
Employing Multiple Methods

Interpreting Evidence:
Making Inferences

Four Stages of Assessment (NCTM 1995)

What support does the *Mathematics Framework* provide for assessment of student learning?

The CA CCSSM make it clear that Mathematical Practice Standards are on an equal footing with the Mathematical Content Standards and, as a result, assessments must measure students' abilities to persevere through solving difficult problems, communicate mathematical thinking, use tools and model with mathematics, use quantities appropriately and attend to precision, and transfer patterns in reasoning and structure to new problems. The *Mathematics Framework* provides guidance to teachers by describing two main categories of assessment practices—*formative assessment* and *summative assessment*—as well as several examples of assessment tools and related strategies, information on grading policies, using scoring rubrics, homework, and the four Claims for Student Learning that are assessed by the Smarter Balanced Assessments.

Instructional Materials to Support the California Common Core State Standards for Mathematics

This chapter provides guidance to districts on the selection of instructional materials for classroom use. It features the Criteria for Evaluating Mathematics Instructional Materials for Kindergarten through Grade Eight (Criteria), which were the basis for the 2014 state adoption and are a useful tool for local educational agencies conducting their own instructional materials evaluations. The Criteria is a comprehensive description of effective instructional programs that are aligned to the CA CCSSM and support the principles of focus, coherence, and rigor. This chapter also includes guidance for local districts on the adoption of instructional materials for students in grades nine through twelve.
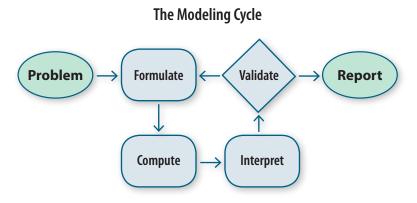
Appendix A: Financial Literacy and Mathematics Education

When students are introduced to financial literacy education early in their academic lives, they can develop a lifelong foundation for making intelligent decisions about how to earn, save, and invest money. The CA CCSSM open many doors for examining and practicing financial literacy topics, especially through the application of the Standards for Mathematical Practice and real-world problems. Appendix A provides guidance and example problems for incorporating financial literacy into mathematics instruction, as well as a list of no-cost resources for classroom use.

Appendix B: Mathematical Modeling

What does the *Mathematics Framework* say about mathematical modeling?

Modeling links classroom mathematics to everyday life, work, and decision making. One of the major changes with the CA CCSSM is the emphasis on mathematical modeling in all grades. In grades K–8 and the higher mathematics courses, modeling is a Standard for Mathematical Practice (MP.4) and helps students realize that mathematics is useful and can be applied to answer questions about the world around them. In the higher mathematics standards, modeling is also considered a conceptual category. Standards marked with a star symbol (★) are modeling standards—meaning they present opportunities for students to examine the world around them using mathematics (refer to the table in Appendix B, Higher Mathematics Modeling Standards in the CA CCSSM).

The Modeling Cycle

What support does the *Mathematics Framework* provide for modeling?

The *Mathematics Framework* provides guidance on creating a modeling course or curriculum that supports "The Modeling Cycle" and offers examples of modeling problems appropriate for upper-middle grades and higher mathematics courses. Modeling with mathematics is much more than simply "word problems," or even just "application problems," and involves making observations about the world around us, forming questions, using mathematics to answer those questions, and refining the resulting "models" to better explain the world. In early grades, students model when they learn that operations like addition and multiplication can solve real-life problems; in higher mathematics, students understand that functions model relationships between quantities that change together. Modeling problems are unfamiliar and original to students; they are memorable due to students taking such an active role in their learning. The level at which students engage in modeling varies by the grade level and the mathematical content, but Modeling is a prominent focus of the CA CCSSM.