

## The Combinatorics of Convex Permutominoes

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**Abstract.** A permutomino of size  $n$  is a polyomino determined by particular pairs  $(\pi_1, \pi_2)$  of permutations of  $n$ , such that  $\pi_1(i) \neq \pi_2(i)$ , for  $1 \leq i \leq n$ . Here we study various classes of convex permutominoes. We determine some combinatorial properties and, in particular, the characterization for the permutations defining convex, directed-convex, and parallelogram permutominoes.

Using standard combinatorial techniques we provide a recursive decomposition for permutations associated with convex permutominoes, and we derive a closed formula for the number of these permutations of size  $n$ .

**Keywords:** Convex polyomino; Permutation; Enumeration; Bijection.

### 1. Convex Polyominoes

In the plane  $\mathbb{Z} \times \mathbb{Z}$  a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations (see Figure 1 (a)). A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells lying on a vertical (horizontal) line.

Polyominoes were introduced by Golomb [19], and then they have been studied in several mathematical problems, such as tilings [2, 18], or games [17] among many others. The enumeration problem for general polyominoes is difficult to solve and still open. The number  $a_n$  of polyominoes with  $n$  cells is

known up to  $n = 56$  [21] and asymptotically, these numbers satisfy the relation  $\lim_n (a_n)^{1/n} = \mu$ ,  $3.96 < \mu < 4.64$ , where the lower bound is a recent improvement of [1].

In order to simplify enumeration problems of polyominoes, several subclasses were defined by combining the two simple notions of *convexity* and *directed growth*. A polyomino is said to be *column convex* (resp. *row convex*) if every its column (resp. row) is connected (see Figure 1 (b)). A polyomino is said to be *convex*, if it is both row and column convex (see Figure 1 (c)). The *area* of a polyomino is just the number of cells it contains, while its *semi-perimeter* is half the number of edges of cells in its boundary. Thus, for any convex polyomino the semi-perimeter is the sum of the numbers of its rows and columns. Moreover, any convex polyomino is contained in a rectangle in the square lattice which has the same semi-perimeter, called *minimal bounding rectangle*.

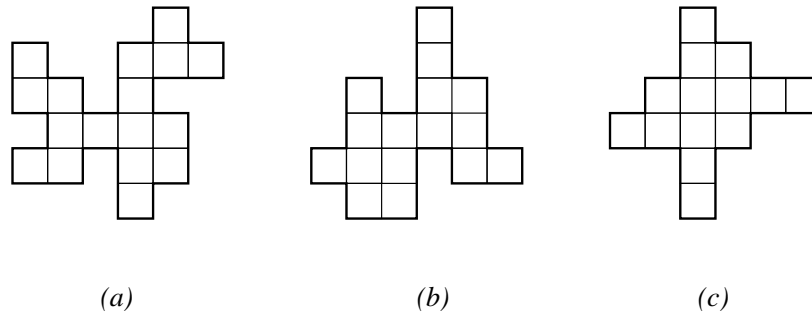


Figure 1: (a) a polyomino; (b) a column convex polyomino which is not row convex; (c) a convex polyomino.

A significant result in the enumeration of convex polyominoes was obtained by Delest and Viennot in [11], where the authors proved that the number  $\ell_n$  of convex polyominoes with semi-perimeter equal to  $n + 2$  is:

$$\ell_{n+2} = (2n + 1)4^n - 4(2n + 1) \binom{2n}{n}, \quad n \geq 2; \quad \ell_0 = 1, \quad \ell_1 = 2. \quad (1)$$

This is sequence A005436 in [22], the first few terms being:

$$1, 2, 7, 28, 120, 528, 2344, 10416, \dots$$

During the last two decades convex polyominoes, and several combinatorial objects obtained as a generalizations of this class, have been studied by various points of view. For the main results concerning the enumeration and other combinatorial properties of convex polyominoes we refer to [4, 5, 6, 8].

There are two other classes of convex polyominoes which will be useful in the paper, the *directed convex* and the *parallelogram* polyominoes. A polyomino is

said to be *directed* when each of its cells can be reached from a distinguished cell, called the root, by a path which is contained in the polyomino and uses only north and east unit steps.

A polyomino is *directed convex* if it is both directed and convex (see Figure 2 (a)). It is known that the number of directed convex polyominoes of semi-perimeter  $n + 2$  is equal to the  $n$ th central binomial coefficient, i.e.

$$b_n = \binom{2n}{n}, \tag{2}$$

sequence A000984 in [22], the first terms being:

1, 2, 6, 20, 70, 252, 924, 3432, 12870, ...

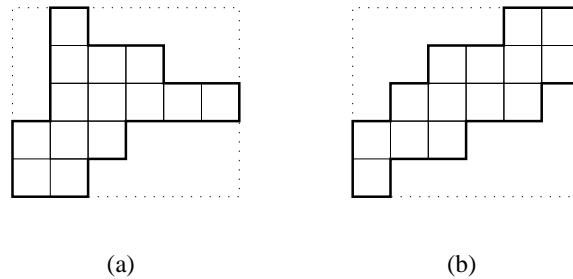


Figure 2: (a) A directed convex polyomino; (b) a parallelogram polyomino.

Finally, *parallelogram* polyominoes are a special subset of the directed convex ones, defined by two lattice paths that use north and east unit steps, and intersect only at their origin and extremity. These paths are called the *upper* and the *lower path* (see Figure 2 (b)). It is known [23] that the number of parallelogram polyominoes having semi-perimeter  $n + 1$  is the  $n$ -th *Catalan number* (sequence A000108 in [22]),

$$c_n = \frac{1}{n + 1} \binom{2n}{n}. \tag{3}$$

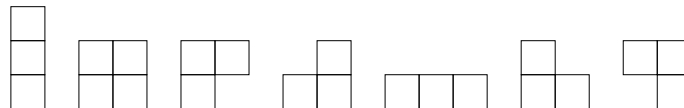


Figure 3: The seven convex polyominoes having semi-perimeter equal to four; the first five from the left are parallelogram ones, the sixth one is directed convex, but not parallelogram.

## 2. Convex Permutominoes

Let  $P$  be a polyomino, having  $n$  rows and columns,  $n \geq 1$ ; we assume without loss of generality that the south-west corner of its minimal bounding rectangle is placed in  $(1, 1)$ . Let  $\mathcal{A} = (A_1, \dots, A_{2(r+1)})$  be the list of its vertices (i.e., corners of its boundary) ordered in a clockwise sense starting from the lowest leftmost vertex.

We say that  $P$  is a *permutomino* if  $\mathcal{P}_1 = (A_1, A_3, \dots, A_{2r+1})$  and  $\mathcal{P}_2 = (A_2, A_4, \dots, A_{2r+2})$  represent two permutations of  $\mathcal{S}_{n+1}$ , where, as usual,  $\mathcal{S}_n$  is the symmetric group of size  $n$ . Obviously, if  $P$  is a permutomino, then  $r = n$ , and  $n + 1$  is called the *size* of the permutomino. The two permutations defined by  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are indicated by  $\pi_1(P)$  and  $\pi_2(P)$ , respectively (see Figure 4).

From the definition any permutomino  $P$  of size  $n$  has the property that, for each abscissa (resp. ordinate) between 1 and  $n$  there is exactly one vertical (resp. horizontal) side in the boundary of  $P$  with that coordinate. It is simple to observe that this property is also a sufficient condition for a polyomino to be a permutomino.

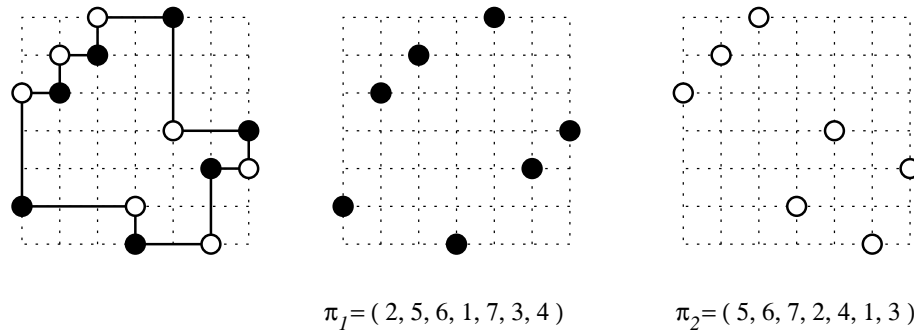


Figure 4: A permutomino of size 7 and its defining permutations.

Permutominoes were introduced by F. Incitti in [20] while studying the problem of determining the  $\tilde{R}$ -polynomials (related with the Kazhdan-Lusztig  $R$ -polynomials) associated with a pair  $(\pi_1, \pi_2)$  of permutations. Concerning the class of polyominoes without holes, our definition (though different) turns out to be equivalent to Incitti's one, which is more general but uses some algebraic notions not necessary in this paper.

Let us recall the main enumerative results concerning convex permutominoes. In [16], using bijective techniques, it was proved that the number of *parallelogram permutominoes* of size  $n + 1$  is equal to  $c_n$  and that the number of *directed convex permutominoes* of size  $n + 1$  is equal to  $\frac{1}{2} b_n$ , where, throughout all the paper,  $c_n$  and  $b_n$  will denote, respectively, the Catalan numbers and the central binomial coefficients. Finally, in [15] it was proved, using the ECO method, that the number of *convex permutominoes* of size  $n + 1$  is:

$$C_{n+1} = 2(n+3)4^{n-2} - \frac{n}{2} \binom{2n}{n} \quad n \geq 1. \quad (4)$$

The first terms of the sequence are 1, 4, 18, 84, 394, 1836, 8468, ... (sequence A126020) in [22]). The generating function for convex permutominoes is

$$C(x) = \frac{2x^2(1-3x)}{(1-4x)^2} - \frac{x^2}{(1-4x)^{3/2}}. \tag{5}$$

The same formula has been obtained independently by Boldi et al. in [3]. The main results concerning the enumeration of classes of convex permutominoes are listed in the table below, where the first terms of the sequences are given starting from  $n = 2$  (i.e. the one cell permutomino defined by  $\pi_1(1) = (1, 2)$ ,  $\pi_2 = (2, 1)$ ), and are taken from [15, 16]:

<i>Class</i>	<i>First terms</i>	<i>Closed form/rec. relation</i>
convex	1, 4, 18, 84, 394, ...	$C_{n+1} = 2(n+3)4^{n-2} - \frac{n}{2} \binom{2n}{n}$
directed convex	1, 3, 10, 35, 126, ...	$D_{n+1} = \frac{1}{2} b_n$
parallelogram	1, 2, 5, 14, 42, 132, ...	$P_{n+1} = c_n$
symmetric (w.r.t. $x = y$ )	1, 2, 4, 10, 22, 54, ...	$Sym_{n+1} = (n+3)2^{n-2} - n \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} - (n-1) \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}$
centered	1, 4, 16, 64, 256, ...	$Q_n = 4^{n-2}$
bi-centered	1, 4, 14, 48, 164, ...	$B_n = 4B_{n-1} - 2B_{n-2}, \quad n \geq 3$
stacks	1, 2, 4, 8, 16, 32, ...	$St_n = 2^{n-2}$

**Notation.** Before ending this section, just a few words of explanation concerning some notations used in the work. If  $\mathcal{L}$  is a generic class of permutominoes, then the set of permutominoes of  $\mathcal{L}$  of dimension  $n$  will be denoted by  $\mathcal{L}_n$ , its cardinality by  $L_n$ , and its generating function  $L(x) = \sum_{n \geq 0} L_n x^n$ . Moreover, the

set of permutations associated with  $\mathcal{L}_n$  will be denoted by  $\tilde{\mathcal{L}}_n = \{ \pi_1(P) : P \in \mathcal{L}_n \}$ , its cardinality by  $\tilde{L}_n$ , and its generating function by  $\tilde{L}(x) = \sum_{n \geq 0} \tilde{L}_n x^n$ .

### 3. Permutations Associated with Convex Permutominoes

Given a permutomino  $P$ , its two defining permutations are denoted by  $\pi_1$  and  $\pi_2$  (see Figure 4). Clearly, any permutomino of size  $n$  uniquely determines two permutations  $\pi_1$  and  $\pi_2$  of  $\mathcal{S}_n$ , with

- i)  $\pi_1(i) \neq \pi_2(i), 1 \leq i \leq n,$
- ii)  $\pi_1(1) < \pi_2(1),$  and  $\pi_1(n) > \pi_2(n),$

but conversely, not all the pairs of permutations  $(\pi_1, \pi_2)$  of  $n$  satisfying i) and ii) define a permutomino (see, for instance, the two cases given in Figure 5).

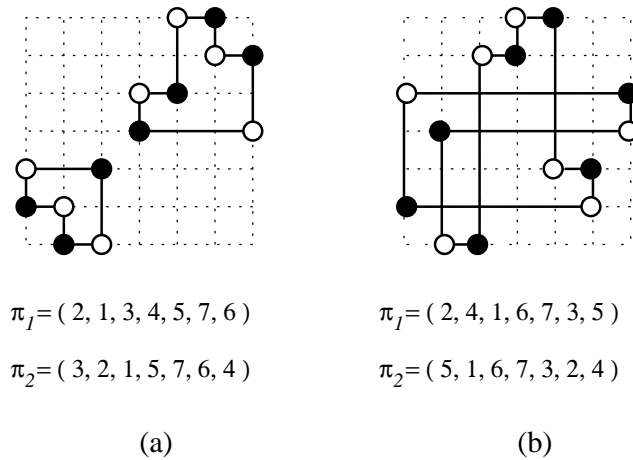


Figure 5: Two permutations  $\pi_1$  and  $\pi_2$  of  $\mathcal{S}_n$ , satisfying i) and ii), do not necessarily define a permutomino, since two problems may occur: (a) two disconnected sets of cells; (b) the boundary crosses itself.

In [16] the authors give a simple constructive proof that every permutation of  $\mathcal{S}_n$  is “associated” with at least one column-convex permutomino.

**Proposition 3.1.** *If  $\pi \in \mathcal{S}_n, n \geq 2,$  then there is at least one column-convex permutomino  $P$  such that  $\pi = \pi_1(P)$  or  $\pi = \pi_2(P)$ .*

For instance, Figure 6 (a) depicts a column convex permutomino defined by the permutation  $\pi_1$  in Figure 5 (b).

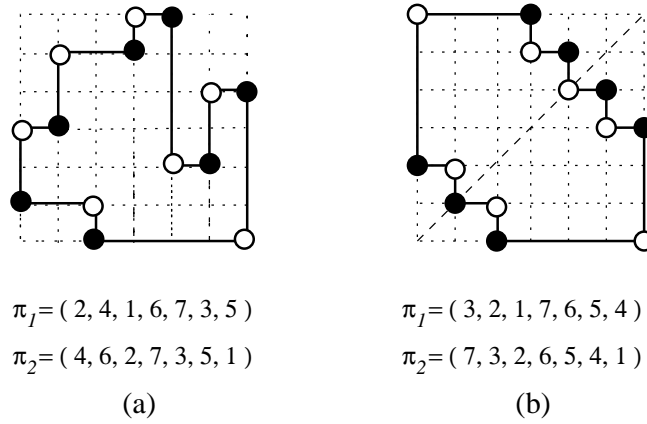


Figure 6: (a) a column convex permutomino defined by the permutation  $\pi_1$  in Figure 5 (b); (b) the symmetric permutomino defined by the involution  $\pi_1 = (3, 2, 1, 7, 6, 5, 4)$ .

If we consider convex permutominoes the statement of Proposition 3.1 does not hold. So in this paper we consider the class  $\mathcal{C}_n$  of convex permutominoes of size  $n$ , and study the problem of giving a characterization for the set of permutations defining convex permutominoes,

$$\{ (\pi_1(P), \pi_2(P)) : P \in \mathcal{C}_n \}.$$

Moreover, let us consider the following subsets of  $S_n$ :

$$\tilde{\mathcal{C}}_n = \{ \pi_1(P) : P \in \mathcal{C}_n \}, \quad \tilde{\mathcal{C}}'_n = \{ \pi_2(P) : P \in \mathcal{C}_n \}.$$

It is easy to prove the following properties:

1.  $|\tilde{\mathcal{C}}_n| = |\tilde{\mathcal{C}}'_n|$ ,
2.  $\pi \in \tilde{\mathcal{C}}_n$  if and only if  $\pi^R \in \tilde{\mathcal{C}}'_n$ , where  $\pi^R$  denotes the *reflection* of  $\pi$ , i.e.  $\pi^R = (\pi_n, \dots, \pi_1)$ , where  $\pi^R = (\pi_n, \dots, \pi_1)$ .

Given a permutation  $\pi \in S_n$ , we say that  $\pi$  is  $\pi_1$ -*associated* (briefly *associated*) with a permutomino  $P$ , if  $\pi = \pi_1(P)$ . With no loss of generality, we will study the combinatorial properties of the permutations of  $\tilde{\mathcal{C}}_n$ , and we will give a simple way to recognize if a permutation  $\pi$  is in  $\tilde{\mathcal{C}}_n$  or not. Moreover we will study the cardinality of this set. In particular we will exploit the relations between the cardinalities of the sets  $\mathcal{C}_n$  and  $\tilde{\mathcal{C}}_n$ .

Let us start with some examples. For small values of  $n$  we have that:

$$\begin{aligned} \tilde{\mathcal{C}}_2 &= \{12\}, \\ \tilde{\mathcal{C}}_3 &= \{123, 132, 213\}, \\ \tilde{\mathcal{C}}_4 &= \{1234, 1243, 1324, 1342, 1423, 1432, 2143, \\ &\quad 2314, 2134, 2413, 3124, 3142, 3214\}. \end{aligned}$$

For any  $\pi \in \tilde{\mathcal{C}}_n$ , let us consider also

$$[\pi] = \{P \in \mathcal{C}_n : \pi_1(P) = \pi\},$$

i.e. the set of convex permutominoes associated with  $\pi$ . For instance, there are 4 convex permutominoes associated with  $\pi = (2, 1, 3, 4, 5)$ , as depicted in Figure 7. We will give a simple way of computing  $[\pi]$ , for any given  $\pi \in \tilde{\mathcal{C}}_n$ . Moreover, since

$$\mathcal{C}_n = \bigcup_{\pi \in \tilde{\mathcal{C}}_n} [\pi]$$

we will prove that the cardinality of  $\tilde{\mathcal{C}}_{n+1}$  is

$$\tilde{\mathcal{C}}_{n+1} = 2(n+2)4^{n-2} - \frac{n}{4} \binom{3-4n}{1-2n} \binom{2n}{n}, \quad n \geq 1, \tag{6}$$

defining the sequence 1, 3, 13, 62, 301, 1450, . . . , not in [22].

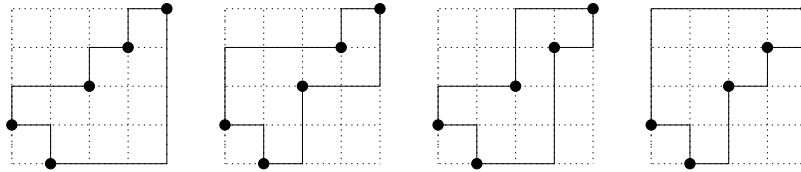


Figure 7: The four convex permutominoes associated with  $(2, 1, 3, 4, 5)$ .

### 3.1 A Matrix Representation of Convex Permutominoes

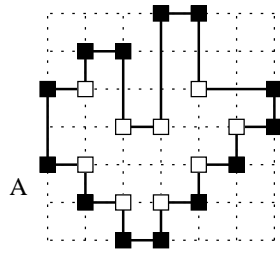
Before going on with the study of convex permutominoes, we would like to point out a simple property of their boundary, related to *reentrant* and *salient points*. So let us briefly recall the definition of these objects.

Let  $P$  be a polyomino; starting from the leftmost point having minimal ordinate, and moving in a clockwise sense, the boundary of  $P$  can be encoded as a word in a four letter alphabet,  $\{N, E, S, W\}$ , where  $N$  (resp.,  $E, S, W$ ) represents a *north* (resp., *east, south, west*) unit step. Any occurrence of a sequence  $NE, ES, SW$ , or  $WN$  in the word encoding  $P$  defines a *salient point* of  $P$ , while



any occurrence of a sequence  $EN$ ,  $SE$ ,  $WS$ , or  $NW$  defines a *reentrant point* of  $P$  (see for instance, Figure 8).

In [10] and successively in [7], in a more general context, it was proved that in any polyomino the difference between the number of salient and reentrant points is equal to 4.



$NNENESSENNNESSEESWSWSWSWNWNW$

Figure 8: The coding of the boundary of a polyomino, starting from  $A$  and moving in a clockwise sense; its salient (resp. reentrant) points are indicated by black (resp. white) squares.

In a convex permutomino of size  $n \geq 2$  the length of the word coding the boundary is  $4(n - 1)$ , and we have  $n + 2$  salient points and  $n - 2$  reentrant points; moreover we observe that a reentrant point cannot lie on the minimal bounding rectangle. This leads to the following remarkable property:

**Proposition 3.2.** *The set of reentrant points of a convex permutomino of size  $n \geq 2$  defines a permutation matrix of dimension  $n - 2$ .*

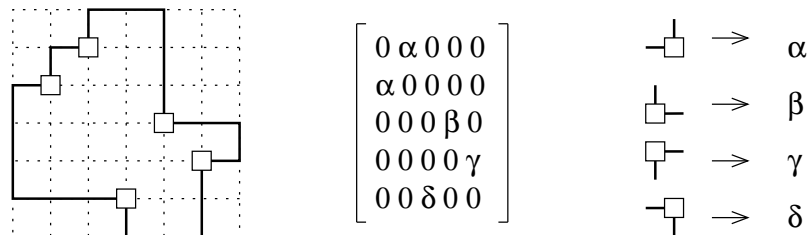


Figure 9: The reentrant points of a convex permutomino uniquely define a permutation matrix in the symbols  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ .

For simplicity of notation, we agree to group the reentrant points of a convex permutomino in four classes; in practice we choose to represent the reentrant

point determined by a sequence  $EN$  (resp.  $SE, WS, NW$ ) with the symbol  $\alpha$  (resp.  $\beta, \gamma, \delta$ ).

Using this notation we can state the following simple characterization for convex permutominoes:

**Proposition 3.3.** *A convex permutomino of size  $n \geq 2$  is uniquely represented by the permutation matrix defined by its reentrant points, which has dimension  $n - 2$ , and uses the symbols  $\alpha, \beta, \gamma, \delta$ , and such that for all points  $A, B, C, D$ , of type  $\alpha, \beta, \gamma$  and  $\delta$ , respectively, we have:*

1.  $x_A < x_B, x_D < x_C, y_A > y_D, y_B > y_C$ ;
2.  $\neg(x_A > x_C \wedge y_A < y_C)$  and  $\neg(x_B < x_D \wedge y_B < y_D)$ ,
3. *the ordinates of the  $\alpha$  and of  $\gamma$  points are strictly increasing, from left to right; the ordinates of the  $\beta$  and of  $\delta$  points are strictly decreasing, from left to right.*

where  $x$  and  $y$  denote the abscissa and the ordinate of the considered point.

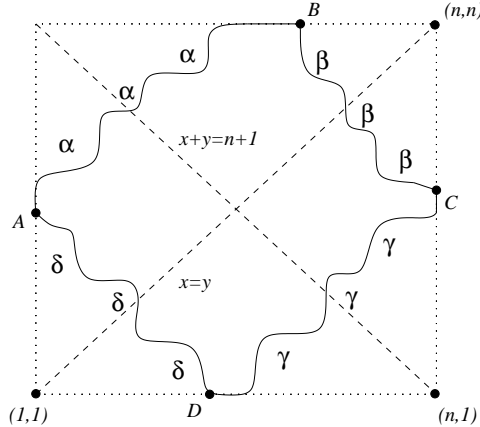


Figure 10: A sketched representation of the  $\alpha, \beta, \gamma$  and  $\delta$  paths in a convex permutomino.

Just to give a more informal explanation, let us consider the following special points on a convex permutomino of size  $n$ :

$$A = (1, \pi_1(1)), \quad B = (\pi_1^{-1}(n), n), \quad C = (n, \pi_1(n)), \quad D = (\pi_1^{-1}(1), 1).$$

The path that goes from  $A$  to  $B$  (resp. from  $B$  to  $C$ , from  $C$  to  $D$ , and from  $D$  to  $A$ ) in a clockwise sense is made only of  $\alpha$  (resp.  $\beta, \gamma, \delta$ ) points, thus it

is called the  $\alpha$ -path (resp.  $\beta$ -path,  $\gamma$ -path,  $\delta$ -path) of the permutomino. The situation is sketched in Figure 10.

The characterization given in Proposition 3.3 implies the following two properties:

- (z1) the  $\alpha$  points are never below the diagonal  $x = y$ , and the  $\gamma$  points are never above the diagonal  $x = y$ .
- (z2) the  $\beta$  points are never below the diagonal  $x + y = n + 1$ , and the  $\delta$  points are never above the diagonal  $x + y = n + 1$ .

### 3.2 Characterization and Combinatorial Properties of $\tilde{\mathcal{C}}_n$

Let us consider the problem of establishing if, given permutation  $\pi \in \mathcal{S}_n$ , there is at least one convex permutomino  $P$  of size  $n$  such that  $\pi_1(P) = \pi$ .

So, let  $\pi$  be a permutation of  $\mathcal{S}_n$ , we define  $\mu$  as the maximal upper unimodal sublist of  $\pi$  ( $\mu$  retains the indexing of  $\pi$ ).

Specifically, if  $\mu$  is denoted by  $(\mu(i_1), \dots, n, \dots, \mu(i_m))$ , then we have the following:

1.  $\mu(i_1) = \mu(1) = \pi(1)$ ;
2. if  $n \notin \{\mu(i_1), \dots, \mu(i_k)\}$ , then  $\mu(i_{k+1}) = \pi(i_{k+1})$  such that
  - i.  $i_k < i < i_{k+1}$  implies  $\pi(i) < \mu(i_k)$ , and
  - ii.  $\pi(i_{k+1}) > \mu(i_k)$ ;
3. if  $n \in \{\mu(i_1), \dots, \mu(i_k)\}$ , then  $\mu(i_{k+1}) = \pi(i_{k+1})$  such that
  - i.  $i_k < i < i_{k+1}$  implies  $\pi(i) < \pi(i_{k+1})$ , and
  - ii.  $\pi(i_{k+1}) < \mu(i_k)$ .

Summarizing we have:

$$\mu(i_1) = \mu(1) = \pi(1) < \mu(i_2) < \dots < n > \dots \mu(i_m) = \mu(n) = \pi(n).$$

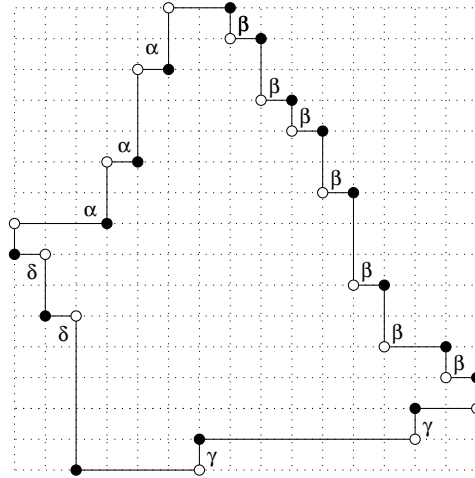
Moreover, let  $\sigma$  denote  $(\sigma(j_1), \dots, \sigma(j_r))$  where:

1.  $\sigma(j_1) = \sigma(1) = \pi(1)$ ,  $\sigma(j_r) = \sigma(n) = \pi(n)$ , and
2. if  $1 < j_k < j_r$ , then  $\sigma(j_k) = \pi(j_k)$  if and only if  $\pi(j_k) \notin \{\mu(i_1), \dots, \mu(i_m)\}$ .

*Example 3.4.* Consider the convex permutomino of size 16 represented in Figure 11. We have

$$\pi_1 = (8, 6, 1, 9, 11, 14, 2, 16, 15, 13, 12, 10, 7, 3, 5, 4),$$

and we can determine the decomposition of  $\pi$  into the two subsequences  $\mu$  and  $\sigma$ :



$$\pi_1 = (8, 6, 1, 9, 11, 14, 2, 16, 15, 13, 12, 10, 7, 3, 5, 4)$$

$$\pi_2 = (9, 8, 6, 11, 14, 16, 1, 15, 13, 12, 10, 7, 5, 2, 4, 3)$$

Figure 11: A convex permutomino  $P$  and the permutations  $(\pi_1(P), \pi_2(P))$ .

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mu$	8	-	-	9	11	14	-	16	15	13	12	10	7	-	5	4
$\sigma$	8	6	1	-	-	-	2	-	-	-	-	-	-	3	-	4

For brevity sake, when there is no possibility of misunderstanding, we use to represent the two sequences omitting the empty spaces, as

$$\mu = (8, 9, 11, 14, 16, 15, 13, 12, 10, 7, 5, 4), \quad \sigma = (8, 6, 1, 2, 3, 4).$$

While  $\mu$  is upper unimodal by construction, here  $\sigma$  turns out to be lower unimodal. In fact from the characterization given in Proposition 3.3 we have that

**Proposition 3.5.** *If  $\pi$  is associated with a convex permutomino then its subsequence  $\sigma$  is lower unimodal.*

The conclusion of Proposition 3.5 is a necessary condition for a permutation  $\pi$  to be associated with a convex permutomino, but it is not sufficient. For instance, if we consider the permutation  $\pi = (5, 9, 8, 7, 6, 3, 1, 2, 4)$ , then  $\mu = (5, 9, 8, 7, 6, 4)$ , and  $\sigma = (5, 3, 2, 1, 4)$  is lower unimodal, but as shown in Figure 12 (a) there is no convex permutomino associated with  $\pi$ . In fact any convex

permutomino associated with such a permutation has a  $\beta$  point below the diagonal  $x + y = 10$  and, correspondingly, a  $\delta$  point above this diagonal. Thus the  $\beta$  and the  $\delta$  paths cross themselves. In order to give a necessary and sufficient

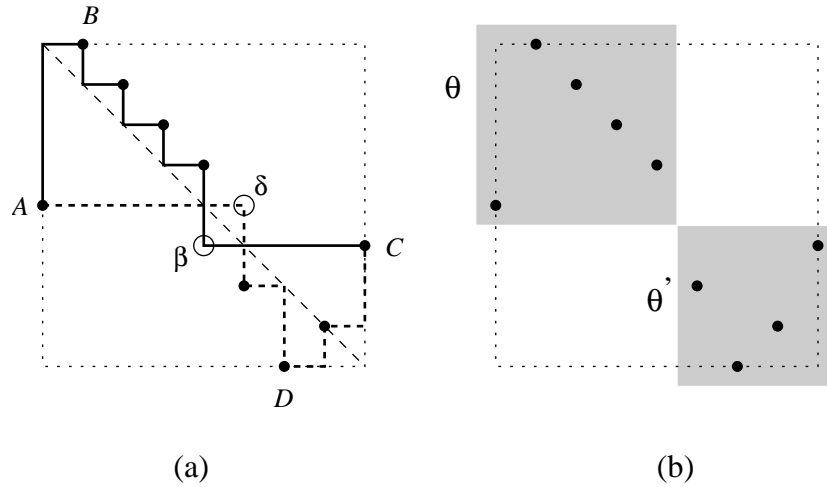


Figure 12: (a) there is no convex permutomino associated with  $\pi = (5, 9, 8, 7, 6, 3, 1, 2, 4)$ , since  $\sigma$  is lower unimodal but the  $\beta$  path passes below the diagonal  $x + y = 10$ . The  $\beta$  point below the diagonal and the corresponding  $\delta$  point above the diagonal are encircled. (b) The permutation  $\pi = (5, 9, 8, 7, 6, 3, 1, 2, 4)$  is the direct difference  $\pi = (1, 5, 4, 3, 2) \ominus (3, 1, 2, 4)$ .

condition for a permutation  $\pi$  to be in  $\tilde{\mathcal{C}}_n$ , let us recall the following definition. Given two permutations  $\theta = (\theta_1, \dots, \theta_m) \in \mathcal{S}_m$  and  $\theta' = (\theta'_1, \dots, \theta'_{m'}) \in \mathcal{S}_{m'}$ , their *direct difference*  $\theta \ominus \theta'$  is a permutation of  $\mathcal{S}_{m+m'}$  defined as

$$(\theta_1 + m', \dots, \theta_m + m', \theta'_1, \dots, \theta'_{m'}).$$

For instance, if  $\theta = (1, 5, 4, 3, 2)$ ,  $\theta' = (3, 2, 1, 4)$ , then  $\theta \ominus \theta' = (5, 9, 8, 7, 6, 3, 1, 2, 4)$  (a pictorial description is given in Figure 12 (b)).

Finally the following characterization holds.

**Theorem 3.6.** *Let  $\pi$  be a permutation of length  $n \geq 1$ . Then  $\pi \in \tilde{\mathcal{C}}_n$  if and only if:*

1.  $\sigma$  is lower unimodal, and
2. there are no two permutations,  $\theta \in \mathcal{S}_m$ , and  $\theta' \in \mathcal{S}_{m'}$ , such that  $m + m' = n$ , and  $\pi = \theta \ominus \theta'$ .

*Proof.* Before starting, we need to observe that in a convex permutomino all the  $\alpha$  and  $\gamma$  points belong to the permutation  $\pi_1$ , thus by **(z1)** they can also lie on the diagonal  $x = y$ ; on the contrary, the  $\beta$  and  $\delta$  points belong to  $\pi_2$ , then by **(z2)** all the  $\beta$  (resp.  $\delta$ ) points must remain strictly above (resp. below) the diagonal  $x + y = n + 1$ .

( $\implies$ ) By Proposition 3.5 we have that  $\sigma$  is lower unimodal. So we have to prove that  $\pi$  may not be decomposed into the direct difference of two permutations,  $\pi = \theta \ominus \theta'$ .

If  $\pi(1) < \pi(n)$  the property is straightforward. So let us consider the case  $\pi(1) > \pi(n)$ , and assume that  $\pi = \theta \ominus \theta'$  for some permutations  $\theta$  and  $\theta'$ . We will prove that if the vertices of polygon  $P$  define the permutation  $\pi$ , then the boundary of  $P$  crosses itself, hence  $P$  is not a permutomino.

So let us assume that  $P$  is a convex permutomino associated with  $\pi = \theta \ominus \theta'$ . We start by observing that the  $\beta$  and the  $\delta$  paths of  $P$  may not be empty. In fact, if the  $\beta$  path is empty, then  $\pi(n) = n > \pi(1)$ , against the hypothesis. Similarly, if the  $\delta$  path is empty, then  $\pi(1) = 1 < \pi(n)$ . Essentially for the same reason, both  $\theta$  and  $\theta'$  must have more than one element.

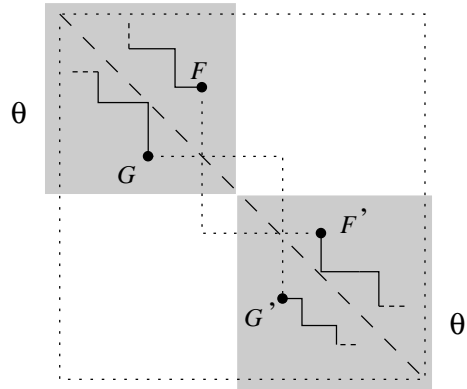


Figure 13: If  $\pi = \theta \ominus \theta'$  then the path encoding the boundary of  $P$  crosses itself.

As we observed, the points of  $\theta$  (resp.  $\theta'$ ) in the  $\beta$  path of  $P$ , are placed strictly above the diagonal  $x + y = n + 1$ . Let  $F$  (resp.  $F'$ ) be the rightmost (resp. leftmost) of these points. Similarly, there must be at least one point of  $\theta$  (resp.  $\theta'$ ) in the  $\delta$  path of  $P$ , placed strictly below the diagonal  $x + y = n + 1$ . Let  $G$  (resp.  $G'$ ) be the rightmost (resp. leftmost) of these points. The situation is schematically sketched in Figure 13.

Since  $F$  and  $F'$  are consecutive points in the  $\beta$  path of  $P$ , they must be connected by means of a path that goes down and then right, and, similarly, since  $G'$  and  $G$  are two consecutive points in the  $\delta$  path, they must be connected by means of a path that goes up and then left. These two paths necessarily cross in at least two points, and their intersections must be on the diagonal  $x + y = n + 1$ .

( $\Leftarrow$ ) Clearly condition 2. implies that  $\pi(1) < n$  and  $\pi(n) > 1$ , which are necessary conditions for  $\pi \in \tilde{\mathcal{C}}_n$ . We start building up a polygon  $P$  such that  $\pi_1(P) = P$ , and then prove that  $P$  is a permutomino. As usual, let us consider the points

$$A = (1, \pi(1)), \quad B = (\pi^{-1}(n), n), \quad C = (n, \pi(n)), \quad D = (\pi^{-1}(1), 1).$$

The  $\alpha$  path of  $P$  goes from  $A$  to  $B$ , and it is constructed connecting the points of  $\mu$  increasing sequence; more formally, if  $\mu(i_l)$  and  $\mu(i_{l+1})$  are two consecutive points of  $\mu$ , with  $\mu(i_l) < \mu(i_{l+1}) \leq n$ , we connect them by means of a path

$$1^{\mu(i_{l+1})-\mu(i_l)} 0^{i_{l+1}-i_l},$$

(where 1 denotes the vertical, and 0 the horizontal unit step). Similarly we construct the  $\beta$  path, from  $B$  to  $C$ , connecting the points of  $\mu$  decreasing sequence, the  $\delta$  path from  $A$  to  $D$ , connecting the points of  $\sigma$  decreasing sequence, and then the  $\gamma$  path from  $D$  to  $C$ , connecting the points of  $\sigma$  increasing sequence. Since the subsequence  $\sigma$  is lower unimodal the obtained polygon is convex (see Figure 14).

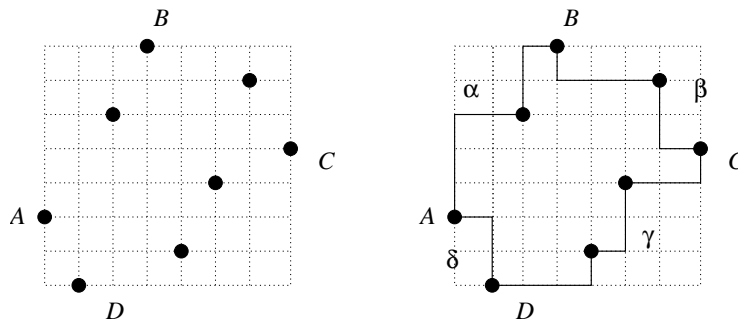


Figure 14: Given the permutation  $\pi = (3, 1, 6, 8, 2, 4, 7, 5)$  satisfying conditions 1. and 2., we construct the  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  paths.

Now we must prove that the four paths we have defined may not cross themselves. First we show that the  $\alpha$  path and the  $\gamma$  path may not cross. In fact, if this happened, there would be a point  $(r, \pi(r))$  in the path  $\gamma$ , and two points  $(i, \pi(i))$  and  $(j, \pi(j))$  in the path  $\alpha$ , such that  $i < r < j$ , and  $\pi(i) < \pi(r) > \pi(j)$  (see Figure 15 (a)). In this case, according to the definition,  $\pi(r)$  should belong to  $\mu$ , and then  $(r, \pi(r))$  should be in the path  $\alpha$ , and not in  $\gamma$ .

Finally we prove that the paths  $\beta$  and  $\delta$  may not cross. In fact, if they crossed, their intersection should necessarily be on the diagonal  $x + y = n + 1$ ; if  $(r, s)$  is the intersection point having minimum abscissa, then the reader can easily check, by considering the various possibilities, that the points  $(i, \pi(i))$  of  $\pi$  satisfy:

$$i \leq r \quad \text{if and only if} \quad \pi(i) \geq s$$

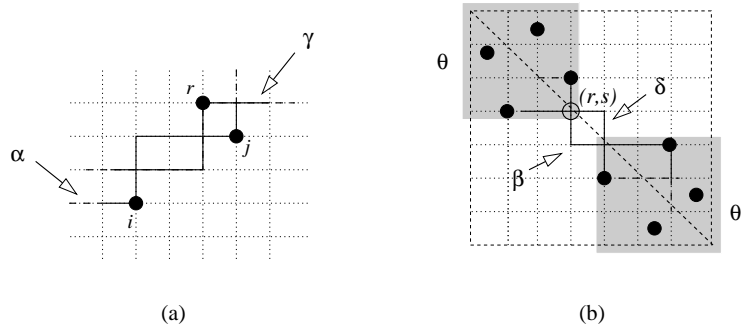


Figure 15: (a) The  $\alpha$  path and the  $\gamma$  path may not cross; (b) The  $\beta$  path and the  $\delta$  path may not cross.

(see Figure 15 (b)). Therefore, setting

$$\theta = \{(i, \pi(i) - s + 1) : i \leq r\}$$

we have that  $\theta$  is a permutation of  $\mathcal{S}_r$ , and letting

$$\theta' = \{(i, \pi(i)) : i > r\}$$

we see that  $\pi = \theta \ominus \theta'$ , against the hypothesis. ■

There is an interesting refinement of the previous general theorem, which applies to a particular subset of the permutations of  $\mathcal{S}_n$ .

**Corollary 3.7.** Let  $\pi \in \mathcal{S}_n$ , such that  $\pi(1) < \pi(n)$ . Then  $\pi \in \tilde{\mathcal{C}}_n$  if and only if  $\sigma$  is lower unimodal.

### 3.3 The Relation between the Number of Permutations and the Number Convex Permutominoes

The characterization given in Theorem 3.6 suggests a possible way to obtain the enumeration of  $\tilde{\mathcal{C}}_n$ . However in this paper we will obtain it by means of a recursive decomposition of this class, which relies on some interesting combinatorial properties of its elements.

Let  $\pi \in \tilde{\mathcal{C}}_n$ , and  $\mu$  and  $\sigma$  defined as above. Let  $\mathcal{F}(\pi)$  (briefly  $\mathcal{F}$ ) denote the set of fixed points of  $\pi$  lying in the increasing subsequence of the sequence  $\mu$  and which are different from 1 and  $n$ . We call the points in  $\mathcal{F}$  the *free fixed points* of  $\pi$ .

For instance, concerning the permutation  $\pi = (2, 1, 3, 4, 7, 6, 5)$  we have  $\mu = (2, 3, 4, 7, 6, 5)$ ,  $\sigma = (2, 1, 5)$ , and  $\mathcal{F}(\pi) = \{3, 4\}$ ; here 6 is a fixed point of  $\pi$  but



it is not on the increasing sequence of  $\mu$ , then it is not free. By definition, the number of free fixed points of a permutation of  $\tilde{\mathcal{C}}_n$  is a number between 0 (for instance, see the permutation associated with the permutomino in Figure 11), and  $n - 2$  (as the identity  $(1, \dots, n)$ ).

**Theorem 3.8.** *Let  $\pi \in \tilde{\mathcal{C}}_n$ , and let  $\mathcal{F}(\pi)$  be the set of free fixed points of  $\pi$ . Then we have:*

$$|[\pi]| = 2^{|\mathcal{F}(\pi)|}.$$

*Proof.* Since  $\pi \in \tilde{\mathcal{C}}_n$  there exists a permutomino  $P$  associated with  $\pi$ . If we look at the permutation matrix defined by the reentrant points of  $P$ , we see that all the free fixed points of  $\pi$  can only be of type  $\alpha$  or  $\gamma$ , while the type of all the other points of  $\pi$  is established. It is easy to check that in any way we set the typology of these fixed points in  $\alpha$  or  $\gamma$  we reach, starting from the matrix of  $P$ , a permutation matrix which defines a permutomino associated with  $\pi$ , and in this way we obtain all the permutominoes associated with the permutation  $\pi$ . ■

Applying Theorem 3.8 we have that the number of convex permutominoes associated with  $\pi = (2, 1, 3, 4, 7, 6, 5)$  is  $2^2 = 4$ , as shown in Figure 16. Moreover, Theorem leads to an interesting property.

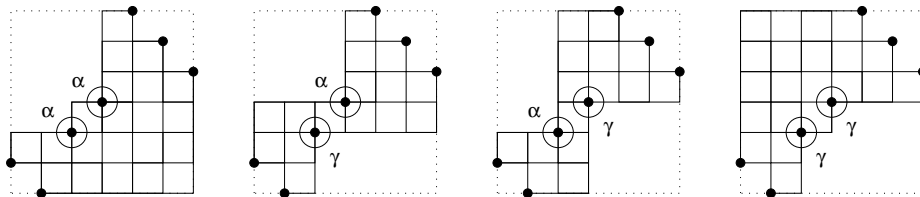


Figure 16: The four convex permutominoes associated with the permutation  $\pi = (2, 1, 3, 4, 7, 6, 5)$ . The two free fixed points are encircled.

**Proposition 3.9.** *Let  $\pi \in \tilde{\mathcal{C}}_n$ , with  $\pi(1) > \pi(n)$ . Then there is only one convex permutomino associated with  $\pi$ , i.e.  $|\pi| = 1$ .*

*Proof.* If  $\pi(1) > \pi(n)$  then all the points in the increasing part of  $\mu$  are strictly above the diagonal  $x = y$ , then  $\pi$  cannot have free fixed points. The thesis is then straightforward. ■

Let us now introduce the sets  $\tilde{\mathcal{C}}_{n,k}$  of permutations having exactly  $k$  free fixed points, with  $0 \leq k \leq n - 2$ . The relations between the cardinalities  $\tilde{\mathcal{C}}_{n,k}$  of  $\tilde{\mathcal{C}}_{n,k}$ ,

$\tilde{C}_n$  of  $\tilde{\mathcal{C}}_n$ , and  $C_n$  of  $\mathcal{C}_n$  are clear:

$$\tilde{C}_n = \sum_{k=0}^{n-2} \tilde{C}_{n,k} \qquad C_n = \sum_{k=0}^{n-2} 2^k \tilde{C}_{n,k}. \tag{7}$$

### 3.4 Some Auxiliary Classes of Convex Permutominoes

Our aim is to determine a decomposition of the permutations of  $\tilde{\mathcal{C}}_{n,k}$  in terms of permutations of smaller dimensions. In order to present this decomposition, let us introduce some combinatorial classes which will help us in our investigation:

1.  $\mathcal{P}_n$  is the set of parallelogram permutominoes;
2.  $\mathcal{D}_n$  is the set of directed convex permutominoes;
3.  $\tilde{\mathcal{P}}_n = \{\pi_1(P) : P \in \mathcal{P}_n\}$  is the set of permutations associated with parallelogram permutominoes;
4.  $\tilde{\mathcal{D}}_n = \{\pi_1(P) : P \in \mathcal{D}_n\}$  is the set of permutations associated with directed convex permutominoes.

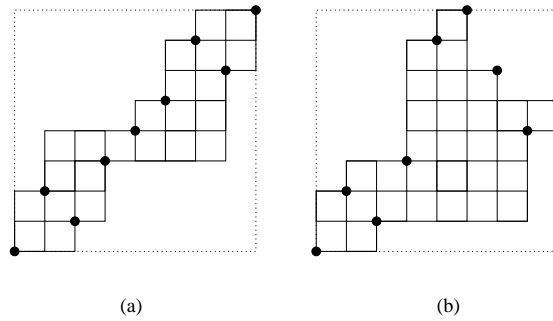


Figure 17: (a) A parallelogram permutomino associated with  $\pi = (1, 3, 2, 4, 5, 6, 8, 7, 9)$ ; (b) a directed convex permutomino associated with  $\pi = (1, 3, 2, 4, 8, 9, 7, 5, 6)$ .

#### 3.4.1 Parallelogram Permutominoes and Permutations

A parallelogram permutomino is simply a convex permutomino where the  $\beta$  and the  $\delta$  paths are empty, thus a permutation  $\pi$  associated with a parallelogram permutomino has a simple characterization:

**Proposition 3.10.** *A permutation  $\pi$  belongs to  $\tilde{\mathcal{P}}_n$  if and only if  $\pi(1) = 1$ ,  $\pi(n) = n$ , and  $\mu$  and  $\sigma$  are both increasing.*

For instance, referring to the permutomino in Figure 17 (a) we have that  $\mu = (1, 3, 4, 5, 6, 8, 9)$ , and  $\sigma = (1, 2, 7, 9)$ . The first cases are:

$$\begin{aligned} \tilde{\mathcal{P}}_2 &= \{12\}; \\ \tilde{\mathcal{P}}_3 &= \{123\}; \\ \tilde{\mathcal{P}}_4 &= \{1234, 1324\}; \\ \tilde{\mathcal{P}}_5 &= \{12345, 12435, 13245, 13425, 13245\}; \end{aligned}$$

The following remarkable property is a direct consequence of Proposition 3.10.

**Proposition 3.11.** *If  $P$  is a parallelogram permutomino, then all the convex permutominoes associated with  $\pi_1(P)$  are parallelogram ones.*

We start by proving that the permutations associated with parallelogram permutominoes are a well-known class of permutations, and they are enumerated by the Catalan numbers. To do this we need recall some definitions.

Let  $n, m$  be two positive integers with  $m \leq n$ , and let  $\pi = (\pi(1), \dots, \pi(n)) \in \mathcal{S}_n$  and  $\nu = (\nu(1), \dots, \nu(m)) \in \mathcal{S}_m$ . We say that  $\pi$  contains the pattern  $\nu$  if there exist indices  $i_1 < i_2 < \dots < i_m$  such that  $(\pi(i_1), \pi(i_2), \dots, \pi(i_m))$  is in the same relative order as  $(\nu(1), \dots, \nu(m))$ . If  $\pi$  does not contain  $\nu$  we say that  $\pi$  is  $\nu$ -avoiding [23]. For instance, if  $\nu = (1, 2, 3)$  then  $\pi = (5, 2, 4, 3, 1, 6)$  contains  $\nu$ , while  $\pi = (6, 3, 2, 5, 4, 1)$  is  $\nu$ -avoiding.

Let us denote by  $\mathcal{S}_n(\nu)$  the set of  $\nu$ -avoiding permutations in  $\mathcal{S}_n$ . It is known that for each pattern  $\nu \in \mathcal{S}_3$  we have that  $|\mathcal{S}_n(\nu)| = c_n$ , the  $n$ th Catalan number [23].

**Theorem 3.12.** *A permutation  $\pi$  belongs to  $\tilde{\mathcal{P}}_n$  if and only if  $\pi(1) = 1$ ,  $\pi(n) = n$ , and  $\pi$  avoids the pattern  $\nu = (3, 2, 1)$ .*

*Proof.* ( $\implies$ ) If  $\pi$  belongs to  $\tilde{\mathcal{P}}_n$  we have, from Proposition 3.10, that  $\pi(1) = 1$ ,  $\pi(n) = n$  and  $\mu$  and  $\sigma$  are both increasing. In particular  $\pi$  trivially avoids the pattern  $(3, 2, 1)$ .

( $\impliedby$ ) Since  $\pi(n) = n$  then, by definition,  $\mu$  is increasing. Now suppose that  $\sigma$  is not increasing, then there exists an index  $i$  and an index  $j > i$ , such that  $\pi(i) > \pi(j)$  with  $\pi(i)$  and  $\pi(j)$  in  $\sigma$ . Since  $\pi(1) = 1$  we have that  $i > 1$  and so, remember  $\pi(i)$  is not in  $\mu$ , there exists an index  $k < i$  with  $\pi(k) > \pi(i)$ . If we take  $(\pi(k), \pi(i), \pi(j))$  we have the pattern  $(3, 2, 1)$ . So also  $\sigma$  must be increasing. From Proposition 3.10 we have the thesis. ■

As a neat consequence of this property we have that, for any  $n \geq 2$ ,

$$|\tilde{\mathcal{P}}_n| = c_{n-2}. \tag{8}$$

Let us denote, as usual, by  $\tilde{\mathcal{P}}_{n,k}$  the set of permutations of  $\tilde{\mathcal{P}}_n$  having exactly  $k$  free fixed points. We observe that in a permutation of  $\tilde{\mathcal{P}}_n$  the free fixed points are all its fixed points except 1 and  $n$ . Clearly,

$$|\tilde{\mathcal{P}}_n| = \sum_{k=0}^{n-2} |\tilde{\mathcal{P}}_{n,k}|, \tag{9}$$

while, from Theorem 3.8 and Proposition 3.11 we have that

$$|\mathcal{P}_n| = \sum_{k=0}^{n-2} 2^k |\tilde{\mathcal{P}}_{n,k}|. \tag{10}$$

The first terms of the sequence are reported in the table below.

$n/k$	0	1	2	3	4	5	...
2	1						
3	0	1					
4	1	0	1				
5	2	2	0	1			
6	6	4	3	0	1		
7	18	13	6	4	0	1	
8	57	40	21	8	5	0	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Obviously, the row sums give the Catalan numbers. In [13] the authors studied the distribution of fixed points in the permutations of  $\mathcal{S}_n(321)$  and established a bijection between the permutations of  $\mathcal{S}_n(321)$  having  $k$  fixed points, and the Dyck paths of length  $2n$  having  $k$  hills. In this way they proved that the number of permutations of  $\mathcal{S}_n(321)$  having no fixed points is given by the *Fine numbers*  $f_n$  (sequence A000957 in [22], see [14])

$$1, 0, 1, 2, 6, 18, 57, 186, 622, 2120, \dots$$

satisfying the relation

$$c_n = 2f_n + f_{n-1},$$

and having as generating function

$$F(x) = \frac{2}{1 + 2x + \sqrt{1 - 4x}}.$$

As a neat consequence they proved that the generating function of the permutations of  $\mathcal{S}_n(321)$  having  $k \geq 1$  fixed points is given by the  $(k+1)$ th convolution of  $F(x)$ , and precisely it is  $x^k F^{k+1}(x)$ . Passing to parallelogram permutominoes, and using Theorem 3.12, we have that

**Proposition 3.13.** *The generating function of the class  $\tilde{\mathcal{P}}_{n,0}$  is*

$$\tilde{P}_0(x) = \frac{2x^2}{1 + 2x + \sqrt{1 - 4x}},$$

while, for all  $k > 0$ , the generating function of the class  $\tilde{\mathcal{P}}_{n,k}$  is

$$\frac{\tilde{P}_0(x)^{k+1}}{x^k} = x^{k+2} F^{k+1}(x).$$

We would like to point out that the result stated in the previous proposition can be reformulated into an interesting combinatorial property of the permutations of  $\tilde{\mathcal{P}}_{n,k}$ , which will be successively generalized to the other classes of permutations.

To do this we introduce the binary operation  $\odot$  of *diagonal sum* (very similar to the well known *direct sum*). Given two permutations  $\pi \in \mathcal{S}_n$ , and  $\pi' \in \mathcal{S}_{n'}$ , such that  $\pi(n) = n$  and  $\pi'(1) = 1$ ,  $\pi \odot \pi' \in \mathcal{S}_{n+n'-1}$  is defined as

$$\pi \odot \pi' = (\pi(1), \dots, \pi(n-1), n, \pi'(2) + (n-1), \dots, \pi'(n') + (n-1)).$$

For instance, we have  $(2, 4, 1, 3, 5) \odot (1, 3, 4, 2) = (2, 4, 1, 3, 5, 7, 8, 6)$ .

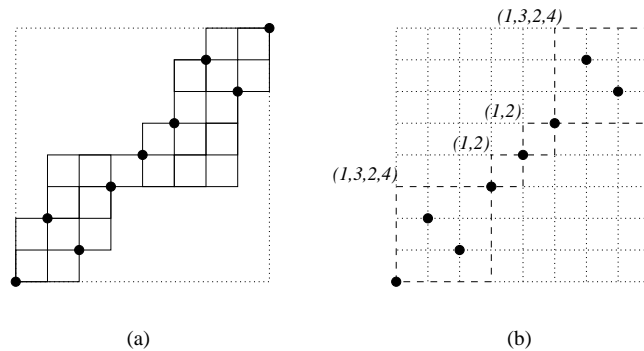


Figure 18: (a) A parallelogram permutomino associated with  $\pi = (1, 3, 2, 4, 5, 6, 8, 7, 9)$ ; (b) the decomposition of  $\pi$  in the diagonal sum of the permutations  $(1, 3, 2, 4) \odot (1, 2) \odot (1, 2) \odot (1, 3, 2, 4)$ .

**Proposition 3.14.** *A permutation  $\pi \in \tilde{\mathcal{P}}_{n,k}$ ,  $k > 0$ , if and only if  $\pi$  can be uniquely decomposed into the diagonal sum of  $k + 1$  permutations,  $\pi = \eta_1 \odot \dots \odot \eta_{k+1}$ , where  $\eta_j \in \tilde{\mathcal{P}}_{n_j,0}$ , and  $n_1 + \dots + n_{k+1} = n + k$ .*

Proposition 3.14 says that each permutation of  $\tilde{\mathcal{P}}_n$  having exactly  $k > 0$  free fixed points is obtained as the diagonal sum of  $k + 1$  permutations associated with parallelogram permutominoes, but having no free fixed points. For instance, looking at Figure 18 we have that

$$\pi = (1, 3, 2, 4, 5, 6, 8, 7, 9) \in \tilde{\mathcal{P}}_{9,4} = (1, 3, 2, 4) \circ (1, 2) \circ (1, 2) \circ (1, 3, 2, 4).$$

Finally, in view of (10) and Proposition 3.13 we can re-obtain the generating function of parallelogram permutominoes according to the size:

$$\sum_{n \geq 2} |\mathcal{P}_n| x^n = \sum_{k \geq 0} 2^k x^{k+2} F^{k+1}(x) = \frac{\tilde{P}_0(x)}{1 - 2xF(x)} = \frac{1 - x - \sqrt{1 - 4x}}{2}. \quad (11)$$

### 4. Directed Convex Permutominoes and Permutations

A directed convex permutomino is simply a convex permutomino where the  $\delta$  path is empty. Hence every permutation  $\pi$  associated with a directed convex permutominoes is characterized by the following property:

**Proposition 4.1.** *A permutation  $\pi$  belongs to  $\tilde{\mathcal{D}}_n$  if and only if  $\pi(1) = 1$  and  $\sigma$  is increasing.*

For instance, referring to the directed convex permutomino in Figure 19 (a) we have that  $\mu = (1, 2, 4, 5, 6, 9, 8, 7)$ , and  $\sigma = (1, 3, 7)$ . The first cases are:

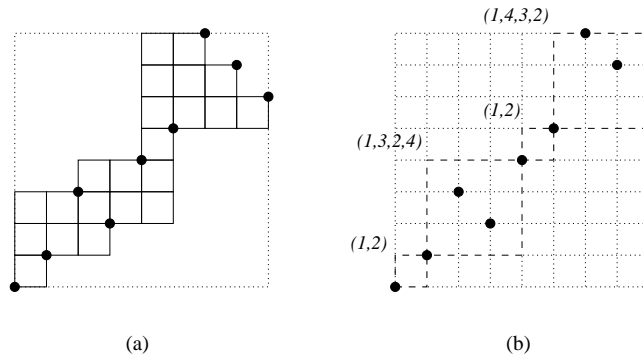


Figure 19: (a) A directed convex permutomino associated with  $\pi = (1, 2, 4, 3, 5, 6, 9, 8, 7)$ ; (b) the decomposition of  $\pi$  in the diagonal sum of the permutations  $(1, 2) \circ (1, 3, 2, 4) \circ (1, 2) \circ (1, 4, 3, 2)$ .

$$\begin{aligned} \tilde{\mathcal{D}}_2 &= \{12\}; \\ \tilde{\mathcal{D}}_3 &= \{123, 132\}; \\ \tilde{\mathcal{D}}_4 &= \{1234, 1243, 1324, 1342, 1423, 1432\}. \end{aligned}$$

Analogously to Proposition 3.11 we can state that:

**Proposition 4.2.** *If  $P$  is a directed convex permutomino, then all the convex permutominoes associated with  $\pi_1(P)$  are directed convex ones.*

Let us denote, as usual, by  $\tilde{\mathcal{D}}_{n,k}$  the set of permutations of  $\tilde{\mathcal{D}}_n$  having exactly  $k$  free fixed points. We have,

$$|\tilde{\mathcal{D}}_n| = \sum_{k=0}^{n-2} |\tilde{\mathcal{D}}_{n,k}|, \quad |\mathcal{D}_n| = \sum_{k=0}^{n-2} 2^k |\tilde{\mathcal{D}}_{n,k}|. \tag{12}$$

The table below shows the cardinalities of  $\tilde{\mathcal{D}}_{n,k}$  for small values of  $n, k$ .

$n/k$	0	1	2	3	4	5	...
2	1						
3	1	1					
4	4	1	1				
5	13	5	1	1			
6	46	16	6	1	1		
7	166	58	19	7	1	1	
8	610	211	71	22	8	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We would like to point out that similarly to Proposition 3.14, for any  $k > 0$  the permutations of  $\tilde{\mathcal{D}}_{n,k}$  can be decomposed into  $k + 1$  permutations having no free fixed points.

**Proposition 4.3.** *Let  $\pi \in \mathcal{S}_n$ ; we have that  $\pi \in \tilde{\mathcal{D}}_{n,k}$ ,  $k > 0$ , if and only if  $\pi$  can be uniquely decomposed into the diagonal sum of  $k + 1$  permutations,  $\pi = \eta_1 \circ \dots \circ \eta_k \circ \zeta$ , where  $\eta_j \in \tilde{\mathcal{P}}_{n_j,0}$ ,  $\zeta \in \tilde{\mathcal{D}}_{r,0}$ , and  $n_1 + \dots + n_k + r = n + k$ .*

*Proof.* By definition a free fixed point is a fixed point lying in the increasing part of  $\mu$ . If  $i$  is a free fixed point of  $\pi$ , then for all  $j < i$  we have  $\pi(j) < i$ , since otherwise  $i$  would not belong to  $\mu$ . Then  $(\pi(1), \dots, \pi(i))$  is a permutation of  $\mathcal{S}_i$  with  $\mu$  and  $\sigma$  both increasing, and  $\pi(1) = 1, \pi(i) = i$ , hence  $(\pi(1), \dots, \pi(i)) \in \tilde{\mathcal{P}}_i$ . Moreover, if we consider the free fixed point  $j$  with maximal ordinate, then  $(\pi(j) - j + 1, \dots, \pi(n) - j + 1)$  is a permutation of  $\mathcal{S}_{n-j+1}$  with  $\mu$  unimodal,  $\sigma$  increasing, and  $\pi(j) = j$ , hence  $(\pi(j) - j + 1, \dots, \pi(i) - j + 1) \in \tilde{\mathcal{D}}_{n-j+1}$ . ■

In particular Proposition 4.3 states that each permutation of  $\tilde{\mathcal{D}}_n$  having exactly  $k > 0$  free fixed points is obtained as the diagonal sum of  $k$  permutations associated with parallelogram permutominoes and one associated with directed convex permutominoes, all having no fixed points. For instance, looking at Figure 18 we have that  $\pi = (1, 3, 2, 4, 5, 6, 8, 7, 9) \in \tilde{\mathcal{P}}_{9,4} = (1, 3, 2, 4) \circ (1, 2) \circ (1, 2) \circ (1, 3, 2, 4)$ . Passing to generating functions, and letting

$$\tilde{D}_k(x) = \sum_{n \geq 2} |\tilde{\mathcal{D}}_{n,k}| x^n$$

by Proposition 4.3 we have that, with  $k > 0$

$$\tilde{D}_k(x) = \frac{\tilde{D}_0(x) \tilde{P}_0^k(x)}{x^k}. \tag{13}$$

Moreover, by (12) we have that the generating function  $D(x)$  of directed convex permutominoes is equal to

$$D(x) = \sum_{k \geq 0} 2^k \tilde{D}_k(x) = \frac{\tilde{D}_0(x)}{1 - \frac{2\tilde{P}_0(x)}{x}}. \tag{14}$$

Since we know from [16] that

$$D(x) = \frac{x}{2} \left( \frac{1}{\sqrt{1-4x}} - 1 \right), \tag{15}$$

from (14) and (15) we have the following

**Proposition 4.4.** *The generating function of the class  $\tilde{\mathcal{D}}_{n,0}$  is*

$$\tilde{D}_0(x) = \frac{x^2 (3\sqrt{1-4x} + 1 - 4x)}{2(x+2)(1-4x)}.$$

The function  $\tilde{D}_0(x)$  defines the sequence A026641 in [22], whose first terms are

$$1, 1, 4, 13, 46, 166, 610, 2269, 8518, 32206, 122464, \dots$$

Finally, the generating function  $\tilde{D}(x)$  of the permutations associated with directed convex permutominoes is given by the sum

$$\tilde{D}(x) = \sum_{k \geq 0} \tilde{D}_k(x) = \frac{x^2}{\sqrt{1-4x}},$$

which leads to the following remarkable result.

**Proposition 4.5.** *For any  $n \geq 0$ , we have  $|\tilde{\mathcal{D}}_{n+2}| = \binom{2n}{n}$ .*



As a generalization of Theorem 3.12, we can also state that the permutations of  $\tilde{\mathcal{D}}_n$  are characterized by the avoidance of four patterns of length 4.

**Theorem 4.6.** *A permutation  $\pi$  belongs to  $\tilde{\mathcal{D}}_n$  if and only if  $\pi(1) = 1$ , and it avoids the patterns  $\nu_1 = (3, 2, 1, 4)$ ,  $\nu_2 = (3, 2, 4, 1)$ ,  $\nu_3 = (4, 2, 1, 3)$ ,  $\nu_4 = (4, 2, 3, 1)$ .*

*Proof.* ( $\Rightarrow$ ) One only has to check that if a permutation  $\pi$  begins with  $\pi(1) = 1$  and contains one of the four patterns  $\nu_1, \nu_2, \nu_3, \nu_4$  then its  $\sigma$  sequence cannot be increasing.

( $\Leftarrow$ ) Let us assume by contradiction that  $\pi \notin \tilde{\mathcal{D}}_n$ , i.e. that the sequence  $\sigma$  is not increasing. Then there are two indices  $i_1$  and  $i_2$  such that  $1 < i_1 < i_2 < n$ , and such that  $\sigma(i_1) > \sigma(i_2)$ . Moreover, there should be two indices  $j, k$  belonging to  $\mu$ , such that:

$j < i_1$ , and  $\pi(j) > \sigma(i_1)$ ; in fact, without such an element,  $i_1$  would belong to the increasing part of  $\mu$  and not to  $\sigma$ ;

$k > i_1$ , and  $\pi(k) > \sigma(i_1)$ ; similarly, without such an element,  $i_2$  would belong to the decreasing part of  $\mu$  and not to  $\sigma$ ;

This leads to four possible configurations which are sketched in Figure 20, and each of them defines one of the four patterns  $\nu_1, \nu_2, \nu_3, \nu_4$ . ■

### 5. The Cardinality of $\tilde{\mathcal{C}}_n$

In order to count  $\tilde{\mathcal{C}}_n$  we provide a further generalization of the statement of Proposition 4.3.

**Proposition 5.1.** *A permutation  $\pi \in \tilde{\mathcal{C}}_{n,k}$ ,  $k > 0$ , if and only if  $\pi$  can be uniquely decomposed into the diagonal sum of  $k+1$  permutations,  $\pi = \zeta_1 \circ \eta_2 \circ \dots \circ \eta_k \circ \zeta_2$ , where:*

1.  $\eta_j \in \tilde{\mathcal{P}}_{n_j,0}$ ,  $2 \leq j \leq k$ ,
2.  $\zeta_2 \in \tilde{\mathcal{D}}_{r,0}$ , and
3.  $\zeta_1 = (\zeta_1(1), \dots, \zeta_1(s))$  is simply the reflection of a permutation  $\zeta'_1$  of  $\tilde{\mathcal{D}}_{s,0}$  with respect to  $x + y = 0$ , precisely  $\zeta'_1 = (n + 1 - \zeta_1(1), \dots, n + 1 - \zeta_1(s)) \in \tilde{\mathcal{D}}_{s,0}$ , and  $r + s + n_2 + \dots + n_k = n + k$ .

*Proof.* It is analogous to that of Propositions 3.14 and 4.3. ■

For instance, looking at Figure 21 we have that the permutation  $\pi = (3, 1, 5, 2, 4, 6, 9, 7, 8, 10, 13, 12, 11) \in \tilde{\mathcal{C}}_{13,2}$ , and it can be decomposed as

$$(3, 1, 5, 2, 4, 6) \circ (1, 4, 2, 3, 5) \circ (1, 4, 3, 2).$$

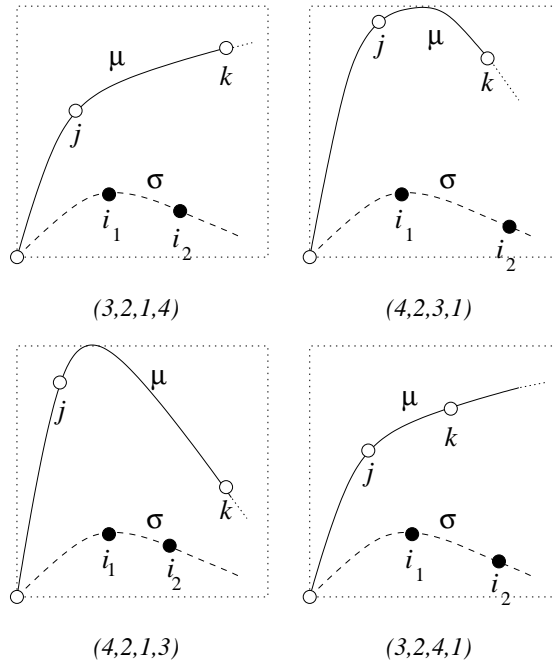


Figure 20: The four forbidden patterns in the permutations of  $\tilde{D}_n$ .

Looking at the figure, we easily observe that the permutation  $\zeta = (3, 1, 5, 2, 4, 6)$  is the reflection according to  $x + y = 0$  of  $\zeta'_1 = (1, 3, 5, 2, 6, 4) \in \tilde{D}_{6,0}$ .

To count the number of elements in  $\tilde{D}_n$  we use essentially the same method used for  $\tilde{D}_n$ . Passing to generating functions, and letting

$$\tilde{C}_k(x) = \sum_{n \geq 2} |\tilde{C}_{n,k}| x^n,$$

by Proposition 5.1 we have that, with  $k > 0$

$$\tilde{C}_k(x) = \frac{\tilde{D}_0^2(x) \tilde{P}_0^{k-1}(x)}{x^k}. \tag{16}$$

Moreover, by (7) we have that the generating function  $C(x)$  of convex per-

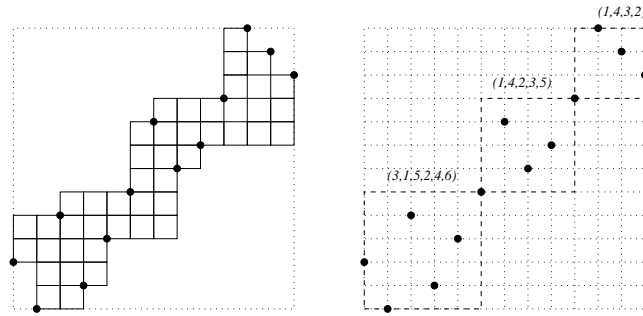


Figure 21: (a) A convex permutomino associated with the permutation  $\pi = (3, 1, 5, 2, 4, 6, 9, 7, 8, 10, 13, 12, 11)$ ; (b) the decomposition of  $\pi$  in the diagonal sum of the permutations  $(3, 1, 5, 2, 4, 6) \circ (1, 4, 2, 3, 5) \circ (1, 4, 3, 2)$ .

mutominoes is equal to

$$\begin{aligned}
 C(x) &= \sum_{k \geq 0} 2^k \tilde{C}_k(x) = \tilde{C}_0(x) + \sum_{k \geq 1} 2^k \tilde{C}_k(x) \\
 &= \tilde{C}_0(x) + \frac{2 \tilde{D}_0^2(x)}{x \left(1 - \frac{2 \tilde{P}_0(x)}{x}\right)}. \tag{17}
 \end{aligned}$$

After some manipulations we obtain that the rightmost summand in (17), i.e. the generating function of convex permutominoes having at least one free fixed point is:

$$\begin{aligned}
 \tilde{K}(x) &= \frac{2 \tilde{D}_0^2(x)}{x \left(1 - \frac{2 \tilde{P}_0(x)}{x}\right)} = \frac{2x \tilde{P}_0(x)}{1 - 4x} \tag{18} \\
 &= 2x^3 + 8x^4 + 34x^5 + 140x^6 + 572x^7 + 2324x^8 + \dots,
 \end{aligned}$$

sequence not in [22]. These numbers  $\tilde{K}_n$  then can be expressed in terms of the convolution of Fine numbers  $f_n$  with the sequence  $4^n$ ,

$$\tilde{K}_{n+3} = 2 \sum_{h=0}^n 4^h f_{n-h}. \tag{19}$$

Moreover from (4) and (19) we can also obtain the following result:

**Proposition 5.2.** *The number of permutations of  $\tilde{\mathcal{C}}_{n+1,0}$  is*

$$2(n+3)4^{n-2} - \frac{n}{2} \binom{2n}{n} - 2 \sum_{h=0}^{n-2} 4^h f_{n-h-1}.$$

generating function	2	3	4	5	6	7	8	...
$C(x)$	1	4	18	84	394	1836	8468	...
$\tilde{C}_0(x)$	1	2	10	50	254	1264	6144	...
$\sum_{k \geq 1} 2^k \tilde{C}_k(x)$		2	8	34	140	572	2324	...
$\sum_{k \geq 1} \tilde{C}_k(x)$		1	3	12	47	186	738	...
$\tilde{C}(x)$	1	3	13	62	301	1450	6881	...

Figure 22: The first terms of the sequences defined by the considered generating functions, starting with  $n = 2$ .

Now we have set all the ingredients necessary to determine the cardinality of  $\tilde{C}_n$ . By Equation (7) we have that the generating function  $\tilde{C}(x)$  of the class  $\tilde{C}_n$  is given by

$$\tilde{C}(x) = \tilde{C}_0(x) + \sum_{k \geq 1} \tilde{C}_k(x),$$

and using (16) and (17) we have that

$$\tilde{C}(x) = C(x) - \left( \sum_{k \geq 1} 2^k \tilde{C}_k(x) - \sum_{k \geq 1} \tilde{C}_k(x) \right). \tag{20}$$

Using the previous results and after some simplifications we get that

$$\sum_{k \geq 1} 2^k \tilde{C}_k(x) - \sum_{k \geq 1} \tilde{C}_k(x) = x^2 \left( \frac{1}{2(1-4x)} - \frac{1}{2\sqrt{1-4x}} \right), \tag{21}$$

and then the  $(n + 2)$ th term of the series is easily found to be

$$\frac{1}{2} \left( 4^n - \binom{2n}{n} \right).$$

Eventually, starting from (20), and using the closed form for the number of convex permutominoes in (4) and the above formula we have

**Proposition 5.3.** *The number of permutations of  $\tilde{C}_{n+1}$  is*

$$2(n+2)4^{n-2} - \frac{n}{4} \binom{3-4n}{1-2n} \binom{2n}{n}, \quad n \geq 1. \tag{22}$$

## 6. Further Work

Here we outline the main open problems and research lines on the class of permutominoes.

1. It would be natural to look for a combinatorial proof of the formula (4) for the number of convex permutominoes and (22) for the number of permutations associated with convex permutominoes. These proofs could perhaps be obtained using the matrix characterization for convex permutominoes provided in Section .
2. We would like to consider the characterization and the enumeration of the permutations associated with other classes of permutominoes, possibly including the class of convex permutominoes. For instance, if we take the class of column-convex permutominoes, we observe that Proposition does not hold. In particular, one can see that, if the permutomino is not convex, then the set of reentrant points does not form a permutation matrix (Figure 23).

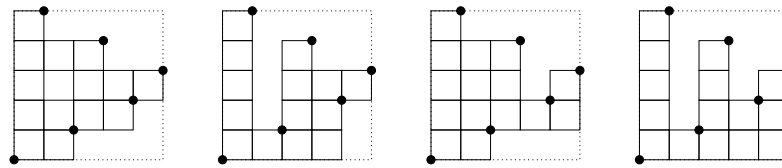


Figure 23: The four column-convex permutominoes associated with the permutation  $(1, 6, 2, 5, 3, 4)$ ; only the leftmost permutomino is convex

Moreover, it might be interesting to determine an extension of Theorem 3.8 for the class of column-convex permutominoes, i.e. to characterize the set of column-convex permutominoes associated with a given permutation. For instance, we observe that while there is one convex permutomino associated with  $\pi = (1, 6, 2, 5, 3, 4)$ , there are four column-convex permutominoes associated with  $\pi$  (Figure 23).

## References

- [1] G. Barequet, M. Moffie, A. Ribó and G. Rote: Counting Polyominoes on Twisted Cylinders, *Integers*, **6** A22 (2006).
- [2] Beauquier, D., Nivat, M.: Tiling the plane with one tile, Proc. of *6th Annual Symposium on Computational geometry*, Berkeley, CA, ACM press 128-138 (1990).
- [3] Boldi, P., Lonati, V., Radicioni, R., Santini, M.: The number of convex permutominoes, Proc. of *LATA 2007, International Conference on Language and Automata Theory and Applications*, Tarragona, Spain, (2007).

- [4] Bousquet-Mélou, M.: A method for the enumeration of various classes of column-convex polygons, *Disc. Math.*, **154**, 1-25 (1996).
- [5] Bousquet-Mélou, M.: Guttman, A. J.: Enumeration of three dimensional convex polygons, *Ann. of Comb.*, **1**, 27-53 (1997).
- [6] Brak, R., Guttman, A. J. and Enting, I. G.: Exact solution of the row-convex polygon perimeter generating function, *J. Phys.*, A **23**, L2319-L2326 (1990).
- [7] Brlek, S., Labelle, G. and Lacasse, A.: A note on a result of daurat and nivat, *Lecture Notes in Computer Science*, Springer Berlin/Heidelberg, **3572**, 189-198 (2005).
- [8] Chang, S.J. and Lin, K.Y.: Rigorous results for the number of convex polygons on the square and honeycomb lattices, *J. Phys. A: Math. Gen.*, **21**, 2635-2642 (1988).
- [9] Conway, J.H. and Lagarias, J.C.: Tiling with polyominoes and combinatorial group theory, *J. Comb. Th. A*, **53**, 183-208 (1990).
- [10] Daurat, A. and Nivat, M.: Salient and reentrant points of discrete sets, *Disc. Appl. Math.*, **151**, 106-121 (2005).
- [11] Delest, M. and Viennot, X.G.: Algebraic languages and polyominoes enumeration, *Theor. Comp. Sci.*, **34**, 169-206 (1984).
- [12] Del Lungo, A., Duchi, E., Frosini, A. and Rinaldi, S.: On the generation and enumeration of some classes of convex polyominoes, *El. J. Comb.*, **11**, #R60 (2004).
- [13] Deutsch, E. and Elizalde, S.: A simple and unusual bijection for Dyck paths and its consequences, *Ann. Comb.*, **7**, 281-297 (2003).
- [14] Deutsch, E. and Shapiro, L.: A survey of the Fine numbers, *Disc. Math.* **241**, 241-265 (2001).
- [15] Disanto, F., Frosini, A., Pinzani, R. and Rinaldi, S.: A closed formula for the number of convex permutominoes, *El. J. Combinatorics* **14**, #R57 (2007).
- [16] Fanti, I., Frosini, A., Grazzini, E., Pinzani, R. and Rinaldi, S.: Polyominoes determined by permutations, *MathInfo 06 Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities* Nancy, DMTCS proc. AG 381-390 (2006).
- [17] Gardner, M.: Mathematical games, *Scientific American*, (1958) Sept. 182-192, Nov. 136-142.
- [18] Golomb, S.W.: *Polyominoes: Puzzles, Patterns, Problems, and Packings*, Princeton Academic Press, 1996.
- [19] Golomb, S. W.: Checker boards and polyominoes, *Amer. Math. Monthly*, **61**, 675-682 (1954).
- [20] Incitti, F.: Permutation diagrams, fixed points and Kazhdan-Lusztig  $R$ -polynomials, *Ann. Comb.*, **10**, 369-387 (2006).
- [21] Jensen, I., Guttman, A.J.: Statistics of lattice animals (polyominoes) and polygons, *J. Phys. A*, **33**, 257-263 (2000).
- [22] Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences, <http://www.research.att.com/~njas/sequences/>
- [23] Stanley, R. P.: *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, Cambridge (1999).