ON A CLASS OF NONLINEAR BINOMIAL SUMS

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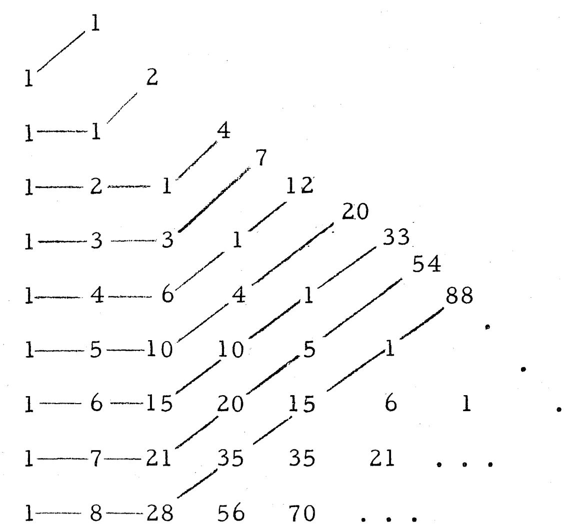
It is known ([2] ; [4] ; [5]) that the Fibonacci numbers may be formed by adding the binomial terms on diagonals of Pascal's Triangle. Recently in this Quarterly V. C. Harris and Carolyn C. Styles [3] generalized the Fibonacci sequence by extending their consideration to sums along straight diagonals of any positive "slope" originating in the first column. As they noted, those sums are special cases of the linear binomial sums investigated by Dickinson [1] . Here we consider a nonlinear generalization of this connection, in which each sum contains a "dogleg" of binomial terms. We then note that these sums obey the same difference equation as the binomial coefficients. From this equation and a set of auxiliary numbers we derive some arithmetic properties, including connections with the Fibonacci numbers, and develop some general recurrences. Because of this close connection with the binomial coefficients, it is not surprising that most of the properties given here stem from corresponding properties of the binomial coefficients.

We define L(n, r), the r-th order nonlinear binomial sum, as the sum of the first r terms of the (n-1)-th row of Pascal's Triangle plus the terms on the rising stairstep diagonal originating at the r-th term. Thus

$$(1) \qquad L(n,r) = \sum_{i=0}^{r-1} \binom{n-1}{i} + \sum_{j=1}^{[\frac{n-r}{2}]} \binom{n-j-1}{j+r-1},$$

where [ ] denotes greatest integer, and the right-most sums is zero if $[\frac{n-r}{2}] < 1$. The sums $L(n, 1) = L(n-1, 2) = F_n$, the n-th Fibonacci number, are those previously considered in [2] , [4] , and [5] . For r = 3 we obtain the following series.



Thus $L(1, 3) = 1$, $L(2, 3) = 2$, $L(3, 3) = 4$, etc. The 4-th order sequence is 1, 2, 4, 8, 15, 27, 47, 80, 134, $\cdots$.

The connection between the Fibonacci numbers and binomial coefficients previously mentioned may be written as

$$F_n = \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n-i-1}{i} .$$

The difference between the nonlinear binomial sums and Fibonacci numbers is therefore

$$F_{n+r-1} - L(n, r) = \sum_{i=0}^{r-1} \binom{n+r-2-i}{i} - \sum_{i=0}^{r-1} \binom{n-1}{i} ,$$

which is a polynomial in n of degree $r-3$ for $r \geq 3$. By evaluating the right side of this equation for small values of r, we find, in addition to $L(n, 1) = L(n-1, 2) = F_n$, that

(2a) $$L(n, 3) = F_{n+2} - 1 ,$$

(2b) $$L(n, 4) = F_{n+3} - n - 1 .$$

Also, since

$$\sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1} ,$$

we see from the definition that

$$L(n, r) = 2^{n-1} \quad (n \leq r) .$$

Let the difference operator $\underset{n}{\Delta}$ be defined by $\underset{n}{\Delta} f(n) = f(n+1) - f(n)$. Then the recurrence relation for the binomial coefficients may be represented as

$$(3) \qquad \underset{n}{\Delta} \binom{n}{r} = \binom{n}{r-1} .$$

From this and the explicit representation in (1), the important difference equation follows that

$$(4) \qquad \underset{n}{\Delta} L(n, r) = L(n, r-1) .$$

Defining the iterated difference operator $\underset{n}{\Delta}^k$ by $\underset{n}{\Delta}^1 f(n) = \underset{n}{\Delta} f(n)$, $\underset{n}{\Delta}^k f(n) = \underset{n}{\Delta} [\underset{n}{\Delta}^{k-1} f(n)]$ for $k > 1$, it is of interest to note that

$$\underset{n}{\Delta}^{r-2} L(n, r) = L(n, 2) = F_{n+1} ,$$

$$\underset{n}{\Delta}^{r-1} L(n, r) = L(n, 1) = F_n .$$

It is apparent that (4) is indeed the same difference equation satisfied by the binomial coefficients in (3), the only change being in the initial values. Thus by using (4) and the easily determined boundary conditions

$$L(n, 1) = F_n , \quad L(1, r) = 1 ,$$

we may construct a table of $L(n, r)$ in which each term is the sum of the term above it and the term above and to the left.

Since the sequence $L(n, 2) = F_{n+1}$ satisfies the recurrence relation

$$L(n+2, 2) = L(n+1, 2) + L(n, 2) ,$$

we take the anti-difference of this r-2 times and obtain the general recurrence relation for r-th order terms as

(5) $$L(n+2, r) = L(n+1, r) + L(n, r) + A(n, r) ,$$

Table of L(n, r)

n \ r	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2
3	2	3	4	4	4	4	4	4	4	4
4	3	5	7	8	8	8	8	8	8	8
5	5	8	12	15	16	16	16	16	16	16
6	8	13	20	27	31	32	32	32	32	32
7	13	21	33	47	58	63	64	64	64	64
8	21	34	54	80	105	121	127	128	128	128
9	34	55	88	134	185	226	248	255	256	256
10	55	89	143	222	319	411	474	503	511	512

where the auxiliary numbers A(n, r) obey

$$\underset{n}{\Delta}^{r-2} A(n, r) = 0$$

with the initial conditions

$$A(n, 1) = A(n, 2) = 0 \quad (n \geq 1); \; A(1, r) = 1 \quad (r \geq 3) .$$

These numbers also obey the binomial recurrence

$$\underset{n}{\Delta} A(n, r) = A(n, r-1) ,$$

so that we may easily construct a table of A(n, r) from the initial conditions using the same rule of formation as that for L(n, r). It appears from this table that while L(n, 1) and L(n, 2) are sequence of the Fibonacci type, the next two obey the slightly more complicated recurrences

$$L(n+2, 3) = L(n+1, 3) + L(n, 3) + 1 ,$$

$$L(n+2, 4) = L(n+1, 4) + L(n, 4) + n .$$

These are readily proved by using equations (2a) and (2b).

Table of A(n, r)

n \ r	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	1	1	1	1	1	1
2	0	0	1	2	2	2	2	2	2	2
3	0	0	1	3	4	4	4	4	4	4
4	0	0	1	4	7	8	8	8	8	8
5	0	0	1	5	11	15	16	16	16	16
6	0	0	1	6	16	26	31	32	32	32
7	0	0	1	7	22	42	57	63	64	64
8	0	0	1	8	29	64	99	120	127	128
9	0	0	1	9	37	93	163	219	247	255
10	0	0	1	10	46	140	256	382	466	502

We may establish from (4) and (5) that the recurrence formula with respect to r is

$$L(n, r) = L(n, r-1) + L(n, r-2) - A(n, r) .$$

From this, with (4) again, it follows that

$$A(n, r) = L(n+1, r-1) - L(n, r) .$$

This last equation may be used to establish that

$$L(n, r) + \sum_{i=0}^{r-1} A(n+i, r-1) = F_{n+r-1} .$$

Taking $n = 1$, we see that the slant sums of the $A(n, r)$ are Fibonacci numbers diminished by a unity, i. e.

$$\sum_{i=1}^{r} A(i, r-i+1) = F_r - 1 .$$

It is also interesting to note that the $A(n, r)$ obey the curious diagonal recurrence

$$A(n+1, r+1) = 2\,A(n, r) + \binom{n-1}{r-2} .$$

The recurrence (5) may be easily extended by induction to

$$L(n, r) = F_{k+1}\,L(n-k, r) + F_k\,L(n-k-1, r) + \sum_{i=1}^{k} F_i\,A(n-i-1, r) \quad (0 \le k < n),$$

and the analogous extended recurrence with respect to r is

$$L(n, r) = F_{k+1}\,L(n, r-k) + F_k\,L(n, r-k-1) - \sum_{i=1}^{k} F_i\,A(n, r-i+1) \quad (0 \le k < n).$$

We remark that setting $r = 1$ in the former recurrence gives the familiar Fibonacci identity

$$F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1} .$$

We may prove by induction that for $r > 1$

$$\sum_{i=1}^{k} L(i, r) = L(k+1, r+1) - 1 ,$$

$$\sum_{i=1}^{k} A(i, r) = A(k+1, r+1) - 1 ,$$

which together imply

$$\sum_{i=1}^{n} [L(i, r) + A(i, r)] = L(n+2, r) - 2 \quad (r > 1) .$$

Finally, we extend the definition of $L(n, r)$ to negative r from (4) by putting

$$L(n, r) = F_{n+r-1} \quad (r \le 0) .$$

With this extension, the readily proved formula

$$L(n, r) = \sum_{i=0}^{k} \binom{k}{i} L(n-k, r-i)$$

if valid for all k such that $0 \le k < n$.

The author would like to thank Vincent C. Harris for his helpful suggestions.

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