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RECURRING SEQUENCES

BY

DOV JARDEN

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RECURRING SEQUENCES

A COLLECTION OF PAPERS

BY

DOV JARDEN

SECOND EDITION
REVISED AND ENLARGED
INCLUDING NUMEROUS NEW FACTORIZATIONS
OF FIBONACCI AND LUCAS NUMBERS
BY JOHN BRILLHART

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9[9].—Dov Jarden, *Recurring Sequences*, Second Edition, Riveon Lematematika, 12 Gat St., Kiryat-Moshe, Jerusalem, 1966, ii + 137 pp. Price $6.

The second edition, which has been produced on a more durable paper, is an enlargement and revision of the first. The enlargement comes from the inclusion of eight new articles, while the revision consists mainly of the inclusion of many new prime factors in the two factor tables in the work.

In general, the book is a collection of short papers by the author on various questions concerning the Fibonacci numbers U_n, their associated sequence V_n, and other recurring sequences. Representative titles are, "Divisibility of U_{mn} by $U_m U_n$ in Fibonacci's sequence," "Unboundedness of the function $[p - (5/p)]/a(p)$ in Fibonacci's sequence," and "The series of inverses of a second order recurring sequence." There is also a large chronological bibliography on recurring sequences.

Among the new articles is one of general interest to Decaphiles, "On the periodicity of the last digits of the Fibonacci numbers," where the period mod 10^d is shown to be 60, 300, and $15 \cdot 10^{d-1}$ for 1, 2, and $d \geqq 3$ final digits.

The two revised factor tables, which were provided by the reviewer, are at present the most extensive in the literature.

Of these, the first table is a special table giving the complete factorization of $5U_n^2 \pm 5U_n + 1$ for odd $n \leqq 77$, the two trinomials being the algebraic factors in

$$V_{5n}/V_n = (5U_n^2 - 5U_n + 1)(5U_n^2 + 5U_n + 1),$$

n odd.

The second table is the general factor table for U_n and V_n with $n \leqq 385$. The overall bound for prime factors is 2^{35} for $n < 300$ and 2^{30} for $300 \leqq m \leqq 385$. It also shows that U_n and V_n are completely factored up to $n = 172$ and $m = 151$ respectively. The table gives as well an indication for the incomplete factorizations whether their cofactors are composite or pseudoprime. The introduction to this table provides the further information that U_n is prime for $n \leqq 1000$ iff $n = 3, 4, 5, 7, 11, 13, 17, 23, 29, 43, 47, 83, 131, 137, 359, 431, 433, 449, 509, 569, 571$, while V_n is prime for $n \leqq 500$ iff $n = 0, 2, 4, 5, 7, 8, 11, 13, 16, 17, 19, 31, 37, 41, 47, 53, 61, 71, 79, 113, 313, 353$. The number U_{359}, which was only known to be a pseudoprime at the time of publication of the tables, has since been shown to be a prime by the reviewer.

John Brillhart

University of Arizona
Tucson, Arizona 85721

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DIVISIBILITY OF U_{mn} BY U_mU_n IN FIBONACCI'S SEQUENCE

Let U_n denote the n-th term of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots \qquad (U_{n+2}=U_{n+1}+U_n)$$

then the following theorem follows easily from known properties of this sequence:

THEOREM 1. Let the greatest common divisor (m,n) of m and n be 1, 2, or 5, then U_{mn} is divisible by U_mU_n.

The object of this note is to show that the converse is also true:

THEOREM 2. If U_{mn} is divisible by U_mU_n, then (m,n)=1, 2, or 5.

The proof of Theorem 2 is based on the following well known results on Fibonacci's sequence.

Let U_a (a>0) be the first term of the Fibonacci sequence divisible by a given prime p. Then a=a(p) is called the rank of apparition of p and is some divisor of p-(5/p), where the symbol (5/p) is Legendre's symbol. Any term U_r of the sequence is divisible by p if and only if r is divisible by a. Let p^π be the highest power of p dividing U_a and let $r=p^\lambda k$, where p does not divide k, then Lucas' "law of repetition" for the Fibonacci sequence states that the highest power of p dividing U_r is $p^{\pi+\lambda+\gamma(r)}$, where

$$\gamma(r) = \begin{cases} 1 & \text{if } p^\lambda=2 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for a fixed p

$$\gamma(rs) \leq \gamma(r)+\gamma(s). \tag{1}$$

To prove Theorem 2 suppose that d, the greatest common divisor of m and n, is different from 1, 2 and 5. Now U_d is not a power of 5, since otherwise by the above paragraph with p=5, a would be 5, d would be divisible by 5 and would contain the same power of 5 as U_a, whereas $U_d>d$ for d>5. Therefore there exists a prime $p\neq5$ dividing U_d since $U_d>1$ for d>2. Then a(p), being a divisor of $p\pm1$, is prime to p and is a factor of d. Let us write

$$d=ah, \quad m=m'd=m'ah, \quad n=n'd=n'ah, \quad mn=m'n'a^2h^2.$$

Finally let p^μ and p^ν be the highest powers of p dividing m and n respectively, then the highest powers of p dividing U_m, U_n, U_mU_n and U_{mn} are

Since, by assumption, $V_mV_n | V_{mn}$, we have $2P+M+N \leq P+M+N$, i.e., $P \leq 0$, which is absurd. Thus d must be 1, which completes the proof.

Analogous theorems are valid for other recurring sequences of second order. In particular, for the sequences (2^n-1) and (2^n+1) we have:

THEOREM 3. $2^{mn}-1$ is divisible by $(2^m-1)(2^n-1)$ if and only if m and n are coprime.

THEOREM 4. $2^{mn}+1$ is divisible by $(2^m+1)(2^n+1)$ if and only if m and n are odd and coprime.

UNBOUNDEDNESS OF THE FUNCTION $[p-(5/p)]/a(p)$ IN FIBONACCI'S SEQUENCE

Let U_n denote the n-th term of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots \qquad (U_{n+2}=U_{n+1}+U_n)$$

Let U_a $(a>0)$ be the first term of the Fibonacci sequence divisible by a given prime p. Then p is said to be a primitive prime factor of U_a; $a=a(p)$ is called the rank of apparition of p and is some divisor of $p-(5/p)$, the symbol $(5/p)$ being Legendre's symbol. This note is devoted to the following theorem.

THEOREM. Let a(p) be the rank of apparition of p. Then $[p-(5/p)]/a(p)$ is an unbounded function of p.

We give for this theorem two related proofs. The first proof is based on Carmichael's theorem that every U_n, $n\neq 1,2,6,12$, contains at least one primitive prime factor p * . The second proof is based on the special and simple case of Carmichael's theorem that every U_q, q an odd prime, contains at least one primitive prime factor p ** , and the special case of Dirichlet's theorem that the arithmetical progression $(3k_1-1)(3k_1+1)\ldots(3k_s-1)(3k_s+1)x+3$, where $k_1,\ldots,k_s$ are integers >0 and $x=1, 2, 3,\ldots$, represents an infinitude of primes.

For the first proof we need the following lemma.

LEMMA. For every set $k_1, k_2, \ldots, k_s$ of integers >0 there exists an integer $n>12$ such that all the numbers $nk_1\pm1, nk_2\pm1, \ldots, nk_s\pm1$ are composite.

PROOF. Put $n=(4k_1-1)(4k_1+1)\ldots(4k_s-1)(4k_s+1)+4$. Then $n>12$, and $nk_i\pm1=\{(4k_1-1)(4k_1+1)\ldots(4k_s-1)(4k_s+1)+4\}k_i\pm1$ is divisible by $4k_i\pm1$, a factor >1 and $<nk_i\pm1$. Hence for all i, $nk_i\pm1$ are composite.

FIRST PROOF OF THE THEOREM. Let p be a primitive prime factor of U_n, $n>12$, the existence of p being assured by Carmichael's theorem. Since $a(p)=n>12>5=a(5)$, the factor p is $\neq 5$. Then $[p-(5/p)]/a(p)=k$ yields: $p=a(p)k+(5/p)=nk\pm1$. Suppose, if possible, that the function $[p-(5/p)]/a(p)=k$ is bounded so that k can take only the values $k_1, k_2, \ldots, k_s$. Then, by the lemma, n can be chosen so that all $nk_i\pm1$ are composite. This contradicts the fact that $nk\pm1$ is a prime p, which proves the theorem.

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* R. D. Carmichael, On the numerical factors of the arithmetic forms $\alpha^n\pm\beta^n$, Annals of Mathematics (2) 15, 1913-4, p. 61-2.

** This case is an immediate result of the well-known relation $(U_m,U_n)=U_{(m,n)}$.

For the second proof we need the following lemma.

LEMMA. For every set $k_1, k_2, \ldots, k_s$ of integers >0 there exists a prime $q>5$ such that all the numbers $qk_1 \pm 1$, $qk_2 \pm 1$, ..., $qk_s \pm 1$ are composite.

PROOF. By Dirichlet's theorem, the arithmetical progression

$$\{(3k_1-1)(3k_1+1)\ldots(3k_s-1)(3k_s+1)\}x+3, \qquad x=1,2,3,\ldots$$

contains at least one prime $q>5$ for let us say, $x=x_q$. But then

$$qk_i \pm 1=\{(3k_1-1)(3k_1+1)\ldots(3k_s-1)(3k_s+1)x_q+3\}k_i \pm 1$$

is divisible by $3k_i \pm 1$, a factor >1 and $<qk_i \pm 1$, since $q \geq 7$. Hence for all i, $qk_i \pm 1$ are composite.

SECOND PROOF OF THE THEOREM. Let q be a prime >5, and let p be a prime factor of U_q. Since $a(p)=q>5=a(5)$, the factor p is $\neq 5$. Then $[p-(5/p)]/a(p)=k$ yields: $p=a(p)k+(5/p)=qk \pm 1$. Suppose, if possible, that the function $[p-(5/p)]/a(p)=k$ is bounded so that k can take only the values $k_1, k_2, \ldots, k_s$. Then by the lemma the prime q can be chosen so that all $qk_i \pm 1$ are composite. This contradicts the fact that $qk \pm 1$ is a prime p, which proves the theorem.

TABLE OF THE RANKS OF APPARITION IN FIBONACCI'S SEQUENCE

Notations: p - a prime. $e=e(p)=1$ if $p=10k \pm 1$, $e=-1$ if $p=10k \pm 3$ or $p=2$, $e(5)=0$. $f=f(p)$ - factorization of p-e. $a=a(p)$ - rank of apparition of p. The factors of $(p-e)/a(p)$ are underlined.

p	f	a	p	f	a	p	f	a	p	f	a
2	3	3	283	$2^2.71$	284	661	$2^2.3.5.11$	55	1087	$2^6.17$	64
3	2^2	4	293	$2.3.7^2$	147	673	2.337	337	1091	2.5.109	1090
5	5	5	307	$2^2.7.11$	44	677	2.3.113	113	1093	2.547	547
7	2^3	8	311	2.5.31	310	683	$2^2.3^2.19$	684	1097	2.3.3.61	183
11	2.5	10	313	2.157	157	691	2.3.5.23	138	1103	$2^4.3.23$	48
13	2.7	7	317	2.3.53	159	701	$2^2.5^2.7$	175	1109	2.2.277	554
17	2.3^2	9	331	2.3.5.11	110	709	2.2.3.59	118	1117	2.13.43	559
19	2.3^2	18	337	2.13^2	169	719	2.359	718	1123	$2^2.281$	1124
23	$2^3.3$	24	347	$2^2.3.29$	116	727	$2^3.7.13$	728	1129	$2.2^2.3.47$	564
29	2.2.7	14	349	2.2.3.29	174	733	2.367	367	1151	2.5.5.23	230
31	2.3.5	30	353	2.3.59	59	739	$2.3^2.41$	738	1153	2.577	577
37	2.19	19	359	2.179	358	743	$2^3.3.31$	248	1163	$2^2.3.97$	1164
41	$2.2^2.5$	20	367	$2^4.23$	368	751	$2.3.5^3$	750	1171	$2.3^2 5.13$	1170
43	$2^2.11$	44	373	2.11.17	187	757	2.379	379	1181	$2^2.5.59$	295
47	$2^4.3$	16	379	$2.3^3.7$	378	761	$2^3.5.19$	95	1187	$2^2.3^3.11$	1188
53	2.3^3	27	383	$2^7.3$	384	769	$2^3.2^5.3$	96	1193	2.3.199	597
59	2.29	58	389	$2^2.97$	97	773	$2.3^2.43$	387	1201	$2.2^3.3.5^2$	600
61	$2^2.3.5$	15	397	2.199	199	787	$2^2.197$	788	1213	2.607	607
67	$2^2.17$	68	401	$2^2.2^2.5^2$	100	797	2.3.7.19	57	1217	2.3.7.29	203
71	2.5.7	70	409	$2.2^2.3.17$	204	809	$2^2.2.101$	202	1223	$2^3.3.3.17$	408
73	2.37	37	419	2.11.19	418	811	$2.3.3^3.5$	270	1229	2.2.307	614
79	2.3.13	78	421	$2^2.3.5.7$	21	821	$2^2.5.41$	205	1231	2.3.5.41	410
83	$2^2.3.7$	84	431	2.5.43	430	823	$2^3.103$	824	1237	2.619	619
89	$2^3.11$	11	433	2.7.31	217	827	$2^2.3^2.23$	828	1249	$2^2.2^3.3.13$	312
97	2.7^2	49	439	2.3.73	438	829	$2^2.3.3.23$	69	1259	2.17.37	1258
101	$2.2.5^2$	50	443	$2^2.3.37$	444	839	2.419	838	1277	2.3.3.71	213
103	$2^3.13$	104	449	$2.2^5.7$	224	853	2.7.61	427	1279	2.3.3.71	426
107	$2^2.3.3^2$	36	457	2.229	229	857	2.3.11.13	429	1283	$2^2.3.107$	1284
109	$2^2.3^3$	27	461	2.2.5.23	46	859	2.3.11.13	78	1289	$2^2.2.7.23$	322
113	2.3.19	19	463	$2^4.29$	464	863	$2^5.3^3$	864	1291	2.3.5.43	430
127	2^7	128	467	$2^2.3^2.13$	468	877	2.439	439	1297	2.11.59	649
131	2.5.13	130	479	2.239	478	881	$2.2^3.5.11$	88	1301	$2^2.5^2.13$	325
137	2.3.23	69	487	$2^3.61$	488	883	$2^2.13.17$	884	1303	$2^3.163$	1304
139	2.3.23	46	491	$2.5.7^2$	490	887	$2^3.3.37$	888	1307	$2^2.3.109$	436
149	$2^2.37$	37	499	2.3.83	498	907	$2^2.227$	908	1319	2.659	1318
151	$2.3.5^2$	50	503	$2^3.3^2.7$	504	911	2.5.7.13	70	1321	$2.2^2.35.11$	660
157	2.79	79	509	2.2.127	254	919	$2.3^2.3.17$	102	1327	$2^4.83$	1328
163	$2^2.41$	164	521	$2^2.2.5.13$	26	929	$2.2^4.29$	464	1361	$2^2.2^2.5.17$	340
167	$2^3.3.7$	168	523	$2^2.131$	524	937	2.7.67	469	1367	$2^3.3^2.19$	1368
173	2.3.29	87	541	$2.2.3.3^2.5$	90	941	2.2.5.47	470	1373	2.3.229	687
179	2.89	178	547	$2^2.137$	548	947	$2^2.3.79$	948	1381	$2^2.3.5.23$	115
181	$2.2.3^2.5$	90	557	$2.3^2.31$	31	953	$2.3^2.53$	53	1399	2.3.233	1398
191	2.5.19	190	563	$2^2.3.47$	188	967	$2^3.11.11$	88	1409	$2^2.2^5.11$	352
193	2.97	97	569	$2.2^2.71$	284	971	2.5.97	970	1423	$2^4.89$	1424
197	$2.3^2.11$	99	571	2.3.5.19	570	977	2.3.163	163	1427	$2^2.3.7.17$	84
199	$2.3^2.11$	22	577	2.17^2	289	983	$2^3.3.41$	984	1429	$2^2.3.7.17$	357
211	2.3.5.7	42	587	$2^2.3.7^2$	588	991	$2.3^2.5.11$	198	1433	2.3.239	717
223	$2^5.7$	224	593	$2.3^3.11$	297	997	2.499	499	1439	2.719	1438
227	$2^2.3.19$	228	599	2.13.23	598	1009	$2^3.2.3^2.7$	126	1447	$2^3.181$	1448
229	2.2.3.19	114	601	$2.2^2.3.5^2$	300	1013	$2.3.13^2$	507	1451	$2.5^2.29$	1450
233	$2.3^2.13$	13	607	$2^5.19$	608	1019	2.509	1018	1453	2.727	727
239	2.7.17	238	613	2.307	307	1021	2.2.35.17	510	1459	2.3^6	1458
241	$2.2^3.3.5$	120	617	2.3.103	309	1031	2.5.103	206	1471	$2.3.5.7^2$	490
251	2.5^3	250	619	2.3.103	206	1033	2.11.47	517	1481	$2.2^2.5.37$	740
257	2.3.43	129	631	$2.3^2.5.7$	630	1039	2.3.173	1038	1483	$2^2.7.53$	212
263	$2^3.3.11$	88	641	$2.2^6.5$	320	1049	$2^2.2.131$	262	1487	$2^4.3.31$	1488
269	$2^2.67$	67	643	$2^2.7.23$	644	1051	$2.3.5^2.7$	1050	1489	$2.2^3.3.31$	744
271	$2.3^3.5$	270	467	$2^3.3^4$	648	1061	2.2.5.53	530	1493	$2.3^2.83$	747
277	2.139	139	653	2.3.109	327	1063	$2^3.7.19$	1064	1499	2.7.107	1498
281	$2.2^2.5.7$	28	659	2.7.47	658	1069	$2^2.3.89$	89	1511	2.5.151	302

INEQUALITIES FOR THE PRODUCT OF TWO FIBONACCI NUMBERS

For Fibonacci numbers defined by $U_1=U_2=1$, $U_n=U_{n-1}+U_{n-2}$ the following equalities hold.

$$U_{a+b}=U_{a+1}U_{b+1}-U_{a-1}U_{b-1}, \qquad U_{a+b-1}=U_aU_b+U_{a-1}U_{b-1}$$

These equalities may be proved by induction on b, b+1. Beginning with U_2 the sequence increases. From these facts the following inequalities may be deduced.

(1) $k,l\neq 2\rightarrow \qquad U_{k+l-2}<U_kU_l<U_{k+l-1}\leftarrow k,l\neq 1$

(2) $k,l\geq 3,\ k+l<k'+l'\rightarrow \qquad U_kU_l<U_{k'}U_{l'}$

(3) $k,l\geq 3\rightarrow \qquad U_kU_l<U_{kl}$

(4) $k,\dots,l\geq 3\rightarrow \qquad U_k\dots U_l<U_{k\dots l}$

(5) $k,l\geq 3,\ kl=k'l',\ |k-l|<|k'-l'|\rightarrow \qquad U_kU_l<U_{k'}U_{l'}$

(6) $k,l\geq 3,\ kl\leq k'l',\ |k-l|/\sqrt{kl}<|k'-l'|/\sqrt{k'l'}\rightarrow U_kU_l<U_{k'}U_{l'}$

Proofs.

(1) $\underline{U_{k+l-2}}=U_{(k-1)+(l-1)}=U_kU_l-U_{k-2}U_{l-2}<\underline{U_kU_l}\leq U_kU_l+U_{k-1}U_{l-1}=\underline{U_{k+l-1}}$.

(2) By (1): $k+l-1<k'+l'-2\rightarrow \underline{U_kU_l}<U_{k+l-1}\leq U_{k'+l'-2}<\underline{U_{k'}U_{l'}}$.

(3) For reasons of symmetry we may suppose that $k\geq l$. Hence, by (2): $k+l\leq k+k=2k<kl<kl+1\rightarrow \underline{U_kU_l}<U_{kl}U_1=\underline{U_{kl}}$.

(4) By induction on n we have by (3): $\underline{U_k\dots U_iU_l}=(U_k\dots U_i)U_l<U_{k\dots i}U_l<\underline{U_{k\dots il}}$.

Lemma. $k,l>0,\ kl=k'l',\ |k-l|<|k'-l'|\rightarrow k+l<k'+l'$.

Proof. This lemma evidently states that the nearer a rectangle, of given area, is to a square, the smaller its circumference.

(5) Lemma and (3).

(6) Denote $k'l'/kl=m$. Hence $(k\sqrt{m})(l\sqrt{m})=k'l'$. From $|k-l|\sqrt{m}<|k'-l'|$ it follows $|k\sqrt{m}-l\sqrt{m}|<|k'-l'|$. Hence by the lemma $k\sqrt{m}+l\sqrt{m}<k'+l'$, thus $k+l<k'+l'$. Hence by (2) the result.

As an application of (3) we shall prove the following two results.

Theorem 1. The greatest primitive divisor of U_{p^e}, $p\neq 5$ being a prime, and $e>1$ being a positive integer, is greater than U_p.

Proof. It may be shown that, for $p\neq 5$, $(p,U_{p^{e-1}})=1$. Hence by the law of repetition of primes in (U_n) we deduce that the greatest imprimitive divisor of U_{p^e} equals $U_{p^{e-1}}$. And by (3) we have $U_{p^{e-1}}U_p<U_{p^e}$, that is $U_{p^e}/U_{p^{e-1}}>U_p$.

Theorem 2. Every U_n, n being a prime-power other than 2 and other than a power of 5, or n being a product $\neq 6$ of two different primes p, q such that $p\nmid U_q$, $q\nmid U_p$, have at least one primitive prime divisor.

Proof. For a prime-power the theorem follows from theorem 1, noting that for $p\neq 2$, $U_p>1$. For $n=pq\neq 6$, p, q being different primes such that $p\nmid U_q$, $q\nmid U_p$, the greatest imprimitive divisor of U_{pq} is U_pU_q. By (3) we have $U_pU_q<U_{pq}$. Hence $U_{pq}/U_pU_q>1$.

LINEAR FORMS OF PRIMITIVE PRIME DIVISORS OF FIBONACCI NUMBERS

1. Introduction. The object of the present note is to establish linear forms to which belong the primitive prime divisors of Fibonacci numbers. The use of linear forms permits a great reduction of the number of tests necessary for factorization. This method has already been used by Lucas[1], who, however, failed to combine his results and therefore obtained linear forms which are weaker than those given in the tables I and II following.

2. The sequences (U_n) and (V_n). Fibonacci's sequence (U_n) is defined by

$$U_1=1,\ U_2=1,\ U_n=U_{n-1}+U_{n-2}.$$

Its associated sequence (V_n) is defined by

$$V_1=1,\ V_2=3,\ V_n=V_{n-1}+V_{n-2}.$$

The first ten Fibonacci numbers are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

The first ten terms of (V_n) are:

1, 3, 4, 7, 11, 18, 29, 47, 76, 123.

We have $U_{2n}=U_nV_n$ [2].

3. Primitive divisors. A divisor $d>1$ of U_n (or V_n) is called *primitive* if it is relatively prime to every U_m (or V_m) with $m<n$. There existsimple rules for obtaining all the non-primitive divisors of U_n, once the factorization of all the U_m with $m|n$ is known. The problem of factorization of a Fibonacci number is thus reduced to the problem of factorization of its *greatest primitive divisor*. Moreover, the following proposition holds:

A. The greatest primitive divisors of U_{2n} and V_n coincide[3]. Therefore the factorization of U_{2n} is equivalent to that of V_n.

4. Linear forms.

I. Every primitive prime divisor p of U_n, for odd $n>5$, has one of the following linear forms:

If $n\equiv$	then $p\equiv$	
1 (mod 10)	1, 8n+1, 14n-1, 18n-1	(mod 20n)
3	1, 6n-1, 16n+1, 18n-1	
7	1, 2n-1, 4n+1, 14n-1	
9	1, 2n-1, 6n-1, 12n+1	
5	1	(mod 4n)

II. Every primitive prime divisor p of V_n (and of U_{2n}) has one of the following linear forms:

If $n\equiv$	then $p\equiv$	
1 (mod 10)	1, 8n+1	(mod 10n)
3	1, 6n+1	
7	1, 4n+1	
9	1, 2n+1	
5	1	(mod 2n)
0 (mod 20)	1	
10	1	(mod 4n)
2	1, 2n-1, 4n+1, 14n-1	(mod 20n)
6	1, 8n+1, 14n-1, 18n-1	
14	1, 2n-1, 6n-1, 12n+1	
18	1, 6n-1, 16n+1, 18n-1	
4	1, 2n-1, 2n+1, 6n-1	(mod 10n)
8	1, 6n-1, 6n+1, 8n-1	
12	1, 2n-1, 4n-1, 4n+1	
16	1, 4n-1, 8n-1, 8n+1	

The proof of I is based on the next two propositions, which have been stated by Lucas:

B. For odd n, every odd prime divisor of U_n is $\equiv 1 \pmod 4$ [4].

C. Every primitive prime divisor p of U_n is $\equiv (\frac{5}{p}) \pmod n$, $(\frac{5}{p})$ being Legendre's symbol [5].

Combining B and C and noting that $(\frac{5}{p})=1$ for $p\equiv\pm1 \pmod{10}$ and $(\frac{5}{p})=-1$ for $p\equiv\pm3 \pmod{10}$, whence for odd $n>5$ $p\equiv(\frac{5}{p}) \pmod{2n}$, we have the table I.

The proof of II is based on the following two propositions:

D. For odd n, every odd prime divisor of V_n is $\equiv \pm1 \pmod{10}$ [6]. For $n\equiv2 \pmod 4$, every odd prime divisor pf V_n is $\equiv$ 1, 3, 9, 27 (mod 40). For $n\equiv0 \pmod 4$, every odd prime divisor of V_n is $\equiv$ 1, 7, 9, 23 (mod 40) [7].

E. Every primitive prime divisor p of V_n is $\equiv (\frac{5}{p}) \pmod{2n}$

(This follows immediately from A and C).

Combining D and E we have the table II.

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1) Comptes Rendus Paris 82 (1876), 167 and American Journal of Mathematics 1 (1878), 298.
2) E. Lucas, Amer. Jour. Math. 1 (1878), 185.
3) P. Bachmann, Niedere Zahlentheorie II (1910), 83.
4) E. Lucas, Amer. Jour. Math. 1 (1878), 200.
5) l. c. 297.
6) l. c. 201.
7) l. c. 201, 212.

The following question arises: can theorems I, II be improved by proving that, beginning with a certain n, p can belong only to some of the classes listed there?

The following theorems give a partial answer to the above question.

THEOREM 1. Every U_q with prime q=7, 11, 13, 17, 23, 29, 31, 37, 43, 47, 49, 53 (mod 60) has at least one prime divisor $p\not\equiv 1$ (mod 20q). In particular:

If q=11, 31 (mod 60) then U_q has at least one prime divisor p=8q+1, 14q-1, 18q-1 (mod 20q).

If q=13, 23, 43, 53 (mod 60) then U_q has at least one prime divisor p=6q-1, 16q+1, 18q-1 (mod 20q).

If q=7, 17, 37, 47 (mod 60) then U_q has at least one prime divisor p=2q-1, 4q+1, 14q-1 (mod 20q).

If q=29, 49 (mod 60) then U_q has at least one prime divisor p=2q-1, 6q-1, 12q+1 (mod 20q).

PROOF. It is well-known that any prime divisor p of U_q, with prime q, is a primitive divisor. Were all the divisors p=1 (mod 20q), then also U_q=1 (mod 20q), thus U_q=1 (mod 10). But, by the periodicity of the sequence (U_n), U_n=3, 7, 9 (mod 10), for any n=7, 11, 13, 17, 23, 29, 31, 37, 43, 47, 49, 53 (mod 60). Hence U_q has at least one prime divisor $p\not\equiv 1$ (mod 20q). The theorem in detail results from I.

THEOREM 2. Every V_q with prime q=7, 11, 19, 23, 31, 43, 47, 59 (mod 60), or with q being a power of 2, has at least one prime divisor $p\not\equiv 1$ (mod 10q). In particular:

If q=11, 31 (mod 60) then V_q has at least one prime divisor p=8q+1 (mod 10q).

If q=23, 43 (mod 60) then V_q has at least one prime divisor p=6q+1 (mod 10q).

If q=7, 47 (mod 60), or if q is a power of 2, then V_q has at least one prime divisor p=4q+1 (mod 10q).

If q=19, 59 (mod 60) then V_q has at least one prime divisor p=2q+1 (mod 10q).

PROOF. It is well-known that any prime divisor p of V_q with prime q, or with q being a power of 2, is a primitive divisor. Were all the divisors p=1 (mod 10q), then also V_q=1 (mod 10q), thus V_q=1 (mod 10). But, by the periodicity of the sequence (V_n), V_n=3, 7, 9 (mod 10), for n=7, 11, 19, 23, 31, 43, 47, 59 (mod 60), or for n being a power of 2. The theorem in detail results from II, noting that V_q=7 (mod 10) for q being a power of 2.

Since, by Dirichlet's theorem, there exist infinitely many primes for any of the forms listed in theorems 1, 2, the question raised above can be answered as follows:

It is impossible to improve I, in that sense that not all the classes different from 1 (mod 20n) may be canceled of no one of the theorems I1, I2, I3, I4. However, it has not been proved, although it is probable, that in no of these theorems, no two classes different from 1 (mod 20n) may be canceled beginning with a certain n, or one class different from 1 (mod 20n), or the class equalling 1 (mod 20n).

It is impossible to improve II, in that sense that the class different from 1 (mod 10n) may not be canceled of no one of the theorems II1, II2, II3, II4. However, it was not proved, although it is probable, that the class equalling 1 (mod 10n) may not be canceled beginning from a certain n.

APPEARANCE OF PRIME FACTORS IN THE SEQUENCE ASSOCIATED WITH FIBONACCI'S SEQUENCE

Let U= 1, 1, 2, 3, 5,... and V= 1, 3, 4, 7, 11,... denote Fibonacci's sequence and the sequence associated with it, in both of which each term is the sum of the two preceding terms. We shall say that a prime p appears as a factor in U (or V) if p divides some term of U (or V). It is known that every prime appears as a factor in U[1], while the primes $\equiv 3, 7, 11, 19 \bmod 20$ appear[2] and the primes $\equiv 13, 17 \bmod 20$ do not appear[3] as factors in V. The question which, and how many of, the primes $\equiv 1, 9 \bmod 20$ appear as factors in V has not been discussed so far and it is the purpose of this note to contribute to an answer for the primes $\equiv 1 \bmod 20$.

Let p be any prime factor of U_n (or V_n) such that p does not divide any U_m (or V_m) with $m<n$. Then p is called a p r i m i t i v e factor of U_n (or V_n). It is known that any primitive factor of U_{2n} is also a primitive factor of V_n and conversely.[4] Hence a prime p appears as a factor in V if and only if p is a primitive factor of a term of U with e v e n index.

We proceed to prove the following

THEOREM 1. Every primitive prime factor of $U_{5(2k+1)}$, where k is any positive integer, is $\equiv 1 \bmod 20$.

The proof is based on the following two propositions:

A. For odd n, every odd prime factor of U_n is $\equiv 1 \bmod 4$.[5]

B. For any positive integer n, every primitive prime factor p of U_n is $\equiv (\frac{5}{p}) \bmod n$, where $(\frac{5}{p})$ is Legendre's symbol.[6]

Noting that $(\frac{5}{p})=1$ for $p\equiv\pm1 \bmod 10$ and $(\frac{5}{p})=-1$ for $p\equiv\pm3 \bmod 10$, we deduce from B:

B'. For odd n, every primitive prime factor $p>5$ of U_n is $\equiv(\frac{5}{p}) \bmod 2n$.

Now let p be a primitive prime factor of $U_{5(2k+1)}$. Then $p>5$ (since $U_5=5$) and by B' we have: $p\equiv(\frac{5}{p}) \bmod 10$, that is $p\equiv\pm1 \bmod 10$. By the above remark we have $(\frac{5}{p})=1$, whence $p\equiv1 \bmod 10$, or $p\equiv1, 11 \bmod 20$. But by A, $p\equiv1, 9, 13, 17 \bmod 20$, whence $p\equiv1 \bmod 20$.

THEOREM 2. Every primitive prime factor p of V_{10k}, where k is any positive integer, is $\equiv1 \bmod 40$.

The proof is based on the following two propositions:

C. For $n\equiv2 \bmod 4$, every odd prime factor of V_n is $\equiv 1, 3, 9, 27 \bmod 40$. For $n\equiv0 \bmod 4$, every odd prime factor of V_n is $\equiv 1, 7, 9, 23 \bmod 40$.[7]

D. Every primitive prime factor p of V_n is $\equiv (\frac{5}{p}) \bmod 2n$. (This follows immediately from B and the preliminary remark about the primitive prime factors of V_n and U_{2n}).

Now let p be a primitive prime factor of V_{10k}. Then by D we have: $p\equiv(\frac{5}{p}) \bmod 20$, whence we deduce, as in the proof of Theorem 1, that $p\equiv1 \bmod 20$, or $p\equiv1, 21 \bmod 40$. Combining this result with C we have $p\equiv1 \bmod 40$.

THEOREM 3. There exists an infinitude of primes $\equiv 1 \bmod 20$ which do not appear as factors in V, as well as an infinitude of primes $\equiv 1 \bmod 40$ which appear as factors in V.

The proof follows, by the preliminary remark, immediately from Theorems 1 and 2 and from the following theorem: Every U_n with $n>12$ and every V_n with $n>6$ has at least one primitive prime factor.[8]

It would appear that among the primes $\equiv 9 \bmod 20$, too, there exists an infinitude of numbers which do not appear in V, as well as an infinitude of numbers which appear in V. If this turned out to be correct, the following question would arise: Is there a number m such that among the primes p with $p\equiv1, 9 \bmod 20$ those belonging to certain classes modulo m appear as factors in V, while those belonging to the remaining classes do not appear as factors in V?

R e f e r e n c e s

1. E. Lucas, Amer. Jour. Math. 1 (1878), 297.
2. S. Kernbaum, Wiadomości Matematyczne (Warsaw), 25 (1921), 52
3. E. Lucas, Amer. Jour. Math. 1 (1878), 298.
4. P. Bachmann, Niedere Zahlentheorie II (1910), 83.
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7. E. Lucas, Ibid., 201, 212.
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THE ALGEBRAIC FACTORS OF V_{5n}/V_n (n ODD) IN THE SEQUENCE (V_n) ASSOCIATED WITH FIBONACCI'S SEQUENCE

Let (U_n) denote Fibonacci's sequence

(1) $U_0=0,\ U_1=1,\qquad U_n=U_{n-1}+U_{n-2}\qquad (n=0,\pm1,\pm2,\dots)$

and (V_n) the associated sequence

(2) $V_0=2,\ V_1=1,\qquad V_n=V_{n-1}+V_{n-2}\qquad (n=0,\pm1,\pm2,\dots)$

or, explicitly,

(3) $U_n=(\alpha^n-\beta^n)/\sqrt{5},\qquad V_n=\alpha^n+\beta^n,\qquad$ where $\alpha=(1+\sqrt{5})/2,\ \beta=(1-\sqrt{5})/2$

The equivalence of (3) to (1), (2) becomes clear when one considers that (3) is valid for n=0,1 and that (3) fulfils the recursion-formula common to (1) and (2).

Using (3) it is easy to verify the following factorization-formula

(4) $V_{5n}/V_n=A_nB_n$, where $A_n=5U_n^2-5U_n+1,\ B_n=5U_n^2+5U_n+1,\ 2\nmid n$

The object of this paper is to prove the following divisibility-properties of A_n, B_n.

THEOREM 1. A_n (n odd) divides A_{mn} for every m terminating (in the decimal system) in 1 or 9; it divides B_{mn} for every m terminating in 3 or 7.

B_n (n odd) divides B_{mn} for every m terminating (in the decimal system) in 1 or 9; it divides A_{mn} for every m terminating in 3 or 7.

In other words, for every odd n and every positive integer k, we have:

A) $A_{(10k+1)n}\equiv0 \pmod{A_n}$ A') $B_{(10k+1)n}\equiv0 \pmod{B_n}$

B) $B_{(10k+3)n}\equiv0 \pmod{A_n}$ B') $A_{(10k+3)n}\equiv0 \pmod{B_n}$

C) $B_{(10k+7)n}\equiv0 \pmod{A_n}$ C') $A_{(10k+7)n}\equiv0 \pmod{B_n}$

D) $A_{(10k+9)n}\equiv0 \pmod{A_n}$ D') $B_{(10k+9)n}\equiv0 \pmod{B_n}$

To prove Theorem 1 we shall use the following results from the theory of the sequences (U_n), (V_n)

(5) $U_{-n}=(-1)^{n+1}U_n$

(6) $U_{m+n}=U_mU_{n-1}+U_{m+1}U_n$

(7) $U_{2n}=U_nV_n$

(8) $U_{2n\pm1}=U_{n\pm1}V_n-(-1)^n$

(9) $U_{3n}=5U_n^3+(-1)^n3U_n$

as well as the following identity

(10) $125U^6-150U^4+25U^3+45U^2-15U+1=(5U^2-5U+1)(25U^4+25U^3-10U^2-10U+1)$

The formulae (5)-(9) may be easily verified by means of (3). However, it seems worth - while proving (4)-(9) without the use of irrational numbers. For this purpose we need the following further formulae:

(11) $V_n=U_{n-1}+U_{n+1}$

(12) $U_{m+n}=U_mV_n-(-1)^nU_{m-n}$

(13) $V_mV_n=V_{m+n}+(-1)^nV_{m-n}$

(14) $V_{n+1}=5U_n-V_{n-1}$

(15) $V_{n+m}=5U_nU_m+(-1)^mV_{n-m}$

(16) $V_n^2=V_{2n}+(-1)^n2$

(17) $V_{2n}=5U_n^2+(-1)^n2$

(18) $V_n^2=5U_n^2+(-1)^n4$

Now, (5), (6), (11), (12), (13), (14) follow by induction on n, n+1, since they are true for n=0,1.

(15) follows by induction on m, m+1, since it is true for m=0, and, by (14), for m=1.

(16) is the case m=n of (13).

(17) is the case m=n of (15).

(18) follows by adding (16), (17).

(7) follows from (6), (11), for m=n.

(8) is the case m=n±1 of (12).

(9) follows from (12), (7), (18). Namely:

$$U_{3n}=U_{2n+n}=U_{2n}V_n-(-1)^nU_n=U_n(V_n^2-(-1)^n)=U_n(5U_n^2+(-1)^n3)=5U_n^3+(-1)^n3U_n.$$

(4) follows from (15), (7), (9). Namely:

$$V_{5n}=V_{3n+2n}=5U_{3n}U_{2n}+V_n=V_n(5U_{3n}U_n+1);$$

$$V_{5n}/V_n=5U_{3n}U_n+1$$
$$=5(5U_n^3-3U_n)U_n+1$$
$$=25U_n^3-15U_n^2+1$$
$$=(5U_n^2-5U_n+1)(5U_n^2+5U_n+1).$$

Lemma. For odd n and arbitrary integral r

(19) $A_{(10k+r)n} \equiv A_{rn} \pmod{V_{5n}}, \qquad B_{(10k+r)n} \equiv B_{rn} \pmod{V_{5n}}$

PROOF. By (7), (8)

$$U_{10n} = U_{5n}V_{5n}, \qquad U_{10n\pm1} = U_{5n\pm1}V_{5n}+1$$

whence

$$U_{10n} \equiv 0 \pmod{V_{5n}}, \qquad U_{10n\pm1} \equiv 1 \pmod{V_{5n}}$$

whence, by (6),

$$\begin{aligned} A_{(10k+r)n} &= 5U^2_{(10k+r)n} - 5U_{(10k+r)n}+1 \\ &= 5(U_{10kn}U_{rn-1}+U_{10kn+1}U_{rn})^2 - 5(U_{10kn}U_{rn-1}+U_{10kn+1}U_{rn})+1 \\ &\equiv 5U^2_{rn} - 5U_{rn}+1 \\ &= A_{rn} \pmod{V_{5n}} \end{aligned}$$

Similarly the congruence with B is proved.

The lemma shows that in order to prove Theorem 1 it suffices to prove

a) $A_{1n} \equiv 0 \pmod{A_n}$	a') $B_{1n} \equiv 0 \pmod{B_n}$
b) $B_{3n} \equiv 0 \pmod{A_n}$	b') $A_{3n} \equiv 0 \pmod{B_n}$
c) $B_{7n} \equiv 0 \pmod{A_n}$	c') $A_{7n} \equiv 0 \pmod{B_n}$
d) $A_{9n} \equiv 0 \pmod{A_n}$	d') $B_{9n} \equiv 0 \pmod{B_n}$

Of these propositions, a and a' are trivial. For r=-3, k=1, we obtain from (19), by (5),

$$A_{7n} = A_{(10-3)n} \equiv A_{-3n} = A_{3n} \pmod{V_{5n}}, \qquad B_{7n} = B_{(10-3)n} \equiv B_{-3n} = B_{3n} \pmod{V_{5n}}$$

that is

(20) $A_{7n} \equiv A_{3n} \pmod{V_{5n}}, \qquad B_{7n} \equiv B_{3n} \pmod{V_{5n}}$

Similarly

(21) $A_{9n} \equiv A_n \pmod{V_{5n}}, \qquad B_{9n} \equiv B_n \pmod{V_{5n}}$

Therefore, by (4),

$$A_{7n} \equiv A_{3n} \pmod{B_n}, \qquad B_{7n} \equiv B_{3n} \pmod{A_n}$$

$$A_{9n} \equiv A_n \pmod{A_n}, \qquad B_{9n} \equiv B_n \pmod{B_n}$$

Thus, it is sufficient to prove that b, b' are true.

Indeed, by (9), (10),

$$\begin{aligned} B_{3n} &= 5U^2_{3n} + 5U_{3n}+1 \\ &= 5(5U^3_n - 3U_n)^2 + 5(5U^3_n - 3U_n)+1 \end{aligned}$$

$$\begin{aligned} &= 125U^6_n - 150U^4_n + 25U^3_n + 45U^2_n - 15U_n + 1 \\ &= (5U^2_n - 5U_n+1)(25U^4_n + 25U^3_n - 10U^2_n - 10U_n + 1) \\ &= A_n(25U^4_n + 25U^3_n - 10U^2_n - 10U_n + 1) \\ &\equiv 0 \pmod{A_n}. \end{aligned}$$

Similarly one proves b'.

THEOREM 2. $(A_n, B_n)=1$.

PROOF. $(A_n,B_n)=(A_n,B_n-A_n)=(5U^2_n-5U_n+1, 10U_n)=1$.

THEOREM 3. For n odd, the consecutive values of A_n, as well as of B_n, form recurring sequences of order 5 with the common scale 1 -11 33 -33 11 -1. The order 5 is exact.

PROOF. $A_n=5U^2_n-5U_n+1$. For n odd the scale of (U_n) (and of $(-5U_n)$) is: -1 3 -1. The scale of (U^2_n) (and of $(5U^2_n)$) is: 1 -8 8 -1. Thus the scale of $(5U^2_n-5U_n)$ is: 1 -11 33 -33 11 -1. Since the sum of the members of the last scale vanishes, it does not change when we add a constant to all the members of the sequence. Thus, the same scale also serves for $(A_n=5U^2_n-5U_n+1)$. A similar proof applies to B_n.

To prove that the order 5 of (A_n) is exact it suffices, by Kronecker's criterion, to show that

$$D=\begin{vmatrix} A_1 & A_3 & A_5 & A_7 & A_9 \\ A_3 & A_5 & A_7 & A_9 & A_{11} \\ A_5 & A_7 & A_9 & A_{11} & A_{13} \\ A_7 & A_9 & A_{11} & A_{13} & A_{15} \\ A_9 & A_{11} & A_{13} & A_{15} & A_{17} \end{vmatrix} \neq 0$$

To do this it suffices to show that $D \not\equiv 0 \pmod m$ for at least one positive integer m. Thus we can replace in D every A_i by r_i where $r_i \equiv A_i \pmod m$. In fact, supposing m=7 we have

$$D\equiv\begin{vmatrix} 1 & -3 & 3 & -3 & -3 \\ -3 & 3 & -3 & -3 & 3 \\ 3 & -3 & -3 & 3 & -3 \\ -3 & -3 & 3 & -3 & 1 \\ -3 & 3 & -3 & 1 & 1 \end{vmatrix} = 4 \not\equiv 0 \pmod 7)$$

Similarly we have for (B_n): $D \equiv 2 \not\equiv 0 \pmod 7$.

CONJECTURE. For n odd, every A_n (n>5) and every B_n have at least one prime divisor, being a primitive divisor of V_{5n} (that is a divisor which does not divide any V_x with $1<x<5n$).

If this conjecture turned out to be correct, it would appear that for n odd every V_{5n} (n>5) have at least t w o primitive prime divisors (so far the existence of one such divisor is known).

$A_n=5U_n^2-5U_n+1=3+V_{2n}-5U_n$	n	$B_n=5U_n^2+5U_n+1=3+V_{2n}+5U_n$
1	1	11
11	3	31
101	5	151
781	7	911
5611	9	5951
39161	11	40051
270281	13	272611
1857451	15	1863551
12744061	17	12760031
87382901	19	87424711
599019851	21	599129311
4105974961	23	4106261531
28143378001	25	28144128251
192899171531	27	192901135711
1322154751061	29	1322159893351
9062194370461	31	9062207833151
62113232767531	33	62113268013311
425730505493801	35	425730597768451
2918000490238361	37	2918000731816531
20000273409331051	39	20000274041790911
137083914639998701	41	137083916295800111
939587132382262661	43	939587136717207031
6440026020705728651	45	6440026032054760351
44140594767846512 81	47	44140595064968020 11
302544139285509881761	49	302544139363297302251
2073668380118888112011	51	2073668380322538222751
14213134521954007123781	53	14213134522487170035511
97418273274625487577901	55	97418273276021326202351
667714778403216083170411	57	667714778406870436132031
4576585175555195800538201	59	4576585175564763020798611
31368381450502288961117801	61	31368381450527336268937411
215002084978010921542925611	63	215002084978076496246124031
1473646213395705311245758301	65	1473646213395876988047533951
10100521408792269610780933781	67	10100521408792719066483062311
69230003648151080875625035211	69	69230003648152257565929645151
474509504128267649899203532561	71	474509504128270730514415233851
3252336525249728629649223095281	73	3252336525249736694804553589211
22291846172619848887956019122251	75	22291846172619870002806798902751
152790586683089255815744470321461	77	152790586683089311095141479169031

FACTORIZATION OF $V_{5n}/V_n = (5U_n^2-5U_n+1)(5U_n^2+5U_n+1)$

Prime subscripts and primitive prime factors are underlined

All factorizations of the primitive divisors of A_n with n = 37, 47, 53-61, 65-73, 77 (except the factor 571 of A_{57}), and of B_n with n = 37, 47-77 (except the factors 941 of B_{47} and 1061 of B_{53}) are due to John Brillhart.

n	$A_n=5U_n^2-5U_n+1$	$B_n=5U_n^2+5U_n+1$
1		11
3	11	31
5	101	151
7	11.71	911
9	31.181	11.541
11	39161	11^2.331
13	11.24571	131.2081
15	151.12301	101.18451
17	11.1158551	12760031
19	87382901	11.191.41611
21	31.911.21211	11.71.767131
23	11.1151.324301	5981.686551
25	28143378001	251.112128001
27	11.271.541.119611	31.181.811.42391
29	1322154751061	11.120196353941
31	311.29138888651	11.823837075741
33	11^2.331.1550853481	31.39161.51164521
35	151.54601.51636551	101.560701.7517651
37	11.265272771839851	2918000731816531
39	31.131.2081.2731.866581	11.1951.24571.37928281
41	1231.111359800682371	11.5741.2170732312961
43	11.1291.66163448516461	431.1721.1266715025281
45	101.18451.221401.15608701	151.12301.3467131047901
47	11.119851.33481417483721	941.6581.8461.842432231
49	491.911.1471.459807660691	11.71.88972241.4353947431
51	31.1021.53551.95881.12760031	11.1158551.162716451241291
53	11.17491.73872456598219381	1061.124021.7627231.14161601
55	101.96453735915470779780 1	151.92401.698211964759801
57	11.191.571.41611.32491.411677941	31.87382901.2069101.119130001
59	552241.8287296987284891561	11.12391.335838031.99979884881
61	86011.30727531.11868899378561	11.2851671040957030569903401
63	11.71.541.631.767131.1051224514831	31.181.911.21211.1983000765501001
65	151.3251.843701.3558039391073701	101.14590556568276009782648851
67	11.918229218981115419161903071	10100521408792719066483062311
69	31.5981.686551.4641631.117169733521	11.1151.4831.324301.3490125311294161
71	474509504128267649899203532561	11.43137227648024611864946839441
73	11.514651.7015301.8942501.9157663121	3252336525249736694804553589211
75	251.751.2251.112128001.46853582653501	28143378001.792081397330050024751
77	11^2.71.331.84100171.582276311.1097233061	911.3851.39161.1112118152741317993 81

ON THE GREATEST PRIMITIVE DIVISORS OF FIBONACCI AND LUCAS NUMBERS WITH PRIME-POWER SUBSCRIPTS

The greatest primitive divisor U'_n of a Fibonacci number U_n is defined as the greatest divisor of U_n relatively prime to every U_x with positive $x < n$.

Similarly, the greatest primitive divisor V'_n ofaLucas number V_n is defined as the greatest divisor of V_n relatively prime to every V_x with non-negative $x < n$.

The first 20 values of the sequence (U'_n) are:

$U'_1 = 1$, $U'_2 = 1$, $U'_3 = 2$, $U'_4 = 3$, $U'_5 = 5$, $U'_6 = 1$, $U'_7 = 13$, $U'_8 = 7$,
$U'_9 = 17$, $U'_{10} = 11$, $U'_{11} = 89$, $U'_{12} = 1$, $U'_{13} = 233$, $U'_{14} = 29$,
$U'_{15} = 61$, $U'_{16} = 47$, $U'_{17} = 1597$, $U'_{18} = 19$, $U'_{19} = 4181$, $U'_{20} = 41$.

As may be seen from these few examples, the growth of the sequence (U'_n) is very irregular. However, some special subsequences of (U'_n) may occur to be increasing sequences. E.g., the subsequence (U'_p), where p ranges over all the primes, is a strictly increasing sequence (since $U'_p = U_p$ and (U_n) is a stictly increasing sequence beginning with n = 2).

Similarly, the subsequence (V'_q), where q ranges over all the odd primes and over all the powers of 2 beginning with 2^2, is a strictly increasing sequence.

The main object of this note is to prove the following inequalities:

(1) $U'_{p^{x+1}} > U'_{p^x}$ (p - a prime, x - a positive integer)

(2) $U'_{2p^{x+1}} > U'_{2p^x}$ (p - a prime, x - a nonnegative integer)

(2*) $V'_{p^{x+1}} > V'_{p^x}$ (p - a prime, x - a nonnegative integer)

In other words: the subsequences (U'_{p^x}) and (U'_{2p^x}) of the sequence (U'_n), as well as the subsequence (V'_{p^x}) of the sequence (V'_n), p being a prime and x = 1,2,3,..., are strictly increasing sequences.

Since (as is well known) the primitive divisors of U_{2n} and V_n ($n \geqslant 1$) coincide, we have: $U'_{2n} = V'_n$ ($n \geqslant 1$), and especially: $U'_{2^{x+1}} = V'_{2^x}$ ($x \geqslant 0$). Hence, (2) and (2*) are equivalent, and, for $p > 2$, also (1) and (2*). Thus it is sufficient to prove (1) for $p \geqslant 2$ and (2*) for $p \neq 2$.

We shall even show the stronger inequalities:

(3) $U'_{p^{x+1}} > U_{p^x}$ (p - a prime, x - a positive integer)

(3*) $V'_{p^{x+1}} > V_{p^x}$ (p - a prime, x - a nonnegative integer)

Since $U_n \geqslant U'_n$, $V_n \geqslant V'_n$, it is obvious that in order to prove (1) for $p \geqslant 2$, and (2*) for $p \neq 2$, it is sufficient to prove (3) for $p \geqslant 2$ and (3*) for $p \neq 2$. However, it may be remarked that for p = 2, (3*) is evidently true, since $V'_{2^x} = V_{2^x}$ and (V_n) is a strictly increasing sequence beginning with n = 1.

The main tools for proving (3) for $p \geqslant 2$ and (3*) for $p \neq 2$, are the following inequalities:

(4) $U_{n^{x+1}} > nU^n_{n^x}$ ($n \geqslant 2$, $x \geqslant 1$)

(5) $V_{n^{x+1}} > V^{n-1}_{n^x}$ ($n \geqslant 2$, $x \geqslant 1$)

In order to prove (3) and (3*), it is sufficient to prove some weaker inequalities then (4) and (5). However, since (4) and (5) are interesting by themselves, we shall prove them. For the proof we shall use the well-known formulae:

(6) $U_n = \frac{1}{\sqrt{5}}\{\alpha^n - (-1)^n\alpha^{-n}\}$ $\qquad \alpha^{n+2} = \alpha^n + \alpha^{n+1}$

(7) $V_n = \alpha^n + (-1)^n\alpha^{-n}$ $\qquad \alpha = \frac{1+\sqrt{5}}{2} > \frac{3}{2}$

as well as the following inequalities:

(8) $\frac{1}{\sqrt{5}}\cdot\frac{6}{7}\cdot 2^n > n \quad (n \geq 3)$

(9) $\frac{1}{2}\alpha^n > U_n \quad (n \geq 2)$

(10) $\frac{6}{5}\alpha^n > V_n \quad (n \geq 2)$

Proof of (8) (by induction). (8) is equivalent to

(8') $6 \cdot 2^n > 7n\sqrt{5} \quad (n \geq 3)$

(8') is valid for n = 3. If (8') is valid for n, then:

$$6\cdot 2^{n+1} = 6\cdot 2^n + 6\cdot 2^n > 7n\sqrt{5} + 7n\sqrt{5} > 7n\sqrt{5} + 7\sqrt{5} = 7(n+1)\sqrt{5}.$$

Proof of (9), (10) (by induction on n and n+1).
(9) is valid for n = 2,3, since

$$\alpha^2 = 1 + \alpha = 1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2} > \frac{3+\sqrt{4}}{2} > 2 = 2U_2,$$

$$\alpha^3 = \alpha + \alpha^2 = \frac{1+\sqrt{5}}{2} + \frac{3+\sqrt{5}}{2} = 2 + \sqrt{5} > 2 + \sqrt{4} = 4 = 2U_3.$$

If

$$\alpha^n > 2U_n,$$
$$\alpha^{n+1} > 2U_{n+1},$$

then also:

$$\alpha^{n+2} = \alpha^n + \alpha^{n+1} > 2(U_n + U_{n+1}) = 2U_{n+2}.$$

(10) may be proven analogously, noting that, by arguments employed in the proof of (9), (10) is valid for n = 2,3, since

$$\frac{6}{5}\alpha^2 > \frac{6}{5}\cdot\frac{3+\sqrt{4}}{2} = 3 = V_2,$$
$$\frac{6}{5}\alpha^3 > \frac{6}{5}\cdot 4 > 4 = V_3$$

Proof of (4).

(1) For n = 2 we have, by (6):

$$U_{2^{x+1}} = \frac{1}{\sqrt{5}}\{\alpha^{2^{x+1}} - \alpha^{-2^{x+1}}\} = \frac{\sqrt{5}}{5}\{\alpha^{2^{x+1}} - \alpha^{-2^{x+1}}\} >$$
$$\frac{2}{5}\{\alpha^{2^{x+1}} - \alpha^{-2^{x+1}}\} > \frac{2}{5}\{\alpha^{2^{x+1}} - (2 - \alpha^{-2^{x+1}})\} =$$
$$\frac{2}{5}\{\alpha^{2^{x+1}} - 2 + \alpha^{-2^{x+1}}\} = 2\left\{\frac{1}{\sqrt{5}}(\alpha^{2^x} - \alpha^{-2^x})\right\}^2 = 2U^2_{2^x}.$$

(2) For $n \geq 3$ we have, by arguments employed in the proof of (9),

$$\alpha^{n^{x+1}} \geq \alpha^{3^2} = (\alpha^3)^3 > 4^3 > 7,$$

i.e.,

$$\frac{\alpha^{n^{x+1}}}{7} > 1.$$

Hence, by (6), (8), (9):

$$U_{n^{x+1}} = \frac{1}{\sqrt{5}}\{\alpha^{n^{x+1}} - (-1)^n\alpha^{-n^{x+1}}\} > \frac{1}{\sqrt{5}}\left\{\alpha^{n^{x+1}} - \frac{\alpha^{n^{x+1}}}{7}\right\} =$$
$$\frac{1}{\sqrt{5}}\cdot\frac{6}{7}\alpha^{n^{x+1}} = \frac{1}{\sqrt{5}}\cdot\frac{6}{7}\cdot 2^n\left(\frac{\alpha^{n^x}}{2}\right)^n > nU^n_{n^x}.$$

Proof of (5). For $n \geq 2$ we have $(n^x-1)/(n-1) = n^{x-1} + n^{x-2} + \dots + 1 \geq n^{x-1} \geq (n-1)^{x-1}$, whence: $n^x - 1 \geq (n-1)^x$. Hence, by (7), (10), and noting that (by arguments employed in the proof of (4), part (2)) $-\alpha^{-n^{x+1}} > -\frac{1}{7}$ we have:

$$V_{n^{x+1}} = \alpha^{n^{x+1}} + (-1)^n\alpha^{-n^{x+1}} \geq \alpha^{n^{x+1}} - \alpha^{-n^{x+1}} >$$
$$\alpha^{n^{x+1}} - \frac{1}{7} > \alpha^{n^{x+1}} - \frac{1}{3}\alpha^{n^{x+1}} = \frac{2}{3}(\alpha^{n^x})^n >$$

$$\frac{1}{\alpha}\left(\alpha^{n^x}\right)^n = \alpha^{n^x-1}\left(\alpha^{n^x}\right)^{n-1} \geqslant \alpha^{(n-1)^x}\left(\alpha^{n^x}\right)^{n-1} \geqslant$$
$$\alpha^{n-1}\left(\alpha^{n^x}\right)^{n-1} > \left(\tfrac{6}{5}\right)^{n-1}\left(\alpha^{n^x}\right)^{n-1} = \left(\tfrac{6}{5}\alpha^{n^x}\right)^{n-1} > V_{n^x}^{n-1}.$$

Remark. In proving the inequalities (4), (5), I was assisted by my son, Moshe, who also noted that (5) cannot be strengthened, analogously to (4), to: $V_{n^{x+1}} > V^n_{n^x}$. Indeed, for $n = 4$, $x = 1$, we have: $V_{4^2} = 2207 < 2401 = 7^4 = V_4^4$.

It may also easily be seen, by (6), (7), that

(11) $$\lim_{x\to\infty} \frac{U_{n^{x+1}}}{nU^n_{n^x}} = \frac{\sqrt{5}^{n-1}}{n} \geqslant 1 \quad (n \geqslant 1)$$

$$\lim_{x\to\infty} \frac{V_{n^{x+1}}}{V^{n-1}_{n^x}} = \infty$$

Proof of (3), (3*). For $p \neq 5$, $(p, U_{p^x}) = 1$. Hence, by the law of repetition of primes in (U_n), the greatest imprimitive divisor of $U_{p^{x+1}}$ is U_{p^x}, whence, by (4):

$$U'_{p^{x+1}} = U_{p^{x+1}} / U_{p^x} > pU^{p-1}_{p^x},$$

hence,

(12) $$U'_{p^{x+1}} > pU^{p-1}_{p^x} \quad (p \geqslant 2,\ x \geqslant 1)$$

For $p = 5$, by the law of repetition of primes in (U_n), the greatest imprimitive divisor of $U_{5^{x+1}}$ is $5U_{5^x}$, whence, by (4):

$$U'_{5^{x+1}} = U_{5^{x+1}} / 5U_{5^x} > U^4_{5^x}$$

hence

(13) $$U'_{5^{x+1}} > U^4_{5^x} \quad (p \geqslant 2,\ x \geqslant 1)$$

For $p \neq 2$, by the law of repetition of primes in (V_n), the greatest imprimitive divisor of $V_{p^{x+1}}$ is V_{p^x}, whence, by (5):

$$V'_{p^{x+1}} = V_{p^{x+1}} / V_{p^x} > V^{p-2}_{p^x} \quad (p > 2,\ x \geqslant 1)$$

hence

(14) $$V'_{p^{x+1}} > V^{p-2}_{p^x} \quad (p > 2,\ x \geqslant 1)$$

Now, (12) and (13) together are stronger then (3) so that (3) is valid a fortiori. (14) is stronger then (3*) (except for the case $x = 0$ in which (3*) simplifies to $V'_p > 1$ which is true) so that (3*) is valid a fortiori.

SUM OF SQUARES OF THE NUMERICAL FUNCTIONS U_n, V_n OF LUCAS

The subject considered here has already been treated by Lucas*, who obtained the formulae (62) - (64'), set out below. However, the way in which Lucas arrived at his formulae is not quite plain from his short exposition, which, besides, suffers from there being certain misprints. In this note we give a fresh account of the same results, keeping as far as possible to the notation of Lucas.

Let a, b denote the roots of the equation

(1) $$x^2 = Px-Q$$

whose coefficients P, Q are coprime, positive or negative, integers, and consider the two numerical functions U, V defined by

(2) $$U_n = (a^n-b^n)/(a-b), \quad V_n = a^n+b^n.$$

There are the following formulae:

(52) $$U_nV_m = U_{m+n}-Q^nU_{m-n}$$

(53) $$V_mV_n = V_{m+n}+Q^nV_{m-n}$$

(53') $$\Delta U_mU_n = V_{m+n}-Q^nV_{m-n}, \quad \Delta = (a-b)^2$$

(54) $$\sum_{k=0}^{n} V_{m+kr}/Q^{kr/2} = V_{(2m+nr)/2}U_{(n+1)r/2}/Q^{nr/2}U_{r/2}$$

(57) $$\sum_{k=1}^{n} V_{m+kr} = (V_{m+r}+Q^rV_{m+nr}-V_{m+(n+1)r}-Q^rV_m)/(1+Q^r-V_r)$$

From these formulae the following further ones can be obtained:

(62) $$\Delta\sum_{k=1}^{n} U^2_{kr}/Q^{kr} = (U_{(2n+1)r}/Q^{nr}U_r)-2n-1$$

(62') $$\Delta\sum_{k=0}^{n-1} U^2_{(2k+1)r}/Q^{(2k+1)r} = (U_{4nr}/Q^{2nr}U_{2r})-2n$$

(63) $$\sum_{k=1}^{n} V^2_{kr}/Q^{kr} = (U_{(2n+1)r}/Q^{nr}U_r)+2n-1$$

(63') $$\sum_{k=0}^{n-1} V^2_{(2k+1)r}/Q^{(2k+1)r} = (U_{4nr}/Q^{nr}U_{2r})+2n$$

(64) $$\Delta\sum_{k=0}^{n} U^2_{m+kr} = \frac{V_{2m+2(n+1)r}-V_{2m}-Q^{2r}(V_{2m+2nr}-V_{2m-2r})}{V_{2r}-Q^{2r}-1} - 2Q^m\frac{Q^{(n+1)r}-1}{Q^r-1}$$

(64') $$\sum_{k=0}^{n} V^2_{m+kr} = \frac{V_{2m+2(n+1)r}-V_{2m}-Q^{2r}(V_{2m+2nr}-V_{2m-2r})}{V_{2r}-Q^{2r}-1} + 2Q^m\frac{Q^{(n+1)r}-1}{Q^r-1}$$

Proof of (62). Putting in (53') m=n=kr, we obtain:

$$\Delta U^2_{kr}=V_{2kr}-Q^{kr}2, \quad \Delta U^2_{kr}/Q^{kr}=(V_{2kr}/Q^{kr})-2, \quad \Delta\sum_{k=1}^{n} U^2_{kr}/Q^{kr}=\sum_{k=1}^{n}(V_{2kr}/Q^{kr})-2n$$

Putting 2r instead of m and r in (54') (after multiplying by $Q^{-r/2}$), we further obtain:

$$= (V_{(n+1)r}U_{nr}/Q^{nr}U_r)-2n$$

and by (52'):

$$= ((U_{(2n+1)r}-Q^{nr}U_r)/Q^{nr}U_r)-2n = (U_{(2n+1)r}/Q^{nr}U_r)-2n-1.$$

Proof of (62'). Putting in (53') m=n=(2k+1)r, we obtain:

$$\Delta U^2_{(2k+1)r}=V_{(2k+1)2r}-Q^{(2k+1)r}2, \quad \Delta U^2_{(2k+1)r}/Q^{(2k+1)r}=(V_{(2k+1)2r}/Q^{(2k+1)r})-2,$$

$$\Delta\sum_{k=1}^{n} U^2_{(2k+1)r}/Q^{(2k+1)r}=\sum_{k=1}^{n}(V_{(2k+1)2r}/Q^{(2k+1)r})-2n$$

Putting 2r instead of m and 4r instead of r in (54) (after multiplying by $Q^{-r/4}$), we further obtain:

$$= (V_{2nr}U_{2nr}/Q^{2nr}U_{2r})-2n$$

and by (52):

$$= (U_{4nr}/Q^{2nr}U_{2r})-2n.$$

Similarly (63), (63') are obtainable from (53), (54), (52).

Proof of (64). Putting in (53') m+kr instead of m and n, we obtain:

$$\Delta U^2_{m+kr} = V_{2m+2kr}-Q^{m+kr}2, \quad \Delta\sum_{k=0}^{n} U^2_{m+kr} = \sum_{k=0}^{n} V_{2m+2kr}-Q^m2\sum_{k=0}^{n} Q^{kr},$$

whence by (57), putting 2m instead of m and 2r instead of r, we obtain:

$$= \frac{V_{2m+2r}+Q^{2r}V_{2m+2nr}-V_{2m+2(n+1)r}-Q^{2r}V_{2m}}{1+Q^{2r}-V_{2r}} + V_{2m} - Q^m2\frac{Q^{(n+1)r}-1}{Q^r-1}$$

whence by (53) we obtain (64).

Similarly (64') is obtainable from (53), (57).

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* E. Lucas, Théorie des Fonctions Numériques Simplement Périodiques, American Journal of Mathematics 1 (1878), 204-206.

THE PRODUCT OF SEQUENCES WITH A COMMON LINEAR RECURSION FORMULA OF ORDER 2

1. PRELIMINARIES. A recurring sequence (R_n) of order r with the *alternating scale* $a_0,\dots,a_r$, where $a_0,\dots,a_r$ are arbitrary complex numbers with $a_0a_r\neq 0$, is a sequence for which

$$\sum_{i=0}^{r}(-1)^i a_i R_{n+i}=0 \quad (n=\dots,-1,0,1,\dots).$$

We consider k-1 (k>1) recurring sequences $(W_n^{(i)})$ (i=1,...,k-1) of order 2 with the common alternating scale a, b, c. Our principal aim is to prove that $(P_n)=(\prod_{i=1}^{k-1}W_n^{(i)})$ is a recurring sequence of order k, and to find its alternating scale s_i (i=0,...,k). 1)

The *fundamental* recurring sequence (U_n) with the alternating scale a, b, c is defined by $U_0=0$, $U_1=1$. We call $\binom{k}{i}_U=\frac{U_kU_{k-1}\dots U_{k-i+1}}{U_1U_2\dots U_i}$, and $\binom{k}{0}_U=1$, a *generalized binomial coefficient* formed from the sequence (U_n).

We denote by (α^n) and (β^n) the two *geometrical* sequences with the alternating scale a, b, c, whence α and β satisfy the equation

$$a-bx+cx^2=0.$$

Since $(\frac{\alpha^n-\beta^n}{\alpha-\beta})$, as a linear combination of (α^n) and (β^n), has evidently the same scale as (U_n), and takes the values 0,1 for n=0,1, we have:

(1) $$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}=\alpha^{n-1}+\alpha^{n-2}\beta+\dots+\beta^{n-1},$$

if $\alpha\neq\beta$. For similar reasons we have for $\alpha=\beta$:

(1') $$U_n=n\alpha^{n-1}.$$

By (1) or (1'),

$$U_k=\alpha^{k-i}U_i+\beta^iU_{k-i}.$$

Multiplying by $\binom{k}{i}_U\frac{1}{U_k}$, we have

$$\binom{k}{i}_U=\alpha^{k-i}\binom{k-1}{i-1}_U+\beta^i\binom{k-1}{i}_U.$$

If $S_{k,i}$ denotes the sum of all products of $i\leq k$ different terms of the sequence

(3) $$\alpha^{k-1},\ \alpha^{k-2}\beta,\dots,\ \beta^{k-1},$$

$(S_{k,0}=1)$, then

(4) $$S_{k,i}=\alpha^{k-1}\beta^{i-1}S_{k-1,i-1}+\beta^iS_{k-1,i}.$$

Indeed, the sum of those products in which α^{k-1} is one of the factors is obviously $\alpha^{k-1}\beta^{i-1}S_{k-1,i-1}$, while the sum of all other products is $\beta^iS_{k-1,i}$.

If $\alpha\beta=1$, then by (2)

(4') $$\binom{k}{i}_U=\alpha^{k-1}\beta^{i-1}\binom{k-1}{i-1}_U+\beta^i\binom{k-1}{i}_U.$$

Since $S_{k,0}=\binom{k}{0}_U$ and $S_{k,k}=(\alpha\beta)^{\binom{k}{2}}=\binom{k}{k}_U$, there follows by (4) and (4') that for $\alpha\beta=1$,

(5) $$S_{k,i}=\binom{k}{i}_U.$$

If $\lambda\neq 0$, then $\lambda^ra_0,\ \lambda^{r-1}a_1,\dots,a_r$ is the alternating scale of $(\lambda^{n-1}R_n)$. Hence the quantities belon ng to $(\overline{W}_n^{(i)}=\lambda^{n-1}W_n^{(i)})$ can be expressed by those belonging to $(W_n^{(i)})$ thus:

$$\overline{a}=\lambda^2a,\ \overline{b}=\lambda b,\ \overline{c}=c;\ \overline{\alpha}=\lambda\alpha,\ \overline{\beta}=\lambda\beta;$$

(6) $$\overline{U}_n=\lambda^{n-1}U_n,\ \binom{k}{i}_{\overline{U}}=\lambda^{i(k-i)}\binom{k}{i}_U;\ \overline{P}=\lambda^{(k-1)(n-1)}P_n,\ \overline{s}_i=\lambda^{(k-1)(k-i)}s_i.$$

2. THEOREM 1. The sequence $(P_n=\prod_{i=1}^{k-1}W_n^{(i)})$ whose n-th term is the product of the n-th terms of k-1 (k>1) recurring sequences $(W_n^{(1)}),\dots,(W_n^{(k-1)})$ with the common alternating scale a, b, c ($ac\neq 0$), is a recurring sequence of order k with the alternating scale

$$s_i=\left(\frac{a}{c}\right)^{\binom{k-i}{2}}\binom{k}{i}_U \quad (i=0,\dots,k),$$

i.e. we have

(7) $$\sum_{i=0}^{k}(-1)^i\left(\frac{a}{c}\right)^{\binom{k-i}{2}}\binom{k}{i}_UP_{n+i}=0.$$

Also

(7') $$\sum_{i=0}^{k}(-1)^ia^{\binom{k-i}{2}}c^{\binom{i}{2}}\binom{k}{i}_{U*}P_{n+i}=0,$$

where (U_n^*) is the fundamental recurring sequence with the alternating scale ac, b, 1.

In case $P_n=\frac{U_n\dots U_{n-k+2}}{U_1\dots U_{k-1}}=\binom{n}{k-1}_U$, (7) becomes

(8) $$\sum_{i=0}^{k}(-1)^i\left(\frac{a}{c}\right)^{\binom{k-i}{2}}\binom{k}{i}_U\binom{n+i}{k-1}_U=0.$$

Some of the simplest cases have been noted in the literature:

(1) For a=c=b-1=1, $(W_n)=(A+(n-1)D)$ is an *arithmetical progression*, $\binom{k}{i}_U=\binom{k}{i}$, and (7) becomes

(9) $$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(A+(n+i-1)D)^{k-1}=0.$$ 2)

The more general formula

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\prod_{j=1}^{k-1}(A_j+(n+i-1)D_j = 0$$

seems to be new.

(2) For A=1, D=0, (9) becomes

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i} = 0.$$ [3]

(3) For a=c=b-1=1, (8) becomes

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n+i}{k-1} = 0.$$ [4]

(4) For -a=b=c=1, $P_n=U_n^2$, i.e. for the squares of the F i b o n a c c i n u m b e r s 0, 1, 1, 2,..., (7) becomes

$$2(U_n^2+U_{n+1}^2) = U_{n-1}^2+U_{n+2}^2.$$ [5]

Already the next formula

$$U_{n+2}^3+U_{n-2}^3 = 6U_n^3+3(U_{n+1}^3-U_{n-1}^3)$$

seems to be new.

PROOF. First let a=c=1, $\alpha\neq\beta$. Then we can write $W_n^{(i)}=A_i\alpha^n+B_i\beta^n$, $P_n=\prod_{i=1}^{k-1}(A_i\alpha^n+B_i\beta^n)$, whence P_n is a linear combination of geometrical progressions with the ratios (3). Consequently (P_n) is a recurring sequence of order k, whose scale consists of the coefficients of the equation

$$s_o-s_1x+s_2x^2+\ldots+(-1)^ks_kx^k = 0,$$

with the roots (3). But, by (5), $s_i=S_{k,i}=\binom{k}{i}_U$, whence (7) for a=c=1, $\alpha\neq\beta$.

If $\alpha=\beta$, i.e. $b=\pm2$, we can say that (7) considered as an algebraical identity for the variable b, with constant k, n, a=c=1, $W_o^{(i)}$, $W_1^{(i)}$ ($U_2,\ldots,U_k$ and $W_n^{(i)},\ldots,W_{n+k}^{(i)}$ having been expressed as polynomials in b) holds always, since it holds for $b\neq\pm2$.

For arbitrary a, c ($ac\neq0$), we put, in (6), $\lambda=(\frac{c}{a})^{1/2}$, so that $\overline{a}=\overline{c}$ and $\overline{s}_i=\binom{k}{i}_{\overline{U}}$. Hence $s_i=\lambda^{-(k-1)(k-i)}\overline{s}_i=\lambda^{-(k-1)(k-i)}\binom{k}{i}_{\overline{U}}=$ $\lambda^{-(k-1)(k-i)+i(k-i)}\binom{k}{i}_U=\lambda^{-2\binom{k-i}{2}}\binom{k}{i}_U$.

Putting $\lambda=c$ we have:

$$s_i = (\tfrac{a}{c})^{\binom{k-i}{2}}\binom{k}{i}_U = (\tfrac{a}{c})^{\binom{k-i}{2}}c^{-i(k-i)}\binom{k}{i}_{U*} = a^{\binom{k-i}{2}}c^{\binom{i}{2}-\binom{k}{2}}\binom{k}{i}_{U*},$$

whence (7').

3. THEOREM 2. For the fundamental recurring sequence (U_n) with the alternating scale a, b, 1, where a and b are integers, every generalized binomial coefficient

$$\binom{k}{i}_U = \frac{U_k\cdots U_{k-i+1}}{U_1\cdots U_i},\quad k\geqslant0,$$

is an integer. [6]

FIRST PROOF. Obviously $\binom{n}{0}_U=1$ and $\binom{n}{1}_U=U_n$ are integers for all $n\geqslant0$. Let $\binom{n}{0}_U,\ldots,\binom{n}{k-2}_U$ be integers for all $n\geqslant0$. Then we show that also $\binom{n}{k-1}_U$ is an integer for all $n\geqslant0$. This follows from (8), when all coefficients $a^{\binom{k-i}{2}}\binom{k}{i}_U$ (including $\binom{k}{k-1}_U=U_k$) are integers, the last one being equal to 1, since $\binom{n}{k-1}_U$ equals 0 for n=0,...,k-2 and 1 for n=k-1.

SECOND PROOF. Again $\binom{k-1}{0}_U=1$ and $\binom{k-1}{1}_U=U_{k-1}$ are integers for all $k\geqslant0$. Supposing that $\binom{k-1}{i-1}_U$ and $\binom{k-1}{i}_U$ are integers, we see by (2), since α and β are algebraic integers for c=1 and integral a and b, and since $\binom{k}{i}_U = \frac{U_k\cdots U_{k-i+1}}{U_1\cdots U_i}$ is rational, that $\binom{k}{i}_U$ is an integer.

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1) More generally, the product of k recurring sequences of order r+1 with a common alternating scale is a recurring sequence of order $\binom{k+r}{r}$, whose scale it would be interesting to determine.

2) I. M. Ryzhik, Tablicy integralov, summ, riadov i proizvedenij, 1943, p. 264, formula 7 for α=-x.

3) Ibidem p.252, formula 10.

4) Ibidem p. 254, formula 36 for h=-1.

5) A. Boutin, Sur la série de Fibonacci, Mathesis (4) 4, 1914, p. 125, formula 2.

6) That $\binom{k}{i}$ is an integer was proved by B. Pascal, Oeuvres, 3, 1908, p. 278-282. Compare L. E. Dickson, History of the Theory of Numbers I, p. 269.

That $\binom{k}{i}_U$ is an integer was proved by P. Bachmann, Niedere Zahlentheorie II, 1910, p. 81 and R. D. Carmichael, On the Numerical Factors of the Arithmetic Forms $\alpha^n\pm\beta^n$, Annals of Mathematics (2) 15, 1913-1914, on p. 40, for c=1 and coprime integers a and b. But their proofs (which differ from our proofs and from each other) are valid for general integers a and b. Carmichael quotes the proof of É. Lucas, Théorie des Fonctions Numériques Simplement Périodiques, American Journal of Mathematics 1, 1878, on p. 203, which is, however, incomplete.

ON THE PERIODICITY OF THE LAST DIGITS OF THE FIBONACCI NUMBERS

THEOREM 1. The last $d \geq 3$ digits of the consecutive Fibonacci numbers repeat periodically every 15.10^{d-1} times.

The proof is based on the following theorems from the theory of the Fibonacci numbers.

NOTATION. $A(n)$ - the period of the Fibonacci sequence relative to n.

$a(n)$ - the least positive subscript of the Fibonacci numbers divisible by n (known as "rank of apparition" of n).

$[a, b, \ldots]$ - the least common multiple of a, b,... .

THEOREM 2. $A(n)$ exists for any whole positive n.

THEOREM 3. If $n = p_1^{d_1} p_2^{d_2} \ldots p_k^{d_k}$ is the canonical decomposition of n into different prime-powers ($p_1, p_2, \ldots, p_k$ being different primes and $d_1, d_2, \ldots, d_k$ being positive integers), then

$$A(n) = [A(p_1^{d_1}), A(p_2^{d_2}), \ldots, A(p_k^{d_k})].$$

THEOREM 4. For any odd prime p and whole positive d,

$$A(p^d) = a(p^d),\ 2a(p^d),\ \text{or}\ 4a(p^d)$$

according as

$$a(p^d) \equiv 2,\quad 0,\ \text{or}\quad \pm 1 \pmod 4).$$

For $d \geq 3$, $A(2^d) = 2a(2^d)$.

THEOREM 5. For $d \geq 3$, $a(2^d) = 3.2^{d-2}$.

For any whole positive d, $a(5^d) = 5^d$.

PROOF OF THEOREM 1. Obviously the problem of determining the period of the sequence of the last d digits of the consecutive Fibonacci numbers is equivalent to the one of determining the period of the Fibonacci sequence relative to 10^d. Now, for any whole positive $d \geq 3$, by the above theorems,

$$\begin{aligned} A(10^d) = A(2^d 5^d) &= [A(2^d), A(5^d)] \\ &= [2a(2^d), 4a(5^d)] \\ &= [2.3.2^{d-2}, 4.5^d] \\ &= 4[3.2^{d-3}, 5^d] \\ &= 4.3.2^{d-3}.5^d \\ &= 15.10^{d-1} \end{aligned}$$

REMARK. It was well-known long ago that the last (units) digits of the consecutive Fibonacci numbers repeat every 60 times. Stephen P. Geller (The Fibonacci Quarterly volume 1, number 2, page 84) found empirically that the last 2, 3, 4, 5, 6 digits of the consecutive Fibonacci numbers repeat periodically every 300, 1500, 15000, 150000, 1500000 numbers respectively.

TABLE OF FIBONACCI NUMBERS

Dedicated to the Memory of Prof. Jekuthiel Ginsburg.

The following table contains the terms of both the sequences (U_n) and (V_n) from n=0 up to n=385, with factorizations as far as known. The table was firstly published in Riveon Lematematika 1 (1946-7), 35-7, 99, up to n=128, then, improved and enlarged up to n=385, in Riveon Lematematika 11 (1957), 70-90, finally, somewhat improved, in the first edition of Recurring Sequences (1958), 18-39.

The essentially new shape of the table presented here is due to John Brillhart, who, since September 1, 1960, furnished the author with truly amazing factorizations, performed by him firstly with the aid of an IBM 701, afterwards with the aid of an IBM 7090 (and perhaps with other computers). Thus the table is now containing all prime factors $<2^{35}$ of U_n and V_n for $n<300$, and all prime factors $<2^{30}$ for greater n, while in special cases factors above the mentioned limits are present. Compare e. g. U_{283}, U_{301}, U_{335}, V_{324}, V_{327}, V_{353}, V_{376}. The factorization of U_n is now complete up to n=171, and that of V_n up to n=151. According to Brillhart U_n is prime for n=3, 4, 5, 7, 11, 13, 17, 23, 29, 43, 47, 83, 131, 137, 431, 433, 449, 509, 569, 571, all other U_n with $6\leq n\leq 1000$ are composite, with the possible exception of U_{359}; V_n is prime for n=0, 2, 4, 5, 7, 8, 11, 13, 16, 17, 19, 31, 37, 41, 47, 53, 61, 71, 79, 113, 313, 353, all other V_n with $3\leq n\leq 500$ are composite.

NOTATIONS. Primitive prime factors, and among the subscripts belonging to V - also natural powers of 2, are fully underlined, primitive divisors of unknown composition are underlined with a broken line. "c" indicates that the preceding number is composite, but no factor is known. "P" indicates that the preceding number is pseudo-prime, i. e., that it satisfies Fermat's congruence for some base.

HISTORY OF THE TABLE

1 Leonardo Pisano (Fibonacci) gave, in 1202, the terms U_2-U_{14} of the sequence afterwards called on his name.

2 Lame G. gave the terms U_2-U_{17}.

3 Lucas E. showed the primality of U_{29}.

4 He factored U_n, n=1,...,60.

5 He tabulated U_1-U_{60}, with factorizations.

6 He gave the primitive factor 127 of V_{64}.

7 Catalan E. gave the first 43 terms of Fibonacci's sequence.

8 Selivanov D. F. showed the primality of U_{29} and factored U_{40}.

9 Bickmore C. E. and Curjel H. W. gave 107 as a factor of U_{108}, and 109 as a factor of U_{54} and U_{108}.

10 Rosace gave U_{100}.

11 Malo E. gave U_0-U_{10} and V_0-V_{10}. He stated that U_{900}=5487...8800 and possesses 188 digits.

12 Picou G. gave U_{101}.

13 Niewiadomski R. tabulated and factored U_{5k}, k=1,...,12, and V_{5k}, k=1,...,10.

14 He gave the values of V_n, n=3, 19, 23, 29, 31, 32, 37, 41, 64, 128.

15 He gave V_{64} and its factor 127. He also gave V_{128} and established the primality of V_{31}.

16 He proved the primality of V_{37}.

17 Escott E.-B. gave $2^7-1=127$, $2^{19}-1=524287$, $2^{31}-1=2147483647$ as factors of V_n, n=64, 2^{18}, 2^{30} respectively.

18 Laisant C. A. tabulated U_n and V_n, up to n=120.

19 Kernbaum S. tabulated U_n up to n=70, with factorizations.

20 He tabulated the ranks of apparition a(p) for prime p up to p=461.

21 Poulet P. gave the factorization of U_n, V_n for all n=62-85 but U_{73}, U_{77}, U_{79}, U_{83}, U_{85}; V_{73}, V_{76}, V_{77}, V_{79}, V_{80}, V_{82}, V_{83}.

22 Kraitchik M. gave the value of $(p\pm1)/a(p)$, for each prime $p<1000$.

23 He gave the factorization of U_n for each odd n=1,...,71, as well as for n=75, 81, 85, 87, 95, 99, 105, 129. He also gave the factorization of V_n for each n=1,...,71 and for other separated values of n. In a special table he gave the factorization of V_{5k}.

24 Jarden D. tabulated U_n, V_n for n=0,...,128, with factorizations.

25 He gave a table of the ranks of apparition of a prime p in (U_n), for each $p\leq1511$.

26 He announced new factorizations of Poulet, Lehmer D. H., and himself, for various terms of (U_n), (V_n).

27 Jarden D. and Katz A. gave factors of (U_n), n=89, 117, 127.

28 Katz A. factored completely U_{117}, V_{73}, V_{108}, and partially V_{109}, V_{128}.

29 Jarden D. factored V_{5n}/V_n, n=1,...,77 (partially).

30 Beeger N. G. W. H. announced Poulet's factorization of V_{91}.

31 Katz A. factored U_n, n=141, 147, 165, 189, and V_n, n=147, 153, 180, 189. He gave the complete factorization of V_{138}.

32 Jarden D. tabulated $U_n\mp1$, $V_n\mp1$, with complete factorization, for n=0,...,61.

33 He gave $V_{210}/2$ with partial factorization.

34 Lehmer D. H. stated the primality of $347502052673|U_{147}$ and of $466415762341|U_{165}$.

U_n	n	Factorization of U_n
17167680177565	65	5.233.14736206161
27777890035288	66	
44945570212853	67	269.116849.1429913
72723460248141	68	
117669030460994	69	2.137.829.18077.28657
190392490709135	70	
308061521170129	71	6673.46165371073
498454011879264	72	
806515533049393	73	9375829.86020717
1304969544928657	74	
2111485077978050	75	$2.5^2.61.3001.230686501$
3416454622906707	76	
5527939700884757	77	13.89.988681.4832521
8944394323791464	78	
14472334024676221	79	157.92180471494753
23416728348467685	80	
37889062373143906	81	2.17.53.109.2269.4373.19441
61305790721611591	82	
99194853094755497	83	99194853094755497
160500643816367088	84	
259695496911122585	85	5.1597.9521.3415914041
420196140727489673	86	
679891637638612258	87	2.173.514229.3821263937
1100087778366101931	88	
1779979416004714189	89	1069.1665088321800481
2880067194370816120	90	
4660046610375530309	91	$13^2.233.741469.159607993$
7540113804746346429	92	
12200160415121876738	93	2.557.2417.4531100550901
19740274219868223167	94	
31940434634990099905	95	5.37.113.761.29641.67735001
51680708854858323072	96	
83621143489848422977	97	193.389.3084989.361040209
135301852344706746049	98	
218922995834555169026	99	2.17.89.197.19801.18546805133
354224848179261915075	100	
573147844013817084101	101	743519377.770857978613
927372692193078999176	102	
1500520536206896083277	103	519121.5644193.512119709
2427893228399975082453	104	
3928413764606871165730	105	2.5.13.61.421.141961.8288823481
6356306993006846248183	106	
10284720757613717413913	107	1247833.8242065050061761
16641027750620563662096	108	
26925748508234281076009	109	827728777.32529675488417
43566776258854844738105	110	
70492524767089125814114	111	2.73.149.2221.1459000305513721
114059301025943970552219	112	
184551825793033096366333	113	677.272602401466814027129
298611126818977066918552	114	
483162952612010163284885	115	5.1381.28657.2441738887963981
781774079430987230203437	116	
1264937032042997393488322	117	2.17.233.29717.135721.39589685693
2046711111473984623691759	118	
3311648143516982017180081	119	13.1597.159512939815855788121
5358359254990966640871840	120	
8670007398507948658051921	121	89.97415813466381445596089
14028366653498915298923761	122	
22698374052006863956975682	123	2.2789.59369.68541957733949701
36726740705505779255899443	124	
59425114757512643212875125	125	$5^3.3001.158414167964045700001$
96151855463018422468774568	126	
155576970220531065681649693	127	27941.5568053048227732210073
251728825683549488150424261	128	

51 John Brillhart, in a correspondence with Dov Jarden, initiated September 1, 1960, furnished numerous new factors of U_n and V_n, summing up his results with a table of primitive prime factors of U_n, n odd, from n=1 up to n=999 (November 30, 1964), and with a table of primitive prime factors of V_n, from n=1 up to n=500 (January 24, 1966). All primitive prime factors not credited above to others, are due to Brillhart.

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U_n	n	Factorization of U_n	V_n	n	Factorization of V_n
0	0		2	0	2
1	1		1	1	
1	2		3	2	3
2	3	2	4	3	2^2
3	4	3	7	4	7
5	5	5	11	5	11
8	6	2^3	18	6	2.3^2
13	7	13	29	7	29
21	8	3.7	47	8	47
34	9	2.17	76	9	$2^2.19$
55	10	5.11	123	10	3.41
89	11	89	199	11	199
144	12	$2^4.3^2$	322	12	2.7.23
233	13	233	521	13	521
377	14	13.29	843	14	3.281
610	15	2.5.61	1364	15	$2^2.11.31$
987	16	3.7.47	2207	16	2207
1597	17	1597	3571	17	3571
2584	18	$2^3.17.19$	5778	18	$2.3^3.107$
4181	19	37.113	9349	19	9349
6765	20	3.5.11.41	15127	20	7.2161
10946	21	2.13.421	24476	21	$2^2.29.211$
17711	22	89.199	39603	22	3.43.307
28657	23	28657	64079	23	139.461
46368	24	$2^5.3^2.7.23$	103682	24	2.47.1103
75025	25	$5^2.3001$	167761	25	11.101.151
121393	26	233.521	271443	26	3.90481
196418	27	2.17.53.109	439204	27	$2^2.19.5779$
317811	28	3.13.29.281	710647	28	$7^2.14503$
514229	29	514229	1149851	29	59.19489
832040	30	$2^3.5.11.31.61$	1860498	30	$2.3^2.41.2521$
1346269	31	557.2417	3010349	31	3010349
2178309	32	3.7.47.2207	4870847	32	1087.4481
3524578	33	2.89.19801	7881196	33	$2^2.199.9901$
5702887	34	1597.3571	12752043	34	3.67.63443
9227465	35	5.13.141961	20633239	35	11.29.71.911
14930352	36	$2^4.3^3.17.19.107$	33385282	36	2.7.23.103681
24157817	37	73.149.2221	54018521	37	54018521
39088169	38	37.113.9349	87403803	38	3.29134601
63245986	39	2.233.135721	141422324	39	$2^2.79.521.859$
102334155	40	3.5.7.11.41.2161	228826127	40	47.1601.3041
165580141	41	2789.59369	370248451	41	370248451
267914296	42	$2^3.13.29.211.421$	599074578	42	$2.3^2.83.281.1427$
433494437	43	433494437	969323029	43	6709.144481
701408733	44	3.43.89.199.307	1568397607	44	7.263.881.967
1134903170	45	2.5.17.61.109441	2537720636	45	$2^2.11.19.31.181.541$
1836311903	46	139.461.28657	4106118243	46	3.4969.275449
2971215073	47	2971215073	6643838879	47	6643838879
4807526976	48	$2^6.3^2.7.23.47.1103$	10749957122	48	2.769.2207.3167
7778742049	49	13.97.6168709	17393796001	49	29.599786069
12586269025	50	$5^2.11.101.151.3001$	28143753123	50	3.41.401.570601
20365011074	51	2.1597.6376021	45537549124	51	$2^2.919.3469.3571$
32951280099	52	3.233.521.90481	73681302247	52	7.103.102193207
53316291173	53	953.55945741	119218851371	53	119218851371
86267571272	54	$2^3.17.19.53.109.5779$	192900153618	54	$2.3^4.107.11128427$
139583862445	55	5.89.661.474541	312119004989	55	$11^2.199.331.39161$
225851433717	56	$3.7^2.13.29.281.14503$	505019158607	56	47.10745088481
365435296162	57	2.37.113.797.54833	817138163596	57	$2^2.229.9349.95419$
591286729879	58	59.19489.514229	1322157322203	58	3.347.1270083883
956722026041	59	353.2710260697	2139295485799	59	709.8969.336419
1548008755920	60	$2^4.3^2.5.11.31.41.61.2521$	3461452808002	60	2.7.23.241.2161.20641
2504730781961	61	4513.555003497	5600748293801	61	5600748293801
4052739537881	62	557.2417.3010349	9062201101803	62	3.3020733700601
6557470319842	63	2.13.17.421.35239681	14662949395604	63	$2^2.19.29.211.1009.31249$
10610209857723	64	3.7.47.1087.2207.4481	23725150497407	64	127.186812208641

35 Duparc H. J. A. reproduced Jarden's table 24 for U_n, n=0,...,78, and Jarden's table 25 for primes not exceeding 431.

36 Katz A. established (independently of Lehmer, 34) the primality of $347502052673|U_{147}$, and gave the factorization 466415762341= $518101.900241|U_{165}$, contrary to Lehmer.

37 Katz A. factored completely V_{74}, and partially U_{133} and U_{171}.

38 Wall D. D. listed all primes less than 2000 which have a period other than the maximum.

39 Basin S. L. and Hoggatt V. E. Jr.listed the first 571 U_n.

40 Alfred B. U. continued the table of Wall in determining the periods of primes beyond 2000 and less than 3000.

41 Brillhart J. announced complete factorizations of U_n, n odd =71, 79, 83, 89-97, thus completing the factorization of U_n up to n=100.

42 Brillhart J. reprinted the above factorizations, adding new complete factorizations of U_{101}, U_{103}, U_{107} and U_{109}.

43 Ward J. K. tabulated U_n up to n=1505 and V_n up to n=1506.

44 Brown J. L. Jr. announced $U_{10,000}$.

45 Bloom D. M. corrected factorizations of U_{57}, U_{67} in Kraitchik's table 23.

46 Wunderlich M. tabulated Fibonacci entry points for all primes up to p=48163, and primes related to Fibonacci entry points up to the entry point 48164.

47 Painter R. checked the result of Brown, 44, on the IBM 1620 and found to be correct except for a mysterious "o", the third digit from the end of the tenth line. This should have been a "1".

48 Lind D., Morris R. A., Shapiro L. D. continued the tables of Wunderlich, 46, up to 100,000.

49 Dresel L. A. G. and Daykin D. E. factored 36 U_n with $102 \leq n \leq 228$.

50 Alfred B. U. tabulated the first hundred U_n and the first fifty V_n, with complete factorizations. He also tabulated entry points and periods of Fibonacci and Lucas sequences for primes less than 270.

V_n	n	Factorization of V_n
38388099893011	65	11.131.521.2081.24571
62113250390418	66	2.3^2.43.307.261399601
100501350283429	67	4021.24994118449
162614600673847	68	7.23230657239121
263115950957276	69	2^2.139.461.691.1485571
425730551631123	70	3.41.281.12317523121
688846502588399	71	688846502588399
1114577054219522	72	2.47.1103.10749957121
1803423556807921	73	151549.11899937029
2918000611027443	74	3.11987.81143477963
4721424167835364	75	2^2.11.31.101.151.12301.18451
7639424778862807	76	7.1091346396980401
12360848946698171	77	29.199.229769.9321929
20000273725560978	78	2.3^2.90481.12280217041
32361122672259149	79	32361122672259149
52361396397820127	80	2207.23725145626561
84722519070079276	81	2^2.19.3079.5779.62650261
137083915467899403	82	3.163.800483.350207569
221806434537978679	83	35761381.6202401259
358890350005878082	84	2.7^2.23.167.14503.65740583
580696784543856761	85	11.3571.1158551.12760031
939587134549734843	86	3.313195711516578281
1520283919093591604	87	2^2.59.349.19489.947104099
2459871053643326447	88	47.93058241.562418561
3980154972736918051	89	179.22235502640988369
6440026026380244498	90	2.3^3.41.107.2521.10783342081
10420180999117162549	91	29.521.689667151970161
16860207025497407047	92	7.253367.9506372193863
27280388024614569596	93	2^2.63799.3010349.35510749
44140595050111976643	94	3.563.5641.4632894751907
71420983074726546239	95	11.191.9349.41611.87382901
115561578124838522882	96	2.1087.4481.11862575248703
186982561199565069121	97	3299.56678557502141579
302544139324403592003	98	3.281.5881.61025309469041
489526700523968661124	99	2^2.19.199.991.2179.9901.1513909
792070839848372253127	100	7.2161.9125201.5738108801
1281597540372340914251	101	809.7879.201062946718741
2073668380220713167378	102	2.3^2.67.409.63443.66265118449
3355265920593054081629	103	619.1031.5257480026438961
5428934300813767249007	104	47.3329.106513889.325759201
8784200221406821330636	105	2^2.11.29.31.71.211.911.21211.767131
14213134522220588579643	106	3.1483.2969.1076012367720403
22997334743627409910279	107	47927441.479836483312919
37210469265847998489922	108	2.7.23.6263.103681.177962167367
60207804009475408400201	109	128621.788071.593985111211
97418273275323406890123	110	3.41.43.307.59996854928656801
157626077284798815290324	111	2^2.4441.146521.1121101.54018521
255044350560122222180447	112	223.449.2207.1154149773784223
412670427844921037470771	113	412670427844921037470771
667714778405043259651218	114	2.3^2.227.26449.29134601.212067587
1080385206249964297121989	115	11.139.461.1151.5981.324301.686551
1748099984655007556773207	116	7.299281.834428410879506721
2828485190904971853895196	117	2^2.19.79.521.859.10526459855555841
4576585175559979410668403	118	3.15247723.100049587197598387
7405070366464951264563599	119	29.239.3571.10711.27932732439809
11981655542024930675232002	120	2.47.1103.1601.3041.23735900452321
19386725908489881939795601	121	199.97420733208491869044199
31368381450514812615027603	122	3.19763.21291929.24848660119363
50755107359004694554823204	123	2^2.4767481.370248451.7188487771
82123488809519507169850807	124	7.743.467729.33758740830460183
132878596168524201724674011	125	11.101.151.251.112128001.28143378001
215002084978043708894524818	126	2.3^3.83.107.281.1427.1461601.764940961
347880681146567910619198829	127	509.5081.487681.13822681.19954241
562882766124611619513723647	128	119809.4698167634523379875583

U_n	n
407305795904080553832073954	129
659034621587630041982498215	130
1066340417491710595814572169	131
1725375039079340637797070384	132
2791715456571051233611642553	133
4517090495650391871408712937	134
7308805952221443105020355490	135
11825896447871834976429068427	136
19134702400093278081449423917	137
30960598847965113057878492344	138
50095301248058391139327916261	139
81055900096023504197206408605	140
131151201344081895336534324866	141
212207101440105399533740733471	142
343358302784187294870275058337	143
555565404224292694404015791808	144
898923707008479989274290850145	145
1454489111232772683678306641953	146
2353412818241252672952597492098	147
3807901929474025356630904134051	148
6161314747715278029583501626149	149
9969216677189303386214405760200	150
16130531424904581415797907386349	151
26099748102093884802012313146549	152
42230279526998466217810220532898	153
68330027629092351019822533679447	154
110560307156090817237632754212345	155
178890334785183168257455287891792	156
289450641941273985495088042104137	157
468340976726457153752543329995929	158
757791618667731139247631372100066	159
1226132595394188293000174702095995	160
1983924214061919432247806074196061	161
3210056809456107725247980776292056	162
5193981023518027157495786850488117	163
8404037832974134882743767626780173	164
13598018856492162040239554477268290	165
22002056689466296922983322104048463	166
35600075545958458963222876581316753	167
57602132235424755886206198685365216	168
93202207781383214849429075266681969	169
150804340016807970735635273952047185	170
244006547798191185585064349218729154	171
394810887814999156320699623170776339	172
638817435613190341905763972389505493	173
1033628323428189498226463595560281832	174
1672445759041379840132227567949787325	175
2706074082469569338358691163510069157	176
4378519841510949178490918731459856482	177
7084593923980518516849609894969925639	178
11463113765491467695340528626429782121	179
18547707689471986212190138251399707760	180
30010821454963453907530667147829489881	181
48558529144435440119720805669229197641	182
78569350599398894027251472817058687522	183
127127879743834334146972278486287885163	184
205697230343233228174223751303346572685	185
332825110087067562321196029789634457848	186
538522340430300790495419781092981030533	187
871347450517368352816615810882615488381	188
1409869790947669143312035591975596518914	189
2281217241465037496128651402858212007295	190
3691087032412706639440686994833808526209	191
5972304273877744135569338397692020533504	192

n	Factorization of U_n
129	2.257.5417.8513.39639893.433494437
131	1066340417491710595814572169
133	13.37.113.3457.42293.351301301942501
135	2.5.17.53.61.109.109441.1114769954367361
137	19134702400093278081449423917
139	277.2114537501.85526722937689093
141	2.108289.1435097.142017737.2971215073
143	89.233.8581.1929584153756850496621
145	5.514229.349619996930737079890201
147	2.13.97.293.421.3529.6168709.347502052673
149	110557.162709.4000949.85607646594577
151	5737.2811666624525811646469915877
153	2.17^2.1597.6376021.7175323114950564593
155	5.557.2417.21701.12370533881.61182778621
157	313.11617.7636481.10424204306491346737
159	2.317.953.55945741.97639037.229602768949
161	13.8693.28657.612606107755058997065597
163	977.4892609.33365519393.32566223208133
165	2.5.61.89.661.19801.86461.474541.518101.900241
167	18104700793.1966344318693345608565721
169	233.337.89909.104600155609.126213229732669
171	2.17.37.113.797.6841.54833.5741461760879844361
173	638817435613190341905763972389505493 c
175	5^2.13.701.3001.141961.17231203730201189308301
177	2.353.2191261.805134061.1297027681.2710260697
179	21481.156089.3418816640903898929534613769 P
181	8689.422453.8175789237238547574551461093 P
183	2.1097.4513.555003497.14297347971975757800833
185	5.73.149.2221.1702945513191305556907097618161
187	89.373.1597.10157807305963434099105034917037 P
189	2.13.17.53.109.421.38933.35239681.955921950316735037
191	3691087032412706639440686994833808526209 c

V_n	n
9107634472711795301329222476	129
14736462133957911496466461 23	130
23844096606669706797795685 99	131
38580558740627618294262147 22	132
62424655347297325092057833 21	133
101005214087924943386319980 43	134
163429869435222268478377813 64	135
264435083523147211864697794 07	136
427864952958369480343075607 71	137
692300036481516692207773401 78	138
1120164989439886172550849009 49	139
1812465025921402864758622411 27	140
2932630015361289037309471420 76	141
4745095041282691902068093832 03	142
7677725056643980939377565252 79	143
12422820097926672841445659084 82	144
20100545154570653780823224337 61	145
32523365252497326622268883422 43	146
52623910407067980403092107760 04	147
85147275659565307025360991182 47	148
137771186066633287428453098942 51	149
222918461726198594453814090124 98	150
360689647792831881882267189067 49	151
583608109519030476336081279192 47	152
944297757311862358218348468259 96	153
1527905866830892834554429747452 43	154
2472203624142755192772778215712 39	155
4000109490973648027327207963164 82	156
6472313115116403220099986178877 21	157
10472422606090051247427194142042 03	158
16944735721206454467527180320919 24	159
27417158327296505714954374462961 27	160
44361894048502960182481554783880 51	161
71779052375799465897435929246841 78	162
116140946424302426079917484030722 29	163
187919998800101891977353413277564 07	164
304060945224404318057270897308286 36	165
491980944024506210034624310585850 43	166
796041889248910528091895207894136 79	167
1288022833273416738126519518479987 22	168
2084064722522327266218414726374124 01	169
3372087555795744004344934244854111 23	170
5456152278318071270563348971228235 24	171
8828239834113815274908283216082346 47	172
14284392112431886545471632187310581 71	173
23112631946545701820379915403392928 18	174
37397024058977588365851547590703509 89	175
60509656005523290186231462994093438 07	176
97906680064500878552083010584799947 96	177
158416336070024168738314473578896386 03	178
256323016134525047290397484163696333 99	179
414739352204549216028711957742592720 02	180
671062368339074263319109441906289054 01	181
1085801720543623479347821399648881774 03	182
1756864088882697742666930841555170828 04	183
2842665809426321222014752241204052602 07	184
4599529898309018964681683082759223430 11	185
7442195707735340186696435323963276032 18	186
12041725606044359151378118406722499462 29	187
19483921313779699338074553730685775494 47	188
31525646919824058489452672137408274956 76	189
51009568233603757827527225868094050451 23	190
82535215153427816316979898005502325407 99	191
133544783387031574144507123873596375859 22	192

n	Factorization of V_n
129	2^2.6709.144481.308311.761882591401
130	3.41.3121.90481.42426476041450801
131	1049.414988698461.5477332620091
132	2.7.23.263.881.967.5281.66529.152204449
133	29.9349.10694421739.215295865045 9
134	3.6163.201912469249.2705622682163
135	2^2.11.19.31.181.271.541.811.5779.42391.119611
136	47.5626278372832919401376548 81
137	541721291.78982487870939058281
138	2.3^2.4969.16561.162563.275449.1043766587
139	30859.253279129.14331800109223159
140	7^2.2161.14503.118021448662479038881
141	2^2.79099591.6643838879.13950955527 1
142	3.283.569.2820403.9799987.35537616083
143	199.521.1957099.2120119.178471438002 1
144	2.769.2207.3167.115561578124838522881
145	11.59.19489.120196353941.1322154751061
146	3.29201.371258578501847272607888 81
147	2^2.29.211.65269.620929.8844991.599786069
148	7.10661921.114087288048701953998401
149	952111.4434539.3263039535803245519
150	2.3^2.41.401.601.2521.570601.87129547172401
151	1511.109734721.217533000184835774779
152	47.124171938195538399220442825 3601 c
153	2^2.19.919.3469.3571.13159.829397682682 9399
154	3.43.281.307.15252467.900164950225760603
155	11.311.3010349.29138888651.823837075741
156	2.7.23.103.1249.102193207.944918421835 51489
157	39980051.16188856575286517818849171
158	3.21803.5924683.14629892449.184715524801
159	2^2.785461.119218851371.4523819299182451
160	641.1087.4481.878132240443974874201601
161	29.139.461.1289.1917511.965840862268529759
162	2.3^5.107.11128427.1828620361.6782976947987
163	1043201.6601501.168645467119223044 5929
164	7.26845714114300270282479059039652 01
165	2^2.11^2.31.199.331.9901.39161.51164521.1550853481
166	3.6464041.253701435384102199658399604 1 c
167	766531.103849927693584542320127327909 P
168	2.47.1103.10745088481.115613939510481515041
169	521.400012422749007152825031617346281 c
170	3.41.67.1361.40801.63443.116146542119540 32961
171	2^2.19^2.229.9349.95419.162451.1617661.7038398989
172	7.1261177119159116467844040459440335 21
173	78889.6248069.16923049609.171246170261359
174	2.3^2.347.97787.528295667.1270083883.5639710969
175	11.29.71.101.151.911.54601.560701.7517651.51636551
176	1409.2207.194585895659829667028972152 2689 c
177	2^2.709.8969.336419.10884439.105117617351706859
178	3.528054453566747229127714911929654 6201
179	359.713991688397005702758767365358485 61 c
180	2.7.23.241.2161.8641.20641.103681.133737637659 86881
181	97379.689124316679237066841012376288819 c
182	3.281.90481.232961.61105786342948865348 08481
183	2^2.14686239709.560074829380 1.533975715909289
184	47.367.164801774562370063308873108076065 43 c
185	11.54018521.265272771839851.291800073181 6531
186	2.3^2.15917507.302073370060 1.859886421593527043
187	199.1871.3571.9056742344085065262650973 90431
188	7.18049.100769.15303763064966619496209144 3041
189	2^2.19.29.211.379.1009.5779.31249.85429.912871.1258740001
190	3.41.2281.4561.29134601.782747561.174795553490801
191	22921.3600855772149025623532127656101493 19 c
192	2.127.383.5662847.6803327.19073614849.186812208641

U_n	n
9663391306290450775010025392525829059713	193
15635695580168194910579363790217849593217	194
25299086886458645685589389182743678652930	195
40934782466626840596168752972961528246147	196
66233869353085486281758142155705206899077	197
107168651819712326877926895128666735145224	198
173402521172797813159685037284371942044301	199
280571172992510140037611932413038677189525	200
453973694165307953197296969697410619233826	201
734544867157818093234908902110449296423351	202
1188518561323126046432205871807859915657177	203
1923063428480944139667114773918309212080528	204
3111581989804070186099320645726169127737705	205
5034645418285014325766435419644478339818233	206
8146227408089084511865756065370647467555938	207
13180872826374098837632191485015125807374171	208
21327100234463183349497947550385773274930109	209
34507973060837282187130139035400899082304280	210
55835073295300465536628086585786672357234389	211
90343046356137747723758225621187571439538669	212
146178119651438213260386312206974243796773058	213
236521166007575960984144537828161815236311727	214
382699285659014174244530850035136059033084785	215
619220451666590135228675387863297874269396512	216
1001919737325604309473206237898433933302481297	217
162114018899219444701881625761731807571877809	218
2623059926317798754175087863660165740874359106	219
4244200115309993198876969489421897548446236915	220
6867260041627791953052057353082063289320596021	221
11111460156937785151929026842503960837766832936	222
17978720198565577104981084195586024127087428957	223
29090180355503362256910111038089984964854261893	224
47068900554068939361891195233676009091941690850	225
76159080909572301618801306271765994056795952743	226
123227981463641240980692501505442003148737643593	227
199387062373213542599493807772079972055335596336	228
322615043836854783580186309282650000354271239929	229
522002106210068326179680117059857997559804836265	230
844617150046923109759866426342507997914076076194	231
1366619256256991435939546543402365995473880912459	232
2211236406303914545699412969744873993387956988653	233
3577855662560905981638959513147239988861837901112	234
5789092068864820527338372482892113982249794889765	235
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197	15761.25795969.162908787637576537632028409653	c
199	397.436782169201002048261171378550055269633	
201	2.269.116849.1429913.5050260704396247169315999021	
203	13.1217.514229.56470541.2586982700656733994659533	
205	5.821.2789.59369.125598581.36448117857891321536401	
207	2.17.137.829.18077.28657.4072353155773627601222196481	
209	37.89.113.57314120955051297736679165379998262001	P
211	22504837.38490197.800972881.80475423858449593021	
213	2.1277.308061521170129.185790722054921374395775013	
215	5.433494437.2607553541.67712817361580804952011621	
217	13.433.557.2417.44269.217221773.2191174861.6274653314021	
219	2.123953.4139537.9375829.86020717.3169251245945843761	
221	233.1597.18455365724971961787396586822078046791921	c
223	4013.108377.251534189.164344610046410138896156070813	P
225	2.5^2.17.61.3001.109441.230686501.11981661982050957053616001	
227	23609.5219534137983025159078847113619467285727377	P
229	457.2749.40487201.6342725572732757535995514095793253	c
231	2.13.29.89.199.421.19801.988681.4332521.9164259601748159235188401	
233	139801.15817028535589262921577191649164698345419253	c
235	5.2971215073.389678426275593986752662955603693114561	
237	2.157.1668481.40762577.92180471494753.7698999052751136773	
239	10037.62141.63617830634057826632388440323309010644033	c
241	11042621.7005329677.1342874889289644763267952824739273	P
243	2.17.53.109.2269.4373.19441.7177905237579946589743592924684177	c
245	5.13.97.141961.6168709.128955073914024460192651484843195641	
247	37.113.233.1913489357079567637602203056753846715378384401	c
249	2.99194853094755497.24599047201225310501731215529292521	c
251	12776523572924732586037033894655031898659556447352249	c
253	89.28657.4322114369.3034387188241996163132401983770604929	c
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196	7^3.14503.3016049.6100804791163473872231629367 P
197	148103434286339167474095648101628263391371 c
198	2.3^3.43.107.307.261399601.11166702227.1076312899454363
199	2389.4503769.36036960414811969810787847118289
200	47.1601.3041.124001.6996001.31604388341748173556001
201	2^2.4021.24994118449.2525130352198123584657999511 c
202	3.547497418496144666543167613835090178297001
203	29^2.59.19489.2748232098283374889444289976282269 P
204	2.7.23.1223.23230657239121.470039965023902754923207
205	11.1231.5741.370248451.2170732312961.111359800682371
206	3.81163.46235392144586222367191440726672730987 P
207	2^2.19.139.461.691.1485571.3643684402534298380040912641
208	2207.7489.45045727.39586709834808244008811690207 P
209	199.419.9349.61176489628586697237626752676097179 c
210	2.3^2.41.83.281.1427.2521.721561.12317523121.140207234004601
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213	2^2.1279.1882921.688846502588399.49258624519847932639
214	3.21401.374929.226981241.96796731322417872953594929 c
215	11.431.1291.1721.6709.144481.1266715025281.66163448516461
216	2.47.1103.3023.19009.447901921.10749957121.48265838239823
217	29.18229.3010349.125024551.11260169813534893704769219 P
218	3.1307.924503867289824805827159934087885660335843 P
219	2^2.439.151549.11899937029.12748437199.145282738021003201
220	7.263.881.967.2161.2800076631444853778881663695403201 P
221	521.3571.8253552568783067669129276267354079318961 c
222	2.3^2.443.11987.55927129.6870470209.8336942267.81143477963
223	209621.191782505151874799799825102831271417475449 P
224	1087.2689.4481.4966336310413757728406317515606275329 P
225	2^2.11.19.31.101.151.181.541.12301.18451.221401.15608701.3467131047901
226	3.6329.2151521.122464427.34040411535767969315747440867 P
227	39499.5098421.1368272321369783215357851670042342001 c
228	2.7.23.62929.307826903.1091346396980401.65494688793368423
229	6871.104990418946773667410736999685208265866007631 P
230	3.41.4969.275449.6933331242503222492307683343778240321 c
231	2^2.29.199.211.4621.9901.229769.9321929.19630381.201562805274601
232	47.463.929.12527.277007.43561231976081277978655158673967 P
233	818757341.6911530261.873757179900549251563653697571 P
234	2.3^3.107.467.21529.90481.12280217041.123944171348849444948627 P
235	11.941.6581.8461.119851.842432231.6643838879.33481417483721
236	7.12743.13687.5974828049.2871307447985313921708888731089
237	2^2.637293949.399660629491.1027912163389.32361122672259149
238	3.67.281.63443.75683.3465148147.58351516230584163679868441 P
239	479.7649.24216191671442408226762026802756956706931169 P
240	2.769.2207.3167.281490241.23725145626561.1939653272832963841
241	1156801.4645999.43219877626484550971962471774087607599 P
242	3.43.307.9490319961894625110532872069721973508136597201 c
243	2^2.19.3079.5779.59779.62650261.120074026624398979403194983601 P
244	7.487.288640467827809263557401213961548917269059680823 c
245	11.29.71.491.911.1471.88972241.599786069.4353947431.459807660691
246	2.3^2.163.800483.350207569.3131999980001698199622556887959401 c
247	521.9349.383839.768548899.2900839194578436063903816717541 P
248	47.1952755969.73483350528661634941003491044929827858529 P
249	2^2.499.221806434537978679.24648343889003317136003221973239 c
250	3.41.401.570601.1353439001.463542291063404760238758581586001 c
251	15061.170179.712841.15636705475517134545061743537722067281 P
252	2.7^2.23.167.503.14503.103681.65740583.2469769366696238955243413447 c
253	139.199.461.5865488412904746102270928952219322103714238401 c
254	3.1523.26487408254541486132499495083633739300801481688547 c
255	2^2.11.31.919.1021.3469.3571.53551.95881.1158551.12760031.162716451241291
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n	Factorization of U_n	
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259	13.73.149.1553.2221.123066918818135422969466420288970740903065 7	c
261	2.17.173.2089.20357.36017.40193.322073.514229.3821263937.6857029027549	
263	4733.93629.9283622964639019423529121698442566463089390281	P
265	5.953.15901.55945741.2540886647968538818904473752068910075661	c
267	2.1069.1665088321800481.792081680350120843691572367894481930 1	c
269	5381.137191509000413860951881501847061521214608667096904 49	c
271	193270471243015279782059101964580241188515112465021394429	c
273	2.13^2.233.421.135721.640457.741469.159607993.14835473303439058865152 73	
275	5^2.89.661.3001.474541.7239101.15806979101.5527278404454199535821801	P
277	3468097888158339286797581652104954628434169971646694834457	c
279	2.17.557.2417.11717.4531100550901.373624834088997895802300093075585 3	c
281	174221.119468273.114205973520041784262049438829321530369345505 7	P
283	10753.825229.15791401.444111888848805843163235784298630863264881	
285	2.5.37.61.113.761.797.29641.54833.67735001.956734616715046328502480330601	
287	13.2789.59369.1981600710018532677967006925074901845705010643822 01	P
289	577.1597.1733.98837.101232653.69891371804579617715291347844255883137	c
291	2.193.389.3084989.361040209.174812390963745482302997072876347538563 21	c
293	7654090467756936378415884538348976340768064993978954512095813	c
295	5.353.1181.35401.75521.160481.737501.2710260697.112096925062539066084 69121	
297	2.17.53.89.109.197.593.4157.19801.1360418597.18546805133.123692430687502422800 33	
299	233.28657.2056992877234275208463485342027139282056040284860517152 1	P
301	13.433494437.6380692745271404734077815684636927896943536596672852 1	
303	2.743519377.770857978613.821246127744216999814751420752635267445501	c
305	5.2441.4513.6101.555003497.132143380343891855589611028370046294780106 61	c
307	613.9143689.115116780181304152598063221920397506646469704378859600 9	c
309	2.617.318889.519121.5644193.512119709.32386142297.883364563627459323040861	
311	837833.6872477.603717553.127223270401321860892580102952310478018380 93	P
313	1877.5009.123149057322573777281207034204319380530029006035226343539 81	c
315	2.5.13.17.61.421.109441.141961.9761221.35239681.8288823481.120570028745492370271501	
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319	89.1913.514229.578029.1435522969.2859900490942608249407328939472547239027 7	c

V_n	n
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257	512653048485188394162163283930413917147479973138989971 c
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260	7^2.103.2161.21183761.57089761.102193207.1932300241.5836312049326721
261	2^2.19.59.349.19489.947104099.121645431297608956949367975807331201
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263	1579.924709.2098741.3001949101336686906107454320302466346629 P
264	2.47.1103.52337681992411201.274295811448428332665070851953601 c
265	11.1061.17491.124021.7627231.14161601.119218851371.73872456598219381
266	3^2.281.978347.29134601.186313849.3336915203.2608509549583653221689
267	2^2.179.3739.22235502640988369.1059215940559134586375464519784009
268	7.4289.6387083201.532023636345822147038743367122454382963889 P
269	13451.12272149989343726401717039290569405150567798203517201 c
270	2.3^4.41.107.2521.11128427.12315241.10783342081.100873547420073756574681
271	59621.899179.8061300232508209254303499776966344362014707611 c
272	2207.4470047.7378607647.70880002682854749400142159467015952505272739743 c
273	2^2.29.79.211.521.859.689667151970161.1895779504507826667970479592081 c
274	3.547.27947.39918080763884273313151499573587120820720882287609 c
275	11^2.101.151.199.331.39161.92401.6982111964759801.964537359154707797801
276	2.7.23^2.253367.3365543.4333249681.9506372193863.62004958987926543420438647
277	1109.5923369.1003666289.1176214902581878448764358665041814178909 c
278	3^2.65609.63749871895972620649953899115136073800470980902829089 P
279	2^2.19.63799.2870911.3010349.35510749.3599504551.379039987657071574441
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282	2.3^2.563.1129.5641.183407723.4632894751907.1568243714391295376547405323 P
283	1699.81904697726943126494463230440143683491348547590190211271 c
284	7^2.32165609929722273666211791549816223828923850302972185505601 P
285	2^2.11.31.191.229.571.9349.32491.41611.95419.2069101.87382901.119130001.411677941
286	3.43.307.5147.90481.31961375809164952327867653664580939435338675283 c
287	29.256579.319973431.370248451.1081998047970095124467105785083573349 c
288	2.1087.4481.270143.11862575248703.49434848723465562366786155433824447 c
289	3571.878516651.795954394282744053474161101150396077552743674481 P
290	3^2.41.347.5801.52201.96281.1270083883.2556385824976729332973207199871721 c
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295	11.709.8969.12391.336419.552241.335838031.99979884881.8287296987284891561
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297	2^2.19.199.991.2179.5779.9901.1513909.220862269.187749394693723383466604449429 c
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299	139.461.521.599.2233531.6875941590920532515070899245710574081024232069 c
300	2.7.23.241.1201.2161.20641.9125201.148669201.5738108801.9602982001.1599014169601
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303	2^2.809.7879.77569.3334819.201062946718741.1587387219272594340274903604341 c
304	607.1823.2207.20063.91807.757169782972914065435430015689333357557665601 c
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306	2.3^3.67.107.409.4283.63443.18531361.66265118449.168778913873539642145017404947 P
307	1229.11739610117429203651282768407085324070169775523763828726810201 P
308	7^2.263.881.967.14503.7872253927.18624389462456074981666789441253366119063 c
309	2^2.619.1031.5257480026438961.2814452349473288674491269177774204148823411 c
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311	34211.2890615644252691924572487628689034423952562309093965400390309 P
312	2.47.1103.3329.740687.71467343.106513889.325759201.252395324361178683936108001 P
313	258899611203303418721656157249445530046830073044201152332257717521
314	3^2.142934683.976922609704830455114855830649608682719829067966633529507 P
315	2^2.11.19.29.31.71.181.211.541.631.911.1009.21211.31249.767131.1051224514831.1983000765501001
316	7.675607.231900742042861225269632036231770950710877804680654044867543 P
317	1774524334209840819478601135200322570341117256884368267719803156571 c
318	2.3^2.1483.2969.14627.346656889.1076012367720403.6640081134555739130972368679147 c
319	59.199.19489.625728071.32447179970327852021339607533382691005310445765831 P
320	127.186812208641.316837008400094222150776737920885469881809351973256961 c

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325	5^2.233.1301.3001.4235401.605416501.880262501.14736206161.492847069677875690583 01	
327	2.653.1746181.827728777.32529675488417.158954614142727267943384636436638045 7	P
329	13.1973.26321.2971215073.127391874411097592672469891375644477141948573020237	P
331	29129.229666866486321202763912280284852008413184976225333705916645024 61	P
333	2.17.73.149.2221.12653.1459000305513721.115508436688780475488413078268664894242277	c
335	5.269.116849.1429913.2040410654589510290615412852018689113300321765114476636 1	
337	673.1783752923237962803666942349768474316048433094733486279369931336172 9	c
339	2.677.149161.258317.272602401466814027129.220987865057977688874221534869142003 3	P
341	89.557.2417.686718062834145927990372608851494464254947439866142023185082001	c
343	13.97.46649.6168709.593641009781799758820843528216578915245805457375675178037	c
345	2.5.61.137.829.1381.18077.28657.186301.2441738887963981.25013864044961447973152814604981	P
347	324097.1434497.317578804297017810849632820740670542053162515231204886263909 7	c
349	1358309.2663569.106841238838703462696574404940158408719142811954606871707526 9	c
351	2.17.53.109.233.29717.135721.2623373.8023861.39589685693.657903216797404903717440980342 57	
353	736357.359802011012573915498603609235632625259749492479918321872573852016 89	P
355	5.4261.75309701.309273161.308061521170129.453748430610423650463292853819795414881	c
357	2.13.421.1429.1597.258469.6376021.159512939815855788121.276538662398362584638816230929 61	
359	475420437734698220747368027166749382927701417016557193662268716376935476241	P
361	37.113.297695973435970582594631907579321477163892921001085193295076858332955181	c
363	2.89.19801.974158134663814455960 89.94905596043359637960818476990353850018366158 01	
365	5.210241.9375829.27583781.86020717.364795723939094556038462329414967577466828961741	c
367	733.17969789.75991753.223133898952823719113768040171047349614978489786992691865 73	c
369	2.17.2789.8117.59369.199261.68541957733949701.936898899104939310314006125714092909358 73	c
371	13.953.207017.55945741.106689145433069236091111846991549277021128640256853245796611 3	P
373	2237.9697.371509.20580649.241642336422695515238330396875615413792846354212011836945 7	P
375	2.5^3.61.3001.9001.169501.230686501.158414167964045700001.411211283995406849171824219359001	c
377	233.514229.229267706221493959508237568880987778484173552777192506048964642455236 01	c
379	757.11889989.799012033863629446107001779701812210353551738609813062636353527061277	c
381	2.27941.18995897.556805304822773221 0073.318545021366982696682842071203909335961765769 3	P
383	1639241.300703446415893757577947980425130220360072332787369911537119082763534986 17	c
385	5.13.89.661.141961.988681.474541.4832521.37807001.75954341.36513554571487776881633473109477461	c

V_n	n
12162770728265582317626551789152812462340990725146376721706364378476	321
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4343325767662069420018267290383923355983240072735967824084581294858480 3	338
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1063072416667491349414207929678038043261366354361402805346626540249634 794799	359
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1783420916989818431634666834082210122316620527717965665015698362483580 3932590082	384
2885635659935574799574835538832430859892853950396188449371461639700317 48466562176 1	385

n	Factorization of V_n	
321	2^2.47927441.809013091.479836483312919.163432894718897814320076670502885071	P
322	3.281.643.4969.275449.770867.25154641.136796138491378769717922579229418853 5321	c
323	3571.9349.95379005710785301231669416509416397896397000350849891560420 1	c
324	2.7.23.647.6263.103681.12340209383.177962167367.173421718166321520831726341471281	
325	11.101.131.151.521.2081.3251.24571.843701.3558039391073701.1459055656827600978264885 1	
326	3.4496239812110895496688641723946600120754312204117277413326859700948 1	c
327	2^2.128621.788071.1358359.802006741.8541593161.593985111211.97389944419638836239	P
328	47.751360126575112691926042052820564621087882021417397688476247509680 1	P
329	29.659.6643838879.4500192981527891986360625004567391475303090666798745179	c
330	2.3^2.41.43.307.1321.2521.817081.3666961.261399601.599968549286568 01.60642572794138104 1	
331	526291.284238550293442429421152997237560751050514642675601126205762603 9	c
332	7.97607.354256280335930696852822377120219563354289329681984216804016290 3	P
333	2^2.19.1999.4441.146521.1121101.14678641.54018521.445660241709738713684642751169927 99	P
334	3.821641.7162963.358901917348421556099213936652614212451002624642335527907	c
335	11.4021.24994118449.918229218981115419161903071.1010052140879271906648306231 1	
336	2.223.449.769.2207.3167.18143.416187743.1368322369.1154149773784223.129252872630958048 1	P
337	21569.340819559.365157515602245993389037020412043685381655238251492049695 1	P
338	3.2027.90481.141283.558727569143205256419132404572064220979037676754412825681	c
339	2^2.41267042784492103747077 1.425742205044275450856459241847795030601698336111	c
340	7.2161.5441.897601.232306572391 21.662554071992381125484871608768998301552526 41	c
341	199.2729.12959.347821.3010349.24968047172542592952969282984682443824284389694771	P
342	2.3^2.227.683.20521.26449.47881.29134601.212067587.2069638565604053338227300806861747	c
343	29.2094359.599786069.1837202669.71971083096389729490209202029340833080374221 99	P
344	47.9303823.178233341166002919406947900436165130536508000472834355294918164 7	c
345	2^2.11.31.139.461.691.1151.4831.5981.324301.686551.1485571.4641631.117169733521.34901253112941 61	
346	3.68014619340568764690338116753350952665397003330448384917711848684619550 81	c
347	662771.49813570218790300646632066495774726229646834657248769033875719531 49	c
348	2.7.23.299281.8344284108795067 21.66431599051105166423027812958421347637000718801	c
349	864343652971263621068393977129950801520895344965840025558372350730689700 1	c
350	3.41.281.401.2801.28001.57061.12317523121.1830525227553199168278865697123947876987201	c
351	2^2.19.79.521.859.5779.65597689.1052645985555841.2110408777664818745937685400328400864 9	P
352	3.1087.4481.75170057074401244349177674001674283467443988279180431037247625792 01	c
353	592429953134577297805108237673547307982868489214813748742645347055736283 71	
354	2.3^2.15247723.10004958719759838 7.3490855311525894593942346160303930322601870428401	P
355	11.688846502588399.4313722764802461186494683944 1.4745095041282676498992035325 61	
356	7.63367.565768471959285714079262248889509474547974219027885983055827845016103	P
357	2^2.29.211.239.919.3469.3571.10711.27932732439809.2037962186691204100928530687800099843828 1	P
358	3.2190049620009999590537720343592848118961897884535638604941195069392187644 01	c
359	719.1648529.896886254071077987926885668123271339039771087126000364568813909649	c
360	2.47.1103.1601.3041.5208481.10749957121.23735900452321.256398714710592962213808203802480 1	P
361	9349.2976959802479390925221662169680105382619858887670870886913693016703803 49	c
362	3.15010823406694912675217519983966076670537479312685305263348655541278023236 01	c
363	2^2.199.9439.9901.2435731.363397849.974207332084918690441 99.11358794087797342561956147 61	P
364	7^2.103.727.14503.193649.800801.102193207.13980206768551366640135929266105636049362846007	c
365	11.151549.514651.7015301.8942501.9157663121.11899937029.3252336525249736694804555892 11	
366	2.3^2.19763.102481.10225307.21291929.21791641.24848660119363.71816349296373557767010814126 83	P
367	2298889.2172430890432626512218324058868794301088329219934595302031321708253386 1	c
368	2207.245087.11079007.13484292549345009218015967701713491137426073107017330576389569	P
369	2^2.19.739.4767481.370248451.1834684796684062744924940447694296317974460292 59	c
370	3.41.1481.11987.81143477963.11939952491306142175725173848307099580979926808298722812 41	c
371	29.6679.119218851371.148238208621737727939036154808965612542574638594693300767321 59	c
372	2.7.23.743.1489.467729.337587408304601 83.98465083280977541404796710798431455619960794209	P
373	2239.400254035815230315691149177295200682735160941001088181098834904823746738439	P
374	3.43.67.307.2243.49369.63443.777870407498836400791008733412392475736147872922982778048 03	c
375	2^2.11.31.101.151.251.751.2251.12301.18451.112128001.281433 78001.46853582653501.79208139733005002475 1	
376	47^2.10779169.15943062077208588967576599806975152756754056224043158985561025416036 7	c
377	59.521.19489.1025323906903817638167400005889614448718257502427374033529691854638280 1	c
378	2.3^4.83.107.281.1427.2267.32507.1461601.11128427.764940961.18788923389546130048317121725489105289	P
379	342912379.468956426231922484189006878505762245432735158374232784872789019462636 31	c
380	7.2161.109134639698040 1.157611487966856395597006905601707783497560682716484779158960 1	c
381	2^2.509.3049.5081.487681.7357111.13822681.19954241.13487637940319141560417976471439835482057 49	P
382	3.55009.41278468010826232294987989146753769112612154470564124978853355844409777308 9	c
383	7901291.1394980570829066247452686592502525962778636366120533881378431145941614866 9	c
384	2.119809.885503.1359361.4698167634523379875583.131607629147911703853902499918379122339841	c
385	11^2.29.71.199.331.911.3851.39161.229769.9321929.84100171.582276311.1097233061.1112118152741317993811	

U'_n	n	φ(n)	U'_n	n	φ(n)
1	1	1	14736206161	65	48
1	2	1	9901	66	20
2	3	2	44945570212853	67	66
3	4	2	4250681	68	32
5	5	4	2053058121	69	44
1	6	2	64681	70	24
13	7	6	308061521170129	71	70
7	8	4	703681	72	24
17	9	6	806515533049393	73	72
11	10	4	54018521	74	36
89	11	10	230686501	75	40
1	12	4	29134601	76	36
233	13	12	4777821694801	77	60
29	14	6	67861	78	24
61	15	8	14472334024676221	79	78
47	16	8	4868641	80	32
1597	17	16	192900153617	81	54
19	18	6	370248451	82	40
4181	19	18	99194853094755497	83	82
41	20	8	118441	84	24
421	21	12	32522917584361	85	64
199	22	10	969323029	86	42
28657	23	22	661078661101	87	56
23	24	8	224056801	88	40
3001	25	20	1779979416004714189	89	88
521	26	12	97921	90	24
5777	27	18	118344378961717	91	72
281	28	12	1368706081	92	44
514229	29	28	4531100550901	93	60
31	30	8	6643838879	94	46
1346269	31	30	1527884938291801	95	72
2207	32	16	2435423	96	32
19801	33	20	83621143489848422977	97	96
3571	34	16	599786069	98	42
141961	35	24	3653720611201	99	60
107	36	12	228811001	100	40
24157817	37	36	573147844013817084101	101	100
9349	38	18	3188011	102	32
135721	39	24	1500520536206896083277	103	102
2161	40	16	10525900321	104	48
165580141	41	40	8288823481	105	48
211	42	12	119218851371	106	52
433494437	43	42	10284720757613717413913	107	106
13201	44	20	11128427	108	36
109441	45	24	26925748508234281076009	109	108
64979	46	22	12962291	110	40
2971215073	47	46	1459000305513721	111	72
1103	48	16	10745088481	112	48
598364773	49	42	184551825793033096366333	113	112
15251	50	20	21850951	114	36
6376021	51	32	337204140427825761	115	88
90481	52	24	440719107401	116	56
53316291173	53	52	20000273725560977	117	72
5779	54	18	2139295485799	118	58
313671601	55	40	1595129398158557888121	119	96
14503	56	24	4974481	120	32
43701901	57	36	974158134663814455960 89	121	110
1149851	58	28	5600748293801	122	60
956722026041	59	58	685419577339497 01	123	80
2521	60	16	3020733700601	124	60
2504730781961	61	60	158414167964045700001	125	100
3010349	62	30	31530241	126	36
35239681	63	36	1555769702205310656816 49693	127	126
4870847	64	32	23725150497407	128	64

U'_n	n	φ(n)
469793567274867421	129	84
6698324881	130	48
1066340417491710595814572169	131	130
261399601	132	40
513626746742783513994 01	133	108
100501350283429	134	66
1114769954367361	135	72
23230657239121	136	64
19134702400093278081449423917	137	136
1026529561	138	44
50095301248058391139327916261	139	138
12317523121	140	48
220702975250559883 21	141	92
688846502588399	142	70
16557761623387534111504801	143	120
10749957121	144	48
34961999693073707989 0201	145	112
1803423556807921	146	72
359316079957723981	147	84
972666870342481	148	72
6161314747715278029583501626149	149	148
226965751	150	40
16130531424904581415797907386349	151	150
1091346396980401	152	72
717532311495056459 3	153	96
2141890304401	154	60
16424697761902088993749801	155	120
1228021041	156	48
28945064194127398549508804210 4137	157	156
32361122672259149	158	78
7106567261110294289821	159	104
23725145626561	160	64
5325384894714727861491234721	161	132
192900153619	162	54
519398102351802715749578685048 8117	163	162
456946384892998 01	164	80
40326773227765201	165	80
221806434537978679	166	82
35600075545958458963222876581316753	167	166
10978677361	168	48
400009475456580321242184872389193	169	156
14783146675081	170	64
392773399061790152736 01	171	108
313195711516578281	172	84
6388174356131903419057639723895054 93	173	172
330539330551	174	56
12079073814871033705119001	175	120
5233768199241 1201	176	80
2288292587779989705334201	177	116
398015497273691805 1	178	88
114631137654914676953405286264297821 21	179	178
10783342081	180	48
30010821454963453907530667147829489881	181	180
689667151970161	182	72
15684190725257406307513801	183	120
240860100364248 6721	184	88
17029455131913055569070976181 61	185	144
2265550275451	186	60
3788862125124360918966178024054801	187	160
1471353168337065 8881	188	92
37216909291681445195521	189	108
694493169660601	190	72
3691087032412706639440686994833808526209	191	190
11862575248703	192	64

U'_n	n	$\varphi(n)$
9663391306290450775010025392525829059713	193	192
186982561199565069121	194	96
88999250837499877681	195	96
358889844987430121	196	84
66233869353085486231758142155705206899077	197	196
3269118441601	198	60
173402521172797813159685037284371942044301	199	198
52361396168994001	200	80
5050260704396247169315999021	201	132
1281597540372340914251	202	100
177789476511755544773333681149178601	203	168
271102433445641	204	64
3758399976002037812130285171971401	205	160
3355265920593054081629	206	102
4072353155773627601222196481	207	132
115509240442846111681	208	96
57314120955051297736679165379998262001	209	180
16271615641	210	48
55835073295300465536628086585786672357234389	211	210
47377115074406862859881	212	104
237254752064134595103404691601	213	140
22997334743627409910279	214	106
176564796682276305310248237411699961	215	168
1114577054219521	216	72
57247659290265530398525693996253373401	217	180
60207804009475408400201	218	108
1626168262624366331113444171121	219	144
59996854928656801	220	80
18455365724971961787396586822078046791921	221	192
729500152756861	222	72
17978720198565577104981084195536024127087428957	223	222
115561554399692896321	224	96
11981661982050957053616001	225	120
412670427844921037470771	226	112
123227981463641240980692501505442003148737643593	227	226
1273237463143801	228	72
322615043836854783580186309282650000354271239929	229	228
1532746093600320481	230	88
9164259601748159235188401	231	120
249728569236429650967601	232	112
2211236406303914545699412969744873993387956988653	233	232
1052645985555841	234	72
389673426275593986752662955603693114561	235	184
1525528391853326470222801	236	116
523621130304502562371359707102101	237	156
71405811821907813561	238	96
39679027332006820581608740953902289377834488152161	239	238
23735900452321	240	64
103881042195729914708510518382775401680142036775841	241	240
97420733208491869044199	242	110
7177905237579946589743592924684177	243	162
10456127150171604205009201	244	120
128955073914024460192651484843195641	245	168
34270978866974851	246	80
1913489357079567637602203056753846715378384401	247	216
11731926972788501024264401	248	120
24599047201225310501731215529292521	249	164
792070839876516006251	250	100
12776523572924732586037033894655031898659556447352249	251	250
1118038473538561	252	72
13114968467410239465917422663539036363453124801	253	220
347880681146567910619198829	254	126
433500170917755760773281881	255	128
562882766124611619513723647	256	128

U'_n	n	$\varphi(n)$
229265413057075367692743352179590077832064383222590237	257	256
234896783637433711	258	84
1911229249245643118715813507087715606224610321	259	216
132413031725367949921	260	96
135956658509092363649293620019958401	261	168
2384409660666970679779568599	262	130
4114000911454431885883343305337966369078499341559272017	263	262
5347545768789841	264	80
40402638589347735759400037131647739113085561	265	208
23024647794636831928201	266	108
7920816803501208436915723678944819301	267	176
3366840469597498112877332681	268	132
738227509931226985782074361439038045655809237648344306069	269	268
1114384154071681	270	72
193270471243015279782059101964580241188515112465021394429	271	270
562627837283291940137654881	272	128
950148272550066932359912199761	273	144
42786495295836948034307560771	274	136
632477172138993049465314748315158527610001	275	200
2810034504249567841	276	88
3468097888158339286797581652104954628434169971646694834457	277	276
112016498943988617255084900949	278	138
43777621810207883451155501905666329601	279	180
118021448662479038881	280	96
23770696554372451866815101694984845480039225387896643963981	281	280
11035148762527994161	282	92
62232491515607091882574410635924603070626544377175485625797	283	282
158169834709423063402269794401	284	140
956734616715046328502480330601	285	144
7405284634925086989050401	286	120
198160071001853267796700692507490184570501064382201	287	240
115561578124833522881	288	96
699259079143875588352875272257068190759451239084227956437	289	272
158918180423302701281401	290	112
17481239096574548230299707287634753856321	291	192
1084112175083244220742296114081	292	144
7654090467756936378415884538348976340768064993978954512095813	293	292
358464620052610891	294	84
34189026373831073512730814477047681275126362360401	295	232
121638965227950458607658558321	296	144
41481113125477462254014744580245308801	297	180
13777118606663328742845309894251	298	148
20569928772342752084634853420271392820560402848605171521	299	264
52364857850613001	300	80
63806927452714047340778156846369278969435365966728521	301	252
36068964779283188188226718906749	302	150
821246127744216999814751420752635267445501	303	200
1241719381955383992204428253601	304	144
196795070965000355893436141405307761691082367379601	305	240
109140441064248061441	306	96
6452389184720949856740872794933738025334109298792472139250504213	307	306
13729736197875056147157601	308	120
5628904698946577348982538355548408297646821	309	204
7465771709957142670086601	310	120
44225333398004061429732838340729878012027363723832270745251370289	311	310
118020310887255809761	312	96
115783425999770513860373944643635095356961600163955231274253486033	313	312
647231311511640322009998617887721	314	156
1176910696561103780033951262721	315	144
349080753536335041580906471401401	316	156
793591407804151926593793042126891128819610710140145037958273777397	317	316
35553283630555147144911	318	104
45396844439396427251479423963858929077218814693965024038001	319	280
562882766124587894363226241	320	128

U'_n	n	$\varphi(n)$
26443870265522619375245858112105515141492892 1	321	212
23872415056900646982889953 61	322	132
2132739020660858611399983843172719723785638705615180222739401	323	288
1240348975528266616 3307	324	108
1447273017164780824250612014185359555218198685 4001	325	240
116140946424302426079917484030 72229	326	162
1812489831821701531824147050779299837088420201	327	216
268457141143002702824790590 3965201	328	160
661562985153688798057239386150242143206941961433874 3385921	329	276
79348675496547601	330	80
66899661538800503153100008124174541530676651724677455196459529218 6469	331	330
16399364800816873667820810352861681	332	164
146152824942313935635489067933341690684753 0881	333	216
79604188924891052809189520789413679	334	166
2040410654589510290615412852018689113300321765114476 6361	335	264
115613939510481515041	336	96
12004657173391489668678522013941832147005954727556362660159637892443617	337	336
400012422749007152825031617346281	338	156
85148441008855090171291848369559006120339667221	339	224
644963618349135574923636721	340	128
68671306283414592799037260885149446425494743986614202318508 2001	341	300
184962542494471817 0779	342	108
276927594653111769492335297477751898173015787811178713802480 13	343	294
12611777119159116467844040459440 33521	344	168
46601078854403627188463425137225652 81	345	176
142843921124318865454716321873105 8171	346	172
147647522703638250328143702791153654140662564470619466815243873234644927 3	347	346
291349997442501259462201	348	112
3865462327928467072415604609040860366007401579690263197296200323999931849	349	348
1188423726917190428210 7001	350	120
13848586593679652194976260087652740134391 12321	351	216
2741715269847000008438217625468801	352	160
26494272942318589069480525788592273303839335703403521573912286394960106973	353	352
114414629388999485266 7101	354	116
450318539016111831326965081696478114596677365861196978 31201	355	280
52805445356674722912771491192965 46201	356	176
102140163618431159432395564928375729 57161	357	192
2563230161345250472903974841636963 3399	358	178
4754204377346982207473680271667493829277014170165571936622687163769354 76241	359	358
1155626927018926 38721	360	96
297695973435970582594631907579321477163892921001085193295076858332955181	361	342
67106236833907426331910944190628905401	362	180
94905596043359637960818476990353850018366 15801	363	220
1423526509223971062035518542241	364	144
211553857469470098855087199754537908360929029378310996198 5761	365	288
78420953626287031537 56901	366	120
223346406617740673564123319000380099530453510206838235072028935074763140370 53	367	366
604822512643898132343564306639160 1281	368	176
15153417194259367647935934603573890941576115863 6801	369	240
774066142359684874594246376881	370	144
2208646682011769404787370122864955668108298772005298838507708 14921	371	312
13687248134919917861641801	372	120
4007788650469974194095938181950950360587940820696032859364853667898835670 55193	373	372
169451649257831571064199721749 6401	374	160
627376215360397612529601866427135826537501	375	200
278341733053995704829636481866939649921	376	184
229267706221493959508237568880987778484173552777192506048964642455 23601	377	336
372040292400144850 13761	378	108
71916849301841794820162763956116726391052481262321753233495337087104278929 56421	379	378
142343441346104975595569144 0401	380	144
6051048415750002506969509651856127743268098491748 5621	381	252
825352151534278163169798980055023254 0799	382	190
49292541820623609906583302538007088755326533087070104115801862234837985426429697	383	382
281441383062305809756861823	384	128
104852536770885187327530985282686231735603659815201	385	240

CONJECTURED INEQUALITIES FOR EULER'S φ-FUNCTION AND FOR FIBONACCI NUMBERS

(1) $\varphi(2n) < \varphi(2n\pm1)$ (1') $U'_{2n} < U'_{2n\pm1}$

(2) $\varphi(6n) < \varphi(6n\pm1,2)$ (2') $U'_{6n} < U'_{6n\pm1,2}$

(3) $\varphi(30n) < \varphi(30n\pm1,2,3,4,5)$ (3') $U'_{30n} < U'_{30n\pm1,2,3,4,5}$

with eventual exceptions called anomalies.

All the inequalities in φ have been checked up to n=10,000 by Glaisher's "Number-Divisor Tables" (Cambridge 1940).

For (1), 137 anomalies have been found up to n=10,000, most of them around numbers divisible by 3.5=15, among them 17 cases of equality, and 150 cases of inequality in the opposite direction. Most of the inequalities appear in pairs, i.e., there are simultaneously: $\varphi(2n)>\varphi(2n+1)$ and $\varphi(2n+1)<\varphi(2n+2)$. Up to n=10,000 there are 64 pairs (among them 1 pair for which all the values are equal, namely $\varphi(5186) =\varphi(5187)=\varphi(5188)$, and 39 single anomalies.

For (2), 30 anomalies have been found up to n=10,000, all of them around numbers divisible by 2.5.7=70, except for $\varphi(6)=\varphi(4)$ and $\varphi(12)=\varphi(10)$.

For (3), no anomalies have been found up to n=10,000. It is impossible to improve (3) generally to have $\varphi(30n)<\varphi(30n\pm k)$ for k>5, since, e.g., $\varphi(2190)=\varphi(2184)$, $\varphi(3240)>\varphi(3234)$, $\varphi(5550)=\varphi(5544)$, $\varphi(6000)>\varphi(6006)$, $\varphi(8310)>\varphi(8316)$, $\varphi(8730)=\varphi(8736)$.

CONJECTURE. Anomalies in $\varphi(n)$ and U'_n occur simultaneously, apart from n=15, for which $\varphi(16)=\varphi(15)$, while $U'_{16}<U'_{15}$.

Since $(U'_m,U'_n)=1$ for any positive m, n, and $U'_n>1$ for $n\neq1$, 2, 6, 12, an anomaly in U'_n for n>1 is always an inequality.

The following anomalies in U'_n have been checked: $U'_1=U'_2$, $U'_3<U'_4$, $U'_{104}>U'_{105}$, $U'_{105}<U'_{106}$, $U'_{164}>U'_{165}$, $U'_{165}<U'_{166}$, $U'_{194}>U'_{195}$, $U'_{255}<U'_{256}$, $U'_{314}>U'_{315}$, $U'_{315}<U'_{316}$, $U'_{495}<U'_{496}$, $U'_{524}>U'_{525}$, $U'_{525}<U'_{526}$, $U'_{584}>U'_{585}$, $U'_{585}<U'_{586}$, $U'_{734}>U'_{735}$, $U'_{735}<U'_{736}$.

TABLE OF ANOMALIES IN $\varphi(n)$ UP TO n=10000

Cases in which one or both of the inequalities $\varphi(2n)<\varphi(2n\pm1)$ do not hold

$\varphi(1)=\varphi(2)$

$\varphi(3)=\varphi(4)$

$\varphi(15)=\varphi(16)$

$\varphi(104)=\varphi(105)$
$\varphi(105)<\varphi(106)$

$\varphi(164)=\varphi(165)$
$\varphi(165)<\varphi(166)$

$\varphi(194)=\varphi(195)$

$\varphi(255)=\varphi(256)$

$\varphi(314)>\varphi(315)$
$\varphi(315)<\varphi(316)$

$\varphi(495)=\varphi(496)$

$\varphi(524)>\varphi(525)$
$\varphi(525)<\varphi(526)$

$\varphi(584)=\varphi(585)$
$\varphi(585)<\varphi(586)$

$\varphi(734)>\varphi(735)$
$\varphi(735)<\varphi(736)$

$\varphi(824)>\varphi(825)$

$\varphi(944)>\varphi(945)$

$\varphi(974)>\varphi(975)$
$\varphi(975)=\varphi(976)$

$\varphi(1154)>\varphi(1155)$
$\varphi(1155)<\varphi(1156)$

$\varphi(1364)>\varphi(1365)$
$\varphi(1365)<\varphi(1366)$

$\varphi(1485)<\varphi(1486)$

$\varphi(1574)>\varphi(1575)$
$\varphi(1575)<\varphi(1576)$

$\varphi(1754)>\varphi(1755)$
$\varphi(1755)<\varphi(1756)$

$\varphi(1784)>\varphi(1785)$
$\varphi(1785)<\varphi(1786)$

$\varphi(1814)>\varphi(1815)$
$\varphi(1815)<\varphi(1816)$

$\varphi(1994)>\varphi(1995)$
$\varphi(1995)<\varphi(1996)$

$\varphi(2144)>\varphi(2145)$
$\varphi(2145)<\varphi(2146)$

$\varphi(2204)=\varphi(2205)$
$\varphi(2205)<\varphi(2206)$

$\varphi(2414)>\varphi(2415)$
$\varphi(2415)<\varphi(2416)$

$\varphi(2474)>\varphi(2475)$
$\varphi(2475)<\varphi(2476)$

$\varphi(2535)<\varphi(2536)$

$\varphi(2624)>\varphi(2625)$
$\varphi(2625)=\varphi(2626)$

$\varphi(2804)>\varphi(2805)$
$\varphi(2805)<\varphi(2806)$

$\varphi(2834)=\varphi(2835)$
$\varphi(2835)<\varphi(2836)$

$\varphi(3003)<\varphi(3004)$

$\varphi(3044)>\varphi(3045)$
$\varphi(3045)<\varphi(3046)$

$\varphi(3134)>\varphi(3135)$

$\varphi(3254)>\varphi(3255)$
$\varphi(3255)=\varphi(3256)$

$\varphi(3314)>\varphi(3315)$
$\varphi(3315)<\varphi(3316)$

$\varphi(3464)>\varphi(3465)$
$\varphi(3465)<\varphi(3466)$

$\varphi(3675)<\varphi(3676)$

$\varphi(3704)>\varphi(3705)$
$\varphi(3705)=\varphi(3706)$

$\varphi(3884)>\varphi(3885)$
$\varphi(3885)<\varphi(3886)$

$\varphi(3927)<\varphi(3928)$

$\varphi(4094)>\varphi(4095)$
$\varphi(4095)<\varphi(4096)$

$\varphi(4124)>\varphi(4125)$
$\varphi(4125)<\varphi(4126)$

$\varphi(4304)>\varphi(4305)$
$\varphi(4305)<\varphi(4306)$

$\varphi(4388)>\varphi(4389)$

$\varphi(4455)<\varphi(4456)$

$\varphi(4485)<\varphi(4486)$

$\varphi(4514)>\varphi(4515)$
$\varphi(4515)<\varphi(4516)$

$\varphi(4724)>\varphi(4725)$
$\varphi(4725)<\varphi(4726)$

$\varphi(4785)<\varphi(4786)$

$\varphi(4845)<\varphi(4846)$

$\varphi(4874)>\varphi(4875)$

$\varphi(4934)>\varphi(4935)$
$\varphi(4935)<\varphi(4936)$

$\varphi(5114)>\varphi(5115)$
$\varphi(5115)<\varphi(5116)$

$\varphi(5144)>\varphi(5145)$
$\varphi(5145)<\varphi(5146)$

$\varphi(5186)=\varphi(5187)$
$\varphi(5187)=\varphi(5188)$

$\varphi(5265)<\varphi(5266)$

$\varphi(5313)<\varphi(5314)$

$\varphi(5354)>\varphi(5355)$
$\varphi(5355)<\varphi(5356)$

$\varphi(5444)>\varphi(5445)$

$\varphi(5564)>\varphi(5565)$

$\varphi(5774)>\varphi(5775)$
$\varphi(5775)<\varphi(5776)$

$\varphi(5864)>\varphi(5865)$

$\varphi(5985)<\varphi(5986)$

$\varphi(6044)>\varphi(6045)$
$\varphi(6045)<\varphi(6046)$

$\varphi(6105)<\varphi(6106)$

$\varphi(6194)>\varphi(6195)$
$\varphi(6195)<\varphi(6196)$

$\varphi(6404)>\varphi(6405)$
$\varphi(6405)<\varphi(6406)$

$\varphi(6434)>\varphi(6435)$
$\varphi(6435)<\varphi(6436)$

$\varphi(6614)>\varphi(6615)$
$\varphi(6615)<\varphi(6616)$

$\varphi(6824)>\varphi(6825)$
$\varphi(6825)<\varphi(6826)$

$\varphi(7034)>\varphi(7035)$
$\varphi(7035)<\varphi(7036)$

$\varphi(7094)>\varphi(7095)$
$\varphi(7095)<\varphi(7096)$

$\varphi(7214)>\varphi(7215)$

$\varphi(7244)>\varphi(7245)$
$\varphi(7245)<\varphi(7246)$

$\varphi(7394)>\varphi(7395)$
$\varphi(7395)<\varphi(7396)$

$\varphi(7454)>\varphi(7455)$
$\varphi(7455)<\varphi(7456)$

$\varphi(7604)>\varphi(7605)$
$\varphi(7605)<\varphi(7606)$

$\varphi(7664)>\varphi(7665)$
$\varphi(7665)<\varphi(7666)$

$\varphi(7754)>\varphi(7755)$

$\varphi(7874)>\varphi(7875)$

$\varphi(7995)<\varphi(7996)$

$\varphi(8084)>\varphi(8085)$
$\varphi(8085)<\varphi(8086)$

$\varphi(8264)>\varphi(8265)$
$\varphi(8265)<\varphi(8266)$

$\varphi(8295)<\varphi(8296)$

$\varphi(8384)>\varphi(8385)$

$\varphi(8415)<\varphi(8416)$

$\varphi(8504)>\varphi(8505)$
$\varphi(8505)<\varphi(8506)$

$\varphi(8714)>\varphi(8715)$
$\varphi(8715)<\varphi(8716)$

$\varphi(8744)>\varphi(8745)$
$\varphi(8745)<\varphi(8746)$

$\varphi(8775)<\varphi(8776)$

$\varphi(8835)<\varphi(8836)$

$\varphi(8924)>\varphi(8925)$
$\varphi(8925)<\varphi(8926)$

$\varphi(9075)<\varphi(9076)$

$\varphi(9134)>\varphi(9135)$
$\varphi(9135)<\varphi(9136)$

$\varphi(9165)<\varphi(9166)$

$\varphi(9344)>\varphi(9345)$
$\varphi(9345)<\varphi(9346)$

$\varphi(9404)>\varphi(9405)$
$\varphi(9405)<\varphi(9406)$

$\varphi(9554)>\varphi(9555)$
$\varphi(9555)<\varphi(9556)$

$\varphi(9734)>\varphi(9735)$
$\varphi(9735)<\varphi(9736)$

$\varphi(9764)>\varphi(9765)$
$\varphi(9765)<\varphi(9766)$

$\varphi(9945)<\varphi(9946)$

$\varphi(9974)>\varphi(9975)$
$\varphi(9975)<\varphi(9976)$

Cases of equality

$\varphi(1)=\varphi(2)$, $\varphi(3)=\varphi(4)$, $\varphi(15)=\varphi(16)$, $\varphi(104)=\varphi(105)$, $\varphi(164)=\varphi(165)$, $\varphi(194)=\varphi(195)$, $\varphi(255)=\varphi(256)$, $\varphi(495)=\varphi(496)$, $\varphi(584)=\varphi(585)$, $\varphi(975)=\varphi(976)$, $\varphi(2204)=\varphi(2205)$, $\varphi(2625)=\varphi(2626)$, $\varphi(2834)=\varphi(2835)$, $\varphi(3255)=\varphi(3256)$, $\varphi(3705)=\varphi(3706)$, $\varphi(5186)=\varphi(5187)=\varphi(5188)$.

TABLE OF ANOMALIES IN $\varphi(n)$ UP TO n=10000

Cases in which one ore more of the inequalities $\varphi(6n)<\varphi(6n\pm1,2)$ do not hold

$\varphi(6)=\varphi(4)$	$\varphi(1542)>\varphi(1540)$	$\varphi(3222)>\varphi(3220)$	$\varphi(5322)>\varphi(5320)$	$\varphi(7698)>\varphi(7700)$
$\varphi(12)=\varphi(10)$	$\varphi(1608)=\varphi(1610)$	$\varphi(3642)>\varphi(3640)$	$\varphi(5388)>\varphi(5390)$	$\varphi(8328)>\varphi(8330)$
$\varphi(72)=\varphi(70)$	$\varphi(2172)=\varphi(2170)$	$\varphi(3852)>\varphi(3850)$	$\varphi(5952)=\varphi(5950)$	$\varphi(8472)>\varphi(8470)$
$\varphi(768)>\varphi(770)$	$\varphi(2382)>\varphi(2380)$	$\varphi(4062)>\varphi(4060)$	$\varphi(6372)>\varphi(6370)$	$\varphi(8682)>\varphi(8680)$
$\varphi(912)=\varphi(910)$	$\varphi(2592)=\varphi(2590)$	$\varphi(4338)=\varphi(4340)$	$\varphi(6648)>\varphi(6650)$	$\varphi(9102)=\varphi(9100)$
$\varphi(1332)=\varphi(1330)$	$\varphi(2658)>\varphi(2660)$	$\varphi(4548)>\varphi(4550)$	$\varphi(7278)>\varphi(7280)$	$\varphi(9312)>\varphi(9310)$

Cases of equality

$\varphi(6)=\varphi(4)$, $\varphi(12)=\varphi(10)$, $\varphi(72)=\varphi(70)$, $\varphi(912)=\varphi(910)$, $\varphi(1332)=\varphi(1330)$, $\varphi(1608)=\varphi(1610)$, $\varphi(2172)=\varphi(2170)$, $\varphi(2592)=\varphi(2590)$, $\varphi(4338)=\varphi(4340)$, $\varphi(5952)=\varphi(5950)$, $\varphi(9102)=\varphi(9100)$.

DIVISIBILITY OF FIBONACCI AND LUCAS NUMBERS BY THEIR SUBSCRIPTS

NOTATIONS:

U_n - Fibonacci numbers $U_0=0$, $U_1=1$, $U_n=U_{n-1}+U_{n-2}$.

V_n - Lucas numbers $V_0=2$, $V_1=1$, $V_n=V_{n-1}+V_{n-2}$.

N - The set of all integers $n>1$ for which $n|U_n$.

$\bar{N}$ - The set of all integers $n>1$ for which $n|V_n$.

$a(n)$ ("rank of apparition of n in U") - the smallest positive subscript $a=a(n)$ for which $n|U_a$.

$\bar{a}(n)$ ("rank of apparition of n in V") - the smallest positive subscript $\bar{a}=\bar{a}(n)$ for which $n|V_{\bar{a}}$.

$n=p^{\pi}r^{\rho}\ldots$ - the canonic factorization of n.

$\tilde{\pi},\tilde{\rho},\ldots$ ("exponents of apparition of $p,r,\ldots$ in U") - the exponents of the highest powers of $p,r,\ldots$ respectively, dividing $U_{a(p)}$, $U_{a(r)},\ldots$ respectively.

$(a,b,\ldots)$ - the greatest common divisor of $a,b,\ldots$.

$[a,b,\ldots]$ - the least positive common multiple of $a,b,\ldots$.

Properties of U_n, V_n used in the proofs of the theorems:

$(a(p),p)=1$.	$(\bar{a}(p),p)=1$.
$n\mid U_n \longleftrightarrow a(n)\mid n$.	$2\nmid\frac{V_n}{n}$, $n\mid V_n \longleftrightarrow \bar{a}(n)\mid n$, $2\nmid\frac{n}{\bar{a}(n)}$.
$a(n)=[a(p)p^{\pi-\tilde{\pi}},a(r)r^{\rho-\tilde{\rho}},\ldots]$.	$\bar{a}(n)=[\bar{a}(p)p^{\pi-\tilde{\pi}},\bar{a}(r)r^{\rho-\tilde{\rho}},\ldots]$.
The prime factors of $a(p)$ are $<p$, if p is a prime $\neq 2,5$.	The prime factors of $\bar{a}(p)$ are $<p$, if p is a prime $\neq 2$.
$5^{\alpha}\mid n \longrightarrow 5^{\alpha}\mid U_n$.	$2\mid n \longrightarrow 4\nmid V_n$.

Theorems A(1), B(1), C(1), F(1) were proved and presented to me by Prof. Theodore Motzkin not later than 1951. Then I also developed the other material given in this paper.

THEOREM A.

(1) $n\in N$ if and only if $[a(p),a(r),\ldots]|n$, i. e., if and only if the rank of apparition of any prime divisor of n also divides n.

(2) $n\in\bar{N}$ if and only if n is an odd multiple of $[\bar{a}(p),\bar{a}(r)\ldots]$, i. e., if and only if the rank of apparition of any prime divisor of n also divides n, and n is not divisible by a power of 2 higher than the powers of 2 dividing the ranks of apparition of the prime factors of n.

PROOF.

(1) First, let $[a(p),a(r),\ldots]|n$. Hence $a(p)|n$, $a(r)|n,\ldots$. On the other hand $p^{\pi-\tilde{\pi}}|n$, $r^{\rho-\tilde{\rho}}|n,\ldots$. But $(a(p),p)=1$, $(a(r),r)=1,\ldots$. Therefore $a(p)p^{\pi-\tilde{\pi}}|n$, $a(r)r^{\rho-\tilde{\rho}}|n,\ldots$. Therefore $[a(p)p^{\pi-\tilde{\pi}},a(r)r^{\rho-\tilde{\rho}},\ldots]|n$. But $[a(p)p^{\pi-\tilde{\pi}},a(r)r^{\rho-\tilde{\rho}},\ldots]=a(n)$. Hence $a(n)|n$. Hence $n|U_n$, i. e., $n\in N$.

Secondly, let $n\in N$, i. e., $n|U_n$. Hence $a(n)|n$. But $a(n)=[a(p)p^{\pi-\tilde{\pi}},a(r)r^{\rho-\tilde{\rho}},\ldots]$. Hence $[a(p)p^{\pi-\tilde{\pi}},a(r)r^{\rho-\tilde{\rho}},\ldots]|n$. Hence $[a(p),a(r),\ldots]|n$.

(2) First, let n be an odd multiple of $[\bar{a}(p),\bar{a}(r),\ldots]$. Hence $\bar{a}(p)|n$, $\bar{a}(r)|n,\ldots$. On the other hand $p^{\pi-\tilde{\pi}}|n$, $r^{\rho-\tilde{\rho}}|n,\ldots$. But $(\bar{a}(p),p)=1$, $(\bar{a}(r),r)=1,\ldots$. Hence $\bar{a}(p)p^{\pi-\tilde{\pi}}|n$, $\bar{a}(r)r^{\rho-\tilde{\rho}}|n,\ldots$. Hence $[\bar{a}(p)p^{\pi-\tilde{\pi}}, \bar{a}(r)r^{\rho-\tilde{\rho}},\ldots]|n$. But n is an odd multiple of $[\bar{a}(p),\bar{a}(r),\ldots]$. Hence n is an odd multiple of $[\bar{a}(p)p^{\pi-\tilde{\pi}},\bar{a}(r)r^{\rho-\tilde{\rho}},\ldots]=\bar{a}(n)$. Therefore $n|V_n$, i. e., $n\in\bar{N}$.

Secondly, let $n\in\bar{N}$, i. e., $n|V_n$. Then $n\neq 2$, since $2\nmid V_2=3$. Hence n is an odd multiple of $\bar{a}(n)$. But $\bar{a}(n)=[\bar{a}(p)p^{\pi-\tilde{\pi}},\bar{a}(r)r^{\rho-\tilde{\rho}},\ldots]$. Hence, at least one of the numbers $a(p)p^{\pi-\tilde{\pi}}$, $a(r)r^{\rho-\tilde{\rho}},\ldots$ is divisible by 2^{α}, where 2^{α} is the highest power of 2 dividing n. But the exponent of apparition of 2 is 1. Hence, it is not $\bar{a}(2)2^{\alpha-1}=3.2^{\alpha-1}$ which is divisible by 2^{α}. Hence we can drop $\bar{a}(2)2^{\alpha-1}$ from the numbers $\bar{a}(p)p^{\pi-\tilde{\pi}}$, $\bar{a}(r)r^{\rho-\tilde{\rho}},\ldots$ and n will be an odd multiple of $[\bar{a}(p)p^{\pi-\tilde{\pi}},\bar{a}(r)r^{\rho-\tilde{\rho}},\ldots]$, where all the numbers p, $r,\ldots$ are odd. Hence n is an odd multiple of $[\bar{a}(p),\bar{a}(r),\ldots]$, where all the numbers p, $r,\ldots$ are odd. But, according to a remark above, $\bar{a}(2)|n$ if $2|n$. Hence n is an odd multiple of $[a(p),a(r),\ldots]$, where p, $r,\ldots$ are all the prime factors of n.

Theorem A yields immediately

THEOREM B.

(1) If $n\in N$ and m is composed of only prime factors of n, then also $mn\in N$.

(2) If $n\in\bar{N}$ and m is composed of only odd prime factors of n, then also $mn\in\bar{N}$.

THEOREM C.

(1) If $n_1\in N$, $n_2\in N$, then also $[n_1,n_2]\in N$. In particular, if $n_1\in N$, $n_2\in N$, and $(n_1,n_2)=1$, then also $n_1n_2\in N$.

(2) If $n_1\in\bar{N}$, $n_2\in\bar{N}$, and n_1, n_2 contain 2 to the same highest power, then also $[n_1,n_2]\in\bar{N}$.

PROOF.

(1) $n_1\in N$, i. e., $n_1|U_n$, implies $n_1|U_{[n_1,n_2]}$. Analogously $n_2|U_{[n_1,n_2]}$. Hence $[n_1,n_2]|U_{[n_1,n_2]}$, i. e., $[n_1,n_2]\in N$.

(2) $n_1\in\bar{N}$, i. e., $n_1|V_{n_1}$ coupled with the assumption that n_1,n_2 contain 2 to the same highest power, implies $n_1|V_{[n_1,n_2]}$. Analogously $n_2|V_{[n_1,n_2]}$. Hence $[n_1,n_2]|V_{[n_1,n_2]}$, i. e., $[n_1,n_2]\in\bar{N}$.

DEFINITION.

(1) $n\in\overset{*}{N}$ is said to be a <u>fundamental</u> number if it is neither a product mn of a number $n\in N$ and a number m composed of only prime factors of n, nor a product $m[n_1,n_2]$ of the least common multiple $[n_1,n_2]$ of two different numbers $n_1\in N$, $n_2\in N$ and a number m composed of only prime factors of n_1n_2.

THEOREM D.

(1) The least positive multiple $p'=qp$ of a prime p belonging to N is a fundamental number.

(2) The least positive multiple $p'=qp$ of a prime p belonging to $\bar{N}$ is a fundamental number.

PROOF.

(1) $p'=qp$ is not a product of a number $n\in N$ and a number m composed of only prime factors of n. Indeed, the assumtion $p'=qp=mn$ implies $p|n$. Since, by assumption, p' is the least positive multiple of p belonging to n, we have $p'\leqq n<mn$, i. e., $p'<mn$, contrary to the assumption $p'=mn$.

$p'=qp$ is not a product $m[n_1,n_2]$ of the least common multiple $[n_1,n_2]$ of two different numbers $n_1\in N$, $n_2\in N$ and a number m composed of only prime factors of n_1n_2. Indeed, the assumption $p'=qp=m[n_1,n_2]$ implies $p|n_1$ or $p|n_2$. By symmetry we may put $p|n_1$. Since, by assumption, p' is the least positive multiple of p belonging to N, we have $p'\leqq n_1<m[n_1,n_2]$, i. e., $p'<m[n_1,n_2]$, contrary to the assumption $p'=m[n_1,n_2]$.

(2) The proof is analogical to that of (1), noting that in the definition the postulates of (2) are analogous to those of (1) and even stronger than those of (1).

THEOREM E.

(1) If $n\in N$, $p\neq 2,5$ is the greatest prime factor of n, α is the exponent of the highest power of p dividing n, then also $n/p^{\alpha}\in N$.

(2) If n $\bar{N}$, $p\neq 3$ is the greatest prime factor of n, α is the exponent of the highest power of p dividing n, then also $n/p^{\alpha}\in\bar{N}$.

PROOF.

(1) According to theorem A(1) if a prime q divides n also $a(q)$ divides n. In particular this is true for any prime factor $q<p$ of n. For $q\neq 2,5$ the prime factors of $a(q)$ are smaller than q, hence they are smaller all the more so than p. For $q=2$, $a(q)=3<p$. Both for $q\neq 2$ and for $q=2$ the prime factors of $a(q)$ are different than p. Hence, if a prime q divides n/p^{α} also $a(q)$ divides n/p^{α}. Hence, by theorem A(1), $n|p^{\alpha}\in N$.

(2) According to theorem A(2) if a prime q divides n also $\bar{a}(q)$ divides n, and n is not divisible by a power of 2 higher than the powers of 2 dividing the ranks of apparition of the prime factors of n. In particular this is true for any prime factor $q \neq p$ of n. For $q \neq 2$ the prime factors of $\bar{a}(q)$ are smaller than q, hence they are all the more smaller than p. For q=2, $\bar{a}(q)=3 \neq p$. Both for $q \neq 2$ and for q=2 the prime factors of $\bar{a}(q)$ are different than p. Hence, if a prime q divides n/p^{α} also $\bar{a}(q)$ divides n/p^{α}. Moreover, n/p^{α} is not divisible by a power of 2 higher than the highest power of 2 dividing the ranks of apparition of the prime factors of n. Hence, by theorem A(2), $n/p^{\alpha} \in \bar{N}$.

THEOREM F.

(1) Every $n \in N$ is divisible either by 5, or by 12, or by 60.

(2) Every $n \in \bar{N}$ is divisible by 6, and is not divisible by 4.

PROOF.

(1) If $2|n$, then, according to theorem A(1), also $a(2)=3|n$, hence also $a(3)=4|n$, hence $3.4=12|n$ and the theorem is valid. We may therefore assume $p|n$, where p is a prime different than 2, 5. In this case the prime factors of a(p) are smaller than p. If p=3, then, according to theorem A(1), $a(3)=4|n$, hence $3.4=12|n$, and the theorem is valid. Suppose the theorem is valid for any number n having a prime factor smaller than p, then it is also valid for a number n divisible by p, since, by theorem A(1), $p|n$ implies $a(p)|n$.

(2) If $2|n$, then, according to theorem A(2), also $\bar{a}(2)=3|n$, hence also $\bar{a}(3)=2|n$, hence $3.2=6|n$. On the other hand $2|n$ implies $4 \nmid V_n$, hence $4 \nmid n$, and the theorem is valid. We may therefore assume $p|n$, where p is a prime different than 2. In this case the prime factors of $\bar{a}(p)$ are smaller than p. If p=3, then, according to theorem A(2), $\bar{a}(3)=2|n$, and it has been already proved that in case $2|n$ the theorem is valid. Suppose the theorem is valid for any number n having a prime factor smaller than p, then it is also valid for any number n divisible by p, since, by theorem A(2), $p|n$ implies $\bar{a}(p)|n$.

THEOREM G.

(1) For any prime p there exists at least one fundamental number $\overset{*}{p}$ divisible by p. The least fundamental number $\overset{*}{p}$ divisible by p is the least common multiple of p, a(p) and the least fundamental numbers belonging to prime factors of a(p).

(2) For any prime 2, $p^{(n)}$, where $p^{(1)}$ are all the primes 3, 107, 11128427, 1828620361, 6782976947987,... with a rank of apparition $\bar{a}(p^{(1)})=2.3^{\alpha}$ ($\alpha \geq 0$), and $p^{(n)}$ are all the primes with a rank of apparition $\bar{a}(p^{(n)})$ containing with every prime $p^{(n-1)}$ also a multiple $mp^{(n-1)} \in \bar{N}$ (e. g., $p^{(2)}=107$, since $\bar{a}(107)=6.3=18$ and $18 \in \bar{N}$), and only for such primes $p^{(n)}$, there exists at least one fundamental number $\overset{*}{p}^{(n)}$ divisible by $p^{(n)}$. The least fundamental number $\overset{*}{p}^{(n)}$ divisible by $p^{(n)}$ is the least common multiple of $p^{(n)}$, $\bar{a}(p^{(n)})$ and the least fundamental numbers belonging to the prime factors of $\bar{a}(p^{(n)})$.

PROOF.

(1) The theorem can be easily verified by computation for p=2, 5. We can therefore assume $p \neq 2$, 5. Then the prime factors of a(p) are smaller than p. Suppose the theorem is valid for all primes smaller than p. Then, the way we have constructed $\overset{*}{p}$ implies, by theorem A(1), that any number $n \in N$ divisible by p is also divisible by $\overset{*}{p}$. Hence $\overset{*}{p}$ is the least multiple of p belonging to N. This, combined with theorem D(1), implies that $\overset{*}{p}$ is a fundamental number, and it is the least fundamental number divisible by p.

(2) First we shall prove that for any prime p=2, $p^{(n)}$ there exists at least one fundamental number $\overset{*}{p}^{(n)}$ divisible by $p^{(n)}$, and it is the least common multiple of $p^{(n)}$, $\bar{a}(p^{(n)})$ and the least fundamental numbers belonging to the prime factors of $\bar{a}(p^{(n)})$. The theorem can be easily verified for 2, 3 by computation. Suppose the theorem is valid for all primes p smaller than $p^{(n)}$. Then, the way we have constructed $\overset{*}{p}^{(n)}$ implies, by theorem A(2), that $\overset{*}{p}^{(n)} \in \bar{N}$. On the other hand it is clear, by theorem A(2), that any n $\bar{N}$ divisible by $p^{(n)}$ is also divisible by $\overset{*}{p}^{(n)}$. Hence, $\overset{*}{p}^{(n)}$ is the least multiple of p belonging to $\bar{N}$. This, combined with theorem D(2), implies that $\overset{*}{p}^{(n)}$ is afundamental number, and it is the least fundamental number divisible by $\overset{*}{p}^{(n)}$.

Secondly, we shall prove that only for primes $p=2$, $p^{(n)}$ there exists at least one fundamental number $\overset{*}{p}{}^{(n)}$ divisible by $p^{(n)}$. It is easily seen, by theorem A(2), that for any prime pfor which a fundamental number $\overset{*}{p}$ exists, the last part of the theorem identifying $\overset{*}{p}$ with the least common multiple of p, $\bar{a}(p)$ and the least fundamental numbers belonging to the prime factors of $\bar{a}(p)$, is valid. From this one can verify by calculation that for $p=7$ (being the least prime grater than 3 that at all appears in V) there is no fundamental number, since otherwise it should be $84 \in \bar{N}$, which is false. Let p be the least prime not belonging to $p^{(n)}$, for which, nevertheless, a least fundamental number $\overset{*}{p}$ exists. The prime factors of $\bar{a}(p)$ are then smaller than p. Hence $\bar{a}(p)$ is not divisible by any prime q not belonging to $p=2$, $p^{(n)}$, since if α is the highest power of p dividing $\overset{*}{p}$, it will be, by theorem E(2), also $\overset{*}{p}/p^{\alpha} \in \bar{N}$. But $q \mid \overset{*}{p}/p^{\alpha}$. On the other hand it follows from theorem E(2) that for any prime $q \neq 2$ appearing as a factor in a number n $\bar{N}$, there exists a least fundamental number $\overset{*}{p}$, since, by theorem E(2), we can delete all the superfluous factors. It would therefore exist for q a fundamental number $\overset{*}{q}$, contrary to the assumption on p. $\bar{a}(p)$ is therefore composed of only prime factors belonging to the set $p^{(n)}$. On the other hand, according to the part of the theorem already proved, for any prime factor of $\bar{a}(p)$ there exists a multiple belonging to $\bar{N}$. Hence p is one of the primes $p^{(n)}$, contrary to the assumption. Consequently the assumption is false and the theorem has thus been proved.

Least fundamental numbers $\overset{*}{p}$ in U belonging to primes $p<100$ arranged according to the increasing magnitude of p

$\overset{*}{p}$	Factorization of $\overset{*}{p}$
12	3.2^2
5	5
168	$7.3.2^3$
660	$11.5.3.2^2$
2184	$13.7.3.2^3$
612	$17.3^2.2^2$
684	$19.3^2.2^2$
552	$23.3.2^3$
4872	$29.7.3.2^3$
1860	$31.5.3.2^2$
25308	$37.19.3^2.2^2$
4920	$41.5.3.2^3$
28380	$43.11.5.3.2^2$
2176	$47.3.2^4$
5724	$53.3^3.2^2$
337716	$59.29.7.3.2^3$
3660	$61.5.3.2^2$
41004	$67.17.3^2.2^2$
59640	$71.7.5.3.2^3$
1847484	$73.37.19.3^2.2^2$
1207752	$79.13.7.3.2^3$
13944	$83.7.3.2^3$
58740	$89.11.5.3.2^2$
114072	$97.7^2.3.2^3$

Least fundamental numbers $\overset{*}{p}$ in V

$\overset{*}{p}$	$\bar{a}(p)$	Factorization of $\overset{*}{p}$
6	2	3.2
1926	18	$107.3^2.2$
64300051206	54	$11128427.107.3^2.2$

Subscripts dividing their terms below 1000 in U

5, 12, 24, 25, 36, 48, 60, 72, 96, 108, 120, 125, 144, 168, 180, 192, 216, 240, 288, 300, 324, 336, 360, 384, 432, 480, 504, 540, 552, 576, 612, 625, 660, 672, 684, 720, 768, 840, 864, 960, 972.

Subscripts dividing their terms in V

6, 18, 54, 162, 486, 1458, 1926, 4374, 5778, 13122, 17334, 39366, 52002, 118098, 156006, 206082, 354294, 468081, 618246, 1062882, 1404054, 1854738, 3188646, 4212162, 5564214, 9565938, 12636486, 16692642, 22050774, 28697814, 37909458, 50077926, 66152322, 86093502, 113728374, 150233778, 198456966.

DIVISIBILITY OF TERMS BY THEIR SUBSCRIPTS IN A SEQUENCE OF SUMS OF EQUAL POWERS

It is well-known that in a wequnce of sums of equal powers

(1) $$V_n = x_1^n + \ldots + x_s^n \quad (s > 3;\quad n=1,2,3,\ldots)$$

satisfying the condition

(2) $$V_1 = x_1 + \ldots + x_s = 0$$

where $x_1,\ldots,x_s$ are the roots of a monic polynomial (i.e. a polynomial with highest coefficient 1) with integral coefficients, there is

(3) $$p \mid V_p$$

for any prime p.

the question whether this property is characteristic for primes has been raised by Perrin[1] in the particular case of the recurring sequence

(4) $$V_0 = 3,\ V_1 = 0,\ V_2 = 2,\ V_{n+3} = V_{n+1} + V_n$$

i.e., in the case s=3 and x_1, x_2, x_3 are the roots of the equation $x^2-x-1=0$, and it was also treated by Malo[2] and Escott[3], without being settled.

Here a negative answer to the question in its general form is given.

For this purpose we shall make use of Carmichael pseudoprimes.

A number P is said to be a Carmichael pseudoprime[4] if it is composite and satisfies

(5) $$a^{P-1} \quad 1 \pmod{P}$$

for any positive integer a prime to P. The least Carmichael pseudoprime is $561 = 3\cdot 11\cdot 17$.

Multiplying (5) sidewise by a, we get

(6) $$a^P \equiv a \pmod{P}$$

Since, as it is well-known, Carmichael pseudoprimes are odd[5], we have, by (6)

$$(-a)^P = -a^P \equiv -a \pmod{P}$$

i.e., the congruence (6) holds for qny integer coprime with P.

Let now $x_1,\ldots,x_s$ be integers satisfying condition (2). $x_1,\ldots,x_s$ are the roots of the monic polynomial $(x-x_1)\ldots(x-x_s)$ of degree s with integral coefficients, and it holds

$$x_1^P \equiv x_1 \pmod{P}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$x_s^P \equiv x_s \pmod{P}$$

for any Carmichael pseudoprime P coprime with $x_1,\ldots,x_s$. Hence by addition we get, by (2),

$$V_P = x_1^P + \ldots + x_s^P \equiv x_1 + \ldots + x_s = 0 \pmod{P}$$

for any Carmichael pseudoprime P coprime with $x_1,\ldots,x_s$.

As I have shown [6] there holds, at the condition (2), even

$$p \mid V_{p^i}$$

for any prime p and any positive integer i. On the other hand it is easily seen by induction that there holds

$$a^{P^i} \equiv a \pmod{P}$$

for any pseudoprime P coprime with a. Hence,by analogously to the above, we get

$$P \mid V_{P^i}$$

for any Carmichael pseudoprime P coprime with $x_1,\ldots,x_s$.

Remark 1. As relating Perrin's sequence (4) it is easily seen that $P|V_P$ for $P=561=3\cdot 11\cdot 17$. Indeed, the length of the period mod 3 in thes sequence is 13. However, $561 \equiv 2 \pmod{13}$ and $V_2=2\not\equiv 0 \pmod 3$ therefore $V_{561}\not\equiv 0 \pmod 3$ and a fortiori $V_{561}\not\equiv 0 \pmod{561}$. Similarly one can decide relating other Carmichael pseudoprimes (compare the table of the distribution of zeros modulo p

Remark 2. Instead of condition (2) one can put

$$V_1 = x_1+\dots+x_s \equiv 0 \pmod P$$

and the considerations remain true for those Carmichael pseudoprimes of which V_1 is a multiple.

Remark 3. Also odd composite numbers n satisfying $a^n \equiv a \pmod n$ for certain values of a that are coprime with n (but not for any value of a) may supply examples for wequences of sums of equal powers (V_n) for which $n|V_n$ for certain composite values of n. The sequence of sums of equal powers of order a+1 for which $x_1=a$, $x_2=\dots=x_{a+1}=-1$ is an example for it.

A MULTIPLICATORY FORMULA FOR THE GENERAL RECURRING SEQUENCE OF ORDER 2

We consider the general recurring sequence of order 2, defined by

(1) $$W_l = aW_{l-2}+bW_{l-1}, \qquad (l=0,\ \pm 1,\ \pm 2,\dots)$$

with arbitrary complex $a\neq 0$, b, W_o, W_1, and also the special case U_l with the same a, b and

(2) $$U_o=0,\quad U_1=1.$$

Our purpose is to establish the formula

(3) $W_{kl}=\sum_{i=o}^{k} u_{k,l,i}W_i$ where $u_{k,l,i}=\binom{k}{i}(aU_{l-1})^{k-i}U_l^{\,i}$ (k=0,1,2,...)

which, for a constant k>0, expresses the values of $W_l'=W_{kl}$ in terms of $W_o,\dots,W_k$.

Putting $W_k^{(m)}=W_{k+m}$, m=0, ±1, ±2,..., which sequence belongs to the same a, b, we have the equivalent formula

(3') $$W_{kl+m} = \sum_{i=o}^{k} u_{k,l,i}W_{i+m} \qquad (m=0,\ \pm 1\ \ \pm 2,\dots)$$

and, in particular,

(3") $$U_{kl} = \sum_{i=o}^{k} u_{k,l,i}U_i, \qquad U_{kl+1} = \sum_{i=o}^{k} u_{k,l,i}U_{i+1}.$$

The formulae (3") were given by H. Siebeck (Journal für Mathematik 33, 1346, 71-76), who, however, considered neither other sequences W_l than U_l and $U_l^{(1)}$, nor general a, b (which he supposes to be relatively prime integers) - his proof being founded on the theory of continued fractions.

We give two direct proofs. While the first proof is based on induction, the second one is heuristic and leads to a generalization of (3) for recurring sequences of any order, to be published separately.

As a common base for both proofs we need the formula

(4) $$W_{k+l} = aU_{l-1}W_k+U_lW_{k+1},$$

which is easily verified by induction with regard to l.

Since W_l can be obtained as a linear combination of any two non-proportional sequences with the same a, b, it is sufficient to prove (3) for any two such sequences, e. g. to prove (3").

<u>Proof of (3")</u>. (3") is evidently true for $k=0$. Let (3") be true for a certain $k\geq 0$; then we shall prove (3") for $k+1$. Indeed, noting that $U_o=\binom{k}{k+1}=\binom{k}{-1}=0$, we have, by (4) and (1),

$$
\begin{aligned}
U_{(k+1)l} &= U_{kl+l} \\
&= aU_{l-1}U_{kl}+U_lU_{kl+1} \\
&= aU_{l-1}\sum_{i=0}^{k}u_{k,l,i}U_i+U_l\sum_{i=0}^{k}u_{k,l,i}U_{i+1} \\
&= \sum_{i=0}^{k+1}\binom{k}{i}(aU_{l-1})^{k-i+1}U_l^{\,i}U_i+\sum_{i=0}^{k+1}\binom{k}{i-1}(aU_{l-1})^{k-i+1}U_l^{\,i}U_i \\
&= \sum_{i=0}^{k+1}u_{k+1,l,i}U_i;
\end{aligned}
$$

$$
\begin{aligned}
U_{(k+1)l+1} &= U_{kl+(l+1)} \\
&= aU_lU_{kl}+U_{l+1}U_{kl+1} \\
&= aU_l\sum_{i=0}^{k}u_{k,l,i}U_i+U_{l+1}\sum_{i=0}^{k}u_{k,l,i}U_{i+1} \\
&= U_l\sum_{i=1}^{k+1}u_{k,l,i-1}(U_{i+1}-bU_i)+(aU_{l-1}+bU_l)\sum_{i=0}^{k+1}u_{k,l,i}U_{i+1} \\
&= \sum_{i=1}^{k+1}\binom{k}{i-1}(aU_{l-1})^{k-i+1}U_l^{\,i}U_{i+1}-b\sum_{i=1}^{k+1}\binom{k}{i-1}(aU_{l-1})^{k-i+1}U_l^{\,i}U_i \\
&= \sum_{i=0}^{k+1}\binom{k}{i}(aU_{l-1})^{k-i+1}U_l^{\,i}U_{i+1}+b\sum_{i=0}^{k+1}\binom{k}{i}(aU_{l-1})^{k-i}U_l^{\,i+1}U_{i+1} \\
&= \sum_{i=0}^{k+1}u_{k+1,l,i}U_{i+1}.
\end{aligned}
$$

<u>Alternative proof of (3)</u>. We consider the two recurring sequences α^l and β^l belonging to the same a, b, whence α, β satisfy the equation

(5) $$x^2=a+bx.$$

By (4), with $k=0$, we have

(6) $$\alpha^l=aU_{l-1}+U_l\alpha.$$

Raising both sides of (6) to the power k we obtain (3) for $W_l=\alpha^l$. The same is true for β^l, which, if $4a+b^2\neq 0$,* that is $\alpha\neq\beta$, proves (3).

As an immediate consequence of the first formula of (3")

$$U_{kl}=U_l\sum_{i=1}^{k}\binom{k}{i}(aU_{l-1})^{k-i}U_l^{\,i-1}U_i$$

it follows that, <u>in case a, b are integers and $l>0$, U_{kl} is divisible by U_l.</u>

The last result, and the main formula (3), for $l\geq 0$, (obtained by the first proof), are seen to be true in an arbitrary number-ring or abstract ring containing 1, provided that $ab=ba$; if 1 is divisible by a, they hold also for $l<0$.

---

* If $4a+b^2=0$, that is $\alpha=\beta$, we can use the sequences α^l and $l\alpha^l$. Indeed, by (4), $k=0$, we have

(7) $$l\alpha^l=\alpha U_l,$$

whence by (6) and $k\binom{k-1}{i-1}=i\binom{k}{i}$,

$$
\begin{aligned}
kl\alpha^{kl} &= k\alpha U_l(aU_{l-1}+U_l\alpha)^{k-1} \\
&= k\sum_{i=1}^{k}\binom{k-1}{i-1}(aU_{l-1})^{k-i}U_l^{\,i}\alpha^i \\
&= \sum_{i=0}^{k}\binom{k}{i}(aU_{l-1})^{k-i}U_l^{\,i}i\alpha^i.
\end{aligned}
$$

We can also say that (3) considered as an algebraical identity for the variable a, with constant b, k, l, W_o, W_1 ($W_2,\dots,W_k$, U_{l-1}, U_l having been expressed by a, b, W_o, W_1), holds always, since it holds for $a\neq -b^2/4$.

A SLOWLY INCREASING SECOND ORDER RECURRING SEQUENCE

Marshall Hall (Slowly increasing arithmetic series, Journal of the London Mathematical Society 8 (1933), 162-166) has given a table of the first 100 terms of a slowly increasing recurring sequence of order 6, with complete factorization. Herewith a similar table of order 2 is given. There are underlined: primitive factors (that is factors which appear for the first time), and subscripts whose corresponding terms contain only primitive factors (namely prime subscripts $n \neq 2,3,5,13$, and the composite subscripts n=4,6,9,10,15,25,26,39,65,(169), which have no proper divisors other than 2,3,5,13). Compare Poulet, La chasse... 38-40.

1607 (ignore - signs)

U_n	n	Factorization $U_n = -(2U_{n-2}+U_{n-1})$	n	Factorization of U_n
0	0			
1	1			
-1	2			
-1	3			
3	4	3		
-1	5			
-5	6	5		
7	7	7		
3	8	3		
-17	9	17		
11	10	11		
23	11	23		
-45	12	3^2.5		
-1	13			
91	14	7.13		
-89	15	89		
-93	16	3.31		
271	17	271		
-85	18	5.17		
-457	19	457		
627	20	3.11.19		
287	21	7.41		
-1541	22	23.67		
967	23	967		
2115	24	3^2.5.47		
-4049	25	4049		
-181	26	181		
8279	27	17.487		
-7917	28	3.7.13.29		
-8641	29	8641		
24475	30	5^2.11.89		
-7193	31	7193		
-41757	32	3.31.449		
56143	33	23.2441		
27371	34	101.271		
-139657	35	7.71.281		
84915	36	3^3.5.17.37		
194399	37	73.2663		
-364229	38	457.797		
-24569	39	79.311		
753027	40	3.11.19.1201		
-703889	41	409.1721		
-802165	42	5.7.13.41.43		
2209943	43	257.8599		
-605613	44	3.23.67.131		
-3814273	45	17.89.2521		
5025499	46	967.5197		
2603047	47	2603047		
-12654045	48	3^2.5.31.47.193		
7447951	49	7^2.97.1567		
17860139	50	11.401.4049		
-32756041			51	271.120871
-2964237			52	3.53.103.181
68476319			53	68476319
-62547845			54	5.17.487.1511
-74404793			55	23.439.7369
199500483			56	3.7.13.29.113.223
-50690897			57	457.110921
-348310069			58	173.233.8641
449691863			59	5783.77761
246928275			60	3^2.5^2.11.19.59.89
-1146312001			61	1951.587551
652455451			62	61.1487.7193
1640168551			63	7.17.41.127.2647
-2945079453			64	3.31.449.70529
-335257649			65	335257649
6225416555			66	5.23.67.331.2441
-5554901257			67	5554901257
-6895931853			68	3.101.137.271.613
18005734367			69	967.18620201
-4213870661			70	7.11.13.71.211.281
-31797598073			71	31797598073
40225339395			72	3^3.5.17.37.47.10079
23369856751			73	23369856751
-103820535541			74	73.2663.534059
57080822039			75	89.151.1049.4049
150560249043			76	3.227.457.607.797
-264721893121			77	7.23.11087.148303
-36398604965			78	5.79.181.311.1637
565842391207			79	4423.127931809
-493045181277			80	3.11.19.31.1201.21121
-638639601137			81	17.487.7937.9719
1624729963691			82	409.1721.2308219
-347450761417			83	6473.53676929
-2902009165965			84	3^2.5.7.13.29.41.43.83.167
3596910688799			85	271.13272733169
2207107643131			86	257.8599.998717
-9400929020729			87	8641.1087944569
4986713734467			88	3.23.67.131.1231.6689
13815144306991			89	340337.40592543
-23788571775925			90	5^2.11.17.89.2521.22679
-3841716838057			91	7.712711.770041
51418860389907			92	3.367.967.5197.9293
-43735426713793			93	929.1303.5023.7193
-59102294066021			94	2603047.22705043
146573147493607			95	191.457.1679209361
-28368559361565			96	3^2.5.31.47.193.449.4993
-264777735625649			97	751943.352124743
321514854348779			98	7^2.13.97.491.1567.6763
208040616902519			99	17.23.2441.217973449
-851070325600077			100	3.11.19.199.401.4049.4201

THE SERIES OF INVERSES OF A SECOND ORDER RECURRING SEQUENCE

Let

(1) $U_0=0,\ U_1=1,\ U_n=PU_{n-1}+U_{n-2}$ (n=2,3,4,...; P an arbitrary, positive real number)

define a second order recurring sequence. Let

(2) $a = \frac{P-\sqrt{\Delta}}{2},\quad b = \frac{P+\sqrt{\Delta}}{2} = -\frac{1}{a} \qquad (\Delta = P^2+4)$

be the roots of the equation

(3) $x^2-Px-1=0$

It is easy to verify that

(4) $U_n = \frac{a^n-b^n}{a-b} = \frac{(-1)^n-a^{2n}}{a^n\sqrt{\Delta}}$

Indeed, (4) holds for n=0,1 and (4) satisfies the recurrence relation in (1); hence (4) also holds for n=2,3,4,...

Since a<b, one obtains $|\frac{a}{b}|<1$; hence the quotient

$$\frac{U_{n+1}}{U_n} = \frac{a^{n+1}-b^{n+1}}{a^n-b^n} = b\,\frac{1-(\frac{a}{b})^{n+1}}{1-(\frac{a}{b})^n}$$

approaches, for increasing n, the limit b, $|b|>1$. Therefore the series of the inverses

(5) $\frac{1}{U_1}+\frac{1}{U_2}+\ldots+\frac{1}{U_n}+\ldots$

converges, and one may inquire whether or not it is possible to express the value of this sum by known functions. This problem was settled by E. Landau[1] in case P=1, that is when $U_0=0$, $U_1=1$, $U_n=U_{n-1}+U_{n-2}$ (n=2,3,4,...) defines the Fibonacci sequence. Landau separates the series (5) into two sub-series

(6) $\frac{1}{U_2}+\frac{1}{U_4}+\ldots+\frac{1}{U_{2n}}+\ldots$

(7) $\frac{1}{U_1}+\frac{1}{U_3}+\ldots+\frac{1}{U_{2n+1}}+\ldots$

and expresses the sums of (6) and (7) respectively by Lambert series and theta series.

The purpose of the present note is to generalize Landau's results to arbitrary positive real values of P. While the proof for the second sum is designed on the same lines as that of Landau, a slightly simpler proof, not involving double series, is given for the first sum.

THEOREM 1. The series $\sum_1^\infty 1/U_{2n}$ converges, and

(8) $$\sum_1^\infty 1/U_{2n} = \sqrt{\Delta}\,(L(a^2)-L(a^4))$$

where

(9) $$L(x) = \sum_1^\infty \frac{x^n}{1-x^n}$$

is Lambert's series.

PROOF. Since Lambert's series $L(x)$ converges for $|x|<1$, and since

$$|a| = \left|\frac{P-\sqrt{P^2+4}}{2}\right| < \left|\frac{P-\sqrt{P^2+4P+4}}{2}\right| = \left|\frac{P-\sqrt{(P+2)^2}}{2}\right| = 1,$$

both Lambert's series in (8) converge absolutely. Therefore we have by (4)

$$\frac{1}{\sqrt{\Delta}}\sum_1^\infty \frac{1}{U_{2h}} = \sum_1^\infty \frac{a^{2h}}{1-a^{4h}} = \sum_1^\infty\left(\frac{a^{2h}}{1-a^{2h}} - \frac{a^{4h}}{1-a^{4h}}\right) = \sum_1^\infty \frac{a^{2h}}{1-a^{2h}} - \sum_1^\infty \frac{a^{4h}}{1-a^{4h}},$$

which proves (8).

THEOREM 2. The series $\sum_0^\infty 1/U_{2h+1}$ converges, and

(10) $$\sum_0^\infty \frac{1}{U_{2h+1}} = \frac{\sqrt{\Delta}}{2}\vartheta_2\left(o\,\middle|\,\frac{4}{\pi i}\log\frac{\sqrt{\Delta}-1}{2}\right)\vartheta_3\left(o\,\middle|\,\frac{4}{\pi i}\log\frac{\sqrt{\Delta}-1}{2}\right).$$

PROOF.

$$\frac{1}{\sqrt{\Delta}}\sum_0^\infty \frac{1}{U_{2h+1}} = \sum_0^\infty \frac{a^{2h+1}}{1+a^{4h+2}} = \sum_0^\infty a^{2h+1}(1-a^{4h+2}+a^{8h+4}-a^{12h+6}+\dots)$$

$$\begin{array}{l|l} = a-a^3+a^5-a^7+a^9-\dots & \\ \quad +a^3 \qquad\qquad -a^9+\dots & \\ \qquad +a^5 \qquad\qquad -\dots & = a+2a^5+a^9+\dots \\ \qquad\quad +a^7 \qquad\quad -\dots & \\ \qquad\qquad +a^9-\dots & \end{array}$$

Since the series is absolutely convergent, one can combine all the terms a^n, where n ranges over all the odd numbers, and one easily sees that a term a^n appears in all the horizontal lines corresponding to a divisor d of n, and with the sign plus or minus, according as n/d is ≡1 or ≡3 (mod 4). Since n/d ranges together with d over all the divisors of n, we have

$$\frac{1}{\sqrt{\Delta}}\sum_0^\infty 1/U_{2h+1} = \sum a^n D(n),$$

where n ranges over all the odd numbers, D(n) denoting the excess of the number of factors of n of the form 4k+1 over the number of those of the form 4k+3. Now, this excess equals double the number of decompositions of n into two squares, excepting the case where n is a square, where 1 must be subtracted. Thus we have

$$\frac{1}{\sqrt{\Delta}}\sum_0^\infty \frac{1}{U_{2h+1}} = 2(1+a^4+a^{16}+a^{36}+\dots)(a+a^9+a^{25}+\dots)-(a+a^9+a^{25}+\dots)$$

$$= (1+2a^4+2a^{16}+2a^{36}+\dots)(a+a^9+a^{25}+\dots).$$

Both brackets are theta series, and we have (10).

---

1) Sur la série des inverses des nombres de Fibonacci, Bulletin de la Société mathématique de France 27 (1899), 298-300.

THIRD ORDER RECURRING SEQUENCES

1. ALGEBRAICAL PROPERTIES. Let us consider the general recurring sequence (W_n) of order 3, defined by

(1) $W_n=aW_{n-3}+bW_{n-2}+cW_{n-1}$ $(n=0, \pm1, \pm2, \ldots)$

with arbitrary complex $a\neq0$, b, c, W_0, W_1, W_2, and also the special cases (U_n), (V_n), defined by:

(2) $U_0=U_1=0,\ U_2=1,\quad U_n=aU_{n-3}+bU_{n-2}+cU_{n-1},$

(3) $V_0=3,\ V_1=c,\ V_2=2b+c^2,\ V_n=aV_{n-3}+bV_{n-2}+cV_{n-1}.$

The following result is due to M. d'Ocagne[1]:

(4') $W_n=aW_{-1}U_n+(W_1-cW_0)U_{n+1}+W_0U_{n+2}.$

Denoting $W_m, W_{m+1}, W_{m+2}, \ldots$ by $W_0, W_1, W_2, \ldots$, that is, beginning the sequence with W_m instead of W_0, we can rewrite (4') as

(4) $W_{m+n}=aW_{m-1}U_n+(W_{m+1}-cW_m)U_{n+1}+W_mU_{n+2}.$

It is easy to prove (4) independently. Indeed, noting that $U_{-1}=1/a$, $U_3=c$, we easily verify that (4) holds for $n=-1, 0, 1$, whence (4) holds generally.

For $W=U$, $m=n$ we have by (2), (4):

(5) $U_{2n}=(2aU_{n-1}+bU_n)U_n+U^2_{n+1}.$

For $W=U$, $m=n+1$ (4) becomes:

(6) $U_{2n+1}=aU^2_n+(2U_{n+2}-cU_{n+1})U_{n+1}.$[2]

The following special cases are of importance:

1) $W_0=0$, 2) $W_1=cW_0$, 3) $b=0$, 4) $c=0$.

1) For $W_0=0$ (4') becomes:

(7) $W_n=aW_{-1}U_n+W_1U_{n+1}\leftrightarrow W_0=0.$ (The implication → follows for n=0)

2) For $W_1=cW_0$ (4') becomes:

(8) $W_n=aW_{-1}U_n+W_0U_{n+2}\leftrightarrow W_1=cW_0.$ (The implication → follows for n=1)

3) For $b=0$ (5) becomes:

(9) $U_{2n}=2aU_{n-1}U_n+U^2_{n+1}\leftrightarrow b=0.$ (The implication → follows for n=2)

If $b=0$ then $V_{-1}=-b/a=0$, $V_{-2}=(b^2-2ac)/a^2=-2c/a$. If $V_{-1}=0$ then $b=-aV_{-1}=0$. Thus, putting $W_n=V_{n-1}$ we have by (7):

(10) $V_{n-1}=-2cU_n+3U_{n+1}\leftrightarrow b=0$, vel $V_{-1}=0$.
(The implication → follows for n=0,3)

If $b=0$ (or, if $V_{-1}=0$), we have by (3): $V_2=2b+c^2=c^2=cV_1$. Putting $W_n=V_{n+1}$ we obtain by (8):

(11) $V_{n+1}=3aU_n+cU_{n+2}\leftrightarrow b=0$, vel $V_{-1}=0$.
(The implication → follows for n=1)

4) For $c=0$ (6) becomes:

(12) $U_{2n+1}=aU^2_n+2U_{n+1}U_{n+2}\leftrightarrow c=0.$ (The implication → follows for n=1)

If $c=0$ then, by (3), $V_1=c=0$, $V_2=2b$. Putting $W_n=V_{n+1}$ we obtain by (7):

(13) $V_{n+1}=3aU_n+2bU_{n+1}\leftrightarrow c=0.$ (The implication → follows for n=1)

If $c=0$ then, by (3), $V_1=c=0=cV_0$. Putting $W_n=V_n$ we obtain by (8):

(14) $V_n=-bU_n+3U_{n+2}\leftrightarrow c=0.$ (The implication → follows for n=1)

Let A, B, C be the roots of the equation

(15) $x^3-cx^2-a=0,$

and let us denote: $\alpha=-1/(A-C)(B-A)$, $\beta=-1/(B-A)(C-B)$, $\gamma=-1/(C-B)(A-C)$. Then:

(16) $$U_n=\begin{cases}\alpha A^n+\beta B^n+\gamma C^n & \leftrightarrow A\neq B\neq C\\ \dfrac{(A-C)nA^{n-1}-(A^n-C^n)}{(A-C)^2} & \leftrightarrow A=B\neq C\\ \frac{1}{2}n(n-1)A^{n-2} & \leftrightarrow A=B=C.\end{cases}$$

(17) $V_n=A^n+B^n+C^n.$

Indeed, since $V_{-1}=-b/a$, $V_0=3$, $V_1=c$ one easily sees that (17) is valid for $n=-1, 0, 1$. Multiplying the equation (15) by x^{n-3} we obtain $x^n=ax^{n-3}+bx^{n-2}+cx^{n-1}$ for any integer n, that is the sequences (A^n), (B^n), (C^n), and therefore also their sum $(A^n+B^n+C^n)$, satisfy the same recursion relation as (V_n), whence (17). Similarly (16) is shown, noting that (16) is valid for $n=0, 1, 2$.

Noting that $ABC=a$, and that for $A\neq B\neq C$ it is $\alpha+\beta+\gamma=U_0=0$, $\alpha A+\beta B+\gamma C=U_1=0$, we easily verify by (17), (16) that

(18) $U_{2n}-U_nV_n=a^nU_{-n},$

(19) $V_{2n}-V^2_n=-2a^nV_{-n},$

(20) $U_{2n+1}-U_{n+1}V_n=a^nU_{-(n-1)},$

(21) $V_{2n+1}-V_{n+1}V_n=a^nV_{-(n-1)}\leftrightarrow c=0.$

(The implication → follows for n=0)

In the case of multiple roots, that is when a is a function of b, c, we can say that the formulae (18) - (21), considered as polynomials in the variable a, with constant b, c (U_n, V_n having been expressed by a, b, c, U_o, U_1, U_2, V_o, V_1, V_2) hold always, since they hold for an infinitude of values of a.

Formula (18) is the counterpart of the formula $U_{2n}-U_nV_n=0$, which is of great importance for the arithmetic of Lucas' second order recurring sequences (U_n), (V_n). It shows that third order recurring sequences have another arithmetic than that of Lucas' sequences. It would be interesting to determine the value of $U_{2n}-U_nV_n$ for appropriate recurring sequences of higher order.

2. ARITHMETICAL PROPERTIES. In what follows we suppose that a, b, c, W_o, W_1, W_2 are integers. Then evidently all the W_n and $a_1^nW_{-n}$ with n 0 are integers. We shall say that a fraction P/Q, where P, Q are integers and $Q\neq 0$, is divisible by an integer p, if P is divisible by p.

It is easily seen that:

(22) If W_o, W_1, W_2 have no common prime divisor coprime with a, then no three consecutive W_n, W_{n+1}, W_{n+2} have a common prime divisor coprime with a.

By (7) we have:

(23) If $W_o=0$, then any common divisor of U_n, U_{n+1} also divides W_n. Any prime p coprime with $aW_{-1}W_1$ which appears in U_n, U_{n+1} to different (positive) highest powers, appears in W_n exactly to the lower of the said highest powers.

By (8) we have:

(24) If $W_1=cW_o$, then any common divisor of U_n, U_{n+1} also divides W_n. Any prime p coprime with $aW_{-1}W_o$ which appears in U_n, U_{n+1} to different (positive) highest powers, appears in W_n exactly to the lower of the said highest powers.

By (10) we have:

(25) If b=0, or if $V_{-1}=0$, then any common divisor of U_n, U_{n+1} also divides V_{n-1}. Any prime p coprime with 6c which appears in U_n, U_{n+1} to different (positive) highest powers, appears in V_{n-1} exactly to the lower of the said highest powers.

By (11) we have:

(26) If b=0, or if $V_{-1}=0$, then any common divisor of U_n, U_{n+2} also divides V_{n+1}. Any prime p coprime with 3ac which appears in U_n, U_{n+2} to different (positive) highest powers, appears in V_{n+1} to the lower of the said highest powers.

By (9), (18), (22) we have:

(27) If b=0, then any prime p coprime with 2a which appears in U_n, U_{n+1}^2 to different (positive) highest powers, appears in U_{2n} exactly to the lower of the said highest powers. If p appears in U_{n+1}^2 to the lower highest power, then p also appears in U_{-n} exactly to the lower highest power.

By (9), (22) we have:

(28) If b=0, then any common divisor of U_{n-1}, U_{n+1} also divides U_{2n}. Any prime p coprime with 2a which appears in U_{n-1}, U_{n+1} to different (positive) highest powers, appears in U_{2n} exactly to the lower of the said highest powers.

By (12), (2), (22) we have:

(29) If c=0, then any common divisor of U_n^2, U_{n+2} also divides U_{2n+1}. Any prime p coprime with 2a which appears in U_n^2, U_{n+1} to different (positive) highest powers, appears in U_{2n+1} exactly to the lower of the said highest powers.

By (13) we have:

(30) If c=0, any common divisor of U_n, U_{n+1} also divides V_{n+1}. Any prime p coprime with 6ab which appears in U_n, U_{n+1} to different (positive) highest powers, appears in V_{n+1} exactly to the lower of the said highest powers.

By (14) we have:

(31) If c=0, then any common divisor of U_n, U_{n+2} also divides V_n. Any prime p coprime with 3b which appears in U_n, U_{n+2} to different (positive) highest powers, appears in V_n exactly to the lower of the said highest powers.

By (12), (20), (22) we have:

(32) If c=0, then any prime p coprime with 2a which appears in U_n^2, U_{n+1} to different (positive) highest powers, appears in U_{2n+1} exactly to the lower of the said highest powers. If p appears in U_n^2 to the lower hihest power, then p also appears in $U_{-(n-1)}$ exactly to the lower highest power.

1) L. E. Dickson, History of the theory of numbers I, 409.

2) For W=U, a=b=c=1, the formulae (4), (5), (6) are due to M. Agronomof, Mathesis (4)4 (1914), 126.

Prime terms and prime-power subscripts are underlined

V_{-n}			$V_n=V_{n-2}+V_{n-3}$	n		$U_n=U_{n-2}+U_{n-3}$		U_{-n}
3	3	3	3	0		0		0
	-1		0	1		0		1
	1	2	2	2		1		-1
2	2	3	3	3		0		1
3	-3	2	2	4		1		0
2^2	4	5	5	5		1		-1
2	-2	5	5	6		1	2	2
	-1	7	7	7	2	2	2	-2
5	5	2.5	10	8	2	2		1
7	-7	$2^2.3$	12	9	3	3		1
2.3	6	17	17	10	2^2	4	3	-3
	-1	2.11	22	11	5	5	2^2	4
2.3	-6	29	29	12	7	7	3	-3
$2^2.3$	12	3.13	39	13	3^2	9		0
13	-13	3.17	51	14	$2^2.3$	12	2^2	4
7	7	$2^2.17$	68	15	2^4	16	7	-7
5	5	$2.3^2.5$	90	16	3.7	21	7	7
2.3^2	-18	7.17	119	17	$2^2.7$	28	3	-3
5^2	25	2.79	158	18	37	37	2^2	-4
$2^2.5$	-20	11.19	209	19	7^2	49	11	11
2	2	277	277	20	5.13	65	2.7	-14
23	23	367	367	21	2.43	86	2.5	10
43	-43	2.3^5	486	22	2.3.19	114		1
$3^2.5$	45	$2^2.7.23$	644	23	151	151	3.5	-15
2.11	-22	853	853	24	$2^3.5^2$	200	5^2	25
3.7	-21	2.5.113	1130	25	5.53	265	$2^3.3$	-24
2.3.11	66	3.499	1497	26	$3^3.13$	351	3^2	9
$2^3.11$	-88	3.661	1983	27	3.5.31	465	2^4	16
67	67	37.71	2627	28	$2^3.7.11$	616	$2^3.5$	-40
	-1	$2^3.3.5.29$	3480	29	$2^4.3.17$	816	7^2	49
87	-87	2.5.461	4610	30	23.47	1081	3.11	-33
2.7.11	154	31.197	6107	31	$2^3.179$	1432	7	-7
5.31	-155	2.5.809	8090	32	7.271	1897	$2^3.7$	56
$2^2.17$	68	7.1531	10717	33	7.359	2513	89	-89
2.43	86	14197	14197	34	3329	3329	2.41	82
241	-241	3.6269	18807	35	$2.3^2.5.7^2$	4410	2.13	-26
3.103	309	2.12457	24914	36	2.23.127	5842	$3^2.7$	-63
223	-223	$2^2.37.223$	33004	37	71.109	7739	5.29	145
2.3^2	-18	43721	43721	38	$2^2.11.233$	10252	$3^2.19$	-171
3.109	327	$2.3.7^2.197$	57918	39	$3^3.503$	13581	$2^2.3^3$	108
$2.5^2.11$	-550	$3^2.5^2.11.31$	76725	40	$3^2.1999$	17991	37	37
$2^2.7.19$	532	37.41.67	101639	41	23833	23833	$2^4.13$	-208
5.41	-205	3.37.1213	134643	42	$2^2.3^2.877$	31572	$2^2.79$	316
3.5.23	-345	$2^2.17.43.61$	178364	43	$2^5.1307$	41824	$3^2.31$	-279
877	877	2.31.37.103	236282	44	5.7.1583	55405	71	71
2.541	-1082	23.31.439	313007	45	$2^2.59.311$	73396	5.7^2	245
11.67	737	2.151.1373	414646	46	11.8839	97229	$2^2.131$	-524
$2^2.5.7$	140	13.29.31.47	549289	47	19.6779	128801	5.7.17	595
2.13.47	-1222	3.242551	727653	48	$3.5^4.7.13$	170625	$2.5^2.7$	-350
3.653	1959	5.7.27541	963935	49	2.5.7.3229	226030	2.3.29	-174
17.107	-1819	2.163.3917	1276942	50	2.149713	299426	769	769
3.199	597	$2^2.422897$	1691588	51	$5.7^2.1619$	396655	3.373	-1119
2.3.227	1362	3.746959	2240877	52	$2^4.3^2.41.89$	525456	$3^3.5.7$	945
3181	-3181	2.3.5.53.1867	2968530	53	3.37.6271	696081	$2^4.11$	-176
2.1889	3778	5.89.8837	3932465	54	59.15629	922111	23.41	-943
2.17.71	-2416	$3^2.7.43.641$	5209407	55	3.407179	1221537	$2^5.59$	1888
$3^2.5.17$	-765	5.1380199	6900995	56	$2^4.19.5323$	1618192	$2^4.3.43$	-2064
7.11.59	4543	$2^4.743.769$	9141872	57	$2^5.13.5153$	2143648	19.59	1121
6959	-6959	2.6055201	12110402	58	59.48131	2839729	13.59	767
2.19.163	6194	37.59.7349	16042867	59	$2^4.5.59.797$	3761840	19.149	-2831
13.127	-1651	2.10626137	21252274	60	7.19.89.421	4983377	$2^4.13.19$	3952
$2^2.1327$	-5308	$3^2.19.61.2699$	28153269	61	3.13.19.59.151	6601569	$5.7^2.13$	-3185
$2.3^4.71$	11502	13.2868857	37295141	62	13.107.6287	8745217	2.3.59	354
7.1879	-13153	11.4491413	49405543	63	2.19.304867	11584946	2.7.257	3598
3.5.523	7845	2.5.6544841	65448410	64	2.7.13.37.43.53	15346786	3.7.17.19	-6783

$\bar{V}_{-n}$			$\bar{V}_n=-\bar{V}_{n-2}+\bar{V}_{n-3}$	n		$\bar{U}_n=-\bar{U}_{n-2}+\bar{U}_{n-3}$		$\bar{U}_{-n}$
3	3	3	3	0		0		0
	1		0	1		0		1
	1	2	-2	2		1		1
2^2	4	3	3	3		0		1
5	5	2	2	4		-1	2	2
2.3	6	5	-5	5		1	3	3
2.5	10		1	6		1	2^2	4
3.5	15	7	7	7	2	-2	2.3	6
3.7	21	2.3	-6	8		0	3^2	9
31	31	2.3	-6	9	3	3	13	13
2.23	46	13	13	10	2	-2	19	19
67	67		0	11	3	-3	$2^2.7$	28
2.7^2	98	19	-19	12	5	5	41	41
$2^4.3^2$	144	13	13	13		1	$2^2.3.5$	60
211	211	19	19	14	2^3	-8	$2^3.11$	88
3.103	309	2^5	-32	15	2^2	4	3.43	129
3.151	453	2.3	-6	16	3^2	9	$3^3.7$	189
$2^3.83$	664	3.17	51	17	$2^2.3$	-12	277	277
7.139	973	2.13	-26	18	5	-5	2.7.29	406
2.23.31	1426	3.19	-57	19	3.7	21	5.7.17	595
2.5.11.19	2090	7.11	77	20	7	-7	$2^3.109$	872
3.1021	3063	31	31	21	2.13	-26	$2.3^2.71$	1278
67^2	4489	2.67	-134	22	$2^2.7$	28	1873	1873
$3^2.17.43$	6579	2.23	46	23	19	19	$3^2.5.61$	2745
2.3.1607	9642	3.5.11	165	24	2.3^3	-54	$3^2.149$	4023
13.1087	14131	$2^2.3^2.5$	-180	25	3^2	9	$2^3.11.67$	5896
2.5.19.109	20710	7.17	-119	26	73	73	8641	8641
$2^4.7.271$	30352	3.5.23	345	27	$3^2.7$	-63	$2^3.1583$	12664
44483	44483	61	-61	28	2^6	-64	$2^7.5.29$	18560
3.31.701	65193	$2^4.29$	-464	29	$2^3.17$	136	3.9067	27201
5.97.197	95545	2.7.29	406	30		1	5.7.17.67	39865
$2^2.3.7.1667$	140028	13.31	403	31	$2^3.5^2$	-200	$3.5^2.19.41$	58425
3.67.1021	205221	2.3.5.29	-870	32	$3^3.5$	135	$2.3^2.67.71$	85626
2.150383	300766	3	3	33	3.67	201	67.1873	125491
2.433.509	440794	19.67	1273	34	5.67	-335	$2^2.45979$	183916
5.47.2749	646015	$3^2.97$	-873	35	2.3.11	-66	2.7.13.1481	269542
11.17.61.83	946781	2.5.127	-1270	36	$2^3.67$	536	31.12743	395033
$3.5^2.7.881$	1387575	2.29.37	2146	37	269	-269	3.7.19.1451	578949
2.5.13.15643	2033590	397	397	38	2.7.43	-602	7.47.2579	848491
3.31.73.439	2980371	$2^3.7.61$	-3416	39	5.7.23	805	$2^2.3.173.599$	1243524
$2.3.11.17^2.229$	4367946	3.11.53	1749	40	$3^2.37$	333	$3^3.67499$	1822473
$2^9.12503$	6401536	3.31.41	3813	41	3.7.67	-1407	$2^2.667741$	2670964
23.41.9949	9381907	5.1033	-5165	42	$2^3.59$	472	$2^3.17.107.269$	3914488
13.1057681	13749853	$2^4.3.43$	-2064	43	$2^2.3.5.29$	1740	37.47.3299	5736961
67.167.1801	20151389	2.67^2	8978	44	1879	-1879	$5^2.336317$	8407925
$2^4.3.13.19.47.53$	29533296	7.443	-3101	45	$2^2.317$	-1268	$3^2.47.29131$	12322413
7.13.475639	43283149	2.5521	-11042	46	7.11.47	3619	2.47.192121	18059374
$2.3^2.191.18451$	63434538	47.257	12079	47	13.47	-611	$3^2.53.55487$	26467299
2.3.59.262621	92967834	3.2647	7941	48	$3^3.181$	-4887	$2^4.3^3.13.6907$	38789712
11.31.37.10719	136250983	$3^2.7.367$	-23121	49	$2.3^2.5.47$	4230	2.7.359.11311	56849086
7.28526503	199685521	2.2069	4138	50	$2^2.1069$	4276	5.3851.4327	83316385
5.29.2018299	292653355	23.31.167	31062	51	$3^2.1013$	-9117	257.557.853	122106097
2.214452169	428904338	27259	-27259	52	2.23	-46	11.16268653	178955183
3.1301.161053	628589859	$2^2.53.127$	-26924	53	59.227	13393	$2^4.3.13.420307$	262271568
2.460621607	921243214	58321	58321	54	47.193	-9071	5.7.41.277.967	384377665
$2^5.13.67.48441$	350147552	5.47	-335	55	89.151	-13439	$2^4.3.13.902777$	563332848
377179.107.157	1978737411	3.5.5683	-85245	56	$2^6.3^3.13$	22464	$2^6.3^2.7.13.19.829$	825604416
$5^4.1699.2731$	2899980625	$2^5$3.13.47	58656	57	$2^4.3.7.13$	4368	7.11.73.215261	1209982081
101269.311.503	4250128177	25.7.1213	84910	58	7.23.223	-35903	67.4271.6197	1773314929
$2^2 11^2$31.415147	6228865588	$3^2$59.271	-143901	59	$2^4$3.13.29	18096	5.11.47253079	2598919345
59.154726207	9128846213	2.13127	-26254	60	7.11.523	40271	2.11.367.471749	3808901426
$23^3$5172328759	13378974390	$11^2$31.61	228811	61	11.4909	-53999	3.5.43.1361.6359	5582216355
2.9803919989	19607839978	71.1657	-117647	62	$5^2.887$	-22175	$2^2.5^2.83.985679$	8181135700
31097.8731901	28736686191	5139.367	-255065	63	25.11.857	94270	2323.67.1296781	11990037126
3436133.53233	42115660581	23773113	346458	64	$2^4 3^2$13.17	-31824	$3^4.401.541001$	17572253481

TABLE OF THE DISTRIBUTION OF ZEROS MOD p IN THE BINARY LINEAR THIRD ORDER RECURRING SEQUENCES

$$(U_0,U_1,U_2) = (\bar U_0,\bar U_1,\bar U_2) = (0,0,1)$$
$$(V_0,V_1,V_2) = (\bar V_0,\bar V_1,-\bar V_2) = (3,0,2)$$
$$U_n = U_{n-2}+U_{n-3} \qquad V_n = V_{n-2}+V_{n-3}$$
$$\bar U_n = -\bar U_{n-2}+\bar U_{n-3} \qquad \bar V_n = -\bar V_{n-2}+\bar V_{n-3}$$

Notations: p - a prime.

P - length of the period mod p.

N - number of zeros in the period mod p.

The zeros were calculated by direct calculation of the periods.

For V, till p=23, the table was given by E. Malo, L'interméd. des Math. 7 (1900), 313. For $\bar U$, the period mod 5 was given by Marshall Hall, Duke Mathematical Journal 4 (1938), 695.

p	2		3	5		7		11		13		17		19		23		29		31	
P	7		13	24		48		120		183		288		180		22	506	871		993	
	V	U		V	U	V	U	V	U	V	U	V	U	V	U	V	U	V	U	V	U
N	3		4	5		6	12	11		9	21	17		6	18	1	22	30	31	36	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	3	3	5	3	7	3	11	3	13	3	10	3	19	3		3	12	3	31	3
	4	7	9	6	11	17	12	19	28	47	20	14	29	61	22		30	29	79	40	27
			13	8	20	23	16	40	38	62	26	15	82	79	47		36	47	85	44	89
				16	24	33	17	63	46	74	43	17	117	121	56		68	72	100	45	107
						39	19	74	67	108	57	43	141	139	60		83	102	108	47	171
							28	80	90	123	61	96	156		61		94	165	147	71	203
							32	89	101	135	62	131	165		63		142	222	166	133	255
							33	93	107	169	64	155	178		82		202	276	188	151	288
							35	94	116		81	170	186		107		219	296	214	215	318
							44	96	120		87	180	197		116		227	308	241	247	327
							48				109	192	218		120		239	332	294	299	331
											118	200	224		121		258	341	305	332	332
											122	211	241		123		328	345	356	362	334
											123	232	275		142		379	346	366	371	358
											125	238	284		167		383	348	400	375	420
											142	255	288		176		407	424	496	376	438
											148				180		431	430	527	378	502
											170						452	445	538	402	534
											179						493	453	555	464	586
											183						502	492	573	482	619
																	506	511	598	546	649
																		533	628	578	658
																		559	691	630	662
																		586	748	663	663
																		650	802	693	665
																		701	822	702	689
																		711	834	706	751
																		745	858	707	769
																		841	867	709	833
																			871	733	865
																				795	917
																				813	950
																				877	980
																				909	989
																				961	993

The subscripts of the zeros (a dash is drawn between 2 consecutive zeros)

p	2		3	5		7		11		13		17		19		23		29		31	
P	7		8	31		57		60		168		288		381		528		840		30	427
	$\bar V$	$\bar U$		$\bar V$	$\bar U$	$\bar V$	$\bar U$	$\bar V$	$\bar U$	$\bar V$	$\bar U$	$\bar V$	$\bar U$	$\bar V$	$\bar U$	$\bar V$	$\bar U$	$\bar V$	$\bar U$	$\bar V$	$\bar U$
N	3		3	6		9	12	5	6	18		17		27		23		29		3	12
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	3	3	5	3	7	3	11	3	10	3	11	3	11	3	11	3	11	3	11	3
	4	7	8	11	8	11	8	20	8	11	8	17	8	12	8	23	8	29	8	21	8
				24	12	20	19	24	35	13	21	26	29	14	23	27	39	30	43		92
				25	18	26	20	40	46	18	47	96	64	19	71	90	52	32	59		109
				27	31	30	22		60	31	56	110	68	34	90	91	58	37	159		135
						39	27			57	57	142	81	82	96	93	86	72	172		167
						45	38			66	59	154	105	101	117	98	146	88	184		196
						49	39			67	64	165	115	107	127	129	163	188	204		274
							41			69	77	184	121	128	128	142	181	201	251		313
							46			74	103	185	130	138	130	148	208	213	257		391
							57			87	112	187	200	139	135	176	222	233	268		427
										113	113	192	214	141	150	236	237	280	290		
										122	115	213	246	146	198	253	262	286	305		
										123	120	248	258	161	217	271	335	297	314		
										125	133	252	269	209	223	298	396	319	379		
										130	159	265	288	228	244	312	419	334	383		
										143	168			234	254	327	428	343	417		
														255	255	352	439	408	487		
														265	257	425	449	412	531		
														266	262	486	461	446	606		
														268	277	509	465	516	655		
														273	325	518	528	560	678		
														288	344			635	705		
														336	350			684	746		
														355	371			707	760		
														361	381			734	812		
																		775	822		
																		789	840		

The subscripts of the zeros(a dash is drawn between 2 consecutive zeros)

SOME SPECIAL PROPERTIES OF U, V, $\bar U$, $\bar V$. The lengths of the periods mod p of U, V ($\bar U$, $\bar V$) are equal, except for p=23 (p=31). $P|p^2+p+1$ (all the odd P in the table) or $P|p^2-1$, except for p=23 (p=31). These exceptional primes are the discriminants of scales of relation of the above recurring sequences. $p|V_{p^\alpha},\bar V_{p^\alpha},\bar V_{11p^\alpha}$ (α=0,1,...).

$p|U_n,U_{n+1}\to p|U_{kn},U_{kn+1},U_{kn+3},U_{kn-4},V_{kn+1}$. $p|V_n,V_{n+1}\to p|V_{n+3},V_{n-5},V_{n-13}$.
$p|\bar U_n,\bar U_{n+1}\to p|\bar U_{kn},\bar U_{kn+1},\bar U_{kn+3},\bar U_{kn-8},\bar V_{kn+1},\bar V_{kn+11}$. $p|\bar V_n,\bar V_{n+1}\to p|\bar V_{n+3},\bar V_{n+8}$.
$31|V_{10n+1}$. $67|V_{11(3n+1)},V_{11(3n+2)},V_{33n-3}$ (N. G. W. H. Beeger, Riveon Lemátematika 5 (1951-2), 12).

CONJECTURES. The number of zeros in the period mod p of V ($\bar V$) does not exeed the number of zeros in the period mod p of U ($\bar U$). If (as for p=31 (p=13,19)) the numbers of zeros in the periods mod p of U,V ($\bar U$,$\bar V$) are equal and if a|P, a denoting the least positive subscript for which $p|U_a,U_{a+1}$ ($p|\bar U_a,\bar U_{a+1}$), then the numbers of zeros till ka+1 inclusively are also equal for any ka<P. For which moduli p the periods of V ($\bar V$) contain a pair of consecutive zeros?

FACTORIZATION FORMULAE FOR FIBONACCI AND LUCAS NUMBERS DECREASED OR INCREASED BY A UNIT

The subject of the following lines is to give a complete set of formulae for the factorization of the numbers $U_n \mp 1$, $V_n \mp 1$, defined as follows:

$$(1)\ U_0=0,\ U_1=1,\ U_n=aU_{n-2}+bU_{n-1}$$

$$(2)\ V_0=2,\ V_1=1,\ V_n=aV_{n-2}+bV_{n-1}$$

where a=b=1 (i. e., for the Fibonacci and Lucas sequences).

The sequences (1), (2), as defined above, are special cases of the general Lucas[1] sequences, where $a \neq 0$, b are arbitrary constants.

Lucas[2] established for his sequences the following formulae:

$$(3)\ U_{k+m}+(-a)^m U_{k-m}= U_k V_m$$

$$(4)\ U_{k+m}-(-a)^m U_{k-m}= V_k U_m$$

$$(5)\ V_{k+m}+(-a)^m V_{k-m}= V_k V_m$$

$$(6)\ V_{k-m}-(-a)^m V_{k-m}=(4a+b^2) U_k U_m$$

$$(7)\ V_{2m}+(-a)^m=U_{3m}/U_m$$

$$(8)\ V_{2m}-(-a)^m=V_{3m}/V_m$$

The formulae (3)-(6) can be proved by double induction. First, they hold for m=1, which follows by induction on k, being valid for k=0, 1 and noting that $U_{-1}=1/a$, $U_2=b$, $V_{-1}=-b/a$, $V_2=2a+b^2$. Then the formulae are proved by induction on m, being valid for m=0, 1. The formulae (7), (8) are easily verified with the aid of the formulae:

$$(9)\ U_n=(\alpha^n-\beta^n)/(\alpha-\beta)=\alpha^{n-1}+\alpha^{n-2}\beta+\ldots+\beta^{n-1},\quad V_n=\alpha^n+\beta^n$$

where α, β are roots of the equation $x^2-bx-a=0$ (if $\alpha=\beta$, then $U_n=n\alpha^{n-1}$, $V_n=2\alpha^n$). The validity of the formulae (9) may be verified by noting that they are valid for n=0, 1 and that they satisfy the recursion relation common to (1), (2).

For a=b=1, i. e., for the Fibonacci and Lucas sequences, we have the following set of formulae, showing that the factorization of numbers of the Fibonacci and Lucas sequences decreased or increased by a unit entirely depends of the factorization of appropriate numbers of the Fibonacci and Lucas sequences:

(10) $U_{4n}-1=U_{2n+1}V_{2n-1}$	(10') $V_{4n}-1=V_{6n}/V_{2n}$
(11) $U_{4n+1}-1=U_{2n}V_{2n+1}$	(11') $V_{4n+1}-1=5U_{2n}U_{2n+1}$
(12) $U_{4n+2}-1=U_{2n}V_{2n+2}$	(12') $V_{4n+2}-1=U_{3(2n+1)}/U_{2n+1}$
(13) $U_{4n+3}-1=U_{2n+2}V_{2n+1}$	(13') $V_{4n+3}-1=V_{2n+2}V_{2n+1}$
(14) $U_{4n}+1=V_{2n+1}U_{2n-1}$	(14') $V_{4n}+1=U_{6n}/U_{2n}$
(15) $U_{4n+1}+1=V_{2n}U_{2n+1}$	(15') $V_{4n+1}+1=V_{2n}V_{2n+1}$
(16) $U_{4n+2}+1=V_{2n}U_{2n+2}$	(16') $V_{4n+2}+1=V_{3(2n+1)}/V_{2n+1}$
(17) $U_{4n+3}+1=V_{2n+2}U_{2n+1}$	(17') $V_{4n+3}+1=5U_{2n+2}U_{2n+1}$

The formulae in U with the subscripts 4n, 4n+1, 4n+2, 4n+3 follows from (3), (4) putting k=2n+1, m=2n-1; k=2n+1, m=2n; k=2n+2, m=2n; k=2n+2, m=2n+1 respectively. Similarly the formulae in V with odd subscripts follow from (5), (6), while the formulae in V with even subscripts follow directly from (7), (8) putting k=2n, m=2n+1.

The sequences $(\overset{\mp}{U}_n=U_n \mp 1)$, $(\overset{\mp}{V}_n=V_n \mp 1)$ are recurring sequences of order 3 with the scale -1 0 2 -1, i. e. they satisfy the homogeneous linear recursion relation: $X_n=-X_{n-3}+2X_{n-1}$. They also satisfy the nonhomogeneous linear recursion relation: $X_n=X_{n-1}+X_{n-2} \mp 1$.

1) E. Lucas, Théorie des fonctions numériques simplement périodiques, American Journal of Mathematics 1 (1878), 184-240, 289-321.

2) Ibidem, pages 199, 202, formulae (45), (52), (53).

(71) have

1610

1611

1612

$\bar{U}_n=U_n-1$	n	
-1	0	
0	1	
0	2	
1	3	
2	4	2
4	5	2^2
7	6	7
12	7	$2^2.3$
20	8	$2^2.5$
33	9	3.11
54	10	2.3^3
88	11	$2^3.11$
143	12	11.13
232	13	$2^3.29$
376	14	$2^3.47$
609	15	3.7.29
986	16	2.17.29
1596	17	$2^2.3.7.19$
2583	18	$3^2.7.41$
4180	19	$2^2.5.11.19$
6764	20	$2^2.19.89$
10945	21	5.11.199
17710	22	2.5.7.11.23
28656	23	$2^4.3^2.199$
46367	24	199.233
75024	25	$2^4.3^2.521$
121392	26	$2^4.3^3.281$
196417	27	13.29.521
317810	28	2.5.61.521
514228	29	$2^2.11.13.29.31$
832039	30	13.29.2207
1346268	31	$2^2.3.7.11.31.47$
2178308	32	$2^2.11.31.1597$
3524577	33	3.7.47.3571
5702886	34	$2.3^4.7.47.107$
9227464	35	$2^3.17.19.3571$
14930351	36	37.113.3571
24157816	37	$2^3.17.19.9349$
39088168	38	$2^3.7.17.19.2161$
63245985	39	3.5.11.41.9349
102334154	40	2.13.421.9349
165580140	41	$2^2.3.5.11.29.41.211$
267914295	42	$3^2.5.11.41.43.307$
433494436	43	$2^2.29.89.199.211$
701408732	44	$2^2.29.211.28657$
1134903169	45	89.139.199.461
1836311902	46	2.47.89.199.1103
2971215072	47	$2^5.3^2.7.23.139.461$
4807526975	48	$5^2.139.461.3001$
7778742048	49	$2^5.3^2.7.11.23.101.151$
12586269024	50	$2^5.3^3.7.23.90481$
20365011073	51	11.101.151.233.521
32951280098	52	2.11.17.53.101.109.151
53316291172	53	$2^2.19.233.521.5779$
86267571271	54	$7^2.233.521.14503$
139583862444	55	$2^2.3.13.19.29.281.5779$
225851433716	56	$2^2.19.5779.514229$
365435296161	57	3.13.29.59.281.19489
591286729878	58	$2.3^3.13.29.41.281.2521$
956722026040	59	$2^3.5.11.31.59.61.19489$
1548008755919	60	59.557.2417.19489
2504730781960	61	$2^3.5.11.31.61.3010349$

$\bar{V}_n=V_n-1$	n	
1	0	
0	1	
2	2	2
3	3	3
6	4	2.3
10	5	2.5
17	6	17
28	7	$2^2.7$
46	8	2.23
75	9	3.5^2
122	10	2.61
198	11	$2.3^2.11$
321	12	3.107
520	13	$2^3.5.13$
842	14	2.421
1363	15	29.47
2206	16	2.1103
3570	17	2.3.5.7.17
5777	18	53.109
9348	19	$2^2.3.19.41$
15126	20	2.3.2521
24475	21	$5^2.11.89$
39602	22	2.19801
64078	23	2.7.23.199
103681	24	103681
167760	25	$2^4.3^2.5.233$
271442	26	2.135721
439203	27	3.281.521
710646	28	2.3.83.1427
1149850	29	$2.5^2.13.29.61$
1860497	30	17.109441
3010348	31	$2^2.11.31.2207$
4870846	32	2.769.3167
7881195	33	3.5.7.47.1597
12752042	34	2.6376021
20633238	35	$2.3^3.107.3571$
33385281	36	3.11128427
54018520	37	$2^3.5.17.19.37.113$
87403802	38	2.797.54833
141422323	39	7.2161.9349
228826126	40	2.23.241.20641
370248450	41	$2.3.5^2.11.13.41.421$
599074577	42	17.35239681
969323028	43	$2^2.3.29.43.211.307$
1568397606	44	2.3.261399601
2537720635	45	5.89.199.28657
4106118242	46	2.137.829.18077
6643838878	47	2.47.139.461.1103
10749957121	48	10749957121
17393796000	49	$2^5.3^2.5^3.7.23.3001$
28143753122	50	2.61.230686501
45537549123	51	3.11.101.151.90481
73681302246	52	2.3.12280217041
119218851370	53	2.5.17.53.109.233.521
192900153617	54	2269.4373.19441
312119004988	55	$2^2.7^2.19.5779.14503$
505019158606	56	2.23.167.65740583
817138163595	57	3.5.13.29.281.514229
1322157322202	58	2.173.3821263937
2139295485798	59	$2.3^2.41.59.2521.19489$
3461452808001	60	107.10783342081
5600748293800	61	$2^3.5^2.11.31.61.557.2417$

(Prime-powers are underlined)

$\mathring{U}_n=U_n+1$	n	
0	-2	
2	-1	2
1	0	
2	1	2
2	2	2
3	3	3
4	4	2^2
6	5	2.3
9	6	3^2
14	7	2.7
22	8	2.11
35	9	5.7
56	10	$2^3.7$
90	11	$2.3^2.5$
145	12	5.29
234	13	$2.3^2.13$
378	14	$2.3^3.7$
611	15	13.47
988	16	$2^2.13.19$
1598	17	2.17.47
2585	18	5.11.47
4182	19	2.3.17.41
6766	20	2.17.199
10947	21	3.41.89
17712	22	$2^4.3^3.41$
28658	23	2.7.23.89
46369	24	89.521
75026	25	2.7.23.233
121394	26	2.7.13.23.29
196419	27	3.233.281
317812	28	$2^2.11.31.233$
514230	29	2.3.5.61.281
832041	30	$3^2.7.47.281$
1346270	31	2.5.61.2207
2178310	32	2.5.61.3571
3524579	33	1597.2207
5702888	34	$2^3.17.19.2207$
9227466	35	$2.3^3.107.1597$
14930353	36	1597.9349
24157818	37	$2.3^3.37.107.113$
39088170	38	$2.3^4.5.11.41.107$
63245987	39	7.37.113.2161
102334156	40	$2^2.29.37.113.211$
165580142	41	2.7.13.421.2161
267914297	42	7.89.199.2161
433494438	43	2.3.13.43.307.421
701408734	44	2.13.139.421.461
1134903171	45	3.43.307.28657
1836311904	46	$2^5.3^3.7.23.43.307$
2971215074	47	2.47.1103.28657
4807526977	48	11.101.151.28657
7778742050	49	$2.5^2.47.1103.3001$
12586269026	50	2.47.233.521.1103
20365011075	51	$3.5^2.3001.90481$
32951280100	52	$2^2.5^2.19.3001.5779$
53316291174	53	2.3.17.53.109.90481
86267571273	54	$3^2.13.29.281.90481$
139583862446	55	$2.7^2.17.53.109.14503$
225851433718	56	2.17.53.59.109.19489
365435296163	57	$7^2.14503.514229$
591286729880	58	$2^3.5.7^2.11.31.61.14503$
956722026042	59	$2.3^2.41.2521.514229$
1548008755921	60	514229.3010349
2504730781962	61	$2.3^2.41.557.2417.2521$

$\mathring{V}_n=V_n+1$	n	
0	-1	
3	0	3
2	1	2
4	2	2^2
5	3	5
8	4	2^3
12	5	$2^2.3$
19	6	19
30	7	2.3.5
48	8	$2^4.3$
77	9	7.11
124	10	$2^2.31$
200	11	$2^3.5^2$
323	12	17.19
522	13	$2.3^2.29$
844	14	$2^2.211$
1365	15	3.5.7.13
2208	16	$2^5.3.23$
3572	17	$2^2.19.47$
5779	18	5779
9350	19	$2.5^2.11.17$
15128	20	$2^3.31.61$
24477	21	3.41.199
39604	22	$2^2.9901$
64080	23	$2^4.3^2.5.89$
103683	24	3.17.19.107
167762	25	2.7.23.521
271444	26	$2^2.79.859$
439205	27	5.13.29.233
710648	28	$2^3.211.421$
1149852	29	$2^2.3.11.31.281$
1860499	30	19.181.541
3010350	31	$2.3.5^2.7.47.61$
4870848	32	$2^6.3.23.1103$
7881197	33	2207.3571
12752044	34	$2^2.919.3469$
20633240	35	$2^3.5.17.19.1597$
33385283	36	53.109.5779
54018522	37	$2.3^3.107.9349$
87403804	38	$2^2.229.95419$
141422325	39	$3.5^2.11.37.41.113$
228826128	40	$2^4.3.31.61.2521$
370248452	41	$2^2.7.29.211.2161$
599074579	42	19.1009.31249
969323030	43	2.5.13.89.199.421
1568397608	44	$2^3.9901.19801$
2537720637	45	3.43.139.307.461
4106118244	46	$2^2.691.1485571$
6643838880	47	$2^5.3^2.5.7.23.28657$
10749957123	48	3.17.19.107.103681
17393796002	49	2.11.47.101.151.1103
28143753124	50	$2^2.31.12301.18451$
45537549125	51	$5^3.233.521.3001$
73681302248	52	$2^3.79.859.135721$
119218851372	53	$2^2.3.19.5779.90481$
192900153619	54	3079.62650261
312119004990	55	2.3.5.13.17.29.53.109.281
505019158608	56	$2^4.3.83.211.421.1427$
817138163597	57	$7^2.59.14503.19489$
1322157322204	58	$2^2.349.947104099$
2139295485800	59	$2^3.5^2.11.31.61.514229$
3461452808003	60	17.19.181.541.109441
5600748293802	61	$2.3^2.41.2521.3010349$

(Prime-powers are underlined)

A SEQUENCE WITH SEPARATE RECURRENCES FOR ALTERNATE TERMS

THEOREM. A sequence W satisfying the pair of second order recurrences

(1) $$W_{2n+2}=aW_{2n+1}+bW_{2n}$$

(2) $$W_{2n+3}=cW_{2n+2}+dW_{2n+1}$$

also satisfies the single fourth order recurrence

(3) $$W_n=(b+ac+d)W_{n-2}-bdW_{n-4}.$$

In particular, for bd=1,

(4) $$W_n=(b+ac+\frac{1}{b})W_{n-2}-W_{n-4}$$

which is symmetrical with regard to W_n and W_{n-4}.

The formula (3) also shows that the sequence W may be separated into two second order recurring sequences, the one consisting of all the terms of W with even subscripts, the other of all the terms with odd subscripts. Each sequence satisfies the common recurrence

(5) $$X_n=(b+ac+d)X_{n-1}-bdX_{n-2}.$$

PROOF. By (1), (2) we have

(1) $W_{2n+2}=aW_{2n+1}+bW_{2n}$ (2) $W_{2n+3}=cW_{2n+2}+dW_{2n+1}$

(2') $aW_{2n+1}=acW_{2n}+adW_{2n-1}$ (1') $cW_{2n+2}=acW_{2n+1}+bcW_{2n}$

(1") $adW_{2n-1}=dW_{2n}-bdW_{2n-2}$ (2") $bcW_{2n}=bW_{2n+1}-bdW_{2n-1}$

By the addition of these equations we have

(3') $W_{2n+2}=(b+ac+d)W_{2n}-bdW_{2n-2}$ (3") $W_{2n+3}=(b+ac+d)W_{2n+1}-bdW_{2n-1}$

(3') and (3") can be summarized together in the single formula (3).

REMARK. In case $\frac{a}{c}=b=d=1$, $\frac{c}{a}=k-2$, (1), (2) supply the recurrence formulae of the sequences (u), (v) investigated by Zevulun Tuchman and Shraga Kalai, Application of recurring sequences for solving Diophantine equations, Riveon Lematematika 5 (1951-2), 23-31. According to the statement above these sequences also fulfill the recurrences (4), (5) in the form ($\overline{4}$) $W_n=kW_{n-2}-W_{n-4}$, ($\overline{5}$) $X_n=kX_{n-1}-X_{n-2}$. By starting from these formulae the authors' work could have been greatly simplified.

In case a=1, b=d=-1, c=z+2, (1), (2) supply recurrences of the sequence (P) investigated by Eri Jabotinsky, The minimal Tarry-Escott problem, Riveon Lematematika 4 (1950), 54 ff.

INVARIANCE OF THE DETERMINANT OF RECURRING SEQUENCES WITH COMMON SCALE

THEOREM 1. For any s recurring sequences of order s

$$W^{(1)}, \dots, W^{(s)}$$

with a common recursion formula

(1) $$W_{n+s}=a_oW_n+a_1W_{n+1}+\dots+a_{s-1}W_{n+s-1} \qquad (a_o\neq 0)$$

the expression

$$\frac{(-1)^{n(s-1)}}{a_o^{n-1}}\begin{vmatrix} W^{(1)}_{n+1} & W^{(1)}_{n+2} & \dots & W^{(1)}_{n+s} \\ W^{(2)}_{n+1} & W^{(2)}_{n+2} & \dots & W^{(2)}_{n+s} \\ \dots & \dots & \dots & \dots \\ W^{(s)}_{n+1} & W^{(s)}_{n+2} & \dots & W^{(s)}_{n+s} \end{vmatrix}=(-1)^{n(s-1)}a_o^{1-n}D$$

is invariant with respect to n.

PROOF. Writing, for brevity, the general row instead of the whole determinant, we obtain

$$(-1)^{n(s-1)}a_o^{1-n}D=(-1)^{n(s-1)}a_o^{-n}|a_oW_{n+1}\ W_{n+2}\ \dots\ W_{n+s}|$$
$$=(-1)^{n(s-1)}a_o^{-n}|a_oW_{n+1}+a_1W_{n+2}+\dots+a_{s-1}W_{n+s}\ W_{n+2}\ \dots\ W_{n+s}|$$
$$=(-1)^{n(s-1)}a_o^{-n}|W_{n+s+1}\ W_{n+2}\ \dots\ W_{n+s}|$$
$$=(-1)^{(n+1)(s-1)}a_o^{-n}|W_{n+2}\ W_{n+3}\ \dots\ W_{n+s+1}|.$$

That is, $(-1)^{n(s-1)}a_o^{1-n}D$ remains unaltered in value for n, n+1, hence it is independent of n.

In particular, we have

THEOREM 2. For any recurring sequence (W_n) of order s with the recursion formula (1), the expression

$$\frac{(-1)^{n(s-1)}}{a_o^{n-1}}\begin{vmatrix} W_n & W_{n+1} & \dots & W_{n+s-1} \\ W_{n+1} & W_{n+2} & \dots & W_{n+s} \\ \dots & \dots & \dots & \dots \\ W_{n+s-1} & W_{n+s} & \dots & W_{n+2s-2} \end{vmatrix}$$

is invariant with respect to n.

HOMOGENEOUS AND NONHOMOGENEOUS RECURSION FORMULAE

THEOREM 1. A sequence W satisfying a homogeneous linear recursion formula

(1) $$a_0W_{n-s}+a_1W_{n-s+1}+\dots+a_{s-1}W_{n-1}+a_sW_n=0, \quad a_0a_s\neq 0$$

of order s but of no lower order, also satisfies a nonhomogeneous linear recursion formula

(2) $$b_0W_{n-s}+b_1W_{n-s+1}+\dots+b_{s-1}W_{n-1}+c=0, \quad b_0b_{s-1}\neq 0$$

of order s-1 with $c\neq 0$ if and only if $a_0+\dots+a_s=0$.

If

$$D=\begin{vmatrix} W_n & W_{n+1} & \dots & W_{n+s-1} \\ W_{n+1} & W_{n+2} & \dots & W_{n+s} \\ \dots & \dots & \dots & \dots \\ W_{n+s-2} & W_{n+s-1} & \dots & W_{n+2s-3} \\ W_{n+s-1} & W_{n+s} & \dots & W_{n+2s-2} \end{vmatrix}$$

and if D_i is the determinant of order s obtained from D by replacing each of the elements in its i-th column by 1, then

$$b_0=D_1, \dots, b_{s-1}=D_s, \quad c=-D.$$

PROOF. In order that the system of s+1 simultaneous homogeneous equations

$$b_0W_n+b_1W_{n+1}+\dots+b_{s-1}W_{n+s-1}+c=0$$
$$b_0W_{n+1}+b_1W_{n+2}+\dots+b_{s-1}W_{n+s}+c=0$$
$$\dots\dots\dots$$
$$b_0W_{n+s-1}+b_1W_{n+s}+\dots+b_{s-1}W_{n+2s-2}+c=0$$
$$b_0W_{n+s}+b_1W_{n+s+1}+\dots+b_{s-1}W_{n+2s-1}+c=0$$

in s+1 unknowns $b_0, \dots, b_{s-1}$, c with $c\neq 0$, be consistent, it is necessary and sufficient that the determinant of the coefficients vanish, that is

$$0=\begin{vmatrix} W_n & W_{n+1} & \dots & W_{n+s-1} & 1 \\ W_{n+1} & W_{n+2} & \dots & W_{n+s} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ W_{n+s-1} & W_{n+s} & \dots & W_{n+2s-2} & 1 \\ W_{n+s} & W_{n+s+1} & \dots & W_{n+2s-1} & 1 \end{vmatrix} = \begin{vmatrix} W_n & W_{n+1} & \dots & W_{n+s-1} & 1 \\ W_{n+1} & W_{n+2} & \dots & W_{n+s} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ W_{n+s-1} & W_{n+s} & \dots & W_{n+2s-2} & 1 \\ 0 & 0 & \dots & 0 & a_0+\dots+a_s \end{vmatrix} =$$

$=D(a_0+\dots+a_s)$.

The second determinant is obtained from the first by (1) multiplying its rows, from the last to the first, by $a_0, \dots, a_s$ respectively, and adding to the last. Since W has, by hypothesis, no homogeneous linear recursion formula of lower order, it follows, by a result of Perrin (Comptes Rendus Paris 119, 1894, 900-3, according to Dickson, History of the theory of numbers I, 410) that $D\neq 0$. Hence $a_0+\dots+a_s=0$, which proves the theorem.

THEOREM 2. A sequence W satisfying a linear recursion formula (2) of order s-1, also satisfies an homogeneous recursion formula (1) of order s, where $a_i=b_i-b_{i-1}$, $b_{-1}=b_s=0$.

PROOF.

$$a_0W_{n-s}+a_1W_{n-s+1}+\dots+a_{s-1}W_{n-1}+a_sW_n=$$
$$(b_0-b_{-1})W_{n-s}+(b_1-b_0)W_{n-s+1}+\dots+(b_{s-1}-b_{s-2})W_{n-1}+(b_s-b_{s-1})W_n=$$
$$(b_0W_{n-s}+b_1W_{n-s+1}+\dots+b_{s-1}W_{n-1}+c)-(b_0W_{n-s+1}+\dots+b_{s-2}W_{n-1}+b_{s-1}W_n+c)=$$
$$0-0=0;\ a_0=b_0\neq 0;\ a_s=-b_{s-1}\neq 0.$$

THEOREM 3. A sequence satisfying an homogeneous linear recursion formula of order s, also satisfies a homogeneous linear recursion formula of any order higher than s.

PROOF. Theorem 2 for c=0.

THEOREM 4. A sequence W satisfying a homogeneous linear recursion formula (1) of order s and also the following linear recursion formula

(3) $$c_0W_{n-s}+c_1W_{n-s+1}+\dots+c_{s-1}W_{n-1}+a_sW_n-c=0, \quad c_0a_s\neq 0$$

of order s, also satisfies a linear recursion formula (2) of order s-1 if $b_0b_{s-1}\neq 0$, and of order less than s-1 if $b_0b_{s-1}=0$, where $b_i=a_i-c_i$.

PROOF.

$$b_0W_{n-s}+\dots+b_{s-1}W_{n-1}+c=$$
$$(a_0-c_0)W_{n-s}+\dots+(a_{s-1}-c_{s-1})W_{n-1}+(c_0W_{n-s}+\dots+c_{s-1}W_{n-1}+a_sW_n)=$$
$$a_0W_{n-s}+\dots+a_sW_n=0.$$

THEOREM 5. A sequence W satisfying a homogeneous linear recursion formula (1) of order s but of no lower order, with $a_0+\dots+a_s\neq 0$, does not satisfy any nonhomogeneous linear recursion formula of any order.

PROOF. By Theorem 1, W does not satisfy any nonhomogeneous recursion formulae of order s-1, and by theorem 2 none of any order less than s-1. Suppose W satisfies a nonhomogeneous recursion formula of order $t \geq s$ and none of order less than t. By theorem 3, W also satisfies a homogeneous linear recursion formula of order t. Therefore, by theorem 4, W satisfies a nonhomogeneous linear recursion formula of order t-1, contrary to hypothesis.

EXAMPLES. Fibonacci's sequence U=0,1,1,2,..., being a recurring sequence of order 2, with the homogeneous linear recursion formula $U_{n-2}+U_{n-1}-U_n=0$ of the non-vanishing scale 1 1 -1, does not satisfy any nonhomogeneous linear recursion formula of any order.

The sequence of natural numbers, satisfying the nonhomogeneous linear recursion formula of order 1: $(n-2)-(n-1)+1=0$, also satisfies the homogeneous linear recursion formula of order 2: $-(n-2)+2(n-1)-n=0$, with the vanishing scale $-1+2-1=0$.

INDEPENDENCE OF THE LENGTH OF A GENERAL PERIOD MODULO m IN A RECURRING SEQUENCE OF THE INITIAL TERMS

Let (W_n) be a recurring sequence of order s defined by

$$W_{n+s} = a_o W_n + a_1 W_{n+1} + \dots + a_{s-1} W_{n+s-1}, \quad a_o \neq 0,$$

where $W_o,\dots,W_{s-1}$ and $a_o,\dots,a_{s-1}$ are integers, and $(W_o,\dots,W_{s-1})=1$. The terms $W_o,\dots,W_{s-1}$ are called i n i t i a l t e r m s , and the coefficients $a_o,\dots,a_{s-1}$ are called s c a l e of the sequence (W_n).

It is well-known that any sequence (W_n) is periodic in respect to any modulus m. Here we show that the length of the period is, in general, independent of the initial terms. In fact we prove the following theorem.

THEOREM. The lengths of the periods modulo m in any two recurring sequences (W_n) and $(\overline{W}_n)$ with a common scale of order s are equal for any m coprime with D and $\overline{D}$, where

$$D=\begin{vmatrix} W_o & W_1 & \dots & W_{s-1} \\ W_1 & W_2 & \dots & W_s \\ \dots & \dots & \dots & \dots \\ W_{s-1} & W_s & \dots & W_{2s-2} \end{vmatrix}, \quad \overline{D}=\begin{vmatrix} \overline{W}_o & \overline{W}_1 & \dots & \overline{W}_{s-1} \\ \overline{W}_1 & \overline{W}_2 & \dots & \overline{W}_s \\ \dots & \dots & \dots & \dots \\ \overline{W}_{s-1} & \overline{W}_s & \dots & \overline{W}_{2s-2} \end{vmatrix}.$$

PROOF. The theorem is an immediate consequence of the following two lemmas.

LEMMA 1. If (W_n) and $(\overline{W}_n)$ are any two sequences of integers such that the general term in $(\overline{W}_n)$ multiplied by an integer D is expressible as a linear combination f with integral coefficients of terms of (W_n), then any period P modulo m in (W_n) coprime with D is also a period modulo m in $(\overline{W}_n)$.

PROOF. Let P be a period modulo m in (W_n). Then

$$D\overline{W}_{n+P}=f_{n+P}\equiv f_n=D\overline{W}_n \pmod m$$

Hence, $(D,m)=1$ implies $\overline{W}_{n+P}\equiv\overline{W}_n \pmod m$.

LEMMA 2. If (W_n) and $(\overline{W}_n)$ are two recurring sequences with a common scale of order s, and if

$$D=\begin{vmatrix} W_o & W_1 & \dots & W_{s-1} \\ W_1 & W_2 & \dots & W_s \\ \dots & \dots & \dots & \dots \\ W_{s-1} & W_s & \dots & W_{2s-2} \end{vmatrix} \neq 0$$

then for any n

$$D\overline{W}_n=D_1W_n+D_2W_{n+1}+\ldots+D_sW_{n+s-1}$$

where D_i is the determinant of order s arising from D when one replaces its i-th column by

$$\overline{W}_o,\ \overline{W}_1,\ \ldots,\ \overline{W}_{s-1}.$$

PROOF. The lemma is a special case of lemma 2 in the paper "Representation of terms of recurring sequences by sums of powers", below, page 74.

REMARK. If $S_n=x_1^n+\ldots+x_s^n$, where $x_1,\ldots,x_s$ are the s roots of the equation $x^s-a_{s-1}x^{s-1}-\ldots-a_o=0$ then (S_n) is a recurring sequence of order s, the initial terms of which can be calculated successively by the formula $S_m+a_1S_{m-1}+a_2S_{m-2}+\ldots+a_{m-1}S_1+ma_m=0$, and we have $S_{kp}\equiv S_k$ (mod p) for any prime p and any integer k. Thus, in order to calculate the period mod p in (S_n) it is, by the last property, convenient to write the period in columns of length p. Then the residues written in the last row equal, in order, to the residues of the first column on the left. This property enables us to check after every p residues whether there is no error in calculation.

Examples. For $S_1=0$, $S_2=2$, $S_3=3$, $S_n=S_{n-2}+S_{n-3}$ the residues are

mod 5

$P=24=5^2-1$

0	0	2	0	2
2	2	4	4	1
3	0	4	3	4
2	2	1	4	3
0	2	3	2	

mod 13

$P=183=13^2+13+1$

0	12	7	12	6	6	5	12	11	4	10	2	1	12	3
2	3	1	5	4	6	9	10	2	6	12	3	5	7	
3	12	9	2	9	11	10	6	8	3	1	11	4	12	
2	2	8	4	10	12	1	9	0	10	9	10	6	6	
5	2	10	7	12	4	6	3	10	9	0	6	9	6	
5	1	4	6	5	10	11	2	3	0	10	8	10	5	
7	4	5	11	9	3	7	12	10	6	9	3	2	12	
10	3	1	0	4	1	4	5	5	9	10	1	6	11	
12	5	9	4	1	0	5	1	5	6	6	11	12	4	
4	7	6	11	0	4	11	4	2	2	6	4	8	10	
9	3	10	4	5	1	9	6	10	2	3	12	5	2	
3	12	2	2	1	4	3	5	7	8	12	2	7	1	
0	2	3	2	5	5	7	10	12	4	9	3	0	12	

mod 19

$P=130=(19^2-1)/2$

0	11	6	11	17	17	9	15	7	5
2	6	3	8	9	11	17	1	9	18
3	11	3	14	0	3	14	3	17	17
2	17	9	0	7	9	7	16	16	4
5	17	11	3	9	14	12	4	7	16
5	9	17	14	7	12	2	0	14	2
7	15	1	3	16	4	0	1	4	1
10	7	9	17	16	7	14	4	2	18
12	5	18	17	4	16	2	1	18	3
17	3	10	1	13	11	14	5	6	
3	12	3	15	1	4	16	5	1	
10	8	9	13	17	8	16	6	5	
1	15	13	16	14	15	11	10	7	
13	1	17	14	18	12	13	11	6	
11	4	8	15	12	4	8	16	12	
14	16	16	11	13	3	5	2	13	
5	5	6	10	11	16	2	3	13	
6	1	5	7	6	12	13	18	6	
0	2	3	2	5	5	7	10	12	

REPRESENTATION OF TERMS OF RECURRING SEQUENCES BY SUMS OF POWERS

Introduction. A sequence $W=(W_n)=W_o, W_1, \ldots, W_n, \ldots$ is said to be a recurring sequence of order s provided s+1 constants

(1) $a_o, a_1, \ldots, a_s \quad (a_o a_s \neq 0)$

(called scale of W) exist, such that the relation

(2) $a_o W_n + a_1 W_{n+1} + \ldots + a_s W_{n+s} = 0$

is satisfied for every n.

In particular, the s sequences

(3) $1, x_i, x_i^2, \ldots, x_i^n, \ldots \quad (i=1,\ldots,s)$

$x_1, \ldots, x_n$ being roots of the equation

(4) $f(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_s x^s = 0$

are recurring sequences of order s. Indeed, by (4) we have

$$f(x)x^n = a_o x^n + a_1 x^{n+1} + a_2 x^{n+2} + \ldots + a_s x^{n+s} = 0$$

which coincides with (2) for $W_j \equiv x^j$.

The object of the following note is to show how the terms of a general recurring sequence W of order s with the scale (1) can be expressed as a linear form in the corresponding terms of (3), or, which is the same, how W_n can be represented as a sum of equal powers of the roots of (4).

Lemma 1. If $W^{(o)}, \ldots, W^{(r)}$ are r+1 recurring sequences of order s ($r \leq s$) with a common scale (1), and if the linear relation

(5) $b_o W_n^{(o)} + \ldots + b_r W_n^{(r)} = 0$

holds for s consecutive values of n ($b_o, \ldots, b_r$ being constants), then it holds for every value of n.

Proof. Suppose (5) holds for n=m,m+1,...,m+s-1 and consider the following s+1 expressions:

$$\begin{array}{l} b_o a_o W_m^{(o)} + \ldots + b_r a_o W_m^{(r)} \\ b_o a_1 W_{m+1}^{(o)} + \ldots + b_r a_1 W_{m+1}^{(r)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_o a_{s-1} W_{m+s-1} + \ldots + b_r a_{s-1} W_{m+s-1}^{(r)} \\ b_o a_s W_{m+s}^{(o)} + \ldots + b_r a_s W_{m+s}^{(r)} \end{array}$$

The first s expressions vanish by the hypothesis of the lemma. The sums of the corresponding terms in each expression vanish by the recurrence relation. Therefore, also, the last expression vanishes, even after division by $a_s \neq 0$. Thus (5) also holds for n=m+s. Similarly one shows that (5) holds for n=m-1. Whence by induction (5) holds generally.

Lemma 2. If $W, W^{(1)}, \ldots, W^{(s)}$ are s+1 recurring sequences of order s with a common scale (1), and if

$$D = \begin{vmatrix} W_o^{(1)} & W_o^{(2)} & \ldots & W_o^{(s)} \\ W_1^{(1)} & W_1^{(2)} & \ldots & W_1^{(s)} \\ \ldots & \ldots & \ldots & \ldots \\ W_{s-1}^{(1)} & W_{s-1}^{(2)} & \ldots & W_{s-1}^{(s)} \end{vmatrix} \neq 0$$

then, for every n, the following linear relation holds

$$DW_n = D_1 W_n^{(1)} + D_2 W_n^{(2)} + \ldots + D_s W_n^{(s)}$$

where D_i is the determinant of order s arising from D on replacing its i-th column by the following terms

$$W_o, W_1, \ldots, W_{s-1}.$$

Proof. Provided $D \neq 0$ a solution in $x_1, \ldots, x_s$ of the said kind for the following set of s equations exists

$$\begin{array}{l} W_o = W_o^{(1)} x_1 + W_o^{(2)} x_2 + \ldots + W_o^{(s)} x_s \\ W_1 = W_1^{(1)} x_1 + W_1^{(2)} x_2 + \ldots + W_1^{(s)} x_s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ W_{s-1} = W_{s-1}^{(1)} x_1 + W_{s-1}^{(2)} x_2 + \ldots + W_{s-1}^{(s)} x_s \end{array}$$

Thus the said linear relation holds for s consecutive values of n. Hence, by lemma 1, it holds for every value of n.

Theorem. If W is a recurring sequence satisfying (2), and if the equation (4) has s distinct roots $x_1, x_2, \ldots, x_s$, then

(6) $DW_n = D_1 x_1^n + D_2 x_2^n + \ldots + D_s x_s^n$

holds, where

$$D = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_s \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_s^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{s-1} & x_2^{s-1} & x_3^{s-1} & \ldots & x_s^{s-1} \end{vmatrix} = (x_2 - x_1) \ldots (x_s - x_1) \ldots (x_s - x_{s-1})$$

is the Vandermonde of the roots and D_i is the determinant of order s obtained from D on replacing its i-th column by the following terms

$$W_o, W_1, \ldots, W_{s-1}.$$

If among the roots there are groups of equal roots, one replaces in (6) every k equal powers of the roots by their consecutive derivatives (beginning with the one of order zero, i.e. the function itself). This process obviously includes the former. In case all roots are equal one obtains a Wronskian.

Proof. The sequence of powers of any root x_i,

(7) $$1,\ x_i,\ x_i^2,\ x_i^3,\ \ldots$$

is, by (4), a recurring sequence satisfying (2). Each k-fold root of f(x) is also a root of f'(x), f"(x),..., $f^{(k-1)}(x)$. Thus each sequence of derivatives of (7) of an order not exeeding k-1 also satisfies (2). The corresponding determinant D is different from zero. Hence, by lemma 2, the result.

ARITHMETICAL PROPERTIES OF SUMS OF POWERS

1. Introduction. For each polynomial,

$$p(x) = x^n + a_1x^{n-1} + \ldots + a_n = \prod_{\nu=1}^{n}(x-x_\nu),$$

we define $s_q(p) = x_1^q + \ldots + x_n^q$ (q=1, 2, 3,...). It is known that if all the a_i's are integers, then so are all the s_i's. We will say that an ordered set of integers,

$$S(1),\ S(2),\ldots,\ S(n),$$

has the property P provided there exists a polynomial p, having integral coefficients, for which $S(i) = s_i(p)$ (i=1, 2,..., n).

The following criterion concerning the sums s_i is due to Jänichen [1]:

The set S has the property P if and only if the congruences

$$\sum_{d|m}\mu(d)S(m/d) \equiv 0 \pmod m \qquad (m=1,\ 2,\ \ldots,\ n)$$

all hold.

2. A generalization. The purpose of the first part of this note is to prove the following generalization of Jänichen's criterion.

THEOREM 1. The set S has the property P if and only if the congruences

(1) $$\sum_{d|m}f(d)S(m/d) \equiv 0 \pmod m \qquad (m=1,\ 2,\ldots,\ n)$$

all hold, where f is an arbitrary integer-valued function satisfying the conditions

(2) $$f(1) = \pm 1,$$

(3) $$\sum_{d|m}f(d) \equiv 0 \pmod m \qquad (m=1,\ 2,\ldots,\ n)$$

In particular, f may be an arbitrary multiplicative function that satisfies (3) whenever m is a power of a prime, such as Möbius' function μ and Euler's function φ [2]. The example f(1)=1, $\sum_{d|m}f(d) = -m$ for m>1, whence f(2)=-3, f(3)=-4, f(6)=0, shows that the conditions (2), (3) do not imply that f is multiplicative.

COROLLARY. If f is an arbitrary integer-valued function satisfying the conditions

(3') $$\sum_{d|m}f(d) \equiv 0 \pmod m \qquad \text{for each } m \geq 1,$$

then, for sums of powers $s_q(p)$,

(1') $$\sum_{d|m}f(d)s_{m/d} \equiv 0 \pmod m \qquad \text{for each } m \geq 1.$$

In order to prove Theorem 1 and the corollary we need the three lemmas that follow. All the functions involved in these lemmas are defined for m=1, 2,..., n, where n is an arbitrary positive integer.

LEMMA 1. If f(m), g(m), R(m) are three functions satisfying the conditions

(4) $$g(1)R(1) = \pm 1,$$

(5) $$\sum_{d|m} g(d)R(m/d) \equiv 0 \pmod m,$$

(6) $$\sum_{d|m} f(d)R(m/d) \equiv 0 \pmod m,$$

then also

(7) $$F(m) = \sum_{d|m} f(d)g'(m/d) \equiv 0 \pmod m,$$

where g' is the Dirichlet reciprocal of g, that is, $\sum_{d|m} g(d)g'(m/d)=$ 1, 0 according as m=1 or m>1.

Proof. Relation (7) is evidently true for m=1. Suppose (7) is true for every divisor d<m of m. Statement (7) implies $f(k)=\sum_{d|k} F(d)g(k/d)$. Hence

$$\begin{aligned} 0 \equiv \sum_{d|m} f(d)R(m/d) &= \sum_{d|m} \sum_{d'|d} F(d')g(d/d')R(m/d) \\ &= \sum_{d|m} \sum_{d'|d} F(m/d)g(d/d')R(d') \\ &= \sum_{d|m} F(m/d) \sum_{d'|d} g(d/d')R(d') \\ &= F(m)g(1)R(1) + \sum_{d|m,\, d<m} F(m/d) \sum_{d'|d} g(d/d')R(d') \\ &= \pm F(m) + \sum_{d|m,\, d<m} (m/d)Q(d).dQ'(d) \equiv \pm F(m) \pmod m \end{aligned}$$

(Q, Q' integer-valued).

LEMMA 2. If f(m), g(m), R(m), S(m) are four functions satisfying the conditions (4), (5), (6) and

(8) $$\sum_{d|m} g(d)S(m/d) \equiv 0 \pmod m,$$

then also

(7') $$\sum_{d|m} f(d)S(m/d) \equiv 0 \pmod m.$$

Proof. By Lemma 1, we have

$$\sum_{d|m} f(d)S(m/d) = \sum_{d|m} F(m/d) \sum_{d'|d} g(d/d')S(d') = \sum_{d|m} (m/d)Q(d).dQ''(d) \equiv 0 \pmod m$$

(Q, Q" integer-valued).

In particular, for R≡±1 Lemma 2 implies

LEMMA 3. If f(m), g(m), S(m) are three functions satisfying the conditions

(4') $$g(1) = \pm 1,$$

(5') $$\sum_{d|m} g(d) \equiv 0 \pmod m,$$

(6') $$\sum_{d|m} f(d) \equiv 0 \pmod m$$

and (8), then also (7') hold.

Proof of Theorem 1 and Corollary. The theorem and corollary follow by Jänichen's criterion from Lemma 3 by, first, putting μ for g and, second, putting f for g and μ for f.

CONVERSE OF THEOREM 1. Suppose that f is a number-theoretic function such that every set S has the property P if and only if (1) holds. Then f satisfies (2) and (3').

Proof. (3') is obvious, taking p(x)=x-1, whence S(m)≡1. To prove (2), suppose f(1)≠±1. Then there exists a prime π dividing f(1). Now we can choose a set S which satisfies (1) but does not have the property P. For example n=π, S(m)=0 for m=1, 2,..., π-1, S(π)=1, which implies that the congruences (1) hold for each m≤π. But, by Jänichen's criterion, this set S does not have the property P, since we have μ(1)S(π)+μ(π)S(1)=1≢0 (mod π).

3. The apparition of prime factors. The aim of the second part of this note is to prove some theorems on the apparition of prime factors in sequences (s_q). From the criterion of Jänichen one can, with I. Schur [3], deduce the following congruences:

(9) $$s_{kp^{\alpha+1}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}}$$

for every prime p and non-negative integral α. The congruences (9) can also be written in the following equivalent form:

(10) $$s_{kp^{\alpha+\beta}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}}$$

for every positive integral β. Indeed, (10) becomes (9) for β=1. Let (10) be true for β. Then by (9)

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha+\beta}} \pmod{p^{\alpha+\beta+1}}$$

and a fortiori

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha+\beta}} \pmod{p^{\alpha+1}},$$

Combining the last congruences with (10), supposed true for β, we have

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}},$$

that is, (10) is true also for $\beta+1$, which establishes (10).

From (10) we immediately deduce the following theorems:

THEOREM 2. If $s_{kp^{\alpha}} \equiv 0 \pmod{p^{\gamma}}$, where p is a prime, γ is a positive integer, α is a non-negative integer and $\gamma<\alpha+1$, then $s_{kp^{\alpha+\beta}} \equiv 0 \pmod{p^{\gamma}}$ for every positive integer β.

In particular, for $\alpha=0$, $\gamma=1$ we have

THEOREM 2.1. If $s_k \equiv 0 \pmod p$, where p is a prime, then $s_{kp^{\beta}} \equiv 0 \pmod p$ for every positive integer β.

THEOREM 2.2. If $s_k=0$ then $s_{kp^{\beta}} \equiv 0 \pmod p$ for every prime p and every positive integer β [4].

CONVERSE OF THEOREM 2.2. If $s_{kp} \equiv 0 \pmod p$ for an infinitude of primes p then $s_k=0$.

Proof. By (10) we have $s_{kp} \equiv s_k \pmod p$. Whence, by the hypothesis of the converse, $s_k \equiv 0 \pmod p$ for an infinitude of primes p. Thus $s_k=0$.

4. Remarks. A sequence (s_q), no term of which vanishes, does not necessarily contain all primes as factors. For example, the sequence $(V_q=\alpha^q+\beta^q)$, where α, β are roots of the equation $x^2-x-1=0$ $(V_1=1,\ V_2=3,\ V_q=V_{q-1}+V_{q-2})$ does not contain all primes as factors [5]. The question as to which primes appear and which do not appear as factors in a sequence (s_q) with no vanishing term seems to be open even in the case of (V_q) [6].

The Theorems 2 and 2.1 can be generalized immediately by putting r for 0 in the congruences.

R e f e r e n c e s

1. W. Jänichen, Über die Verallgemeinerung einer Gauss'schen Formel aus der Theorie der höheren Kongruenzen, Sitzungsberichte der Berliner Mathematischen Gesellschaft 20 (1921), 23-29.

2. The expressions $\sum_{d|m}\mu(d)s_{m/d}$, $\sum_{d|m}\varphi(d)s_{m/d}$ have an interpretation in the theory of ordered partitions. Compare Th. Motzkin, Ordered and Cyclic Partitions, Riveon Lematematika 1 (1946-7), 61-67.

3. I. Schur, Arithmetische Eigenschaften der Potenzsummen einer algebraischen Gleichung, Compositio Mathematica 4 (1937), 432-44.

4. The special case $k=\beta=1$ was found by E. B. Escott, American Mathematical Monthly 15 (1908), 187; L. E. Dickson, ibidem, 209. Particular cases of his result appear already in E. Lucas, A. F., Congrès de Clermont-Ferrand (1876), 2; R. Perrin, L'intermed. des Math. 6 (1899), 76-77; E. Malo, ibidem 7 (1900), 280-282, 312-314; E. B. Escott, ibidem 8 (1901), 63-64.

5. E. Lucas, American Journal of Mathematics 1 (1878), 298.

6. See above, pp. 14-15.

EMBEDDING OF A QUADRATIC RECURRING SEQUENCE
IN A LINEAR SECOND ORDER RECURRING SEQUENCE

THEOREM 1. The quadratic recurring sequence

(1) $a_1,\ a_2=a_1^2-2,\ a_3=a_2^2-2,\ \ldots,\ a_n=a_{n-1}^2-2,\ \ldots$

is a subsequence of the linear second order recurring sequence

(2) $S_0=2,\ S_1=a_1,\ \ldots,\ S_n=a_1S_{n-1}-S_{n-2},\ \ldots$

and

(3) $a_n=S_{2^{n-1}}$.

PROOF. By induction on 1, 1+1 it is easily verified that

(4) $S_kS_1=S_{k+1}+S_{k-1}$

since it is true for 1=0, 1. Hence, for k=1,

(5) $S_{2k}=S_k^2-2$

and in particular

(6) $S_{2^n}=S_{2^{n-1}}^2-2$.

Now we have $a_1=S_1$, and by induction on n: $a_n=a_{n-1}^2-2=S_{2^{n-2}}^2-2=S_{2^{n-1}}$, that is (3).

For $a_1=3$, 4 (1) becomes:

(1.3) 3,7,47,2207,4870847,23725150497407,562882766124611619513723647,...

(1.4) 4,14,194,37634,1416317954,2005956546822746114,...

These sequences have been applied to test the composition of numbers of form 2^p-1, where p is an odd prime.

THEOREM 2. For terms of the sequence (1), where a_1 is a positive integer, we have

(7) $a_{n+1} \equiv -2 \pmod{a_n}$

(8) $a_{n+k} \equiv 2 \pmod{a_n}$ for k=2, 3, ...

PROOF. (7) is an immediate consequence of the definition (1). (8) is proved by induction, as follows: the assumption $a_{n+k} \equiv 2 \pmod{a_n}$ implies: $a_{n+k+1}=a_{n+k}^2-2 \equiv 2 \pmod{a_n}$. But $a_{n+2}=a_{n+1}^2-2=(a_n^2-2)^2-2 \equiv 2 \pmod{a_n}$. Hence (8).

THEOREM 3. Any two terms of (1) are coprime or have 2 as their greatest common divisor, according as a_1 is odd or even.

PROOF. Theorem 2.

DIVISIBILITY PROPERTIES OF RECURRING SEQUENCES CONTAINING VANISHING TERMS

We consider the general recurring sequence (W) of order $s\geq2$ defined by

(1) $W_n=a_1W_{n-s}+a_2W_{n-s+1}+\dots+a_sW_{n-1}$ $(n=0, \pm1, \pm2,\dots)$

with arbitrary fixed complex $a_1\neq0$, $a_2,\dots, a_s$, W_0, $W_1,\dots, W_{s-1}$, and also the special case (U) with the same $a_1,\dots, a_s$ and

(2) $U_0=U_1=\dots=U_{s-2}=0$, $U_{s-1}=1$.

The following statement holds.

(3) $$\begin{aligned} W_{m+n}&=a_1W_{m-1}U_n\\ &+(W_{m+s-2}-a_sW_{m+s-3}-a_{s-1}W_{m+s-4}-\dots-a_3W_m)U_{n+1}\\ &+(W_{m+s-3}-a_sW_{m+s-4}-a_{s-1}W_{m+s-5}-\dots-a_4W_m)U_{n+2}\\ &+\dots\\ &+(W_{m+1}-a_sW_m)U_{n+s-2}\\ &+W_mU_{n+s-1}.\end{aligned}$$

Indeed, (3) can be written as follows:

(3') $$\begin{aligned} W_{m+n}=W_{m+s-1}U_n&-W_{m+s-2}a_sU_n-W_{m+s-3}a_{s-1}U_n-\dots-W_ma_2U_n\\ &+W_{m+s-2}U_{n+1}-W_{m+s-3}a_sU_{n+1}-\dots-W_ma_3U_{n+1}\\ &+W_{m+s-3}U_{n+2}-\dots-W_ma_4U_{n+2}\\ &+\dots\\ &+W_mU_{n+s-1}.\end{aligned}$$

Summing (3') by columns we easily see that (3') holds for $n=0, 1, 2,\dots, s-1$, whence (3') holds generally.

The formula (3) for m=0 is due to M. d'Ocagne (according to Dickson, History I, 409).

Putting in (3) $W\equiv U$, $m=(k-1)n+r$, we have:

(4) $$\begin{aligned} U_{kn+r}&=a_1U_{(k-1)n+r-1}U_n+A_1U_{n+1}+A_2U_{n+2}+\dots+A_{s-2}U_{n+s-2}\\ &+U_{(k-1)n+r}U_{n+s-1},\end{aligned}$$

where the A's are polynomials in a's and U's.

In particular, for $s\geq4$, $k=2$, $r=2, 3,\dots, s-2$, (4) becomes:

(5) $U_{2n+r}=B_1(r)U_{n+1}+B_2(r)U_{n+2}+\dots+B_{s-2}(r)U_{n+s-2}$,

where the B's are polynomials in a's and U's.

Putting $m=W_0=0$ in (3), we have:

(6) $W_n=C_0U_n+C_1U_{n+1}+C_2U_{n+2}+\dots+C_{s-2}U_{n+s-2}$,

where the C's are polynomials in a's and W's.

It is supposed now that $a_1,\dots,a_s$ and $W_0,\dots,W_{s-1}$ are integers and that $(W_0,\dots,W_{s-1})=1$. Then evidently all the W_n and $a_1^nW_{-n}$ with $n\geq0$ are integers. We shall say that a fraction P/Q, where P, Q are integers and $Q\neq0$, is divisible by an integer p, if P is divisible by p.

It is well-known that (W), being a recurring sequence, is periodic with respect to any modulus. From this, and noting that zero is divisible by any positive integer, we immediately deduce that

(7) Any positive integer is a divisor of an infinitude of terms in a recurring sequence one term of which vanishes.

Noting that $U_0=U_1=\dots=U_{s-2}=0$, we have:

(8) For any positive integer p there exists a positive integer n such that p is a common divisor of U_n, $U_{n+1},\dots, U_{n+s-2}$.

(9) Any common divisor of U_n, $U_{n+1},\dots, U_{n+s-2}$ coprime with a_1 also divides U_{kn}, $U_{kn+1},\dots, U_{kn+s-2}$ for any integer k.

Indeed, putting in (4) $r=0, 1, 2,\dots, s-2$ successively, we see that (9) holds for k if it holds for k-1. But (9) holds evidently for k=1, and hence it holds for any $k\geq1$. Since (U), for a modulus coprime with a_1, is also periodic backwards, it also follows that (9) holds for $k<1$.

From (5) we have:

(10) For $s\geq4$, any common divisor of U_n, $U_{n+1},\dots, U_{n+s-3}$ also divides U_{2n}, $U_{2n+1},\dots, U_{2n+s-4}$.

From (6) and (9) we have:

(11) If $W_t=0$, then any divisor of U_n, $U_{n+1},\dots, U_{n+s-2}$ also divides W_{kn+t}.

DEFINITION. The least positive integer n such that U_n, $U_{n+1},\dots, U_{n+s-2}$ are all divisible by a positive integer p we shall call the r a n k o f a p p a r i t i o n of p in (U).

(12) Any common divisor p of W_m, U_n, $U_{n+1},\dots, U_{n+s-2}$ also divides W_{m-n}.

For, by (9), p also divides U_{-n}, $U_{-n+1},\dots, U_{-n+s-2}$. Hence, replacing n by -n in (3), we obtain (12).

(13) If n is the rank of apparition of a positive integer p coprime with a_1 in (U), then any number N such that $U_N, U_{N+1}, \ldots, U_{N+s-2}$ are all divisible by p is a multiple of n.

For, since by assumption $U_n, U_{n+1}, \ldots, U_{n+s-2}$ are all divisible by p, we have by (9) that $U_{kn}, U_{kn+1}, \ldots, U_{kn+s-2}$ are all divisible by p. Now suppose, if possible, that $N=kn+r$, where $0<r<n$. Then, putting in (12) $W=U$ and successively $m=kn+r, kn+r+1, \ldots, kn+r+s-2$ we have that $U_r, U_{r+1}, \ldots, U_{r+s-2}$ are all divisible by p, which is impossible, since $0<r<n$, and n is the rank of apparition of p in (U).

LEMMA. Let n be the rank of apparition of a positive integer p coprime with a_1 in (U), and let P be the length of the period modulo p. Then $n<P/(d-1)$, where $d\geq 2$, implies $n\leq P/d$.

For, if it is supposed that $n=(P/d)+r$, where $r>0$, we have by assumption $(P/d)+r=n<P/(d-1)$, whence $r<P/d(d-1)$. Thus, by (9): $U_{dr+t}\equiv U_{P+dr+t}=U_{dn+t}\equiv 0 \pmod p$, that is, $U_{dr+t}\equiv 0 \pmod p$ for $t=0, 1, 2, \ldots, s-2$, which is impossible, by the meaning of rank of apparition, since $dr=(d-1)r+r<(P/d)+r=n$, that is $dr<n$.

(14) If n is the rank of apparition of a positive integer p coprime with a_1 in (U), and if P is the length of the period modulo p, then n is a divisor of P.

For, should n not be a divisor of P, we could, by repeated use of the lemma, deduce that $n<P/d$ for any positive integer $d\geq 2$, that is $n\leq 0$, which is impossible, by the meaning of rank of apparition.

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NEW FORMULAE FOR FIBONACCI AND LUCAS NUMBERS

The Fibonacci numbers U_n and the Lucas numbers V_n are defined by:

(1) $$U_0=0,\ U_1=1,\ U_n=U_{n-1}+U_{n-2}$$

(1) $$V_0=2,\ V_1=1,\ V_n=V_{n-1}+V_{n-2}$$

These numbers satisfy the following relations:

(3) $$U_{k+1}=\tfrac{1}{2}(U_k+V_k)$$

(4) $$U_{k-1}=-\tfrac{1}{2}(U_k-V_k)$$

(5) $$V_{k+1}=\tfrac{1}{2}(5U_k+V_k)$$

(6) $$V_{k-1}=\tfrac{1}{2}(5U_k-V_k)$$

Proof of (3)-(6) by induction on k, k+1, as valid for k=0, 1.

To prove (9)-(10) the following relations will be used:

(7) $$-\binom{a}{2i+1}=\binom{a}{2i}-\binom{a+1}{2i+1}$$

(8) $$-\binom{a}{2i}=\binom{a+1}{2i}+\binom{a}{2i-2}-\binom{a+1}{2i-1}$$

(7) is the for our purposes adapted addition-formula for binomial coefficients $\binom{a}{b}+\binom{a}{b+1}=\binom{a+1}{b+1}$. (8) is obtainable from (7) as follows:

By (7)
$$\binom{a}{2i}=\binom{a+1}{2i}-\binom{a}{2i-1}$$
$$-\binom{a}{2i-1}=\binom{a}{2i-2}-\binom{a+1}{2i-1}$$

Adding side-wise we get (8).

The purpose of the present paper is to prove for Fibonacci and Lucas numbers the following formulae which seem to be new. These formulae were presented to me without proof by Prof. Theodore Motzkin, not later than 1941, and already then I have essentially proved them, as given below.

The symbol [n] denotes the greatest positive integer not greater than n.

THEOREM. For $k\geq 1$ the following formulae hold:

(9) $$U_k=\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}U_{i+[i/2]}$$

(10) $$V_k=-\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}V_{i+[i/2]}$$

PROOF (by induction on k). The theorem is valid, by (1), (2), for k=1. If it is valid for k, then also for k+1. Indeed, by (3),

$$U_{k+1}=\tfrac{1}{2}\{U_k+V_k\}$$

By the induction-hypothesis

$$=\tfrac{1}{2}\{\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}U_{i+[i/2]}$$

$$-\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}V_{i+[i/2]}\}$$

$$=\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}\tfrac{1}{2}\{U_{i+[i/2]}-V_{i+[i/2]}\}$$

By (4)

$$=-\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}U_{i+[i/2]-1}$$

Split the sum into two sums, one for even values of i, the other for odd, putting 2i and 2i+1 for i

$$=\sum_{i=0}^{k-1}(-1)^i\binom{k+i-1}{2i}U_{3i-1}-\sum_{i=0}^{k-1}(-1)^i\binom{k+i}{2i+1}U_{3i}$$

Exclude the value i=0 from the first sum, inorder to avoid in the sequel binomial coefficients $\binom{a}{b}$ with negative b

$$=U_{-1}+\sum_{i=1}^{k-1}(-1)^i\binom{k+i-1}{2i}U_{3i-1}-\sum_{i=0}^{k-1}(-1)^i\binom{k+i}{2i+1}U_{3i}$$

Split the first sum into three sums by (8) and return the value U_{-1} to the first partial sum

$$=\sum_{i=0}^{k}(-1)^i\binom{k+i}{2i}U_{3i-1}+\sum_{i=1}^{k}(-1)^i\binom{k+i-1}{2i-2}U_{3i-1}-\sum_{i=1}^{k}(-1)^i\binom{k+i}{2i-1}U_{3i-1}$$

and the second sum into two sums by (7)

$$+\sum_{i=0}^{k}(-1)^i\binom{k+i}{2i}U_{3i} \qquad -\sum_{i=0}^{k}(-1)^i\binom{k+i+1}{2i+1}U_{3i}$$

Put in the second and third sums i+1 for i

$$=\sum_{i=0}^{k}(-1)^i\binom{k+i}{2i}U_{3i-1}-\sum_{i=0}^{k-1}(-1)^i\binom{k+i}{2i}U_{3i+2}+\sum_{i=0}^{k-1}(-1)^i\binom{k+i+1}{2i+1}U_{3i+2}$$

$$+\sum_{i=0}^{k}(-1)^i\binom{k+i}{2i}U_{3i} \qquad -\sum_{i=0}^{k}(-1)^i\binom{k+i+1}{2i+1}U_{3i}$$

Add the first, second and fourth sums, and the third and fifth

$$=-\sum_{i=0}^{k}(-1)^i\binom{k+i}{2i}U_{3i} \qquad +\sum_{i=0}^{k}(-1)^i\binom{k+1+i}{2i+1}U_{3i+1}$$

$$=\sum_{i=0}^{2(k+1)-1}(-1)^{(i-1)(i-2)/2}\binom{(k+1)+[(i-1)/2]}{i}U_{i+[i/2]}$$

By (5)

$$V_{k+1}=\tfrac{1}{2}\{5U_k+V_k\}$$

By the induction-hypothesis

$$=\tfrac{1}{2}\{5\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}U_{i+[i/2]}$$

$$-\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}V_{i+[i/2]}\}$$

$$=\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}\tfrac{1}{2}\{5U_{i+[i/2]}-V_{i+[i/2]}\}$$

By (6)

$$=\sum_{i=0}^{2k-1}(-1)^{(i-1)(i-2)/2}\binom{k+[(i-1)/2]}{i}V_{i+[i/2-1]}$$

It is no more necessary to continue the calculations in V, since the fourth sum is similar, except for the sign, to the fourth sum in U. Since in the sequel of the calculations in U no use was made of the initial values of U, all the calculations will be valid if one puts in them V for U, and consequently we get a formula entirely similar to the final formula in U, but with an opposite sign, and this is exactly what should be obtained. The proof of the theorem is thus completed.

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