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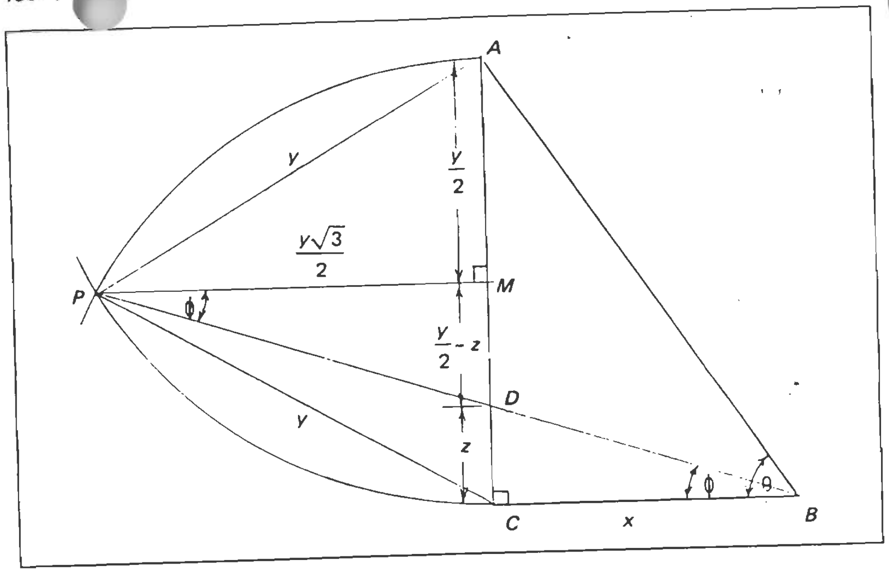

Figure 3.

Thus,

$$\frac{y}{x} = \frac{3\left(\frac{z}{x}\right) - \left(\frac{z}{x}\right)^3}{1 - 3\left(\frac{z}{x}\right)^2} \tag{2}$$

Since $m\angle MPD = m\angle CBD$,

$$\tan\phi = \frac{\frac{y}{2} - z}{\frac{y\sqrt{3}}{2}} = \frac{z}{x} \tag{3}$$

Upon solving for z and substituting in equation (3), we have

$$\tan\phi = \frac{y\sqrt{3}}{3y + 2x\sqrt{3}} \tag{4}$$

Thus, we wish to determine whether or not the equation below is an identity.

$$\frac{y}{x} = \frac{\frac{3y\sqrt{3}}{3y + 2x\sqrt{3}} - \left(\frac{y\sqrt{3}}{3y + 2x\sqrt{3}}\right)^3}{1 - 3\left(\frac{y\sqrt{3}}{3y + 2x\sqrt{3}}\right)^2} \tag{5}$$

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This simplifies to:

$$\frac{y}{x} = \frac{y(2y^2 + 3\sqrt{3}xy + 3x^2)}{x(2x^2 + 3\sqrt{3}xy + 3y^2)} \tag{6}$$

Therefore, we conclude that $\overline{PB}$ trisects $\angle B$ only when $x = y$ ($\overline{AC} = \overline{BC}$); that is, our conjecture holds true only for isosceles right triangles!

Note that the value of the expression $\frac{2y^2 + 3\sqrt{3}xy + 3x^2}{2x^2 + 3\sqrt{3}xy + 3y^2}$ ranges from $\frac{2}{3}$ to $\frac{3}{2}$ as the ratio of y to x ranges from 0 to ∞. In retrospect, it is clear that if $\overline{BC}$ is held fixed and $\overline{AC}$ approaches 0, then $\overline{PB}$ "almost" bisects segment $\overline{AC}$ and $\frac{m\angle PBC}{m\angle ABC}$ approaches $\frac{1}{2}$.

STARS, HEXES, TRIANGULAR NUMBERS, AND PYTHAGOREAN TRIPLES

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Figurate numbers such as the Star numbers, $S_n = 6n(n - 1) + 1$, and the Hex numbers, $H_m = 3m(m - 1) + 1$, have been defined [1]. A table of the first 15,000 Stars and Hexes has been calculated [2] and certain of their properties will be discussed in a forthcoming book [3].

In this article, we answer the question of when $S_n = H_m$. This problem is equivalent to the problem of determining when one triangular number, $T_k = k(k + 1)/2$ is twice another. It is also equivalent to determining the Pythagorean triples x, y, z such that $y = x + 1$. A table of solutions is given in this article.

If $H_m = S_n$ then $m(m - 1)/2 = n(n - 1)$. Thus, $H_m = S_n$ is equivalent to $T_{m-1} = 2T_{n-1}$, where $T_m = m(m + 1)/2$ is a triangular number. Clearly, for $m = 4$, $T_3 = 6$ and for $n = 3$, $T_2 = 3$, so $T_3 = 2T_2$. Therefore, $H_4 = S_3 = 37$. Also, $H_1 = S_1 = 1$. The problem is to find other solutions.

If $m(m - 1)/2 = n(n - 1)$, then $m^2 - m = 2n^2 - 2n$ and $n = (1 + \sqrt{2m^2 - 2m + 1})/2$. In order for n to be a positive integer, we need $2m^2 - 2m + 1$ to be the square of an odd integer, say z^2. So, we need $(m - 1)^2 + m^2 = z^2$. But $(m - 1)^2 + m^2 = z^2$ is precisely the Pythagorean theorem wherein one side of the triangle is one unit longer than the shorter side. This problem has been solved [4].

Table 1. Stars and Hexes

P	M	N	Z	$H_m = S_n$	T_{m-1}	T_{n-1}
–	1	1	–	1	0	0
1	4	3	5	37	6	3
2	21	15	29	1261	210	105
3	120	85	169	42841	7140	3570
4	697	493	985	1455337	242556	121278
5	4060	2871	5741	49438621	8239770	4119885
6	23661	16731	33461	1679457781	279909630	139954815
7	137904	97513	195025	57052125937	9508687656	4754343828
8	803761	568345	1136689	1938092824081	323015470680	161507735340
9	4684660	3312555	6625109	65838103892821	10973017315470	5486508657735
10	27304197	19306983	38613965	2236557439531837	372759573255306	186379786627653

A6062 Star/Hex numbers

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It is shown in reference [4] that a solution is given by $m - 1 = 2q_pq_{p+1}$ for p even and $m = 2q_pq_{p+1}$ for p odd where $q = ((1+\sqrt{2})^p - (1-\sqrt{2})^p)/2\sqrt{2}$ and $p = 1, 2, 3, \ldots,$ a dummy variable. For example, for $p = 1$, $q_1 = 1$, $q_2 = 2$, $m = 4$. In this case, $n = 3$ and $T_3 = 2T_2$ (as above), and $H_4 = S_3$ (as above). For $p = 2$, $q_2 = 2$, $q_3 = 5$, $m = 21$, $n = 15$, $T_{20} = 2T_{14}$, and $H_{21} = S_{15}$. Rather than perform the calculations for n, it can be shown that $n = (1 + q_{2p+1})/2$ for all p. It can also be shown that $q_p = 2q_{p-1} + q_{p-2}$ which sequence defines the Bell numbers. For further details see reference [4].

The numbers in Table 1 show the first ten Star and Hex numbers which satisfy $H_m = S_n$. They also show the corresponding solutions to $T_{m-1} = 2T_{n-1}$ and the hypotenuse z of the corresponding Pythagorean triangles where, as seen in the text, m is the longer side (by unity). The table can be examined for congruence and other relationships according to the whim of the reader.

REFERENCES

1. M. Gardner, *Scientific American, 231:*1, p. 116, July 1974.
2. H. J. Hindin, unpublished, but available from the author.
3. M. Gardner, private communication, November 1, 1978.
4. T. W. Forget and T. A. Larkin, Pythagorean Triads of the Form x, $x + 1$, z Described by Recurrence Sequences, *Fibonacci Quarterly, 6:*3, pp. 94-104, June 1968 (a typographical error on page 102 is corrected in Table 1 of this article).

PANDIAGONAL PRIME MAGIC SQUARES OF ORDER 4

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All pandiagonal fourth-order magic squares can be written in the form

$F + y$	$G + x$	$G - x$	$F - y$
$G - w$	$F + z$	$F - z$	$G + w$
$F + x$	$G + y$	$G - y$	$F - x$
$G + z$	$F - w$	$F + w$	$G - z$