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Crux Mathematicorum

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Table of Contents

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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER

No.91

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This column marks the first anniversary of my taking over this Corner from Murray Klamkin. I am painfully aware of some of the errors that have crept through, but I hope, with the continued contributions from the readership, to carry on providing interesting Olympiad problems and solutions. I would particularly like to thank those who submitted problem sets and solutions this past year, even though we haven't yet published all the contributions. The list of contributors includes H. Abbott, H. Alzer, P. Andrews, B. Arbel, Francisco Bellot, Aage Bondesen, Curtis Cooper, George Evagelopoulos, Chris Fisher, J.T. Groenman, R.K. Guy, Walther Janous, Murray Klamkin, Andy Liu, M. Molloy, Sister J. Monk, John Morvay, Richard Nowakowski, Bob Prielipp, Josef Rita i Coura, Daniel Ropp, Cecil Rousseau, M. Selby, Robert E. Shafer, Bruce Shawyer, D.J. Smeenk, Dan Sokolowsky, Dim. Vathis, and Edward T.H. Wang.

* * *

We begin with five sets of problems forwarded to us by Andy Liu. These are problems from the *Kürschak Competitions*, Hungary, for the years 1982 to 1986. This competition is an annual four hour "open book" contest.

<u>1982.1</u>. A cube of integral dimensions is given in space so that all four vertices of one of the faces are lattice points. Prove that the other four vertices are also lattice points.

<u>1982.2</u>. Prove that for any integer $k > 2$, there exist infinitely many positive integers n such that the least common multiple of

$$n, n + 1, n + 2, \ldots, n + k - 1$$

is greater than the least common multiple of

$$n + 1, n + 2, \ldots, n + k.$$

<u>1982.3</u>. The set of integers is coloured with 100 colours in such a way that all the colours are used and the following is true: for any choice of intervals $[a,b]$ and $[c,d]$ of equal length and with integral endpoints, if a

and c have the same colour and b and d have the same colour, then the intervals $[a,b]$ and $[c,d]$ are identically coloured, in that for any integer x, $0 \leq x \leq b - a$, the numbers $a + x$ and $c + x$ are of the same colour. Prove that -1982 and 1982 are of different colours.

*

1983.1. Let x, y, and z be rational numbers satisfying

$$x^3 + 3y^3 + 9z^3 - 9xyz = 0.$$

Prove that $x = y = z = 0$.

1983.2. Prove that $f(2) \geq 3^n$ where the polynomial

$$f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + 1$$

has non-negative coefficients and n real roots.

1983.3. Given are $n + 1$ points $P_1, P_2, \ldots, P_n$ and Q in the plane, no three collinear. For any two different points P_i and P_j, there is a point P_k such that the point Q lies inside the triangle $P_iP_jP_k$. Prove that n is an odd number.

*

1984.1. Writing down the first four rows of Pascal's triangle in the usual way and then adding up the numbers in vertical columns, we obtain seven numbers as shown below. If we repeat this procedure with the first 1024 rows of the triangle, how many of the 2047 numbers thus obtained will be odd?

```
      1
    1   1
  1   2   1
1   3   3   1
-------------
1 1 4 3 4 1 1
```

1984.2. $A_1B_1A_2$, $B_1A_2B_2$, $A_2B_2A_3$, ..., $B_{13}A_{14}B_{14}$, $A_{14}B_{14}A_1$, and $B_{14}A_1B_1$ are equilateral rigid plates that can be folded along the edges A_1B_1, B_1A_2, A_2B_2, ..., $A_{14}B_{14}$, and $B_{14}A_1$, respectively. Can they be folded so that all 28 plates lie in the same plane?

1984.3. Given are n integers, not necessarily distinct, and two positive integers p and q. If the n numbers are not all distinct, choose two equal ones. Add p to one of them and subtract q from the other. If there are still equal ones among the n numbers, repeat this procedure. Prove that after a finite number of steps, all n numbers are distinct.

*

1985.1. The convex $(n+1)$-gon $P_0P_1\ldots P_n$ is partitioned into $n - 1$

triangles by $n - 2$ non-intersecting diagonals. Prove that the triangles can be numbered from 1 to $n - 1$ such that for $1 \leq i \leq n - 1$, P_i is a vertex of the triangle numbered i.

1985.2. Let n be a positive integer. For each prime divisor p of n, consider the highest power of p which does not exceed n. The sum of these powers is defined to be the power-sum of n. Prove that there exist infinitely many positive integers which are less than their respective power-sums.

1985.3. Let each vertex of a triangle be reflected across the opposite side. Prove that the area of the triangle determined by the three points of reflection is less than 5 times the area of the original triangle.

*

1986.1. Prove that three rays from a given point contain three face diagonals of a cuboid if and only if the rays include pairwise acute angles such that their sum is 180°.

1986.2. Let n be an integer greater than 2. Find the maximum value for h and the minimum value for H such that for any positive numbers $a_1, a_2, \ldots, a_n$,

$$h < \frac{a_1}{a_1 + a_2} + \frac{a_2}{a_2 + a_3} + \ldots + \frac{a_n}{a_n + a_1} < H.$$

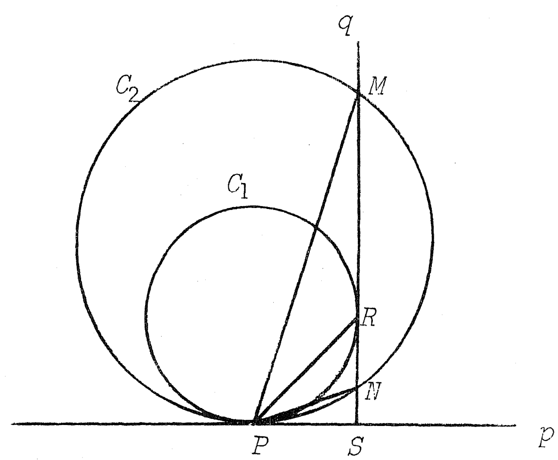
1986.3. A and B play the following game. They arbitrarily select k of the first 100 positive integers. If the sum of the selected numbers is even, then A wins. If their sum is odd, then B wins. For what values of k is the game fair? [*Editor's note*: In this form the question is not completely clear. Do A and B each independently choose k numbers? All at once, or one at a time? Are repetitions allowed, and if so, how are they handled?]

* * *

We return to solutions to problems from past years submitted in 1987.

1. [1985: 168] *1984 Dutch Olympiad.*

Two circles C_1 and C_2 with radii r_1 and r_2, respectively, are tangent to the line p at point P. All other points of C_1 are inside C_2. Line q is perpendicular to p at point S, is tangent to C_1 at point R, and intersects C_2 at points M and N, with N between R and S, as shown in the figure.



(a) Prove that PR bisects $\angle MPN$.

(b) Compute the ratio $r_1 : r_2$ if, moreover, it is given that PN bisects $\angle RPS$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

(a) Let O_1 denote the center of circle C_1. Then

$$\angle OPS = \angle PSR = \angle SRO_1 = 90^\circ .$$

Thus O_1PSR is a square and $\angle RPS = \angle PRS = 45^\circ$. Also, $\angle PMS$ and $\angle NPS$ are equal since p is tangent to C_2 at P and $\angle PMS$ is subtended by the arc PN. Now

$$\angle RPN = \angle RPS - \angle NPS = 45^\circ - \angle NPS;$$

$$\angle MPR = 180^\circ - \angle PMR - \angle PRM$$

$$= 180^\circ - \angle PMS - 135^\circ = 45^\circ - \angle NPS.$$

Thus $\angle RPN = \angle MPR$, or PR bisects $\angle MPS$.

(b) If PN bisects $\angle RPS$, then $\angle NPS = \angle PMS = 22.5^\circ$. Also MN is a chord of C_2 at distance r_1 from the center of C_2 (and hence C_1) and so

$$MN = 2\sqrt{r_2^2 - r_1^2} .$$

Hence,

$$NS = PS \tan 22.5^\circ = r_1(\sqrt{2} - 1)$$

$$MS = MN + NS = 2\sqrt{r_2^2 - r_1^2} + r_2(\sqrt{2} - 1)$$

$$= PS \tan 67.5^\circ = r_1(\sqrt{2} + 1).$$

Thus

$$\frac{r_1}{r_2} = \frac{\sqrt{2}}{2} .$$

*

1. [1985: 170] *34th Bulgarian Mathematical Olympiad (3rd Stage).*

If k and n are positive integers, prove that

$$(n^4 - 1)(n^3 - n^2 + n - 1)^k + (n + 1)n^{4k-1}$$

is divisible by $n^5 + 1$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

The given expression is a polynomial in n, say $f_k(n)$, with integer coefficients. Since $x^5 + 1 = 0$ has distinct complex roots it suffices to show that if $x^5 + 1 = 0$ then $f_k(x) = 0$. Now

$$x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1).$$

Since $f_k(-1) = 0$ by inspection, we suppose that

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

Then $x^4 = x^3 - x^2 + x - 1$ and so

$$f_k(x) = (x^4 - 1)(x^4)^k + (x + 1)x^{4k-1} = x^{4k-1}(x^5 + 1) = 0.$$

This completes the proof.

Editor's note: Edward T.H. Wang of Wilfrid Laurier University pointed out a more straightforward proof by induction.

4. [1985: 170] *34th Bulgarian Mathematical Olympiad (3rd Stage).*

Let a_n and b_n be positive integers satisfying the relation

$$a_n + b_n\sqrt{2} = (2 + \sqrt{2})^n, \quad n = 1,2,3,\dots .$$

Prove that $\lim_{n\to\infty}(a_n/b_n)$ exists, and find this limit.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

By expanding $(2 + \sqrt{2})^n = a_n + b_n\sqrt{2}$, since a_n, b_n are integers, we see that

$$a_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} 2^{n-2k}(\sqrt{2})^{2k}$$

$$b_n = \sum_{k=0}^{[n/2]} \binom{n}{2k+1} 2^{n-2k-1}(\sqrt{2})^{2k}$$

where $[x]$ is the greatest integer less than or equal to x, and we set the convention that $\binom{n}{n+1} = 0$. From this it follows that

$$a_n - b_n\sqrt{2} = (2 - \sqrt{2})^n.$$

Thus

$$a_n = \frac{1}{2}[(2 + \sqrt{2})^n + (2 - \sqrt{2})^n],$$
$$b_n = \frac{1}{2\sqrt{2}}[(2 + \sqrt{2})^n - (2 - \sqrt{2})^n].$$

Therefore

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{2} \left(\frac{1 + \left(\frac{2 - \sqrt{2}}{2 + \sqrt{2}}\right)^n}{1 - \left(\frac{2 - \sqrt{2}}{2 + \sqrt{2}}\right)^n} \right) = \sqrt{2}$$

since $\left|\frac{2 - \sqrt{2}}{2 + \sqrt{2}}\right| < 1$.

6. [1985: 170] *34th Bulgarian Mathematical Olympiad (3rd Stage).*

Five given points in the plane have the following property: of any four of them, three are the vertices of an equilateral triangle.

(a) Prove that four of the five points are the vertices of a rhombus with an angle equal to 60°.

(b) Find the number of equilateral triangles having their vertices among the given five points.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

(a) Label the points A, B, C, D, E. Some three of the points, say A, B, C, form an equilateral triangle. If neither D nor E is the vertex of an equilateral triangle together with vertices A and B, B and C, or C and A, then by consideration of the three sets of four points $\{D,E,A,B\}$, $\{D,E,B,C\}$, and $\{D,E,C,A\}$, we see that two of the three triangles DEA, DEB, DEC are equilateral. Without loss suppose that ΔDEA and ΔDEB are equilateral. Then A,D,B,E form the vertices of a rhombus with an angle of 60°. On the other hand, if either D or E, say D, is the third vertex of an equilateral triangle with A and B then $ADBC$ is a rhombus with interior angle 60°.

(b) By relabelling if necessary, we may assume that $ADBC$ is a rhombus with $\angle ADB = 60^\circ = \angle ACB$. Notice that ΔABE cannot be equilateral as then $E = C$ or $E = D$. Consider now $\{B,C,D,E\}$. Since ΔCDB is not equilateral we must have some one of ΔBDE, ΔBCE or ΔCDE equilateral. In the first two cases we may (by relabelling) assume that ΔBDE is equilateral to obtain Figure 1. But then there is no equilateral triangle for $\{A,C,D,E\}$. Thus we conclude that ΔCDE is equilateral to give the configuration of Figure 2, after relabelling if necessary. Inspection shows that there are three equilateral triangles ΔABC, ΔABD and ΔCDE. It is also clear that such a configuration satisfies the conditions of the problem.

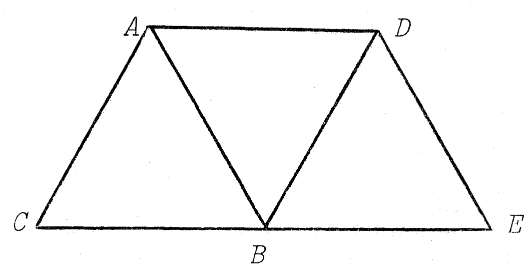


Figure 1

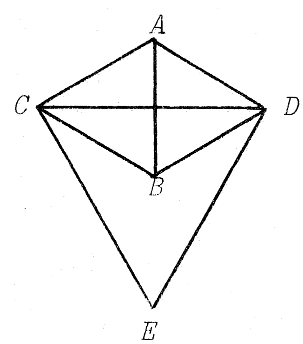


Figure 2

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1. [1985: 212] *16th Austrian Mathematical Olympiad (Final Round).*

Determine all quadruples (a,b,c,d) of nonnegative integers such that

$$a^2 + b^2 + c^2 + d^2 = a^2b^2c^2.$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, and independently by John Morvay, Dallas, Texas, U.S.A.

Let (a,b,c,d) be such a quadruple. If all three of a, b, c are odd, then

$a^2 \equiv b^2 \equiv c^2 \equiv 1 \bmod 8$, since the residues of x^2 modulo 8 are 0, 1 and 4. But then $d^2 \equiv 1 - 1 - 1 - 1 \equiv 6 \bmod 8$, which is impossible. Thus we may suppose that a is even. Then $a^2 \equiv 0 \bmod 4$, and so $b^2 + c^2 + d^2 \equiv 0 \bmod 4$. This is impossible if any of b, c or d are odd. Thus a, b, c, d are all even.

We now prove that $a/2^n$, $b/2^n$, $c/2^n$, $d/2^n$ are all integers for $n \geq 1$ by induction. From this it is immediate that $(a,b,c,d) = (0,0,0,0)$ is the only solution to the problem.

That the four quotients are integers for $n = 1$ was shown in the first paragraph. Suppose then that $a/2^n$, $b/2^n$, $c/2^n$ and $d/2^n$ are integers. The equation $a^2 + b^2 + c^2 + d^2 = a^2b^2c^2$ is equivalent to

$$(a/2^n)^2 + (b/2^n)^2 + (c/2^n)^2 + (d/2^n)^2 = 2^{4n}(a/2^n)^2(b/2^n)^2(c/2^n)^2.$$

Thus

$$(a/2^n)^2 + (b/2^n)^2 + (c/2^n)^2 + (d/2^n)^2 \equiv 0 \bmod 8.$$

As before, we deduce that $a/2^n$, $b/2^n$, $c/2^n$ and $d/2^n$ are all even. This gives $a/2^{n+1}$, $b/2^{n+1}$, $c/2^{n+1}$ and $d/2^{n+1}$ all integers to complete the induction step.

2. [1985: 212] *16th Austrian Mathematical Olympiad (Final Round).*
For $n = 1,2,3,\ldots$ let

$$f(n) = 1^n + 2^{n-1} + 3^{n-2} + \ldots + (n - 1)^2 + n.$$

Determine

$$\min_{n\geq 1} \frac{f(n + 1)}{f(n)}.$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

We calculate

$$\begin{aligned} f(1) &= 1, & f(2) &= 3, \\ f(3) &= 8, & f(4) &= 22, \\ f(5) &= 65, & f(6) &= 209, \\ f(7) &= 732, & f(8) &= 2780, \end{aligned}$$

and suspect that

$$\min_{n\geq 1} \frac{f(n + 1)}{f(n)} = \frac{f(3)}{f(2)} = \frac{8}{3}.$$

To show this it suffices to show that $\dfrac{f(n + 1)}{f(n)} \geq 3$ for $n \geq 5$.

Now

$$\begin{aligned} f(n + 1) &= \sum_{k=1}^{n+1} k^{n+2-k} \\ &\geq 1 + 2^n + 3^{n-1} + 4^{n-2} + 5^{n-3} + \sum_{k=6}^{n} 3k^{n+1-k} \end{aligned}$$

(if $n \geq 5$), and since

$$3f(n) = 3 + 3\cdot 2^{n-1} + 3\cdot 3^{n-2} + 3\cdot 4^{n-3} + 3\cdot 5^{n-4} + \sum_{k=6}^{n} 3k^{n+1-k},$$

it suffices to show that for $n \geq 5$,

$$1 + 2^n + 3^{n-1} + 4^{n-2} + 5^{n-3} \geq 3 + 3\cdot 2^{n-1} + 3\cdot 3^{n-2} + 3\cdot 4^{n-3} + 3\cdot 5^{n-4},$$

or equivalently

$$4^{n-3} + 2\cdot 5^{n-4} \geq 2^{n-1} + 2.$$

But $2n - 6 \geq n - 1$ and $n - 4 > 0$ for $n \geq 5$, so

$$4^{n-3} + 2\cdot 5^{n-4} = 2^{2n-6} + 2\cdot 5^{n-4} > 2^{n-1} + 2,$$

completing the proof.

3. [1985: 212] *16th Austrian Mathematical Olympiad (Final Round).*

A line intersects the sides (or sides produced) BC, CA, AB of a triangle ABC in the points A_1, B_1, C_1, respectively. The points A_2, B_2, C_2 are symmetric to A_1, B_1, C_1 with respect to the midpoints of BC, CA, AB, respectively. Prove that A_2, B_2, and C_2 are collinear.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

For any point P in the plane of triangle ABC, we let $\mathbf{P}$ denote the vector whose head is at P and whose tail is at C. Since A_1, B_1, C_1 lie on BC, CA, AB, respectively,

$$\mathbf{A}_1 = c_1\mathbf{B}$$
$$\mathbf{B}_1 = c_2\mathbf{A}$$
$$\mathbf{C}_1 = \mathbf{A} + c_3(\mathbf{B} - \mathbf{A})$$

for some constants c_1, c_2, c_3. Since any 3 points P, Q, R are collinear if and only if

$$(\mathbf{R} - \mathbf{P}) \times (\mathbf{Q} - \mathbf{P}) = 0$$

we must have

$$\begin{aligned} 0 &= (\mathbf{B}_1 - \mathbf{A}_1) \times (\mathbf{C}_1 - \mathbf{A}_1) \\ &= (c_2\mathbf{A} - c_1\mathbf{B}) \times [(1 - c_3)\mathbf{A} + (c_3 - c_1)\mathbf{B}] \\ &= [c_2(c_3 - c_1) + c_1(1 - c_3)](\mathbf{A} \times \mathbf{B}) \\ &= (c_1 - c_1c_2 - c_1c_3 + c_2c_3)(\mathbf{A} \times \mathbf{B}). \end{aligned} \qquad (1)$$

The midpoints of AB, BC, CA are points whose corresponding vectors are $(\mathbf{A} + \mathbf{B})/2$, $\mathbf{B}/2$, $\mathbf{A}/2$, respectively. By definition

$$\mathbf{A}_2 = \mathbf{B} - \mathbf{A}_1 = (1 - c_1)\mathbf{B}$$
$$\mathbf{B}_2 = \mathbf{A} - \mathbf{B}_1 = (1 - c_2)\mathbf{A}$$
$$\mathbf{C}_2 = \mathbf{A} + \mathbf{B} - \mathbf{C}_1 = c_3\mathbf{A} + (1 - c_3)\mathbf{B}.$$

Now A_2, B_2, C_2 are collinear if and only if

$$(\mathbf{B}_2 - \mathbf{A}_2) \times (\mathbf{C}_2 - \mathbf{A}_2) = 0$$
$$\Longleftrightarrow [(1 - c_2)\mathbf{A} - (1 - c_1)\mathbf{B}] \times [c_3\mathbf{A} + (c_1 - c_3)\mathbf{B}] = 0$$

$$\iff [(1 - c_2)(c_1 - c_3) + (1 - c_1)c_3](A \times B) = 0$$

$$\iff (c_1 - c_1c_2 - c_1c_3 + c_2c_3)(A \times B) = 0.$$

Since this last condition is just (1), A_2, B_2, C_2 are indeed collinear.

4. [1985: 212] *16th Austrian Mathematical Olympiad (Final Round).*

Determine all natural numbers n such that the equation

$$a_{n+1}x^2 - 2x\sqrt{a_1^2 + a_2^2 + \dots + a_{n+1}^2} + a_1 + a_2 + \dots + a_n = 0$$

has real solutions for all real $a_1, a_2, \dots, a_{n+1}$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

If $a_{n+1} = 0$, the given equation has the real solution

$$x = \frac{\sum_{i=1}^{n} a_i}{2\left[\sum_{i=1}^{n+1} a_i^2\right]^{1/2}},$$

unless this denominator is zero, in which case each $a_i = 0$ and any real x is a solution.

If $a_{n+1} \neq 0$, the equation will have a real solution just in case the discriminant is nonnegative, or equivalently,

$$\sum_{i=1}^{n+1} a_i^2 \geq a_{n+1} \sum_{i=1}^{n} a_i,$$

that is,

$$\left[a_{n+1} - \frac{1}{2}\sum_{i=1}^{n} a_i\right]^2 + \sum_{i=1}^{n} a_i^2 - \frac{1}{4}\left[\sum_{i=1}^{n} a_i\right]^2 \geq 0.$$

This holds for all real $a_1, \dots, a_{n+1}$, $a_{n+1} \neq 0$, if and only if

$$\sum_{i=1}^{n} a_i^2 - \frac{1}{4}\left[\sum_{i=1}^{n} a_i\right]^2 \geq 0. \qquad (*)$$

In particular, (*) must hold for $(a_1, \dots, a_n) = (1, \dots, 1)$, so

$$n - \frac{1}{4}n^2 \geq 0.$$

Necessarily then, $n \leq 4$. Conversely, if $n \leq 4$, Cauchy's inequality gives

$$\left[\sum_{i=1}^{n} a_i\right]^2 \leq \left[\sum_{i=1}^{n} 1^2\right]\left[\sum_{i=1}^{n} a_i^2\right] \leq 4\sum_{i=1}^{n} a_i^2$$

for all $a_1, \dots, a_n$. This is evidently equivalent to (*), so the values are $n = 1, 2, 3, 4$.

5. [1985: 212] *16th Austrian Mathematical Olympiad (Final Round).*

Let $\{a_n\}$ be a sequence of natural numbers satisfying

$$a_n = \sqrt{(a_{n-1}^2 + a_{n+1}^2)/2}$$

for all $n \geq 1$. Prove that the sequence is a constant one.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

We square both sides of the equation and rearrange terms to obtain

$$a_{n+1}^2 - a_n^2 = a_n^2 - a_{n-1}^2.$$

Thus

$$a_n^2 - a_0^2 = \sum_{i=0}^{n-1} (a_{n-i}^2 - a_{n-i-1}^2) = \sum_{i=0}^{n-1} (a_1^2 - a_0^2),$$

or

$$a_n^2 = n(a_1^2 - a_0^2) + a_0^2. \qquad (*)$$

If $a_1^2 < a_0^2$, then $a_n^2 < 0$ for n sufficiently large, a contradiction.

If $a_1^2 > a_0^2$, we have $a_{n+1}^2 > a_n^2$. Since each a_i is an integer we have then $a_{n+1} \geq a_n + 1$, and so (a_n) is a strictly increasing, unbounded sequence of integers. But the inequality $a_{n+1} \geq a_n + 1$ implies

$$1 + 2a_n \leq a_{n+1}^2 - a_n^2 = a_1^2 - a_0^2$$

for all n, contradicting the unboundedness of (a_n).

Hence $a_1^2 = a_0^2$, and so by $(*)$ $a_n = a_0$ for all n.

6. [1985: 213] *16th Austrian Mathematical Olympiad (Final Round).*

Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$x^2 f(x) + f(1 - x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Suppose f is such a function. Replace x by $1 - x$ in the equation to obtain

$$(1 - x)^2 f(1 - x) + f(x) = 2(1 - x) - (1 - x)^4.$$

We add this equation to $[-(1 - x)^2]$ times the given equation, finding

$$\begin{aligned}(1 - x^2(1 - x)^2)f(x) &= 2(1 - x) - (1 - x)^4 - (1 - x)^2(2x - x^4)\\ &= (x - 1)(x + 1)(x^2 - x + 1)(x^2 - x - 1).\end{aligned}$$

The left-hand side of this equation is $(1 + x - x^2)(1 - x + x^2)f(x)$ and so

$$(1 + x - x^2)(1 - x + x^2)(f(x) + x^2 - 1) = 0.$$

Now $1 - x + x^2 \neq 0$ for $x \in \mathbb{R}$, thus $f(x) = 1 - x^2$ unless $1 + x - x^2 = 0$, i.e. unless $x = \dfrac{1 \pm \sqrt{5}}{2}$. Setting $x = \dfrac{1 + \sqrt{5}}{2}$ or $x = \dfrac{1 - \sqrt{5}}{2}$ in the original equation yields

$$\left[\frac{3+\sqrt{5}}{2}\right] f\left[\frac{1+\sqrt{5}}{2}\right] + f\left[\frac{1-\sqrt{5}}{2}\right] = \frac{-5-\sqrt{5}}{2}. \qquad (1)$$

Conversely, suppose $f: \mathbb{R} \to \mathbb{R}$ satisfies $f(x) = 1 - x^2$ for $x \neq \frac{1 \pm \sqrt{5}}{2}$ and also satisfies equation (1). Then, if $x \neq \frac{1 \pm \sqrt{5}}{2}$, also $1 - x \neq \frac{1 \pm \sqrt{5}}{2}$, and the functional equation is satisfied since

$$x^2(1 - x^2) + [1 - (1 - x)^2] = 2x - x^4.$$

If $x = \frac{1 \pm \sqrt{5}}{2}$, the given equation holds, by (1). Thus, any such f will indeed solve the problem.

*   *   *

In the next issue we hope to finish the solutions that we have received for problems posed in 1985 and publish a list of problems whose solutions have not yet been discussed. The next Corner will also mark the return to a mixture of solutions submitted some time ago and those submitted recently. Keep those solutions coming!

*   *   *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before August 1, 1988, although solutions received after that date will also be considered until the time when a solution is published.

1301. *Proposed by George Szekeres, University of New South Wales, Kensington, Australia.*

Given a positive rational number $q = a/b$ and an odd positive integer n, find a polynomial, with integer coefficients written in a simple closed form, that has $q^{1/n} + q^{-1/n}$ as a root. (See Crux 1187 [1988: 30].)

1302. *Proposed by Mihaly Bencze, Brasov, Romania.*

Suppose $\alpha_k > 0$ for $k = 1,2,\ldots,n$ and $\sum_{k=1}^{n} \tanh^2\alpha_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sinh \alpha_k} \geq n \sum_{k=1}^{n} \frac{\sinh \alpha_k}{\cosh^2\alpha_k} .$$

1303. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC and $A_1B_1C_1$ be two triangles with sides a, b, c and a_1, b_1, c_1 and inradii r and r_1, and let P be an interior point of ΔABC. Set $AP = x$, $BP = y$, $CP = z$. Prove that

$$\frac{a_1x^2 + b_1y^2 + c_1z^2}{a + b + c} \geq 4rr_1 .$$

1304. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

If p, q, r are the real roots of

$$x^3 - 6x^2 + 3x + 1 = 0,$$

determine the possible values of

$$p^2q + q^2r + r^2p$$

and write them in a simple form.

1305. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be an acute triangle with circumcenter O. Let P_1, Q_1 ($Q_1 \neq A_1$) denote the intersection of A_1O with A_2A_3 and with the circumcircle, respectively, and define P_2, Q_2, P_3, Q_3 analogously. Prove that

(a) $$\frac{\overline{OP_1}\cdot\overline{OP_2}\cdot\overline{OP_3}}{\overline{P_1Q_1}\cdot\overline{P_2Q_2}\cdot\overline{P_3Q_3}} \geq 1;$$

(b) $$\frac{\overline{OP_1}}{\overline{P_1Q_1}} + \frac{\overline{OP_2}}{\overline{P_2Q_2}} + \frac{\overline{OP_3}}{\overline{P_3Q_3}} \geq 3;$$

(c) $$\frac{\overline{A_1P_1}\cdot\overline{A_2P_2}\cdot\overline{A_3P_3}}{\overline{P_1Q_1}\cdot\overline{P_2Q_2}\cdot\overline{P_3Q_3}} \geq 27.$$

1306. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Ellipses

$$\frac{x^2}{a_i^2} + \frac{y^2}{b_i^2} = 1 , \qquad i = 1,2,\ldots,n,$$

all satisfy the condition

$$\frac{1}{a_i^2} + \frac{1}{b_i^2} = 3.$$

Prove that the ellipses all pass through the same point.

1307. *Proposed by Jordi Dou, Barcelona, Spain.*

Let A', B', C' be the intersections of the bisectors of triangle ABC with the opposite sides, and let A'', B'', C'' be the midpoints of $B'C'$, $C'A'$, $A'B'$ respectively. Prove that AA'', BB'', CC'' are concurrent.

1308. *Proposed by Seung-Jin Bang, Seoul, Korea.*

Find $f(x,y)$ and $g(x,y)$ such that

$$\text{(i) } \frac{\partial f}{\partial x} = \frac{1}{g}\cdot\frac{\partial g}{\partial y}, \qquad \text{(ii) } \frac{\partial f}{\partial y} = -\frac{1}{g}\cdot\frac{\partial g}{\partial x},$$

and

$$\text{(iii) } g(x,y)\sin f(x,y) = x$$

all hold.

1309. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let ABC be a triangle with circumcircle Γ, and let DEF be the triangle formed by the lines tangent to Γ at A, B, C. Call a triangle $A'B'C'$ a *circumcevian triangle* if for some point P, A' is the point other than A where the line AP meets Γ, and similarly for B' and C'. Prove that DEF is perspective with every circumcevian triangle.

1310. *Proposed by Robert E. Shafer, Berkeley, California.*

Let

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \frac{N_n}{D_n}$$

where N_n and D_n are positive integers having no common divisor. Find all primes $p \geq 5$ such that $p|N_{p-4}$.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1110*. [1986: 13; 1987: 170] *Proposed by M.S. Klamkin, University of Alberta.*

How may different polynomials $P(x_1,x_2,\ldots,x_m)$ of degree n are there for which the coefficients of all the terms are 0's or 1's and

$$P(x_1,x_2,\ldots,x_m) = 1 \quad \text{whenever} \quad x_1 + x_2 + \ldots + x_m = 1?$$

I. *Partial solution by Len Bos and Bill Sands, University of Calgary, Calgary, Alberta.*

Let $f(n,m)$ be the required number of polynomials. We will investigate the case $m = 2$ and will show that

$$f(n,2) \geq \frac{1}{n+1}\binom{2n}{n} ,$$

the nth Catalan number.

Let $P(x,y)$ be a polynomial of degree n with the required properties. Then $P(x,y) = 1$ whenever $x + y = 1$, so it must be true that

$$P(x,y) = (x + y - 1)q(x,y) + 1 \tag{1}$$

for some polynomial $q(x,y)$ with integer coefficients. We shall count all those possible $q(x,y)$ whose coefficients are also all 0 or 1.

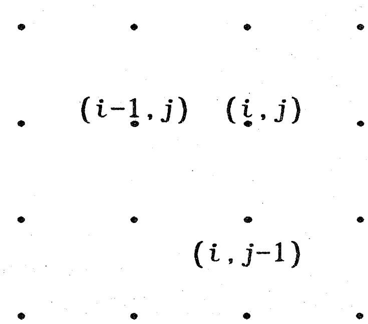
Let q be such a polynomial. Then its terms are monomials of the form x^iy^j where $i,j \in \{0,1,\ldots,n-1\}$ and $i + j \leq n - 1$. We will identify the collection of these monomials with the corresponding subset of lattice points

$$R_q = \{(i,j)\,|\,x^iy^j \text{ is a monomial in } q\}.$$

Thus R_q is a subset of $\{(i,j) \in \mathbb{Z}^2\,|\,0 \leq i,\ 0 \leq j,\ i + j \leq n - 1\}$ and contains at least one (i,j) with $i + j = n - 1$. The next two lemmas establish important properties of R_q.

Lemma 1. If $(i,j) \in R_q$ where $i + j > 0$, then either $(i - 1,j) \in R_q$ or $(i,j - 1) \in R_q$ (or both). In particular, if $(i,0) \in R_q$ then $(i - 1,0) \in R_q$ for $i > 0$, and similarly for $(0,j)$.

Proof. If $(i,j) \in R_q$ and $i + j > 0$ then x^iy^j is a monomial in q. Thus $(x + y - 1)q(x,y)$, when multiplied out, will contain a term $-x^iy^j$. By (1), it must therefore also contain at least one term $+x^iy^j$, which can only happen if $x^{i-1}y^j$ or x^iy^{j-1} were monomials in q, that is, if $(i - 1,j)$ or $(i,j - 1) \in R_q$. □



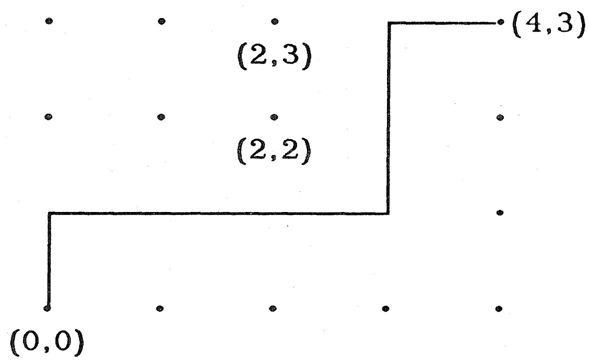
In terms of lattice points, this lemma says that if a lattice point is in R_q, then at least one of its neighbours to the left of or below it must also be in R_q.

Our other lemma is a sort of converse.

Lemma 2. If $(i - 1,j) \in R_q$ and $(i,j - 1) \in R_q$, then $(i,j) \in R_q$.

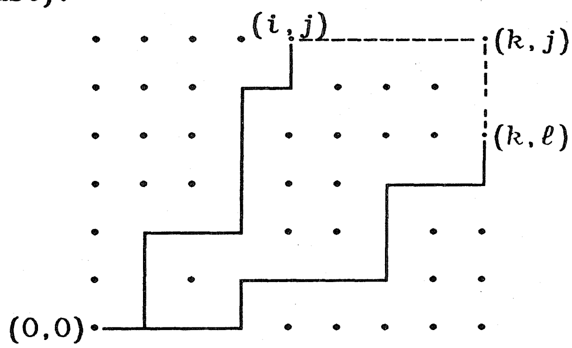
Proof. We have that $x^{i-1}y^j$ and x^iy^{j-1} are both monomials in q. Then $(x + y - 1)q(x,y)$ when multiplied out will contain *two* terms x^iy^j. By (1), it must also contain a term $-x^iy^j$, which implies that $(i,j) \in R_q$. □

Now suppose $(i,j) \in R_q$. By applying Lemma 1 repeatedly, we obtain a descending path of lattice points in R_q from (i,j) to $(0,0)$. By always moving left from a lattice point rather than down, whenever we have a choice, we obtain what we call the *left path* of (i,j). Similarly by moving down instead of left whenever possible, we obtain the *right path* of (i,j). The diagram shows a possible left path of $(4,3)$. All lattice points on the path are in R_q, but the position of the path tells us that $(2,3)$ and $(2,2)$ are *not* in R_q.



Clearly the left path and right path of (i,j) do not cross, although they may meet (and do, at their endpoints at least).

Let (i,j) and (k,ℓ) be in R_q, where we assume that $i < k$ and $j > \ell$. Consider the left path of (i,j) and the right path of (k,ℓ). Extend them to paths beginning at (k,j) by adding horizontal and vertical edges, respectively. The extended paths then enclose a region of lattice points.

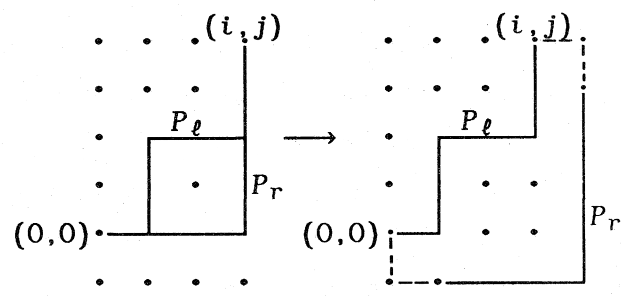


Claim: *Every lattice point inside this region is in* R_q.

This follows by repeated applications of Lemma 2, starting at the bottom left of the region and working up and to the right.

Now it can easily be seen that R_q must coincide with the region of lattice points bounded by the left and right paths of some lattice point (i,j), where $i + j = n - 1$. Furthermore we claim that any such region corresponds to a polynomial $q(x,y)$ such that $P(x,y)$, defined by (1), is a polynomial satisfying the problem. We need only show that $P(x,y)$ has coefficients 0 or 1. Multiplying out $(x + y - 1)q(x,y)$, we need only show that any negative term $-x^iy^j$, $i + j > 0$, is offset by at least one term $+x^iy^j$, and that if two terms x^iy^j occur then also a term $-x^iy^j$ will occur. But this follows from the construction of the region much as in the proofs of Lemmas 1 and 2.

Thus to count all the polynomials $q(x,y)$ we must count the number of pairs of lattice paths P_ℓ and P_r from $(0,0)$ to (i,j), $i + j = n - 1$, which do



not cross and have length $n - 1$. By moving path P_r one unit to the right and one unit down, adding in new common endpoints as shown, and moving both paths to start at (0,0) again, we see that such pairs of paths correspond to those pairs of paths from (0,0) to (i,j), where $i + j = n + 1$, which *do not meet* (except at their endpoints), and which have length $n + 1$. The number of such pairs of paths, over all choices of i, j satisfying $i \geq 0$, $j \geq 0$, $i + j = n + 1$, is known to be the Catalan number

$$\frac{1}{n+1}\binom{2n}{n}$$

(see J. Levine, Note on the number of pairs of non-intersecting routes, *Scripta Mathematica* 24 (1959) 335-338). This number is then a lower bound for $f(n,2)$. Unfortunately it is not the exact answer, since putting

$$q(x,y) = 1 + x + y + 2xy + x^2y + xy^2 + x^2y^2$$

(which has a coefficient not equal to 0 or 1) into (1) yields

$$P(x,y) = x^2 + y^2 + x^2y + xy^2 + x^3y + xy^3 + x^2y^2 + x^3y^2 + x^2y^3,$$

a polynomial with all coefficients 0 or 1. We do believe, however, that $f(n,2)$ *can* be calculated, and, as a possible first step, make the following conjecture:

any $q(x,y)$ *suitable for* (1) *has all coefficients* 0, 1, *or* 2.

II. *Partial solution by P. Penning, Delft, The Netherlands.* (Adapted by the editor to refer to *I* above.)

We show that

$$f(n,m) \geq m^{n-1},$$

thus answering the editor's request [1987: 170] for a proof that $f(n,m) \geq 1$ for each n and m.

A special case of the allowable "regions" in part *I* is that of a single path from (0,0) to (i,j), where $i + j = n - 1$. A similar argument to that in *I* shows more generally that if P is a path of length $n - 1$ from $(0,0,\ldots,0)$ to $(i_1,i_2,\ldots,i_m)$ in $\mathbb{Z}^m$, where $\sum_{j=1}^{m} i_j = n - 1$, then the lattice points on P will correspond to monomials whose sum is a polynomial $q(x_1,\ldots,x_m)$ such that

$$P(x_1,\ldots,x_m) = (x_1 + \ldots + x_m - 1)q(x_1,\ldots,x_m) + 1$$

has all coefficients 0 or 1. To construct such a path, we merely choose a sequence of $n - 1$ elements from $x_1,\ldots,x_m$, repetition allowed, each corresponding to one of the m "directions" the path can take (starting at $(0,\ldots,0)$). The number of these sequences is m^{n-1}.

Example: $m = 5$, $n = 4$. Choose sequence x_2, x_4, x_3. Then

$$q(x_1,x_2,x_3,x_4,x_5) = 1 + x_2 + x_2x_4 + x_2x_4x_3,$$

so

$$\begin{aligned} P(x_1,\ldots,x_5) &= (1 + x_2 + x_2x_4 + x_2x_4x_3)(x_1 + x_2 + x_3 + x_4 + x_5 - 1) + 1 \\ &= x_1 + x_3 + x_4 + x_5 + x_2(x_1 + x_2 + x_3 + x_5) \\ &\quad + x_2x_4(x_1 + x_2 + x_4 + x_5) + x_2x_3x_4(x_1 + x_2 + x_3 + x_4 + x_5). \end{aligned}$$

Examples showing $f(n,m) \ge 1$ for all n and m were also received from LEN BOS, University of Calgary; and the proposer.

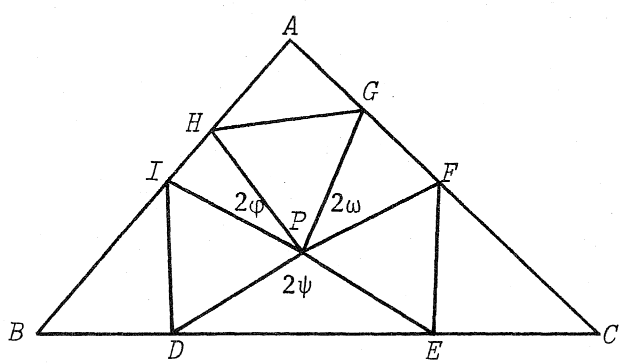
* * *

<u>1174</u>. [1986: 205] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Suppose ABC is an acute triangle. Prove that there is a point P inside ABC and points D, E on BC; F, G on CA; and H, I on AB such that GPH, IPD, and EPF are congruent equilateral triangles.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let's go the other way round. Suppose we have a point P and three congruent, possibly overlapping equilateral triangles GPH, IPD, and EPF, making the (directed!) angles $\angle HPI = 2\varphi$, $\angle DPE = 2\psi$, $\angle FPG = 2\omega$.



Case 1. $\varphi, \psi, \omega \ge 0$.

Then

$$\angle PHI = \angle PIH = 90^\circ - \varphi, \text{ etc.},$$

and thus

$$\angle AHG = 180^\circ - 60^\circ - (90^\circ - \varphi) = \varphi + 30^\circ = \angle BID, \text{ etc.}$$

Finally,

$$\begin{aligned} \angle A &= 180^\circ - (\omega + 30^\circ) - (\varphi + 30^\circ) = 120^\circ - \omega - \varphi \\ \angle B &= 120^\circ - \varphi - \psi \\ \angle C &= 120^\circ - \psi - \omega, \end{aligned} \tag{1}$$

yielding

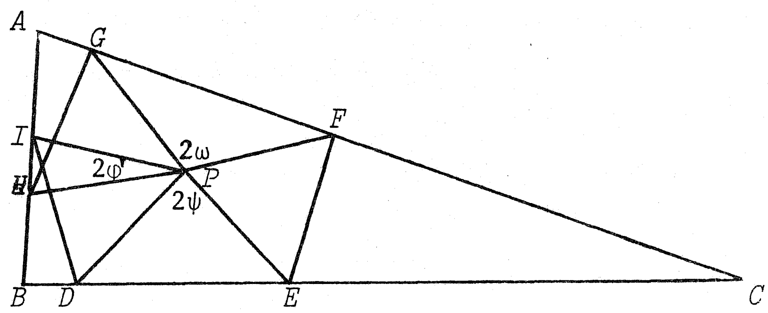
$$\varphi = \angle C - 30^\circ, \quad \psi = \angle A - 30^\circ, \quad \omega = \angle B - 30^\circ. \tag{2}$$

Thus in this case we must have $\min\{\angle A,\angle B,\angle C\} \ge 30^\circ$. Then if we construct the three equilateral triangles as above, using φ, ψ, ω defined in (2), and extend edges DE, FG, HI to form a triangle, by (1) we obtain a triangle similar to

ΔABC. Hence the required points will exist for ΔABC.

Case 2. One of φ, ψ, ω is < 0.

Let $\varphi < 0$. Then



$\angle IPH = 2\varphi'$, $\varphi' = -\varphi > 0$.

As in Case 1,

$\angle PHI = \angle HIP = 90^\circ - \varphi'$

and thus

$\angle GHA = 30^\circ - \varphi' = 30^\circ + \varphi = \angle BID$.

Hence (1) and (2) again hold, and we are done as in Case 1.

Case 3. Two of φ, ψ, ω are < 0.

If say $\omega < 0$ and $\varphi < 0$, then $\angle DPE > 180^\circ$, and P would be outside ΔABC. However, $\angle A > 120^\circ$ from (1), so ΔABC would not be acute.

II. *Comment by Clifford Gardner, Austin, Texas and Jack Garfunkel, Flushing, N.Y.*

We were intrigued by this point P, and suspected that it may have some special property. Our suspicion was justified. The point P turns out to be the Miquel point associated with any equilateral triangle inscribed in triangle ABC. Let Q, R, S be the midpoints of DE, FG, HI respectively; then the proof that P is this Miquel point depends on showing that $QR = RS = SQ$. Proofs of this abound. For instance, the problem was given as B1 of the 1967 Putnam examination, and a proof using complex numbers can be found on page 737 of the 1968 *American Mathematical Monthly*.

We conclude with a final comment. One of the reasons why Morley's Theorem is so popular is the surprise element. An equilateral triangle "mysteriously" emerges by drawing angle trisectors. A similar element of surprise exists here. The Miquel point which is the result of drawing three intersecting circles turns out to be the same point from which congruent equilateral triangles are drawn to the sides of a triangle.

Also solved by JORDI DOU, *Barcelona, Spain;* J.T. GROENMAN, *Arnhem, The Netherlands;* CLIFFORD GARDNER, *Austin, Texas and* JACK GARFUNKEL, *Flushing, N.Y.;* D.J. SMEENK, *Zaltbommel, The Netherlands;* GEORGE TSINTSIFAS, *Thessaloniki, Greece; and the proposer.*

Several solvers noted that the result holds for any triangle with no angle $\geq 120^\circ$. *Tsintsifas also showed that* P *is the Miquel point for equilateral triangles inscribed in* ΔABC, *but credited the conjecture to Garfunkel.*

* * *

1175. [1986: 205] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that if α, β, γ are the angles of a triangle,

$$-2 < \sin 3\alpha + \sin 3\beta + \sin 3\gamma \le \frac{3}{2}\sqrt{3}.$$

Comment by M.S. Klamkin, University of Alberta.

I had set the same problem for the 1981 U.S.A. Mathematical Olympiad (see [1981: 141]). Incidentally, the lower bound is easy to obtain. At least one angle, say α, is $\le 60°$. Then since $\sin 3\alpha \ge 0$, $\sin 3\beta \ge -1$, and $\sin 3\gamma \ge -1$, we obtain the lower bound -2. There is equality only for the degenerate triangle $0°$, $90°$, $90°$. For the upper bound, including more general results where 3α is replaced by $n\alpha$, etc., see Crux 715 and its solutions [1983: 58-62].

Also solved by GEORGE EVAGELOPOULOS, *Athens, Greece;* RICHARD I. HESS, *Rancho Palos Verdes, California;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* EDWIN M. KLEIN, *University of Wisconsin, Whitewater, Wisconsin;* KEE-WAI LAU, *Hong Kong;* J. WALTER LYNCH, *Georgia Southern College, Statesboro, Georgia;* D.S. MITRINOVIC *and* J.E. PECARIC, *University of Belgrade, Yugoslavia;* V.N. MURTY, *Penn State University, Middletown, Pennsylvania;* M. PARMENTER, *Memorial University of Newfoundland, St. John's;* BOB PRIELIPP, *University of Wisconsin-Oshkosh; and the proposer.*

The editor apologizes for including this problem when (as also pointed out by several of the above) it had appeared before.

* * *

1176. [1986: 205] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.* (Dedicated to Léo Sauvé.)

Let n be squarefree such that

$$n = r^2 + s^2 = t^2 + u^2$$

where r, s, t, u are positive integers. Prove that

$$2n(n + rt + su)$$

is a square if and only if $r = t$ and $s = u$.

Solution by Kee-wai Lau, Hong Kong.

Denote the positive integer $2n(n + rt + su)$ by m. Clearly if $r = t$ and $s = u$, m equals the square $(2n)^2$.

We now suppose that m is a square. We first note that

$$m > 2n^2. \qquad (1)$$

Since $n = r^2 + s^2 = t^2 + u^2$ we see that

$$m = n[4n - (r - t)^2 - (s - u)^2]. \qquad (2)$$

Because n is squarefree, m is a square if and only if

$$4n - (r - t)^2 - (s - u)^2 = k^2 n \qquad (3)$$

where k is a positive integer. From (2) and (3) we get

$$m = k^2 n^2. \qquad (4)$$

From (1) and (4) we deduce that

$$k > 1. \qquad (5)$$

Now (3) can be rewritten as

$$n(4 - k^2) = (r - t)^2 + (s - u)^2. \qquad (6)$$

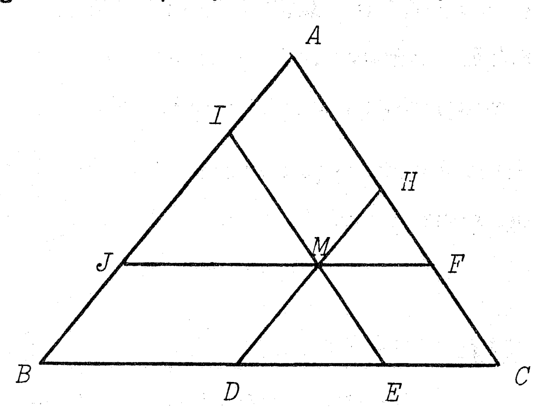
Since the right hand side of (6) is non-negative, it follows from (5) that $k = 2$. Thus from (6) $r = t$ and $s = u$ as required.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MIKE PARMENTER, Memorial University of Newfoundland, St. John's; and the proposer. One incorrect solution was received.

*   *   *

<u>1177</u>. [1986: 205] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

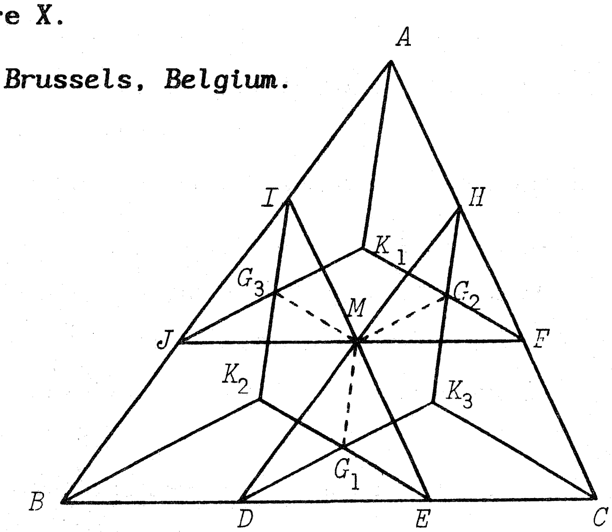
ABC is a triangle and M an interior point with barycentric coordinates $(\lambda_1,\lambda_2,\lambda_3)$. Lines HMD, JMF, EMI are parallel to AB, BC, CA respectively as shown. The centroids of triangles DME, FMH, IMJ are denoted G_1, G_2, G_3 respectively. Prove that



$$[G_1G_2G_3] = \frac{(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)[ABC]}{3},$$

where $[X]$ stands for the area of figure X.

Solution by C. Festraets-Hamoir, Brussels, Belgium.



Let K_1, K_2, K_3 be the centroids of the triangles AJF, BEI, CDH respectively.

The medians FK_1, MG_3, EK_2 drawn respectively in the homothetic triangles AFJ, IMJ, IEB are parallel. Similarly, $JK_1 \parallel MG_2 \parallel DK_3$ and $IK_2 \parallel MG_1 \parallel HK_3$. Thus $MG_3K_1G_2$, $MG_2K_3G_1$, and $MG_1K_2G_3$ are parallelograms, and

$$[G_1G_2G_3] = \frac{1}{2}[G_1K_2G_3K_1G_2K_3]$$

$$= \frac{1}{2}\{[ABC] - ([JK_1A] + [AK_1F] + [HK_3C] + [CK_3D] + [EK_2B] + [BK_2I]) + [IG_3J] + [HG_2F] + [EG_1D]\}$$

$$= \frac{1}{2}\{[ABC] - \frac{2}{3}([AJF] + [CHD] + [BEI]) + \frac{1}{3}([MJI] + [MHF] + [MED]\}$$

$$= \frac{1}{2}\{[ABC] - \frac{2}{3}((1 - \lambda_1)^2 + (1 - \lambda_3)^2 + (1 - \lambda_2)^2)[ABC] + \frac{1}{3}(\lambda_3^2 + \lambda_2^2 + \lambda_1^2)[ABC]\}.$$

Now, using $\lambda_1 + \lambda_2 + \lambda_3 = 1$,

$$[G_1G_2G_3] = \frac{1}{6}[ABC](1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2)$$

$$= \frac{1}{6}[ABC]\{1 - (\lambda_1 + \lambda_2 + \lambda_3)^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1\}$$

$$= \frac{1}{3}[ABC](\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1).$$

Also solved by J.T. GROENMAN, *Arnhem, The Netherlands*; RICHARD I. HESS, *Rancho Palos Verdes, California*; WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria*; MURRAY S. KLAMKIN, *University of Alberta; and the proposer.*

As pointed out by one reader, the question should perhaps have read "normalized *barycentric coordinates*".

* * *

<u>1178</u>. [1986: 206] *Proposed by Gary Gislason, University of Alaska, Fairbanks, Alaska, and M.S. Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated to Léo Sauvé.)

Determine pairs of functions (F,G) such that

$$(F\circ G)' = F\circ G' + F'\circ G$$

where $\circ$ denotes composition and ' denotes differentiation.

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The given relation can also be written in the form

$$F'(G(x))G'(x) = F(G'(x)) + F'(G(x)),$$

i.e.

$$F'(G(x))[G'(x) - 1] = F(G'(x)). \qquad (1)$$

Thus, e.g.,

$$G(x) = x + a, \quad F \text{ differentiable with } F(1) = 0,$$

where $a \in \mathbb{R}$, satisfy (1). Or:

$$F(x) = ax - a, \quad G \text{ arbitrary},$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$, satisfy (1).

II. *Solution by the proposers.*

Assuming $F'(x)$ has an inverse function $H(x)$, we give an implicit parametric representation for G and x in terms of G'.

By hypothesis,

$$F'(G) = F(G')/(G' - 1).$$

Thus

$$G = H[F(G')/(G' - 1)]. \tag{1}$$

Since G is differentiable, we obtain

$$G' = H'[F(G')/(G' - 1)]\left[\frac{(G' - 1)F'(G')dG'/dx - F(G')dG'/dx}{(G' - 1)^2}\right].$$

Then

$$x = \int \frac{H'[F(G')/(G' - 1)][(G' - 1)F'(G') - F(G')]}{G'(G' - 1)^2}\,dG. \tag{2}$$

(1) and (2) give the parametric representation for G and x.

For example, when $F(x) = e^x$ we get

$$G = G' - \ell n(G' - 1)\ , \quad x = \ell n(G'^2/(G' - 1)),$$

and so

$$G(x) = \frac{e^x \pm \sqrt{e^{2x} - 4e^x}}{2} - \ell n\left[\frac{e^x \pm \sqrt{e^{2x} - 4e^x}}{2} - 1\right].$$

Also solved by GLEN E. MILLS, Colonial Senior High, Orlando, Florida. Both Mills and the proposers gave Janous' second example.

* * *

1179. [1986: 206] *Proposed by Jack Garfunkel, Flushing, New York.*

Squares are erected outwardly on each side of a quadrilateral $ABCD$.

(a) Prove that the centers of these squares are the vertices of a quadrilateral $A'B'C'D'$ whose diagonals are equal and perpendicular to each other.

(b)* If squares are likewise erected on the sides of $A'B'C'D'$, with centers A'', B'', C'', D'', and this procedure is continued, will quadrilateral $A^{(n)}B^{(n)}C^{(n)}D^{(n)}$ tend to a square as n tends to infinity?

Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

(a) This is an old theorem of von Aubel [*Editor's aside*: does anyone have a reference?]. For completeness, we include a proof (see also [1] or [2]). Let A, B, C, D be complex number representations of the vertices. Then it follows easily that

$$2A' = A + B + i(A - B), \quad \text{etc.}$$

Thus

$$2(A' - C') = A + B - C - D + i(A + D - B - C)$$

and

$$2(B' - D') = B + C - A - D + i(A + B - C - D),$$

so that

$$B' - D' = i(A' - C'),$$

which proves the theorem.

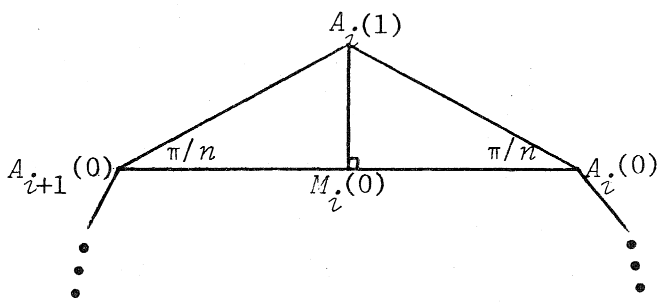
As noted by Kelly in [1], $ABCD$ need not be simple or convex, as long as one uses the same sense of rotation throughout the construction.

(b) We will show the answer is in the affirmative in a more general context. Let $A_1(0), A_2(0), \ldots, A_n(0)$ be complex numbers in a plane representing the n vertices of a given n-gon, simple or not, convex or not. For simplicity, our origin will be the centroid of the vertices, so that

$$A_1(0) + A_2(0) + \ldots + A_n(0) = 0.$$

For each edge $A_i(0)A_{i+1}(0)$, let $M_i(0)$ be its midpoint. Then a new vertex $A_i(1)$ is constructed so that

$$\overline{M_i(0)A_i(1)} = \tan\frac{\pi}{n}\,\overline{M_i(0)A_i(0)}$$



and such that ray $M_i(0)A_i(1)$ is along ray $M_i(0)A_i(0)$ after it has been rotated 90° counterclockwise about $M_i(0)$ (we could just as well use a clockwise rotation).

It is to be noted that if the initial polygon was regular, then the new one $A_1(1), A_2(1), \ldots, A_n(1)$ will also be regular and symmetrically circumscribed about P. We claim that in any case, if this procedure is continued, n-gon $A_1(m), A_2(m), \ldots, A_n(m)$ will approach a regular n-gon as m tends to infinity.

Since

$$M_j(m) = \frac{A_j(m) + A_{j+1}(m)}{2}$$

it follows from the above construction that

$$2A_j(m+1) = A_j(m) + A_{j+1}(m) + i\lambda(A_j(m) - A_{j+1}(m)) \tag{1}$$

for $j = 1, 2, \ldots, n$ ($A_{n+1} = A_1$), where $\lambda = \tan(\pi/n)$. Incidentally, by adding all the equations (1) from $j = 1$ to n, it follows that the centroids of all the iterated polygons are the same. We now make the transformations

$$A_j(m) = \omega^{j-1}(1 + i\lambda)^m B_j(m), \quad j = 1, 2, \ldots, n, \tag{2}$$

where $\omega = e^{2\pi i/n}$. Equations (1) now simplify to

$$2(1 + i\lambda)B_j(m+1) = (1 + i\lambda)B_j(m) + \omega(1 - i\lambda)B_{j+1}(m)$$

or

$$B_j(m+1) = \frac{B_j(m) + B_{j+1}(m)}{2} \qquad (3)$$

since $1 + i\lambda = \omega(1 - i\lambda)$. Adding up all the equations (2) over j, we get that

$$\sum_{j=1}^{n} B_j(m+1) = \sum_{j=1}^{n} B_j(m) = \text{constant} = \sum_{j=1}^{n} B_j(0).$$

Since (3) is a contraction mapping (just take the real parts and imaginary parts separately), each $B_j(m)$ approaches the same limit $\Sigma B_j(0)/n = L$. Then from (2), $A_j(m)/(1 + i\lambda)^m$ approaches the regular polygon with vertices $\omega^{j-1}L$, $j = 1,2,\ldots,n$. Hence the polygon $A_1(m),A_2(m),\ldots,A_n(m)$ approaches a regular polygon, in the sense that its shape approaches regularity. The given problem corresponds to the special case when $n = 4$.[1]

Reference:

[1] P.J. Kelly, Von Aubel's quadrilateral theorem, *Mathematics Magazine* 39 (1966) 35-37.

[2] J. MacNeill, A vector method, *Math. Gazette* no.456 (June 1987) 143-144.

Also solved (both parts) by JORDI DOU, Barcelona, Spain; DANIEL B. SHAPIRO, Ohio State University, Columbus, Ohio; and G. SZEKERES, University of New South Wales, Kensington, Australia. Part (a) (only) solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN PEDOE, University of Minnesota; and the proposer.

Other references to part (a) given by solvers were: A. Schild, *On some properties of the quadrangle, in* Two-year College Mathematics Readings, MAA, *1981, pp. 40-47; J.R. Musselman, solution of Advanced problem* 4034, Amer. Math. Monthly 50 (1943) 459; *and exercise 8.1, p. 44 of D. Pedoe,* A Course of Geometry for Colleges and Universities, *Dover.*

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<u>1180</u>. [1986: 206] *Proposed by J.R. Pounder, University of Alberta, Edmonton, Alberta.* (Dedicated to Léo Sauvé.)

(a) It is well known that the Simson line of a point P on the circumcircle of a triangle T envelopes a deltoid ("Steiner's hypocycloid") as P varies. Show that this is true for an oblique Simson line as well. (An *oblique* Simson line of $T = ABC$ is the line passing through the points A_1, B_1, C_1 chosen on edges BC, CA, AB respectively so that the lines PA_1, PB_1, PC_1 make equal angles (say θ), in the same sense of rotation, with BC, CA, AB

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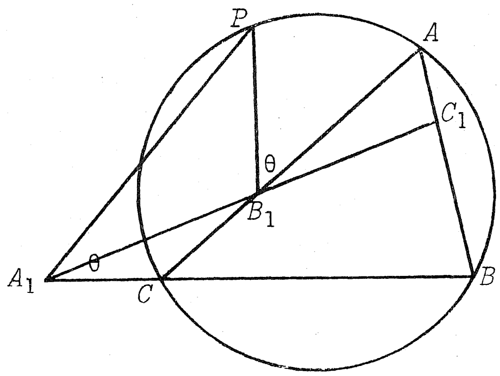
[1][And is therefore, the editor feels compelled to say, a 4-gon conclusion.]

respectively. The usual Simson line occurs when $\theta = 90^\circ$.)

(b)* Given such an "oblique" deltoid for T, locate a triangle T' similar to T such that the "normal" deltoid for T' and the oblique deltoid for T coincide.

Solution of (a) *by the proposer.*



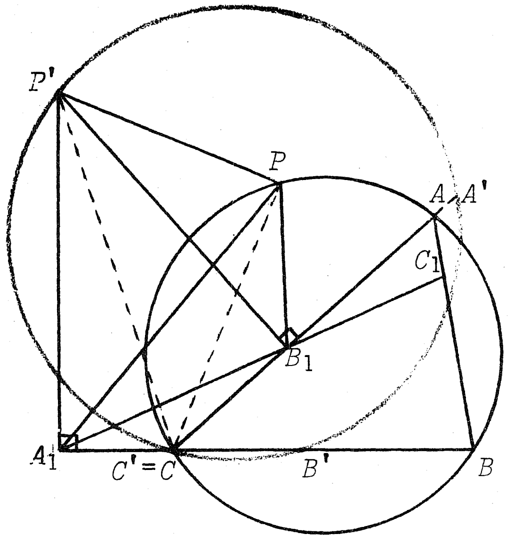
For completeness let us first establish directly that any point P on the circumcircle of a given triangle ABC has an oblique Simson line. Given angle θ, draw line $A_1B_1C_1$ so that $\angle PA_1C = \angle PB_1A = \theta$. Since P, A_1, C, B_1 are concyclic we have

$$\angle PB_1A_1 = \angle PCA_1 = \angle PAB = \angle PAC_1,$$

so that P, A, C_1, B_1 are concyclic, and hence

$$\angle PC_1A = \angle PB_1A = \theta.$$

Thus $A_1B_1C_1$ is an oblique Simson line for ΔABC.



Next we show that $A_1B_1C_1$ is a "normal" Simson line with respect to a certain triangle different from ΔABC but having two of its sides along CA and CB, its third side being determined uniquely by θ. At A_1 and B_1 draw perpendiculars to BC and AC respectively, meeting at P'. The circle through A_1, B_1, C contains both P and P'. Hence $\angle CPP' = 90^\circ$ and $\angle PP'C = \theta$, i.e., CP' is obtained from CP by a rotation through the fixed angle $90^\circ - \theta$ and a magnification in the fixed ratio $\csc\theta$. The locus of P' is therefore the image under this dilatation of the circumcircle Γ of ΔABC, $A_1B_1C_1$ being the normal Simson line of P' with respect to a triangle $A'B'C'$ that is completely determined.

Remark. For (b), it is easy to show that the orientations of the oblique and normal deltoids for the *same* triangle differ by $(90^\circ - \theta)/3$, but I have no *geometrically* simple recipe for the centre of the dilatation required to make them coincide.

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<u>1181</u>. [1986: 241] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo

Sauvé.)

Let x, y, z be real numbers such that

$$xyz(x + y + z) > 0,$$

and let a, b, c be the sides, m_a, m_b, m_c the medians and F the area of a triangle. Prove that

(a) $|yza^2 + zxb^2 + xyc^2| \geq 4F\sqrt{xyz(x + y + z)}$.

(b) $|yzm_a^2 + zxm_b^2 + xym_c^2| \geq 3F\sqrt{xyz(x + y + z)}$.

Solution by G. Tsintsifas, Thessaloniki, Greece.

The formula of Leibniz

$$(\lambda_1 + \lambda_2 + \lambda_3)^2R^2 \geq \lambda_2\lambda_3a^2 + \lambda_3\lambda_1b^2 + \lambda_1\lambda_2c^2$$

(see item 14.1 of O. Bottema et al, *Geometric Inequalities*), for

$$\lambda_1 = yza^2, \quad \lambda_2 = zxb^2, \quad \lambda_3 = xyc^2,$$

together with the well-known fact

$$F = \frac{abc}{4R},$$

gives part (a). For the triangle with sides m_a, m_b, m_c, taking in mind that its area is $3F/4$, the above formula (a) is transformed to (b).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; and the proposers.

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<u>1182</u>. [1986: 241] *Proposed by Peter Andrews and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.* (Dedicated to Léo Sauvé.)

Let $a_1, a_2, \ldots, a_n$ denote positive reals where $n \geq 2$. Prove that

$$\frac{\pi}{2} \leq \tan^{-1}\frac{a_1}{a_2} + \tan^{-1}\frac{a_2}{a_3} + \ldots + \tan^{-1}\frac{a_n}{a_1} \leq \frac{(n - 1)\pi}{2}$$

and for each inequality determine when equality holds.

Solution by Peter Watson-Hurthig, Columbia College, Burnaby, B.C.

If $n = 2$ then

$$\frac{\pi}{2} = \tan^{-1}\frac{a_1}{a_2} + \tan^{-1}\frac{a_2}{a_1} = \frac{(2 - 1)\pi}{2}$$

since the two terms in the sum are complementary angles.

For $n > 2$, set $a_{n+1} = a_1$ and $a_{n+2} = a_2$. Because the sum is composed of at least three positive terms, for the sum to be less than $\pi/2$ it is obviously necessary that the sum of each pair of consecutive terms

$$S_i = \tan^{-1}\frac{a_i}{a_{i+1}} + \tan^{-1}\frac{a_{i+1}}{a_{i+2}},$$

$1 \leq i \leq n$, be less than $\pi/2$. But because $\tan^{-1}$ is an increasing function,

$$S_i < \tan^{-1}\frac{a_i}{a_{i+1}} + \tan^{-1}\frac{a_{i+1}}{a_i} = \frac{\pi}{2}$$

if and only if $a_i < a_{i+2}$, for $1 \le i \le n$. Therefore if n is odd we have

$$a_1 < a_3 < \ldots < a_n < a_{n+2} = a_2 < \ldots < a_{n-1} < a_{n+1} = a_1,$$

and if n is even we have similarly

$$a_1 < a_3 < \ldots < a_{n-1} < a_{n+1} = a_1,$$

both of which are impossible. Therefore $S_i \ge \pi/2$ for at least one i, so the sum must be strictly greater than $\pi/2$ if $n > 2$.

For the second inequality (where $n > 2$) consider the following three cases:

(i) If two or more of the terms $\tan^{-1}(a_i/a_{i+1})$ are less than $\pi/4$, then since

$$0 < \tan^{-1}\frac{a_i}{a_{i+1}} < \frac{\pi}{2}, \quad 1 \le i \le n,$$

we have

$$\sum_{i=1}^{n} \tan^{-1}\frac{a_i}{a_{i+1}} < \frac{(n-2)\pi}{2} + \frac{2\pi}{4} = \frac{(n-1)\pi}{2}.$$

(ii) If only one of the terms, say $\tan^{-1}(a_n/a_1)$, is less than $\pi/4$ and all the other terms are greater than or equal to $\pi/4$, then we would have

$$a_n < a_1 \ge a_2 \ge \ldots \ge a_n$$

and therefore

$$\tan^{-1}\frac{a_n}{a_1} + \tan^{-1}\frac{a_1}{a_2} \le \tan^{-1}\frac{a_n}{a_1} + \tan^{-1}\frac{a_1}{a_n} = \frac{\pi}{2}$$

(since $a_2 \ge a_n$ and $\tan^{-1}$ is increasing), and once again the sum will be less than $(n-1)\pi/2$.

(iii) If all the terms are greater than or equal to $\pi/4$ then

$$a_1 \ge a_2 \ge \ldots \ge a_n \ge a_1$$

which means that

$$a_1 = a_2 = \ldots = a_n,$$

and the sum is $n\pi/4$ which is less than $(n-1)\pi/2$ if $n > 2$.

Therefore both inequalities hold strictly for all $n > 2$ and are equalities when $n = 2$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN and A. MEIR, University of Alberta; KEE-WAI LAU, Hong Kong; and the proposers.

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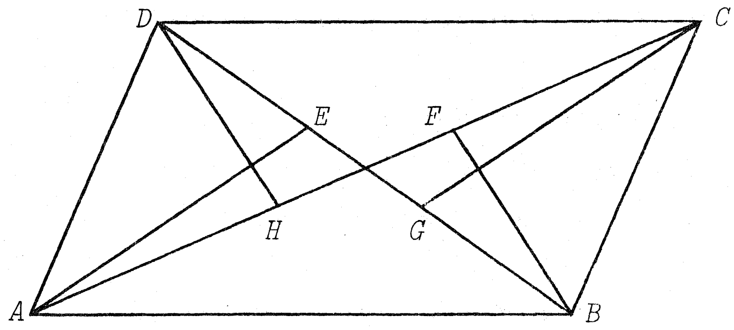
<u>1183</u>. [1986: 241] *Proposed by Roger Izard, Dallas, Texas.*

Let $ABCD$ be a convex quadrilateral and let points E, G lie on BD and F, H lie on AC such that AE, BF, CG, DH bisect angles DAB, ABC, BCD, CDA respectively. Suppose that $AE = CG$ and $BF = DH$. Prove that $ABCD$ is a parallelogram.

Solution by the proposer.

We first claim that if $AD \cdot DC = AB \cdot BC$ or $BA \cdot AD = BC \cdot CD$ then $ABCD$ is a parallelogram.



Suppose for instance that

$$AD \cdot DC = AB \cdot BC.$$

Then since CG and AE are bisectors,

$$\frac{DG}{GB} = \frac{DC}{CB} = \frac{BA}{AD} = \frac{BE}{ED} \quad (= \lambda)$$

and thus $DE = GB$. The two circles of Apollonius determined by base BD (or DB) and ratio λ have equal radii and have chords AE and CG which by assumption are equal in length. Also BD is normal to both circles, and thus $\angle AED = \angle BGC$. Therefore $\Delta AED \equiv \Delta BGC$, and so $AD = BC$. From $AD \cdot DC = AB \cdot BC$ we have $AB = CD$, so $ABCD$ is a parallelogram.

Thus we only have to prove that $AD \cdot DC = AB \cdot BC$ or that $BA \cdot AD = BC \cdot CD$. Suppose without loss of generality that

$$BA \cdot AD < BC \cdot CD \tag{1}$$

and

$$AB \cdot BC < AD \cdot DC. \tag{2}$$

It is well-known that if a triangle has sides a, b, c, then the bisector from angle A has length t satisfying

$$t^2 = bc\left[1 - \frac{a^2}{(b + c)^2}\right].$$

Thus from $AE = CG$ and $BF = DH$ we obtain

$$BA \cdot AD \cdot \left[1 - \frac{(BD)^2}{(BA + AD)^2}\right] = BC \cdot CD \cdot \left[1 - \frac{(BD)^2}{(BC + CD)^2}\right] \tag{3}$$

and

$$AB \cdot BC \cdot \left[1 - \frac{(AC)^2}{(AB + BC)^2}\right] = AD \cdot DC \cdot \left[1 - \frac{(AC)^2}{(AD + DC)^2}\right]. \tag{4}$$

From (1) and (3) follows

$$BA + AD > BC + CD \tag{5}$$

and from (2) and (4) we obtain

$$AB + BC > AD + DC. \tag{6}$$

Adding (5) and (6),

$$2AB + BC + AD > 2CD + BC + AD,$$

so

$$AB > CD.$$

But this implies $AD < BC$ by (1) and $BC < AD$ by (2), which is a contradiction.

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1184. [1986: 242] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a nonequilateral triangle and let O, I, H, F denote the circumcenter, incenter, orthocenter, and the center of the nine-point circle, respectively. Can either of the triangles OIF or IFH be equilateral?

Comment by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.

In 1968 I proposed essentially the same problem, that no three of the four points O, I, H, F can form an equilateral triangle. See problem E2139 of the *American Math. Monthly* (solution in vol. 76 (1969), p.1066).

Also solved by WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* D.J. SMEENK, *Zaltbommel, The Netherlands; and the proposer.*

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1186. [1986: 242] *Proposed by Svetoslav Bilchev, Technical University and Emilia Velikova, Mathematicalgymnasium, Russe, Bulgaria.*

If a, b, c are the sides of a triangle and s, R, r the semiperimeter, circumradius, and inradius, respectively, prove that

$$\Sigma(b + c - a)\sqrt{a} \ge 4r(4R + r)\sqrt{\frac{4R + r}{3Rs}}$$

where the sum is cyclic over a, b, c.

Combination of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Murray S. Klamkin, University of Alberta.

We first prove the inequality

$$\sqrt{3}\ \Sigma \cos A \ge \frac{2F}{R^2}, \qquad (1)$$

where F is the area of the triangle. Indeed, as

$$\Sigma \cos A = \frac{R + r}{R},$$

and $F = rs$, (1) reads equivalently

$$s \le \frac{\sqrt{3}R(R + r)}{2r},$$

which is true since

$$s \le \frac{3\sqrt{3}R}{2}$$

(see 5.3 of Bottema et al, *Geometric Inequalities*) and $3 \le (R + r)/r$, i.e. $2r \le R$. Now (1) can be read as

$$\sqrt{3}\sum\frac{b^2+c^2-a^2}{2bc} \geq \frac{2F}{R^2},$$

i.e., using $4RF = abc$,

$$\sqrt{3}\,\Sigma(b^2+c^2-a^2)a \geq \frac{16F^2}{R}. \qquad (2)$$

Again using $4RF = abc$ and

$$16F^2 = 2\,\Sigma\, b^2c^2 - \Sigma\, a^4,$$

(2) becomes

$$\sqrt{3}\,\Sigma(b^2+c^2-a^2)a \geq \frac{(2\,\Sigma\, b^2c^2 - \Sigma\, a^4)^{3/2}}{abc}. \qquad (3)$$

Since $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are the sides of a triangle, (3) implies

$$\sqrt{3}\,\Sigma(b+c-a)a \geq \frac{(2\,\Sigma\, bc - \Sigma\, a^2)^{3/2}}{\sqrt{abc}}. \qquad (4)$$

Now we have the known relation

$$\Sigma\, bc = r(4R+r) + s^2 = r(4R+r) + \frac{\Sigma\, a^2 + 2\,\Sigma\, bc}{4}$$

so that

$$2\,\Sigma\, bc = 4r(4R+r) + \Sigma\, a^2.$$

Thus (4) becomes

$$\sqrt{3}\,\Sigma(b+c-a)a \geq 4r(4R+r)\sqrt{\frac{4r(4R+r)}{abc}}$$

which with $abc = 4Rrs$ yields the result.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and the proposers.

The stronger inequality (2) above was obtained by both Janous and Klamkin.

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1187. [1986: 242] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find a polynomial with integer coefficients that has $2^{1/5} + 2^{-1/5}$ as a root.

I. *Solution by Bruce Shawyer, Memorial University of Newfoundland, St. John's.*

Let $x = 2^{1/5} + 2^{-1/5}$ so that

$$x^3 = 2^{3/5} + 2^{-3/5} + 3(2^{1/5} + 2^{-1/5}) = 2^{3/5} + 2^{-3/5} + 3x,$$

or

$$2^{3/5} + 2^{-3/5} = x^3 - 3x.$$

Also

$$x^5 = 2 + 2^{-1} + 5(2^{3/5} + 2^{-3/5}) + 10(2^{1/5} + 2^{-1/5})$$
$$= 5/2 + 5(x^3 - 3x) + 10x,$$

whence $2^{1/5} + 2^{-1/5}$ is a root of

$$P(x) = 2x^5 - 10x^3 + 10x - 5.$$

It is easy to adapt this argument to find a polynomial with integer coefficients that has $q^{1/n} + q^{-1/n}$ as a root, where q is a nonzero rational and n is an odd positive integer.

II. *Generalization by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let m and n be positive integers. Obviously, $t = m^{1/n}$ is a solution of

$$t^n + t^{-n} = m + 1/m. \tag{1}$$

Therefore the transformation

$$x = t + 1/t \tag{2}$$

will turn (1) into a polynomial with $m^{1/n} + m^{-1/n}$ as a root. From (2) we have

$$t = \frac{x \pm \sqrt{x^2 - 4}}{2},$$

so (1) becomes

$$\frac{(x + \sqrt{x^2 - 4})^n}{2^n} + \frac{2^n}{(x + \sqrt{x^2 - 4})^n} = m + \frac{1}{m}$$

or

$$(x + \sqrt{x^2 - 4})^n + (x - \sqrt{x^2 - 4})^n = 2^n(m + \frac{1}{m})$$

or

$$x^n + \binom{n}{2}x^{n-2}(x^2 - 4) + \binom{n}{4}x^{n-4}(x^2 - 4)^2 + \ldots = 2^{n-1}(m + \frac{1}{m}).$$

Multiplying by m, we obtain a polynomial with integer coefficients having $m^{1/n} + m^{-1/n}$ as a root. The given problem corresponds to the case $m = 2$, $n = 5$.

For a related problem see 86-3 and its solution in *Math. Intelligencer* 8 (1986) 31, 33.

Also solved by JACK GARFUNKEL, *Flushing, N.Y.*; J.T. GROENMAN, *Arnhem, The Netherlands*; RICHARD I. HESS, *Rancho Palos Verdes, California*; WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria*; MICHAEL JOSEPHY, *Universidad de Costa Rica, San José, Costa Rica*; FRIEND H. KIERSTEAD JR., *Cuyahoga Falls, Ohio*; KEE-WAI LAU, *Hong Kong*; J. WALTER LYNCH, *Georgia Southern College, Statesboro, Georgia*; J.A. MCCALLUM, *Medicine Hat, Alberta*; LEROY F. MEYERS, *The Ohio State University*; M.M. PARMENTER, *Memorial University of Newfoundland, St. John's*; GEORGE SZEKERES, *University of New South Wales, Kensington, Australia*; EDWARD T.H. WANG, *Wilfrid Laurier University, Waterloo, Ontario*; KENNETH M. WILKE, *Topeka, Kansas; and the proposer. All solvers obtained the same solution, by virtually the same method.*

One solver noted that such a polynomial must exist, since the sum of algebraic numbers is algebraic. Two solvers noted that the polynomial $P(x)$ given above has no rational roots by Eisenstein's criterion.

Szekeres found a polynomial with integer coefficients (given in a simple closed form) which has $a^{1/n} + a^{-1/n}$ as a root, where a is a given positive rational and n an odd positive integer. This has been included as problem 1301 in this issue.

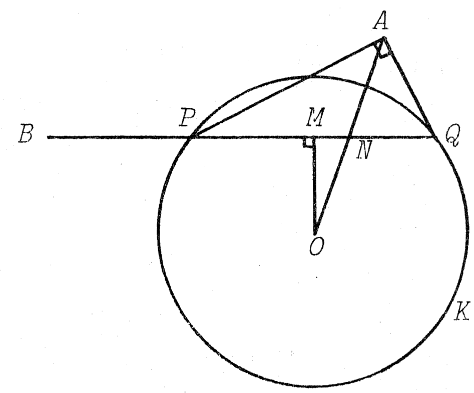
* * *

<u>1188</u>. [1986: 242] *Proposed by Dan Sokolowsky, Williamsburg, Virginia.*

Given a circle K and distinct points A, B in the plane of K, construct a chord PQ of K such that B lies on the line PQ and $\angle PAQ = 90^\circ$.

Solution by George Tsintsifas, Thessaloniki, Greece.

Let K have centre O and radius R. If PQ is the required chord and M is its midpoint, then the power of the point B with respect to K is



$$\overline{BO}^2 - R^2 = \overline{BP}\cdot\overline{BQ}$$
$$= (\overline{BM} - \overline{AM})(\overline{BM} + \overline{AM})$$
$$= \overline{BM}^2 - \overline{AM}^2,$$

so

$$\overline{AM}^2 + \overline{OM}^2 = R^2. \qquad (1)$$

From (1) we conclude that M lies on the circle $(N, \frac{1}{2}\sqrt{2R^2 - \overline{OA}^2})$, where N is the midpoint of the segment OA. Also M obviously lies on the circle of diameter OB. The intersection of these two loci gives the position of M and hence the solution.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; DAN PEDOE, Minneapolis, Minnesota; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Two readers pointed out that (as can be seen from the above solution) the construction is not always possible.

* * *

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