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Rationals r(n) = A006232(n)/A006233(n)

e.g.f: 1/(ln(1+x)/x)

r(n): n=0..30:

[1, 1/2, -1/6, 1/4, -19/30, 9/4, -863/84, 1375/24, -33953/90, 57281/20,
-3250433/132, 1891755/8, -13695779093/5460, 24466579093/840, -132282840127/360,
240208245823/48, -111956703448001/1530, 4573423873125/4, -30342376302478019/1596,
56310194579604163/168, -12365722323469980029/1980, 12365722323469980029/1980,
-20953816286242674495191/8280, 4380881778942163832799/80,
-101543126947618093900697699/81900, 192060902780872132330221667/6552,
-1092286933245454564213092649/1512, 20750321774769671189228515625/112,
-1718089509598695642524656240811/3480, 1092041494691940355778302728249/80]

This sequence of signed rationals r(n) (called Cauchy numbers of the first kind in OEIS) coincides with the so called a-sequence (see below) for the Sheffer (in this case Jabotinsky) matrix Stirling2 A048993.

This sequence r(n) = a(n) determines a recurrence relation for S2(n,m) using all entries in the previous row numbered n-1:

S2(n,m) = (n/m)*sum(binomial(m-1+j,j)*a(j)*S2(n-1,m-1+j),j=0..n-m), n>=1, m>=1.

E.g.: 3 = S2(3,2) = (3/2)*(1*1*1 + 2*(1/2)*1) = 3;

7 = S2(4,2) = (4/2)*(1*1*1 + 2*(1/2)*3 + 3*(-1/6)*1) = 7.

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Introduction to A- and Z- sequences for Riordan matrices and a- and z- sequences for Sheffer matrices

(special lower triangular infinite matrices):

The A- and Z-sequences for Riordan matrices are considered in the papers:

D.G. Rogers, Pascal Triangles, Catalan Numbers and Renewal Arrays, Discrete Math. 22(1978)301-310,

D. Merlini, D.G. Rogers, R. Sprugnoli and M.C. Verri, On some alternative characterizations of Riordan

arrays, Can. J. Math, 49(1997)301-320,

R.Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132(1994)267-290.

For Riordan matrices and the Riordan group see the paper:

L.V. Shapiro, S. Getu. W.-J.Woan, and L. Woodson, The Riordan Group, Discrete Appl. Math. 34(1991)229-239.

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Summary on A- and Z-sequences for Riordan matrices:

A Riordan matrix R=(G,F) (in our notation) with an o.g.f. G(x) with G(0)=1 and an invertible o.g.f.

F(x)=x*Fhat(x) with Fhat(0)=1 is defined by its matrix elements R(n,m):=[(x^n)] G_m(x) with the o.g.f.

for column nr. m>=0 given by G_m(x) = G(x)*F(x)^m = G(x)*(x*Fhat(x))^m.

The o.g.f. of the row polynomials R(n,x):= sum(R(n,m)*x^m,m=0..n) is

R(z,x):= sum(R(n,x)*(z^n)) = G(z)/(1-x*z*Fhat(z)).

A Riordan matrix (coefficient matrix of the polynomials) is infinite lower triangular: R(n,m)=0 if n<m.

Every Riordan matrix satisfies the following recurrence relations:

(a) For the first column m=0 numbers:

    R(n,0)= sum(Z(j)*R(n-1,j),j=0..n-1), n>=1; R(0,0):=1 (by convention).

(b) For the columns m>=1:

    R(n,m) = sum(A(j)*R(n-1,m-1+j),j=0..n-m), n>=1, m>=1.

The o.g.f.s for the Z- and A-sequences are obtained from G and F of the Riordan matrix as follows:

A(y):=sum(a(j)*y^j,j=0..infty) = Fhat(Finv(y))= y/Finv(y) with F(x)=x*Fhat(x) and Finv is the compositional inverse of F.

Z(y):=sum(z(j)*y^j,j=0..infty) = (1- 1/G(Finv(y)))/Finv(y).

Conversely, the o.g.f.s G and F of the Riordan matrix R are determined from the o.g.f.s A(y) and Z(y) as follows. First, Fhat(x)=A(F(x)) is used to either find f(x) directly from a(y) or a corollary to Lagrange's inversion theorem is employed to give F_j := [x^j]F(x) = diff(A(t)^n,t$(n-1)|_{t=0}, n>=1 and

F(0):=0.

Then G(x) is found from G(x)=1/(1-Z(F(x))).

The proof works for both directions. See the quoted references and the hints given below for the Sheffer case.

Example: Pascal's triangle A007318 R=P=(G(x)=1/(1-x),F(x)=x/(1-x)) with the A-sequence generated by

A(y)= Fhat(Finv(y)) = 1+y and the Z-sequence generated by Z(y)=1.

This leads to the obvious recurrences for P(n,m) and P(n,0).

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a- and z-sequences are the analoga of A- and Z-sequences for Sheffer matrices.

For Sheffer matrices (polynomials) and the Sheffer group see the book:

S. Roman, Umbral calculus, Academic Press, 1984.

The notation (g=gR,f=fR) of this book translates as follows to our notation S=(g,f) for a Sheffer matrix:

gR(t)= 1/g(finv(t))), fR(t)= finv(t), with the compositional inverse finv(t) of f(x).

Conversely, g(x)=1/gR(fRinv(x)), f(x)=fRinv(x), with the compositional inverse fRinv(x) of fR(t).

For the subgroup of the Sheffer group (1,f) called Jabotinsky subgroup, see the paper:

D. E. Knuth, Convolution polynomials, The Mathematica J., 2(1992)67-78.

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A Sheffer matrix S=(g,f) with e.g.f. g(x) with g(0):=1 and an invertible e.g.f. f(x) with f(0)=0

is defined by its matrix elements S(n,m):=[(x^n)/n!] g_m(x) with the e.g.f. for column No. m>=0 given

by g_m(x)=g(x)(f(x)^m/m!).

The e.g.f. of the row polynomials s(n,x):=sum(S(n,m)*x^m,m=0..n) is

s(z,x):= sum(s(n,x)*(z^n)/n!) = g(z)*exp(x*f(z)).

A Sheffer matrix (coefficient matrix of the polynomials) is infinite lower triangular: S(n,m)=0 if n<m.

Every Sheffer matrix satisfies the following recurrence relations:

(a) For the first column m=0 numbers:

S(n,0)= n*sum(z(j)*S(n-1,j),j=0..n-1), n>=1; S(0,0):=1 (by convention).

(b) For the columns m>=1:

S(n,m) = (n/m)*sum(binomial(m-1+j,m-1)*a(j)*S(n-1,m-1+j),j=0..n-m), n>=1, m>=1.

The e.g.f.s for the z- and a-sequences are obtained from g and f of the Sheffer matrix as follows:

a(y):=sum(a(j)*(y^j)/j!,j=0..infty) = fhat(finv(y)) = y/finv(y) with f(x)=x*fhat(x) and finv is the

compositional inverse of f.

z(y):=sum(z(j)*(y^j)/j!,j=0..infty) = (1- 1/g(finv(y)))/finv(y).

Conversely, the e.g.f.s g and f of the Sheffer matrix S are determined from the e.g.f.s a(y) and z(y) as

follows. First, f(x)=x*a(f(x)) is used to either find directly f(x) from a(y) or a corollary to Lagrange's inversion theorem is employed to give

f_j := [(x^j)/j!]f(x) = diff(a(t)^n,t$(n-1)|_{t=0}, n>=1 and f(0):=0.

Then g(x)=1/(1-z(f(x))).

The proof works for both directions.

(a) Insert the recurrence for S(n,0) into g(x)=1 + sum(S(n,0)*(x^n)/n!,n=1..infty),

interchange the sums (formal power series here), building the e.g.f. g_j(x) and use its Sheffer structure. This produces g(x)= 1+x*g(x)z(f(x)). From this one finds g(x)=1/(1-x*z(f(x))) or

z(y) = (1- 1/g(finv(y)))/finv(y).

This argument can be reversed.

(b) Insert the recurrence for S(n,m) into

g_m(x)= 0 + sum(S(n,m)*(x^n)/n! ,n=1..infty),

interchange the sums (formal power series), finding the e.g.f. g_{m-1+j}(x) and use its Sheffer structure. The factorials are rearranged to produce g_m(x)*(x*a(f(x)))/f(x). This shows that

a(f(x))=fhat(x) with fhat(x)=f(x)/x.

This argument can also be reversed.

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Note: This recurrences (a) and (b) are not always the simplest one for S(n,m).

E.g. Stirling2 = A048993, which has z(y)=0 from g(x)=1 (this is what one expects for the first m=0 column) but finv(y)=ln(1+y) leading to a(y)=1/(ln(1+y)/y), which generates the sequence A006232(n)/A006233(n). Hence all entries of the previous row starting with S2(n-1,m-1) are needed for S2(n,m).

The usual recurrence used for S2(n,m) needs only to terms of the previous row. See the recurrence for Sheffer polynomials given as next item.
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There is also a recurrence for the row polynomials s(n,x):=sum(S(n,m)*x^m,m=0..n) for every Sheffer matrix S=(g,f). In the general case it uses formal series expansion employing a corollary of Legendre's inversion theorem.

s(n,x) = (x+(ln(g(finv(t))))')/finv'(t)|_{t -> d_x} s(n-1,x), n>=1; s(0,x)=1.

Here ' denotes derivative w.r.t. t, finv is the compositional inverse of f and d_x=d/dx is the derivative w.r.t. x (powers of t should to be replaced by powers of d_x).

This formula is the rewritten version of S. Roman's book (op. cit.) p. 50, Corollary.

The proof uses the fact that finv(d_x) s(z,x) =finv(f(z)) s(z,x) = z s(z,x) with the e.g.f. s(z,x) for the

row polynomials given above, and d_x=d/dx is the derivative w.r.t. x. This follows from del_x^k s(z,x) = f(z)^k s(z,x) together with del_z s(z,x) = (ln(g(z))' +x*f'(z))*s(z,x) with ' denoting differentiation w.r.t. z, and del_x, resp. del_z stands for the partial derivative w.r.t. x, resp. z.

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In the Stirling2 case, with finv(t)=ln(1+t) and g(t)=1 this recurrence becomes

S2(n,x) =x*(1 + d_x)*S2(n-1,x), n>=1, S2(0,x)=1, with the row polynomials S2(n,x):=sum(A048993(n,m),m=0..n).

Comparing coefficients of powers of x leads to the known three term recurrence

S2(n,m) = S2(n-1,m-1) + m*S2(n-1,m). The inputs are: S(0,0)=1, S(n,-1)=0 and S(n,m)=0 if n<m.

################################### e.o.f. #######################################