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GRADUATE SCHOOL

On the structure of Bruhat order

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Nathan Paul Reading

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY.

Victor Reiner, Adviser

April 2002

Acknowledgments

The author wishes to thank his advisor, Vic Reiner, for many helpful conversations, as well as Lou Billera, Francesco Brenti, Kyle Calderhead, Richard Ehrenborg, Jonathan Farley, Joseph Kung, Margaret Readdy, Michelle Wachs and Marcel Wild for helpful comments. The author was partially supported by the Thomas H. Shevlin Fellowship from the University of Minnesota Graduate School and by NSF grant DMS-9877047. Some of the material in Chapter 4 will appear in the journal Order, under the title "Order Dimension, Strong Bruhat Order and Lattice Properties for Posets," and is used here by permission of Kluwer Academic Publishers.

for Cambria

Abstract

This thesis establishes fundamental enumerative and order-theoretic results about the Bruhat order on a Coxeter group, also known as the strong Bruhat order or Chevalley-Bruhat order. One set of results stems from a structural recursion on intervals in the Bruhat order. The recursion gives the isomorphism type of a Bruhat interval in terms of smaller intervals, using basic geometric operations which preserve the Eulerian property and PL sphericity, and have a simple effect on the cd-index. This leads to a new inductive proof that Bruhat intervals are PL spheres as well a recursive formula for the cd-index of Bruhat intervals. This recursive formula leads to a proof that the cd-indices of Bruhat intervals span the space of cd-polynomials. The structural recursion is used to construct Bruhat intervals which are the face lattice of the duals of stacked polytopes. We conjecture that these dual stacked polytopes constitute the upper bound for the cd-indices of Bruhat intervals. As a special case of the conjecture, we show that the flag h-vectors of lower Bruhat intervals are bounded above by the flag h-vectors of Boolean algebras (i. e. simplices).

We determine the order dimension of the Bruhat order on finite Coxeter groups of types A, B and H. The order dimension is determined using a generalization of a theorem of Dilworth: $\dim(P) = \mathrm{width}(\mathrm{Irr}(P))$, whenever P satisfies a simple order-theoretic condition called here the dissective property (or "clivage" in [32, 39]). The result for dissective posets follows from an upper bound and lower bound on the dimension of any finite poset. The dissective property is related, via MacNeille completion, to the distributive property of lattices. We show a similar connection between quotients of the Bruhat order with respect to parabolic subgroups and lattice quotients.

Contents

List of Figures

Chapter 1

Introduction

1.1 Bruhat order

Coxeter groups and Bruhat order (sometimes called "strong Bruhat order") lie in the intersection of several important research areas of modern mathematics, including Lie theory, algebraic geometry, combinatorial group theory, hyperplane arrangements and the theory of posets and lattices. In this thesis, we present new enumerative and order-theoretic results on the Bruhat order.

Readers not familiar with Coxeter groups should concentrate on the symmetric group S_n of permutations of the numbers $\{1, 2, \ldots, n\}$. We will follow an example in S_4. Elements of S_4 can be written as permutations, using "one-line notation." For example, write 3412 for $\begin{pmatrix} 1234 \\ 3412 \end{pmatrix}$, the permutation mapping $1 \mapsto 3,\ 2 \mapsto 4,\ 3 \mapsto 1$ and $4 \mapsto 2$. In general, a Coxeter group W has a set S of special generators called Coxeter generators. In the case of S_4, the Coxeter generators are $S = \{(12), (23), (34)\}$. The notation (12) refers to the permutation which switches the elements 1 and 2 and fixes all other elements. For convenience, set $r := (12)$, $s := (23)$ and $t := (34)$.

We will write elements of a Coxeter group W as words in S. There are infinitely many ways to write an element as a word. For example, the permutation $3412 \in S_4$ can be written inefficiently in many ways:

$$rsrsrtrs = rsrsts = srttts,$$

etc. Given an element w of S_4, the length $l(w)$ of w is defined to be the length of a smallest possible word for w, and a word of that length is called a reduced word for w. In S_n, the length of an element is exactly its inversion number. The inversion number of a permutation $\pi \in S_n$ is the number of pairs $i < j$ in $\{1, 2, \ldots, n\}$ such that j occurs to the left of i in the one-line notation for π. For example, the element $3412 \in S_4$ has 3 occurring before 1 and 2 as well as 4 occurring before 1 and 2, so the length of 3412 is 4. There are two reduced words for 3412:

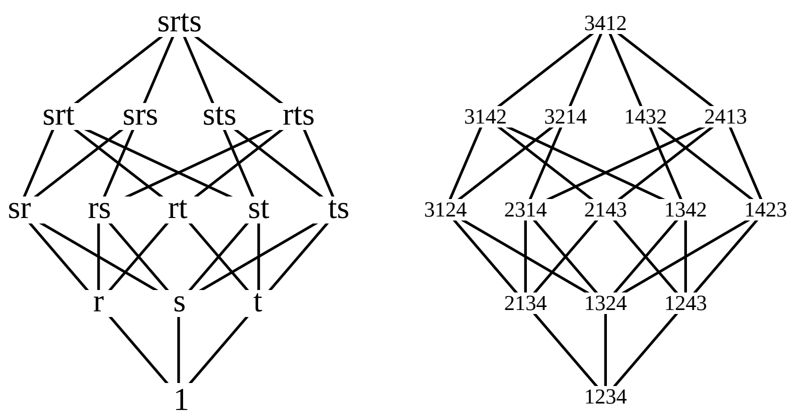
$$srts = strs.$$

Bruhat order on S_n is defined in terms of "undoing" inversions. If $l(u) < l(w)$ and u and w differ by interchanging two entries in one-line notation, then $u \leq w$ in Bruhat order, and more generally, $u \leq w$ in Bruhat order if w can be changed to u by a sequence of such interchanges, each decreasing the length. So for example, we can change 3412 to 1324 by interchanging elements two at a time:

$$3412 \to 3142 \to 3124 \to 1324,$$

and each of these interchanges decreases the number of inversions, so $1324 \leq 3412$. Figure 1.1 shows the Bruhat interval $[1, srts]$ or $[1234, 3412]$ in S_4.

Figure 1.1: The interval $[1, srts]$ or $[1234, 3412]$ in S_4.



The remaining sections in this chapter are informal summaries of Chapters 3 and 4. These summaries aim to be as non-technical as possible, and precise definitions are put off until Chapter 2. More rigorous summaries are given at the beginning of Chapters 3 and 4.

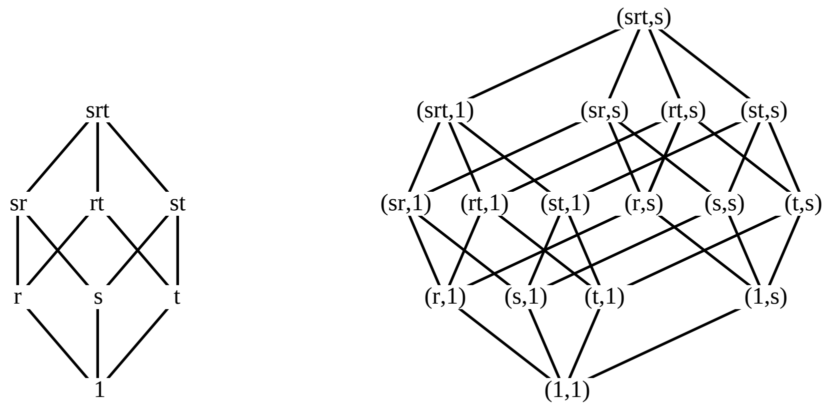
1.2 Summary of Chapter 3

The results presented in Chapter 3 derive from a structural recursion on intervals in the Bruhat order. This recursion, although developed independently, has some resemblance to work by DuCloux [19] and by Dyer [21]. Specifically, the recursion gives the isomorphism type of a Bruhat interval in terms of smaller intervals, using the basic geometric operations of pyramid, vertex shaving and zipping.

The pyramid operation on a partially ordered set (poset) replaces each element x of the poset with a pair $(x, 1)$ and (x, s), as shown in Figure 1.2. The resulting poset is called $\text{Pyr}(P)$. For readers familiar with products of posets, the pyramid operation on a poset P is the product of P with a two-element chain.

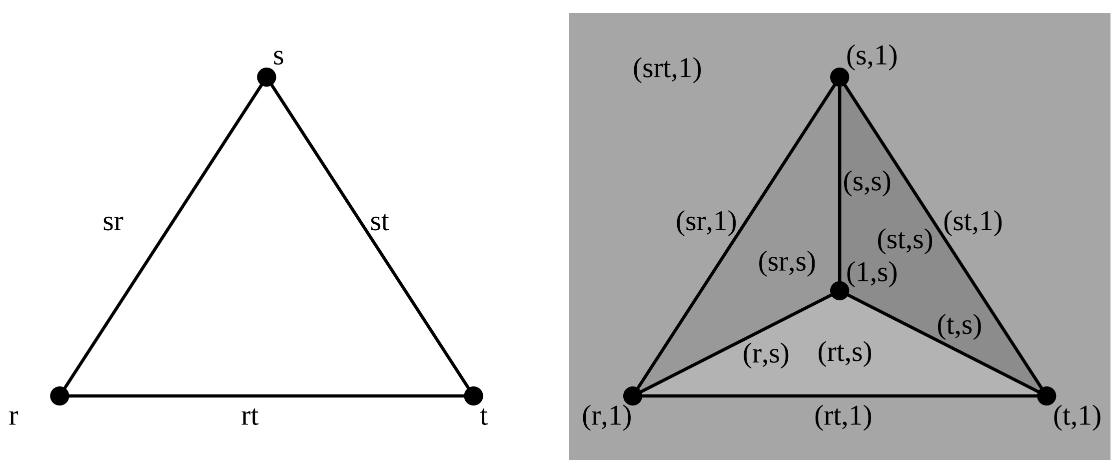
The pyramid operation gets its name from a corresponding operation on polytopes or spherical cell complexes. In the case of polytopes, the pyramid operation turns, for example, a square into a square pyramid. In a way that is made precise in Section 2.4.6, Bruhat intervals correspond to spheres. Figure 1.3 shows the sphere corresponding to $[1, srt]$, and also the pyramid operation

Figure 1.2: The pyramid operation. On the left is the Bruhat interval $[1, srt]$ in S_4, and on the right is $\mathrm{Pyr}([1, srt])$.



applied to that sphere. The right-hand picture represents a projection of the sphere onto the plane, with the shaded rectangle denoting an unbounded region.

Figure 1.3: The pyramid operation on spheres. On the left is sphere corresponding to $[1, srt]$, and on the right is the pyramid of that sphere.



The zipping operation takes three elements of a poset and "zips" them—identifies them to make a single new element. There are technical conditions on these triples of elements which are given in Section 3.2. We will illustrate the zipping operation with an example of the structural recursion on Bruhat intervals. Starting with the Bruhat interval $[1, srt]$ in S_4, we construct the interval $[1, srts]$. Naively, we would like to replace each element in x with a pair of elements x and xs. However, that assumes that the length of the element xs is actually greater than the length of x. In the case of the element $s \in [1, srt]$, ss is the identity element, and has length 1. We can rescue this naive start using the zipping operation. We start with $[1, srt]$ and formally multiply by s, by performing a pyramid operation, as shown in Figure 1.2. The elements $(s, 1)$, $(1, s)$, and (s, s) form a zipper, and when this zipper is zipped, the result is a poset isomorphic to $[1, srts]$. Figure 1.4 illustrates the zipping operation, and Figure 1.5 illustrates the corresponding operation on spheres.

Figure 1.4: The left picture shows the zipper $((s, 1), (1, s), (s, s))$ lightened in $\mathrm{Pyr}([1, srt])$. In the right picture, the zipper has been zipped, and the result is the poset $[1, srts]$.

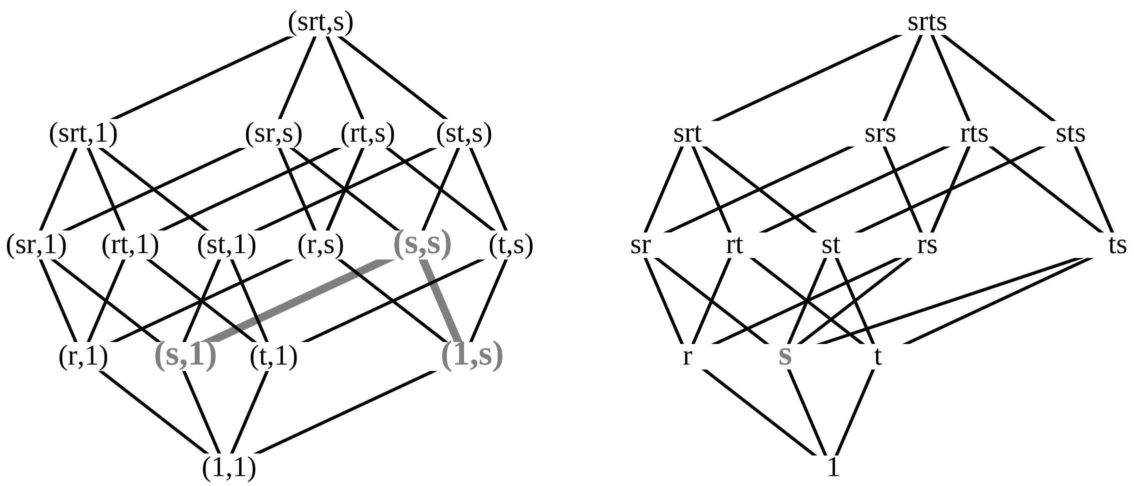
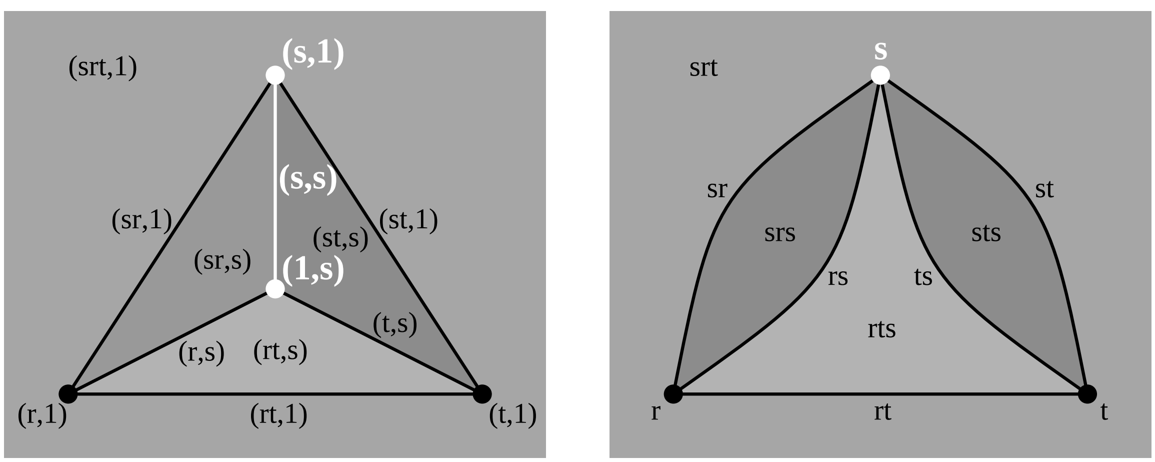
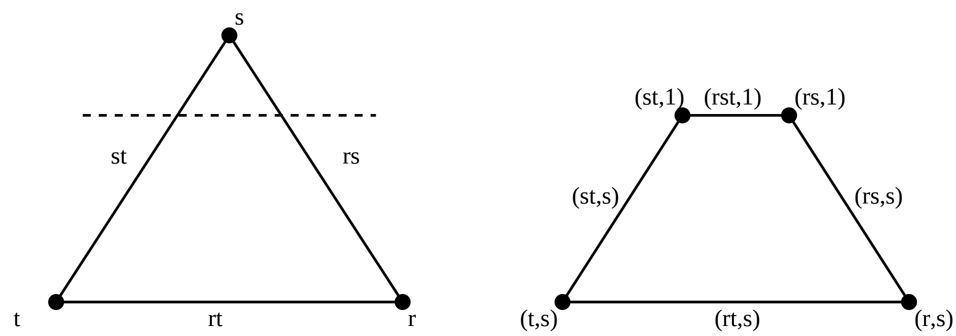


Figure 1.5: Spheres corresponding to the posets in Figure 1.4.



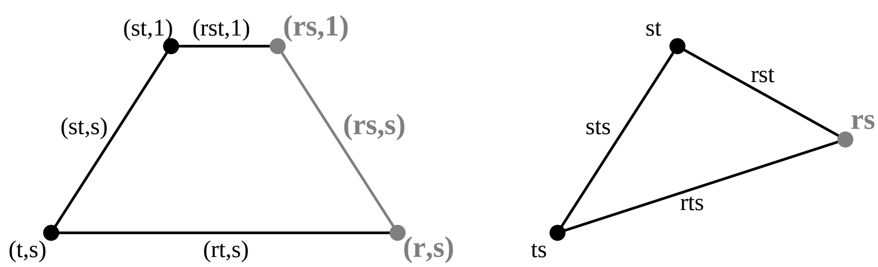
In general, when $l(us) > l(u)$ and $l(ws) > l(w)$, to construct the interval $[u, ws]$ from $[u, w]$ one first performs the pyramid operation. Then one zips one zipper for each $v \in [u, w]$ with $l(vs) < l(v)$. In order to construct all Bruhat intervals, one must also be able to construct $[us, ws]$ as well. To accomplish this, one needs the operation of shaving a vertex. The name of this operation refers to polytopes: one creates a new facet which cuts off some vertex, as pictured in Figure 1.6. The operation of shaving the vertex v is denoted $\mathcal{S}_v$ and will be formally defined in Sections 2.3, 2.4 and 3.4.

Figure 1.6: Shaving the vertex s from the sphere associated to the interval $[1, rst]$.



To construct $[us, ws]$ from $[u, w]$, one first shaves the vertex us and then zips one zipper for each $v \in (us, w)$ with $l(vs) < l(v)$. For example, to construct $[s, rsts]$ from $[1, rst]$, first shave the vertex s, as shown in Figure 1.6. Then zip the zipper $(rs, 1)$, (r, s), (rs, s), as shown in Figure 1.7.

Figure 1.7: The left picture shows a zipper in the sphere corresponding to $[1, rst]$. In the right picture, the zipper has been zipped, giving a sphere which corresponds to the Bruhat interval $[s, rsts]$.



Given a poset that corresponds to a sphere, the operations of pyramid, vertex shaving and zipping each produce a new poset which corresponds to a sphere. Thus the structural recursion constitutes a new proof that Bruhat intervals are spheres. To be correct, we should actually say "PL sphere" instead of "sphere" everywhere in this paragraph. The definition of PL sphere will be given in Section 2.4.

The recursion also leads to a recursive formula for the cd-index of Bruhat intervals. The cd-index is a polynomial in non-commuting variables c and d, which encodes a large amount of enumerative information about Bruhat intervals. Specifically, replacing c by $a + 2b$ and d by $ab + ba + 2bb$ gives the flag-index, a polynomial in non-commuting variables a and b which counts the chains in the poset according to the ranks they visit. A chain in a poset is a set of elements $x_1, x_2, \ldots, x_k$ with $x_1 < x_2 < \ldots < x_k$. Each monomial in the flag index counts a certain set of chains in an interval

$[u,w]$. An a in the i^{th} position in an ab-monomial means that the coefficient of that monomial is counting chains which have no element at rank $l(u)+i$, and a b indicates that the coefficient is counting chains containing an element of length $l(u)+i$.

We will illustrate with an example. The poset $[1, srts]$ in Figure 1.1 has cd-index $c^3+2cd+dc$, which means its flag-index is:

$$a^3+4a^2b+5aba+10ab^2+3ba^2+10bab+10b^2a+20b^3.$$

There are 3 chains which contain an element of length 1 but no elements of lengths 2 or 3, so the coefficient of ba^2 is 3. Each of these chains consists of a single element, and the three chains are $\{r\}$, $\{s\}$ and $\{t\}$. There are 10 chains which contain elements of lengths 1 and 3 but not of length 2, and thus the coefficient of bab in the flag-index is 10. For example, some of these chains are $\{s, srt\}$ and $\{t, sts\}$. The set $\{t, srs\}$ is not a chain, because $t \not< srs$. There are 20 chains which contain elements of lengths 1, 2 and 3 so the coefficient of bbb in the flag-index is 20. Some of these chains are $s < sr < srt$ and $t < ts < sts$.

The poset operations of pyramid, vertex shaving and zipping have nice effects on the cd-index—for precise statements, see Propositions 2.5.2 and 2.5.3 and Theorem 3.2.6. Thus the structural recursion leads to the following formula, where Ψ is the cd-index. The following is an abbreviated version of Theorem 3.1.1.

Theorem 3.1.1. *Let $u < us$, $w < ws$ and $u \leq w$.*

If $us \notin [u,w]$, then $\Psi_{[u,ws]} = \text{Pyr}\Psi_{[u,w]}$, and $\Psi_{[us,ws]} = \Psi_{[u,w]}$.

If $us \in [u,w]$, then

$$\begin{aligned}\Psi_{[u,ws]} &= \text{Pyr}\Psi_{[u,w]} - \sum_{v\in(u,w):vs<v} \Psi_{[u,v]}\cdot d\cdot \Psi_{[v,w]}\\ \Psi_{[us,ws]} &= \mathcal{S}_{us}\Psi_{[u,w]} - \sum_{v\in(us,w):vs<v} \Psi_{[us,v]}\cdot d\cdot \Psi_{[v,w]}\end{aligned}$$

So, for example, $\Psi_{[1,srt]} = c^2+d$. For $u \leq w$, whenever $l(w) = l(u)+1$, $\Psi_{[u,w]} = 1$ and whenever $l(w) = l(u)+2$, $\Psi_{[u,w]} = c$. Since $1 < s$, $srt < srts$, $1 < srts$ and $s \in [1, srts]$, the cd-index of $[1, srts]$ is

$$\begin{aligned}\Psi_{[1,srts]} &= \text{Pyr}\Psi_{[1,srt]} - \Psi_{[1,s]}\cdot d\cdot \Psi_{[s,srt]}\\ &= c^3+2cd+2dc-1\cdot d\cdot c\\ &= c^3+2cd+dc\end{aligned}$$

Theorem 3.1.1 allows us to make some progress towards characterizing the cd-indices of Bruhat intervals. First, we are able to show that the linear span of cd-indices of Bruhat intervals is the entire linear span of cd-polynomials. Also, Stanley [56] conjectured that the coefficients of the cd-index are non-negative for a large class of posets including Bruhat intervals. Assuming non-negativity, Theorem 3.1.1 motivates a conjectural upper bound on the cd-coefficients. The idea is that since zippings would reduce the cd-index, the maximum cd-index would be attained on intervals which can be constructed without zipping.

A polytope is said to be *dual stacked* if it can be obtained from a simplex by a series of vertex-shavings. So for example, every polygon is dual stacked. Using the structural recursion, we are able to construct Bruhat intervals which are dual stacked. That is, the structural recursion starts with a 1-element poset and performs a series of pyramid operations to obtain a simplex, and then a series of vertex shavings, with no zippings. The following conjecture is supported by computer calculations:

Conjecture 3.1.2. *The coefficientwise maximum of all cd-indices $\Phi_{[u,v]}$ with $l(u) = k$ and $l(v) = n$ is attained on a Bruhat interval which is isomorphic to a dual stacked polytope of dimension $n-k-1$ with n facets.*

As discussed in Section 3.5, Conjecture 3.1.2 depends on the non-negativity conjecture and is complicated by some technical difficulties. The situation is slightly better for lower intervals:

Theorem 3.8.2. *Assuming the non-negativity of cd-coefficients of Bruhat intervals, for all $w \in W$,*

$$\Phi_{[1,w]} \leq \Phi_{B_{l(w)}}.$$

Here B_n is the Boolean algebra of rank n. It is not true that the cd-index of general intervals is less than that of the Boolean algebra of appropriate rank. For example, $[1324, 3412]$ is the face lattice of a square, with $\Phi_{[1324,3412]} = c^2 + 2d$. However, $\Phi_{B_3} = c^2 + d$.

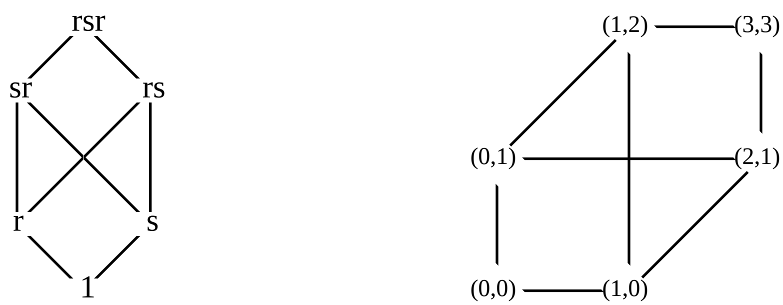
Without appealing to a conjecture, we can prove a weaker inequality, namely Theorem 3.8.3 which states that the flag h-vectors of lower Bruhat intervals are bounded above by the flag h-vectors of Boolean algebras. For definitions, see Section 2.5.

1.3 Summary of Chapter 4

Chapter 4 is organized around the problem of determining the order dimension of Bruhat order on a finite Coxeter group. Here we mean Bruhat order on the entire group—determining the order dimension of intervals in Coxeter groups appears to be much harder. We obtain closed formulas for the order dimensions of the finite Coxeter groups of types A and B, and values for the order dimensions of the exceptional groups of type H. The results are obtained via a generalization of Dilworth's Theorem on the order dimension of distributive lattices.

We will explain this generalization of Dilworth's result, beginning with an explanation of order dimension. Given a finite poset P, for large enough d we can realize the elements $x \in P$ as vectors $(x_1, x_2, \ldots, x_d)$ in $\mathbb{R}^d$. The goal is to choose vectors so that $x \leq y$ in P if and only if $x_i \leq y_i$ for each i. The order dimension is the smallest d for which this is possible. As an example, consider the Bruhat order on S_3, shown in Figure 1.8. The elements of S_3 are described as words in the Coxeter

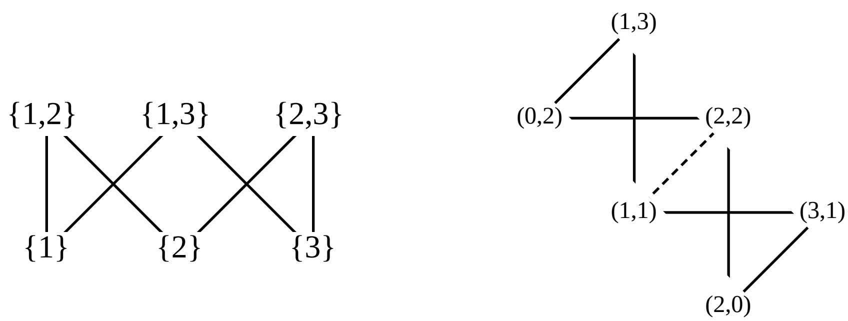
Figure 1.8: The Bruhat order on S_3 and an embedding in $\mathbb{R}^2$.



generators r and s. The picture on the right side of Figure 1.8 shows the poset as a set of vectors in $\mathbb{R}^2$.

The "standard example" of a poset of dimension n is the set of subsets of $\{1, 2, \dots n\}$ of cardinality 1 or $n-1$, ordered by inclusion. Figure 1.9 shows the standard example of a poset of dimension 3, along with a failed attempt to realize it in $\mathbb{R}^2$.

Figure 1.9: The standard example of a poset of dimension 3, and a failed attempt to realize it in $\mathbb{R}^2$. The dotted line indicates an order relation in the "realization" which does not appear in the poset.



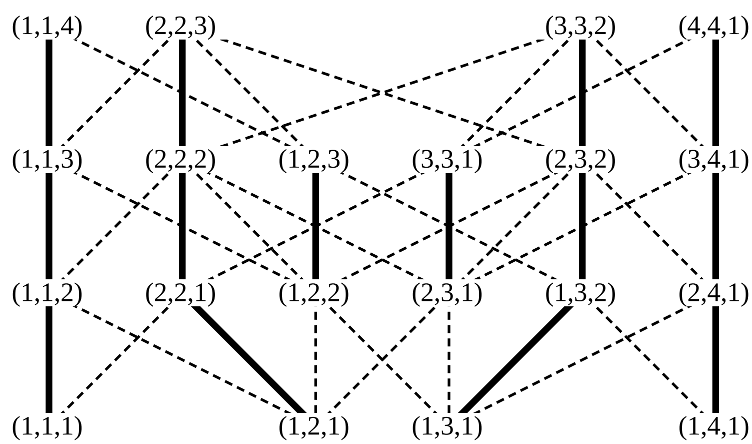
An element x in a poset P is a dissector if there exists an element $\beta(x)$ such that the set $\{p \in P : p \not\geq x\}$ is equal to $\{p \in P : p \leq \beta(x)\}$. For example, in the poset of Figure 1.8, the elements r, s, rs and sr are all dissectors, and the others are not. Given a set S of elements of P, if the set $\{p \in P : p \geq s$ for all $s \in S\}$ has a unique minimum element, this element is called the join of S and denoted $\vee S$. For example, in the poset of Figure 1.1, the element rts is the join of the set $\{rs, s, t\}$. Although one typically speaks of join-irreducibles of a lattice, the usual definition works for general posets: An element x is called join-irreducible if $x = \vee S$ implies $x \in S$. So for example in the poset of Figure 1.8, the element rs is join-irreducible, because the set $\{1, r, s\}$ of elements strictly below rs has no join. The element 1 is not join-irreducible, because it is $\vee\emptyset$, and rsr is also not join-irreducible. It is easily proven that a dissector must be join-irreducible, so r, s, rs and sr are all join-irreducible. A poset in which every join-irreducible is a dissector is called dissective.

An antichain in a poset P is a set of elements of P with no order relations among them. So, for example, the set $\{sr, rs, t\}$ is an antichain in the poset of Figure 1.1. The *width* of a poset is the size of a largest antichain. Dilworth [18] showed that the width of a poset is also equal to the smallest number w such that P can be partitioned into w chains. The following theorem is the key to the order dimension result. A finite lattice is distributive if and only if it is dissective, so the theorem generalizes another theorem of Dilworth on distributive lattices.

Theorem 4.1.2. *If P is a dissective poset then* $\dim(P) = \text{width}(\text{Irr}(P))$.
Here, $\text{Irr}(P)$ is a poset whose elements are the join-irreducibles of P, such that $x \leq y$ in $\text{Irr}(P)$ if and only if $x \leq y$ in P. We call this the subposet of P induced by the join-irreducibles. For example, in Figure 1.8, $\text{Irr}(P)$ is obtained by deleting the elements 1 and rsr from P. The width of $\text{Irr}(P)$ is 2, so the dimension of P is 2.

It was previously known [32, 39] that Bruhat order on Coxeter groups of types A, B and H are dissective. (The Coxeter groups of type A are exactly the symmetric groups S_n.) Theorem 4.1.2 reduces the order dimension calculation to the calculation of the width of a certain subposet of the Bruhat order. The poset $\text{Irr}(S_n)$ can be nicely characterized as a "subrectangle" order, a sort of two-dimensional version of the subword characterization of Bruhat order. This leads to an elegant symmetric chain decomposition which determines the width of $\text{Irr}(S_n)$. Figure 1.10 shows $\text{Irr}(S_5)$ with a symmetric chain decomposition. The labels on the elements describe the rectangles and will be explained in Section 4.8.

Figure 1.10: A symmetric chain decomposition of $\text{Irr}(S_5)$. The chains in the decomposition are the solid lines, and the other order relations are dotted.

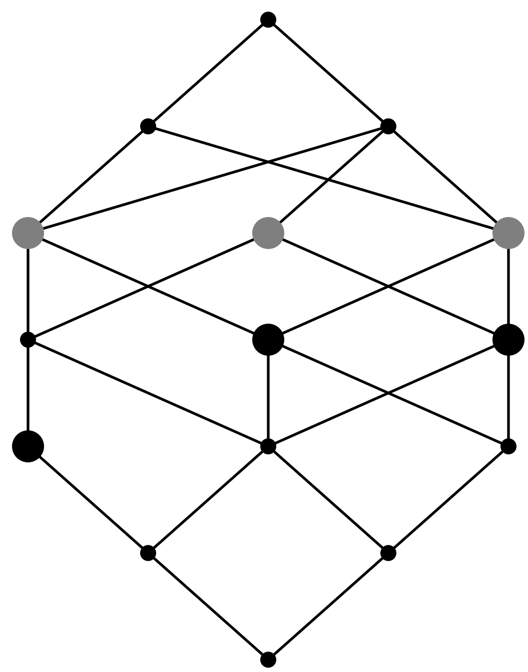


Theorem 4.1.2 follows immediately from more general bounds on order dimension. Here $\text{Dis}(P)$ is the subposet of P induced by the dissectors.

Theorem 4.1.5. *For a finite poset P,* $\text{width}(\text{Dis}(P)) \leq \dim(P) \leq \text{width}(\text{Irr}(P))$.

We now briefly digress to sketch a simple proof of Theorem 4.1.5, beginning with the lower bound. The width of $\mathrm{Dis}(P)$ is the size of a largest antichain in $\mathrm{Dis}(P)$. Given any antichain A in $\mathrm{Dis}(P)$, let $\beta(A) := \{\beta(a) : a \in A\} \subseteq (P)$. It is easily verified that $A \cup \beta(A)$, as an induced subposet of P, is a standard example of dimension $|A|$ as exemplified in Figure 1.11. Thus $\mathrm{width}(\mathrm{Dis}(P)) \leq \dim(P)$.

Figure 1.11: A poset P, with a set A of pairwise three incomparable dissectors, indicated by the large black dots. The set $\beta(A)$ is indicated by large gray dots, and $A \cup \beta(A)$ is a standard example of a poset of dimension three.

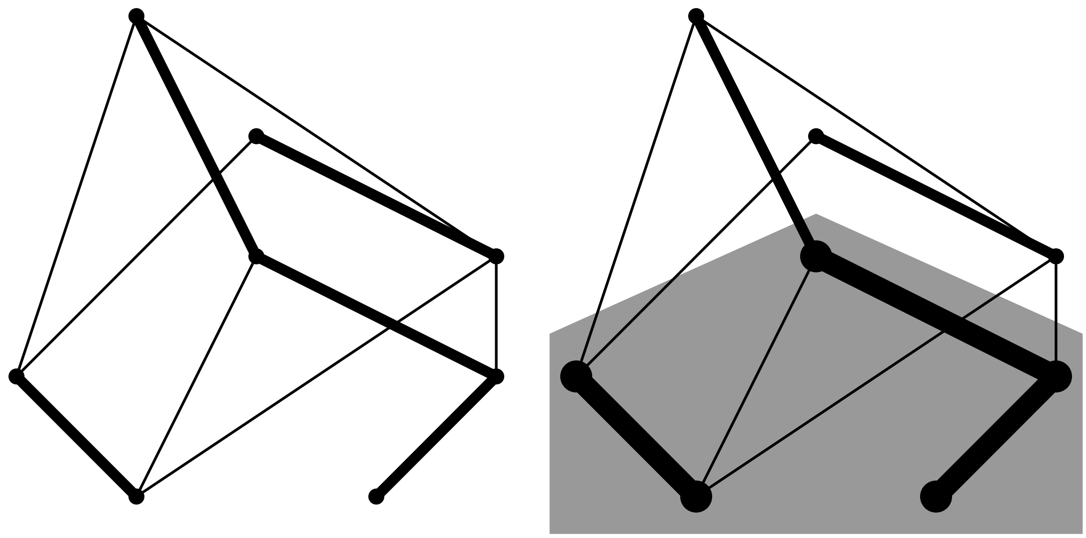


The upper bound is also simple. It is easily checked that any element x of P has $x = \vee I_x$, where I_x is the set of join-irreducible elements weakly below x. The set I_x is an order ideal in $\mathrm{Irr}(P)$, meaning that if $k \in I_x$, $j \in \mathrm{Irr}(P)$ and $j \leq k$, then $j \in I_x$. An order ideal is thus completely described by its maximal elements. It is also easily verified that $x \leq y$ if and only if $I_x \subseteq I_y$.

The width of $\mathrm{Irr}(P)$ is equal to the size of the smallest decomposition of P into chains. The left picture of Figure 1.12 shows $\mathrm{Irr}(P)$ for the poset P from Figure 1.11, along with a decomposition of $\mathrm{Irr}(P)$ into chains. An order ideal is completely determined by the number of elements it contains from each chain in the decomposition. Thus an element $x \in P$ can be encoded as a vector, by letting $x_i := |I_x \cup C_i|$, where C_i is the i^{th} chain in the decomposition. Since I_x is an order ideal, knowing $|I_x \cup C_i|$ is equivalent to knowing which elements are in $I_x \cup C_i$. Thus $x \leq y$ if and only if $I_x \subseteq I_y$ if and only if $x_i \leq y_i$ for all i. The right picture of Figure 1.12 shows an order ideal in Irr(P). If the chains are numbered C_1, C_2, C_3 from left to right, the order ideal shown corresponds to the vector $(2,\ 3,\ 0)$. Thus $\dim(P) \leq \mathrm{width}(\mathrm{Irr}(P))$.

Chapter 4 also contains related results having to do with generalizing lattice properties and constructions to arbitrary posets. In particular, Bruhat orders seem to have several lattice-like

Figure 1.12: On the left is Irr(P), for the poset P of Figure 1.11, with the thick lines indicating a decomposition of Irr(P) into three chains. The right picture is an order ideal in Irr(P) which contains respectively 2, 3 and 0 elements from the three chains.



properties. One such property, the dissective property, was already mentioned. The dissective property of posets is related to the distributive property of lattices by way of MacNeille completion, which takes an arbitrary finite poset P and constructs the smallest lattice which contains P as a subposet. Specifically [39] a poset is dissective if and only if its MacNeille completion is distributive. Thus Bruhat orders of types A and B have a non-lattice version of the distributive property. In a similar vein, we define a notion of order congruence which corresponds via MacNeille completion to lattice congruence. Quotients of the Bruhat order with respect to parabolic subgroups are one example of quotients with respect to an order congruence.

Chapter 2

Preliminaries

This chapter is divided into five sections, devoted respectively to posets, Coxeter groups and Bruhat order, polytopes, CW and PL topology and the cd-index. In each of these sections, the aim is to present the most basic background material. Readers familiar with the topic of a section may wish to skip that section, or to read it superficially for conventions of notation and terminology. Some less well-known material on these topics, including new material, is presented in Chapters 3 and 4.

2.1 Posets

This section contains background information about posets. The poset terminology and notation used in this thesis is standard and generally agrees with [55] or [58], where proofs of the basic results quoted here can be found. A *partially ordered set* or *poset* is a set P, with a reflexive, antisymmetric, transitive relation $\leq$ (sometimes $\leq_P$). Throughout this thesis, all posets considered are finite. A partial order is a *total* or *linear order* if for any x and y, either $x \leq y$ or $y \leq x$. Given $x, y \in P$, x *covers* y ("$x \gtrdot y$") if $x > y$ and if there is no $z \in P$ with $x > z > y$. Two elements $x, y \in P$ are *incomparable* (denoted $x \parallel y$) if neither $x \leq y$ nor $y \leq x$. Given $x \in P$, define

$$\begin{aligned}
D(x) &:= \{y \in P : y < x\} \\
U(x) &:= \{y \in P : y > x\} \\
D[x] &:= \{y \in P : y \leq x\} \\
U[x] &:= \{y \in P : y \geq x\} \\
I(x) &:= \{y \in P : y \parallel x\}.
\end{aligned}$$

If $D(x) = \emptyset$ then x is called *minimal*, and if $U(x) = \emptyset$ then x is called *maximal*. If P has a unique minimal element, it is denoted $\hat{0}$, and if there is a unique maximal element, it is called $\hat{1}$. If $I(x) = \emptyset$, then x is a *pivot* (sometimes called a *bottleneck*).

A finite poset can be described pictorially by means of a *Hasse diagram*. In a Hasse diagram, elements are vertices, and edges are cover relations, where lesser element in the cover occurs lower on the page. All other order relations can be deduced from the Hasse diagram by transitivity. Figure 2.1 in Section 2.2 shows two examples of Hasse diagrams.

If $x \leq y$ in P, the *closed interval* $[x, y]_P$ is $\{z \in P : x \leq z \leq y\}$, and the *open interval* $(x, y)_P$ is $\{z \in P : x < z < y\}$. The subscript P will usually be omitted. Similarly there are "half-open" intervals $(x, y]$ and $[x, y)$.

Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$ and let $[k, n]$ denote the set $\{k, k+1, k+2, \ldots, n\}$.

A *subposet* of P is a subset $S \subseteq P$, together with the partial order induced on S by $\leq_P$. Often this is referred to as an *induced subposet*. A *chain* is a totally ordered subposet of P. A chain is *unrefinable* if whenever x covers y in the chain, x also covers y in P, and *maximal* if it is not properly contained in any other chain. An *antichain* is a subposet of P whose elements are pairwise incomparable. The *width* of a poset is the size of a largest antichain. Dilworth [18] showed that the width of a poset is also equal to the smallest number w such that P can be partitioned into w chains. An *order ideal* is $I \subseteq P$ such that if $x \in I$ and $y \leq x$, then $y \in I$. A *principal* order ideal is an ideal $I = D[x]$ for some $x \in P$. The dual of a poset P is the same set of elements, with the reversed partial order. A poset is *self-dual* if it is isomorphic to its dual. A *dual order ideal* or *order filter* is a subset of P which is an order ideal in the dual of P.

A poset is *graded* if every maximal chain has the same number of elements. A *rank* function on P is the unique function such that $\mathrm{rank}(x) = 0$ for any minimal element x, and $\mathrm{rank}(x) = \mathrm{rank}(y) + 1$ if $x{>}y$. The *rank number* $R_r(P)$ is the number of elements of P of rank r. A ranked poset is called *Sperner* if its width is equal to the maximum of its rank numbers. A *symmetric chain* in a graded poset is an unrefinable chain with bottom and top elements x and y, such that $\mathrm{rank}(x) = n - 1 - \mathrm{rank}(y)$, where n is the cardinality of a maximal chain. A *symmetric chain decomposition* is a decomposition of P into symmetric chains. If P has a symmetric chain decomposition, then it is Sperner. (In fact it is *strongly Sperner*, but that definition is not needed here.) Given two posets P and Q, form their *(direct or Cartesian) product* $P \times Q$. The underlying set is the ordered pairs (p, q) with $p \in P$ and $q \in Q$, and the partial order is $(p, q) \leq_{P \times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$. Later we will need the fact that a product of two chains has a symmetric chain decomposition. The product of P with a two-element chain is called the pyramid $\mathrm{Pyr}(P)$.

Given posets P and Q, a map $\eta : P \to Q$ is *order-preserving* if $a \leq_P b$ implies $\eta(a) \leq_Q \eta(b)$. A poset Q is an *extension* of P if the two are equal as sets, and if $a \leq_P b$ implies $a \leq_Q b$. A *linear extension* is an extension which is also a total order. Every poset has a linear extension, and a poset P is the intersection (as relations) of all linear extensions of P. The *order dimension* $\dim(P)$ is the smallest number d such that P is the intersection of d linear extensions of P. Equivalently the order dimension is the smallest d so that P can be embedded as a subposet of $\mathbb{N}^d$ with componentwise partial order. The "standard example" of a poset of dimension n is the set of subsets of $[n]$ of cardinality 1 or $n-1$, ordered by inclusion.

A poset is called a *lattice* if for any two elements x and y, there is a unique minimal element in $\{z : z \geq x, z \geq y\}$, and a unique maximal element in $\{z : z \leq x, z \leq y\}$, These elements are called respectively the *join* $x \vee y$ and the *meet* $x \wedge y$.

2.2 Coxeter groups and Bruhat order

In this section we give the definition of a Coxeter group and the (strong) Bruhat order on a Coxeter group, as well as one alternate characterization of the Bruhat order.

A *Coxeter system* is a pair (W, S), where W is a group, S is a set of generators, and W is given by the presentation $(st)^{m(s,t)} = 1$ for all $s, t \in S$, with the requirements that:

(i) $m(s, s) = 1$ for all $s \in S$, and

(ii) $2 \leq m(s, t) \leq \infty$ for all $s \neq t$ in S.

In other words, each generator is of order two, the generators are distinct, and they satisfy the "pairwise order relations" given by (ii), or no relation if $m(s, t) = \infty$. The Coxeter system is called *universal* if $m(s, t) = \infty$ for all $s \neq t$. We will refer to a "Coxeter group," W with the understanding that a generating set S has been chosen such that (W, S) is a Coxeter system. In what follows, W or (W, S) will always refer to a fixed Coxeter system, and w will be an element of W.

Examples of finite Coxeter groups include the symmetric group, other Weyl groups of root systems, and symmetry groups of regular polytopes. We will continue to follow the example of the symmetric group S_4 introduced at the beginning of Chapter 1. In this example, W is the symmetric group S_4 and S is the set containing the transpositions $r := (1\,2)$, $s := (2\,3)$ and $t := (3\,4)$. In the language of Coxeter groups, S_4 is the Weyl group A_3, or the Coxeter system $(A_3, \{r, s, t\})$, with $m(r, s) = m(s, t) = 3$ and $m(r, t) = 2$.

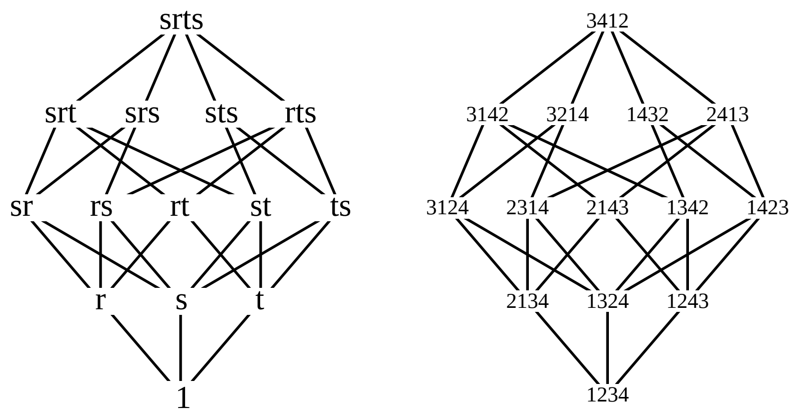
Call a word $w = s_1 s_2 \cdots s_k$ with letters in S a *reduced* word for w if k is as small as possible. Call this k the *length* of w, denoted $l(w)$. We will use the symbol "1" to represent the empty word, which corresponds to the identity element of W. Given any words a_1 and a_2 and given words $b_1 = stst\cdots$ with $l(b_1) = m(s, t)$ and $b_2 := tsts\cdots$ with $l(b_2) = m(s, t)$, the words $a_1 b_1 a_2$ and $a_1 b_2 a_2$ both stand for the same element. Such an equivalence is called a *braid move*. A theorem of Tits says that given any two reduced words a and b for the same element, a can be transformed into b by a sequence of braid moves.

There are several equivalent definitions of Bruhat order. See [16] for a discussion of the equivalent formulations. One definition is by the "Subword Property." Fix a reduced word $w = s_1 s_2 \cdots s_k$. Then $v \leq_B w$ if and only if there is a reduced subword $s_{i_1} s_{i_2} \cdots s_{i_j}$ corresponding to v such that $1 \leq i_1 < i_2 < \cdots < i_j \leq k$. We will write $v \leq w$ for $v \leq_B w$ when the context is clear. Figure 2.1 is an example of an interval in the Bruhat order on S_4.

Bruhat order is ranked by length. The element w covers the elements which can be represented by reduced words obtained by deleting a single letter from a reduced word for w. For example, let w be the permutation $4321 \in S_4$, and choose the reduced word $rsrtsr$. Deleting single letters produces the words $srtsr$, $rrtsr$, $rstsr$, $rsrsr$, $rsrtr$, and $rsrts$. However, not all of these are reduced. Corresponding reduced words are, respectively, $srtsr$, tsr, $rstsr$, s, rst, and $rsrts$. Thus 4321 covers $srtsr = 4312$, $rstsr = 4231$, and $rsrts = 3421$.

A finite Coxeter group W has an element w_0 of maximal length which is an involution, and which gives rise an anti-automorphism $w \mapsto w_0 w$ of Bruhat order. The map $w \mapsto w_0 w w_0$ is

Figure 2.1: The interval $[1, srts]$ or $[1234, 3412]$ in $(S_4, \{r, s, t\})$.



an automorphism which permutes the generators S of W. For S_n the maximal element is the permutation $n(n-1)\cdots 21$.

We will need the "lifting property" of Bruhat order, which can be proven easily using the Subword Property.

Proposition 2.2.1. *If $w > sw$ and $su > u$, then the following are equivalent:*

(i) $w > u$

(ii) $sw > u$

(iii) $w > su$

2.3 Polytopes

This section contains basic information about polytopes. A set C in $\mathbb{R}^d$ is convex if for any $x, y \in C$, the straight line segment with endpoints x and y is contained in C. Given a set $S \subseteq \mathbb{R}^d$, the *convex hull* $\mathrm{conv}(S)$ of S is the intersection of all convex sets containing S. A *hyperplane* in $\mathbb{R}^d$ is the solution set of the equation $a \cdot x = b$, for some fixed vectors a and b, and the corresponding *closed halfspace* is the solution set of the inequality $a \cdot x \leq b$. A *(convex) polytope* $\mathcal{P}$ is the convex hull of a finite number of points. Equivalently—although it takes some work to show it—a convex polytope is a bounded set which is the intersection of a finite number of closed halfspaces. An *affine subspace* of $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ which can be written as $L + v := \{x + v : x \in L\}$ for some linear subspace L and some vector $v \in \mathbb{R}^d$. The *affine span* of a set $S \subseteq \mathbb{R}^d$ is the intersection of all affine subspaces containing S. The *dimension* of a polytope $\mathcal{P}$ is the dimension of its affine span.

A hyperplane $a \cdot x = b$ is called a *supporting* hyperplane of $\mathcal{P}$ if $a \cdot x \leq b$ for every point in $\mathcal{P}$. A *face* of $\mathcal{P}$ is any intersection of $\mathcal{P}$ with a supporting hyperplane. In particular $\emptyset$ is a face of $\mathcal{P}$, and any face of $\mathcal{P}$ is itself a convex polytope. By convention, $\mathcal{P}$ is considered to be a face of itself. A *facet* of $\mathcal{P}$ is a face whose dimension is one less than the dimension of $\mathcal{P}$. The *face lattice* of $\mathcal{P}$

is the set of faces of $\mathcal{P}$ partially ordered by inclusion. A priori, this is just a poset, but it is easily checked that it is a lattice, where the meet operation is intersection. Two polytopes are of the same *combinatorial type* if their face lattices are isomorphic.

We will need two geometric constructions on polytopes, the pyramid operation Pyr and the vertex-shaving operation $\mathcal{S}_v$. Given a polytope P of dimension d, $\mathrm{Pyr}(P)$ is the convex hull of the union of P with some vector v which is not in the affine span of $\mathcal{P}$. This is unique up to combinatorial type. In Section 2.1, a pyramid operator was defined on posets. The face poset of $\mathrm{Pyr}(\mathcal{P})$ is just the pyramid of the face poset of $\mathcal{P}$.

Consider a polytope P and a chosen vertex v. Let $H = \{a \cdot x = b\}$ be a hyperplane that separates v from the other vertices of P. In other words, $a \cdot v > b$ and $a \cdot v' < b$ for all vertices $v' \neq v$. Then the polytope $\mathcal{S}_v(P) = P \cap \{a \cdot x \leq b\}$ is called the *shaving* of P at v. This is unique up to combinatorial type. Every face of P—except v—corresponds to a face in $\mathcal{S}_v(P)$, and in addition for every face of P strictly containing v, there is an additional face of one lower dimension in $\mathcal{S}_v(P)$. In Section 2.4 we describe how this operator can be extended to regular CW spheres, and in Section 3.4 we describe the corresponding operator on posets.

2.4 CW complexes and PL topology

This section provides background material on CW complexes and PL topology which will be useful in Section 3.2. More details about CW complexes, particularly as they relate to posets, can be found in [7]. Additional details about PL topology can be found in [10, 51].

A set of n points in $\mathbb{R}^d$ is *affinely independent* if the smallest affine subspace containing them has dimension $n-1$. A *simplex* (plural: *simplices*) is a polytope which is the convex hull of an affinely independent set S of points. The faces of the simplex are the affine hulls of subsets of S. A *geometric simplicial complex* Δ is a finite collection of simplices (called *faces* of the complex) such that

(i) If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$.

(ii) If $\sigma, \tau \in \Delta$ and $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a face of σ and of τ.

The zero-dimensional faces are called *vertices* . The *underlying space* $|\Delta|$ of Δ is the union in $\mathbb{R}^d$ of the faces of Δ.

An *abstract simplicial complex* Δ on a finite *vertex-set* V is a collection of subsets of V called *faces*, with the following properties:

(i) Every singleton is a face.

(ii) Any subset of a face is another face.

Given a geometric simplicial complex Δ with vertices V, the collection

$$\{F \subseteq V : \text{the convex hull of } F \text{ is in } \Delta\}$$

is an abstract simplicial complex. The process can be reversed: given an abstract simplicial complex Δ, there is a construction which produces a geometric simplicial complex whose underlying abstract simplicial complex is exactly Δ. This *geometric realization* is unique up to homeomorphism, so it makes sense to talk about the topology of an abstract simplicial complex. Two geometric simplicial complexes are *combinatorially isomorphic* if their underlying abstract simplicial complexes are isomorphic. If two complexes are combinatorially isomorphic then their underlying spaces are homeomorphic, but the converse is not true.

Given simplicial complexes Δ and Γ, say Γ is *subdivision* of Δ if $|\Gamma| = |\Delta|$ and if every face of Γ is contained in some face of Δ. A simplicial complex is a *PL d-sphere* if it admits a simplicial subdivision which is combinatorially isomorphic to some simplicial subdivision of the boundary of a $(d+1)$-dimensional simplex. A simplicial complex is a *PL d-ball* if it admits a simplicial subdivision which is combinatorially isomorphic to some simplicial subdivision of a d-dimensional simplex.

We now quote some results about PL balls and spheres. Some of these results appear topologically obvious but, surprisingly, not all of these statement are true with the "PL " deleted. This is the reason that we introduce PL balls and spheres, rather than dealing with ordinary topological balls and spheres. Theorem 2.4.3 is known as Newman's Theorem.

Theorem 2.4.1. *[10, Theorem 4.7.21(i)] Given two PL d-balls whose intersection is a PL $(d-1)$-ball lying in the boundary of each, the union of the two is a PL d-ball.* □

Theorem 2.4.2. *[10, Theorem 4.7.21(ii)] Given two PL d-balls whose intersection is the entire boundary of each, the union of the two is a PL d-sphere.* □

Theorem 2.4.3. *[10, Theorem 4.7.21(iii)] The closure of the complement of a PL d-ball embedded in a PL d-sphere is a PL d-ball.* □

Given two abstract simplicial complexes Δ and Γ, let $\Delta * \Gamma$ be the *join* of Δ and Γ, a simplicial complex whose vertex set is the disjoint union of the vertices of Δ and of Γ, and whose faces are exactly the sets $F \cup G$ for all faces F of Δ and G of Γ. Let B^d stand for a PL d-ball, and let S^d be a PL d-sphere.

Proposition 2.4.4. *[51, Proposition 2.23]*

$$B^p * B^q \cong B^{p+q+1} \tag{2.1}$$

$$S^p * B^q \cong B^{p+q+1} \tag{2.2}$$

$$S^p * S^q \cong S^{p+q+1} \tag{2.3}$$

□

Here $\cong$ stands for PL homeomorphism, which we won't define. In particular, $B^p * B^q$ is a PL ball, etc.

The most important examples, for our purposes, of simplicial complexes are the order complexes of finite posets. Given a poset P the order complex $\Delta(P)$ is the abstract simplicial complex whose vertices are the elements of P and whose faces are the sets of elements which induce totally ordered subposets. In other words, the faces are the chains of P, and $\Delta(P)$ is sometimes called the *chain complex* of P. The order complex of an interval $[x, y]$ will be written $\Delta[x, y]$, rather than $\Delta([x, y])$, and similarly $\Delta(x, y)$, and so forth.

When P is a poset with a $\hat{0}$ and a $\hat{1}$ then statements about the topology of P are understood to apply to the order complex of $(\hat{0}, \hat{1}) = P - \{\hat{0}, \hat{1}\}$. Thus for example, the statement that "P is a PL sphere" means that $\Delta(\hat{0}, \hat{1})$ is a PL sphere. The following proposition follows immediately from [10, Theorem 4.7.21(iv)]:

Proposition 2.4.5. *If P is a PL sphere then any interval $[x, y]_P$ is a PL sphere.* □

An *open cell* is any topological space isomorphic to an open ball. A *CW complex* Ω is a Hausdorff topological space with a decomposition as a disjoint union of cells, such that for each cell e, the homeomorphism mapping an open ball to e is required to extend to a continuous map from the closed ball to Ω. The image of this extended map is $\overline{e}$, which is called a *closed cell*, specifically the *closure* of e. The *face poset* of Ω is the set of closed cells, together with the empty set, partially ordered by containment. The *k-skeleton* of Ω is the union of the closed cells of dimension k or less. A CW complex is *regular* if all the closed cells are homeomorphic to closed balls.

Call P a CW poset if it is the face poset of a regular CW complex Ω. It is well known that in this case Ω is homeomorphic to $\Delta(P - \{\hat{0}\})$. Björner [7] showed that

Theorem 2.4.6. *A non-trivial poset P is a CW poset if and only if*

(i) P has a minimal element $\hat{0}$, and

(ii) For all $x \in P - \{\hat{0}\}$, the interval $[\hat{0}, x]$ is a sphere. □

Given a CW poset, Björner constructs a complex $\Omega(P)$ recursively by constructing the $(k-1)$-skeleton, and then attaching k-cells in a way that agrees with the order relations in P.

The polytope operations Pyr and $\mathcal{S}_v$ can also be defined on regular CW spheres. Both operations preserve PL sphericity via Theorem 2.4.2. We give informal descriptions which are easily made rigorous. Consider a regular CW d-sphere Ω embedded as the unit sphere in $\mathbb{R}^{d+1}$. The new vertex in the Pyr operation will be the origin. Each face of Ω is also a face of $\text{Pyr}(\Omega)$ and for each nonempty face F of Ω there is a new face F' of $\text{Pyr}(\Omega)$, described by

$$F := \{v \in \mathbb{R}^{d+1} : 0 < |v| < 1, \frac{v}{|v|} \in F\}.$$

The set $\{v \in \mathbb{R}^{d+1} : |v| > 1\} \cup \{\infty\}$ is also a face of $\text{Pyr}(\Omega)$ (the "base" of the pyramid) where ∞ is the point at infinity which makes $\mathbb{R}^{d+1} \cup \{\infty\}$ a $d+2$-sphere.

Consider a regular CW sphere Ω and a chosen vertex v. Adjoin a new open cell to make Ω', a ball of one higher dimension. Choose S to be a small sphere $|x - v| = \epsilon$, such that the only vertex inside the sphere is v and the only faces which intersect S are faces which contain v. (Assuming some nice embedding of Ω in space, this can be done.) Then $\mathcal{S}_v(\Omega)$ is the boundary of the ball obtained by intersecting Ω' with the set $|x - v| \geq \epsilon$. As in the polytope case, this is unique up to combinatorial type. Every face of Ω, except v, corresponds to a face in $\mathcal{S}_v(\Omega)$, and for every face of Ω strictly containing v, there is an additional face of one lower dimension in $\mathcal{S}_v(\Omega)$.

Given a poset P with $\hat{0}$ and $\hat{1}$, call P a regular CW sphere if $P - \{\hat{1}\}$ is the face poset of a regular CW complex which is a sphere. In other words, P is a regular CW sphere if every lower interval of P is a sphere. In light of Proposition 2.4.5 and Theorem 2.4.6, if P is a PL sphere, it is also a CW sphere (but not conversely). Section 3.4 describes a construction on posets which corresponds to $\mathcal{S}_v$.

2.5 The cd-index of an Eulerian poset

In this section we give the definition of Eulerian posets, flag f-vectors, flag h-vectors, and the cd-index, and quote results about the cd-indices of polytopes.

The *Möbius function* $\mu\colon \{(x,y)\colon x \leq y \text{ in } P\} \to \mathbb{Z}$ is defined recursively as follows:

$$\mu(x,x) = 1, \text{ for all } x \in P$$

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z) \text{ for all } x < y \text{ in } P.$$

The importance of the Möbius function derives from the following, whose simple proof can be found, for example, in [55].

Proposition 2.5.1 (Möbius Inversion Formula). *Let $f, g : P \to \mathbb{C}$. Under suitable conditions on P (for example, if P is finite) then*

$$g(x) = \sum_{y \leq x} f(y)$$

if and only if

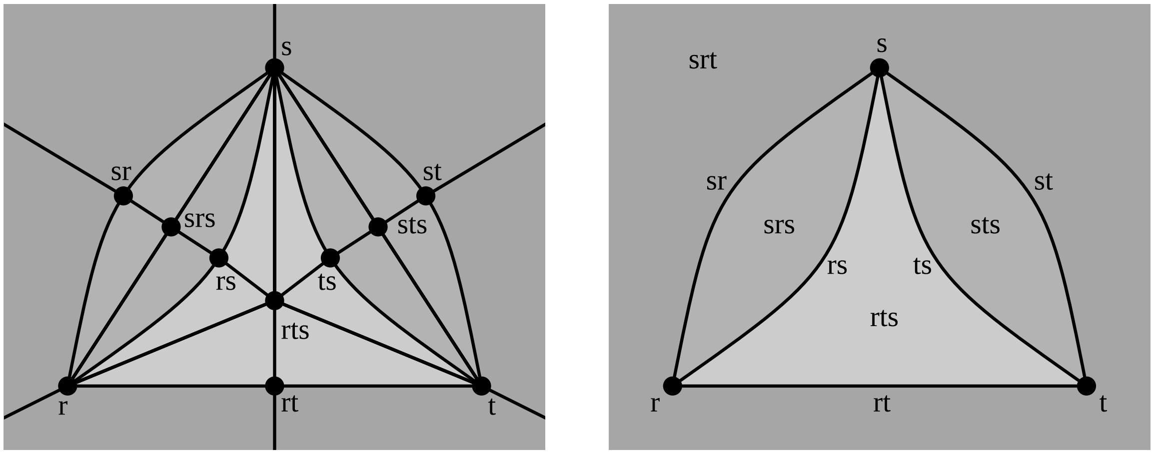
$$f(x) = \sum_{y \leq x} \mu(y,x) g(y).$$

The dual statement—obtained by replacing "$\leq$" by "$\geq$" everywhere—also holds. □

A poset P is *Eulerian* if $\mu(x,y) = (-1)^{\text{rank}(y) - \text{rank}(x)}$ for all intervals $[x,y] \subseteq P$. The resulting simple form of the Möbius inversion formula is the key to the proofs of special properties of Eulerian posets. It can be shown that P is Eulerian if and only if every interval $[x,y]$ with $\text{rank}(x,y) \geq 1$ has an equal number of elements of even rank and elements of odd rank [55, Exercise 3.69.a]. For a survey of Eulerian posets, see [57].

Verma [59] gives an inductive proof that Bruhat order is Eulerian, by counting elements of even and odd rank. Rota [50] proved that the face lattice of a convex polytope is an Eulerian poset (See also [40]). A more general example of an Eulerian poset P is a CW sphere. In [7], Björner showed that Bruhat intervals are CW spheres. Figure 2.2 shows the order complex $\Delta(1, srts)$, and the CW complex $\Omega[1, srts]$. The faces of $\Omega[1, srts]$ and the vertices of $\Delta(1, srts)$ are labeled with elements of the interval. Note that the order complex is the barycentric subdivision of the CW sphere.

Figure 2.2: The order complex $\Delta(1, srts)$ and associated CW complex $\Omega[1, srts]$. The element srt is the unbounded region in $\Omega[1, srts]$, or the vertex at infinity in $\Delta(1, srts)$.



We now proceed to define the cd-index. Let P be a graded poset, rank $n+1$, with a minimal element $\hat{0}$ and a maximal element $\hat{1}$. For a chain C in $P - \{\hat{0}, \hat{1}\}$, define $\operatorname{rank}(C) = \{\operatorname{rank}(x) : x \in C\}$. For any $S \subseteq [n]$, define

$$\alpha_P(S) = \#\{\text{chains } C \subseteq P : \operatorname{rank}(C) = S\}.$$

The function $\alpha_P : 2^{[n]} \to \mathbb{N}$ is called the *flag f-vector*, because it is a refinement of the *f-vector*, which counts the number of elements of each rank.

Define a function $\beta_P : 2^{[n]} \to \mathbb{N}$ by

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T), \tag{2.4}$$

or equivalently, by inclusion-exclusion

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T). \tag{2.5}$$

The function β is called the *flag h-vector* because of its relation to the usual *h-vector*.

Bayer and Billera [2] proved a set of linear relations on the flag f-vector of an Eulerian poset, which they called "Generalized Dehn-Sommerville relations," because their proof follows the proof of the Dehn-Sommerville relations for h-vectors of simplicial polytopes. They also proved that the Generalized Dehn-Sommerville relations—now commonly called the Bayer-Billera relations—and the relation $\alpha_P(\emptyset) = 1$ are the complete set of affine relations satisfied by flag f-vectors of all Eulerian posets.

Let $\mathbb{Z}\langle a, b\rangle$ be the vector space of *ab-polynomials* —polynomials over non-commuting variables a and b with integer coefficients. Subsets $S \subseteq [n]$ can be represented by monomials $u_S = u_1 u_2 \cdots u_n \in \mathbb{Z}\langle a, b\rangle$, where $u_i = b$ if $i \in S$ and $u_i = a$ otherwise. Define ab-polynomials Υ_P and Ψ_P to encode the flag f-vector and flag h-vector respectively.

$$\Upsilon_P(a,b) := \sum_{S\subseteq[n]} \alpha_P(S)u_S \tag{2.6}$$

$$\Psi_P(a,b) := \sum_{S\subseteq[n]} \beta_P(S)u_S. \tag{2.7}$$

The polynomial Ψ_P is commonly called the *ab-index*. There is no standard name for Υ_P, but here we will call it the *flag index*. It is easy to show that Equation (2.4) is equivalent to $\Upsilon_P(a-b,b) = \Psi_P(a,b)$.

The interval $[1234, 3412]$, shown in Figure 2.1 has flag index:

$$\Upsilon_{[1234,3412]} = a^3 + 4a^2b + 5aba + 10ab^2 + 3ba^2 + 10bab + 10b^2a + 20b^3$$

Replacing a by $a-b$, one obtains:

$$\Psi_{[1234,3412]} = a^3 + 3a^2b + 4aba + 2ab^2 + 2ba^2 + 4bab + 3b^2a + b^3$$

Encoding β_P in a polynomial allows a simplification of the complicated Bayer-Billera relations. Let $c = a+b$ and $d = ab+ba$ in $\mathbb{Z}\langle a,b\rangle$. The flag f-vector of a graded poset P satisfies the Bayer-Billera relations if and only if $\Psi_P(a,b)$ can be written as a polynomial in c and d with integer coefficients, called the *cd-index* of P. This surprising fact was proven by Bayer and Klapper [3], who credit J. Fine with suggesting it. The cd-index is *monic*, meaning that the coefficient of c^n is always 1. The existence and monicity of the cd-index constitute the complete set of affine relations on the flag f-vector of an Eulerian poset. Setting the degree of c to be 1 and the degree of d to be 2, the cd-index of a poset of rank $n+1$ is homogeneous of degree n. It is easy to show that the number of cd-monomials of degree $n-1$ is F_n, the n^{th} Fibonacci number, with $F_1 = F_2 = 1$. Thus the affine span of flag f-vectors of Eulerian posets of degree n is $F_n - 1$ [2].

Continuing the previous example, we have:

$$\Psi_{[1234,3412]} = c^3 + 2cd + dc.$$

The literature is divided on notation for the cd-index, due to two valid points of view as to what the ab-index is. If one considers Ψ_P to be a polynomial function of non-commuting variables a and b, one must consider the cd-index to be a different polynomial function in c and d, and give it a different name, typically Φ_P. On the other hand, if Ψ_P is a vector in a space of ab-polynomials, the cd-index is the same vector, which happens to be written as a linear combination of monomials in c and d. Thus one would call the cd-index Ψ_P. We will use either notation, depending on which aspect of the cd-index we need to discuss: in particular, when we talk about inequalities on the coefficients of the cd-index, we must use Φ_P.

Aside from the existence and monicity of the cd-index, there are no additional affine relations on flag f-vectors of polytopes. Bayer and Billera [2] and later Kalai [36] gave a basis of polytopes whose flag f-vectors span $\mathbb{Z}\langle c,d\rangle$. A non-constructive proof of the existence of such a basis can be found in [5]. In the same paper there is a proof that no additional affine relations hold for the flag f-vectors of zonotopes, and a proof that the integer span of cd-indices of zonotopes is $\mathbb{Z}\langle c, 2d\rangle$.

Much is also known about bounds on the coefficients of the cd-index of a polytope. A bound on the cd-index implies bounds on α and β, because α and β can be written as positive combinations of coefficients of the cd-index. The first consideration is the non-negativity of the coefficients. Stanley [56] conjectured that the coefficients of the cd-index are non-negative whenever P is a homology sphere (or in other words when P is a *Gorenstein* poset*). He also showed that the coefficients of Φ_P are non-negative for a class of CW-spheres which includes convex polytopes.

Ehrenborg and Readdy described how the cd-index is changed by the poset operations of pyramid and vertex shaving. The following is a combination of Propositions 4.2 and 6.1 of [25].

Proposition 2.5.2. *Let P be a graded poset and let a be an atom. Then*

$$\Psi_{\mathrm{Pyr}(P)} = \frac{1}{2}\left(\Psi_P \cdot c + c \cdot \Psi_P + \sum_{x \in P, \hat{0} < x < \hat{1}} \Psi_{[\hat{0},x]} \cdot d \cdot \Psi_{[x,\hat{1}]}\right) \tag{2.8}$$

$$\Psi_{\mathcal{S}_a(P)} = \Psi_P + \frac{1}{2}\left(\Psi_P \cdot c - c \cdot \Psi_P + \sum_{a < x < \hat{1}} \Psi_{[a,x]} \cdot d \cdot \Psi_{[x,\hat{1}]}\right). \tag{2.9}$$

Ehrenborg and Readdy also defined a derivation on cd-indices and used it to restate the formulas in Proposition 2.5.2. The derivation G (called G' in [25]) is defined by $G(c) = d$ and $G(d) = dc$. The following is a combination of Theorem 5.2 and Proposition 6.1 of [25].

Proposition 2.5.3. *Let P be a graded poset and let a be an atom. Then*

$$\begin{aligned} \Psi_{\mathrm{Pyr}(P)} &= c \cdot \Psi_P + G(\Psi_P) \\ \Psi_{\mathcal{S}_a(P)} &= \Psi_P + G(\Psi_{[a,\hat{1}]}). \end{aligned}$$

Chapter 3

Recursions for Bruhat intervals

3.1 Main results

In this chapter, we establish a fundamental structural recursion on intervals in the Bruhat order on Coxeter groups. The recursion gives the isomorphism type of a Bruhat interval in terms of smaller intervals, using some basic geometric operations, namely the operations of pyramid, vertex shaving and a "zipping" operation. These operations preserve the Eulerian property and PL sphericity, and have a simple effect on the cd-index. Thus we obtain a new inductive proof that Bruhat intervals are PL spheres as well as recursions for the cd-index of Bruhat intervals:

Theorem 3.1.1. *Let $u < us$, $w < ws$ and $u \leq w$.*

If $us \notin [u, w]$, then $\Psi_{[u,ws]} = \mathrm{Pyr}\Psi_{[u,w]}$, and $\Psi_{[us,ws]} = \Psi_{[u,w]}$.

If $us \in [u, w]$, then

$$\begin{aligned}
\Psi_{[u,ws]} &= \mathrm{Pyr}\Psi_{[u,w]} - \sum_{v\in(u,w):vs<v} \Psi_{[u,v]} \cdot d \cdot \Psi_{[v,w]} \\
&= \frac{1}{2}\left(\Psi_{[u,w]} \cdot c + c \cdot \Psi_{[u,w]} + \sum_{v\in(u,w)} \sigma_s(v)\Psi_{[u,v]} \cdot d \cdot \Psi_{[v,w]}\right) \\
\Psi_{[us,ws]} &= \mathcal{S}_{us}\Psi_{[u,w]} - \sum_{v\in(us,w):vs<v} \Psi_{[us,v]} \cdot d \cdot \Psi_{[v,w]} \\
&= \Psi_{[u,w]} + \frac{1}{2}\left(\Psi_{[us,w]} \cdot c - c \cdot \Psi_{[us,w]} + \sum_{v\in(us,w)} \sigma_s(v)\Psi_{[us,v]} \cdot d \cdot \Psi_{[v,w]}\right).
\end{aligned}$$

Here $\sigma_s(v) := l(vs) - l(v)$. This recursive formula leads to a proof that the cd-indices of Bruhat intervals span the space of cd-polynomials, and motivates a conjectured upper bound for the cd-indices of Bruhat intervals. The structural recursion is used to construct Bruhat intervals which are the face lattice of the duals of stacked polytopes [37]. Based on computer calculations and on the formulas in Theorem 3.1.1, we conjecture:

Conjecture 3.1.2. *The coefficientwise maximum of all cd-indices* $\Phi_{[u,v]}$ *with* $l(u) = k$ *and* $l(v) = d+k+1$ *is attained on a Bruhat interval which is isomorphic to a dual stacked polytope of dimension* d *with* $d+k+1$ *facets.*

The conjectured lower bound (non-negativity) on the coefficients of the cd-indices of Bruhat intervals was made, in a more general setting, by Stanley [56]. We show that if the conjecture on non-negativity holds, then the cd-index of any lower Bruhat interval is bounded above by the cd-index of a Boolean algebra. Since the flag h-vectors of Bruhat intervals are non-negative, we are able to prove that the flag h-vectors of lower Bruhat intervals are bounded above by the flag h-vectors of Boolean algebras.

The chapter is organized as follows: In Section 3.2, the zipping operation is introduced, and its basic properties are proven. Section 3.3 contains the definition of, and a basic proposition about order-projections, which will be used in Section 3.4 to prove the structural recursion. Section 3.5 gives constructions of Bruhat intervals which are isomorphic to the face lattices of certain convex polytopes, namely dual stacked polytopes and simplices. Section 3.6 contains the proof of Theorem 3.1.1 and Section 3.7 uses Theorem 3.1.1 to determine the affine span of cd-indices of Bruhat intervals. In Section 3.8, there is a discussion of conjectured bounds on the coefficients of the cd-index of a Bruhat interval, and in Section 3.9, recursions on other poset invariants are derived from Theorem 3.1.1.

3.2 Zipping

In this section we introduce the zipping operation and prove some of its important properties. In particular, zipping will be part of a new inductive proof that Bruhat intervals are spheres and thus Eulerian. A *zipper* in a poset P is a triple of distinct elements $x, y, z \in P$ with the following properties:

(i) z covers x and y but covers no other element.

(ii) $z = x \vee y$.

(iii) $D(x) = D(y)$.

Call the zipper *proper* if z is not a maximal element. If (x, y, z) is a zipper in P and $[a, b]$ is an interval in P with $x, y, z \in [a, b]$ then (x, y, z) is a zipper in $[a, b]$.

Given P and a zipper (x, y, z) one can "zip" the zipper as follows: Let xy stand for a single new element not in P. Define $P' = (P - \{x, y, z\}) \cup \{xy\}$, with a binary relation called $\preceq$, given by:

$$\begin{array}{ll} a \preceq b & \text{if } a \leq b \\ xy \preceq a & \text{if } x \leq a \text{ or if } y \leq a \\ a \preceq xy & \text{if } a \leq x \text{ or (equivalently) if } a \leq y \\ xy \preceq xy & \end{array}$$

For convenience, $[a, b]$ will always mean the interval $[a, b]_\leq$ in P and $[a, b]_\preceq$ will mean an interval in P'. In each of the following propositions, P' is obtained from P by zipping the proper zipper (x, y, z), although some of the preceding results are true even when the zipper in not proper.

Proposition 3.2.1. *P' is a poset under the partial order $\preceq$.*

Proof. One sees immediately that $\preceq$ is reflexive and that antisymmetry holds in $P' - \{xy\}$. If $xy \preceq a$ and $a \preceq xy$, but $a \neq xy$, then $a \in P - \{x, y, z\}$. We have $a \leq x$ and $a \leq y$. Also, either $x \leq a$ or $y \leq a$. By antisymmetry in P, either $a = x$ or $a = y$. This contradiction shows that $a = xy$. Transitivity follows immediately from the transitivity of P except perhaps when $a \preceq xy$ and $xy \preceq b$. In this case, $a \leq x$ and $a \leq y$. Also, either $x \leq b$ or $y \leq b$. In either case, $a \leq b$ and therefore $a \preceq b$. $\square$

Proposition 3.2.2. *If $a \preceq xy$ then $\mu_{P'}(a, xy) = \mu_P(a, x) = \mu_P(a, y)$. If $a \preceq b \in P'$ with $a \neq xy$, then $\mu_{P'}(a, b) = \mu_P(a, b)$.*

Suppose $[a, b]_\preceq$ is any non-trivial interval in P'. If $a \not\preceq xy$, then $[a, b]_\preceq = [a, b]_\leq$. If $b = xy$, then $[a, b]_\preceq \cong [a, x]_\leq$. If $b \neq xy$ and $b \not> z$, then $[a, b]_\leq$ does not contain both x and y, and we obtain $[a, b]_\preceq$ from $[a, b]_\leq$ by replacing x or y by xy if necessary. Thus in the proofs that follow, one needs only to check two cases: the case where $a \prec xy$ and $b > z$ and the case where $a = xy$.

Proof of Proposition 3.2.2. Let $a \preceq b$ with $a \neq xy$. One needs only to check the case where $a \prec xy$ and $b > z$. This is done by induction on the length of the longest chain from z to b. If $b{>}z$ then

$$\begin{aligned}
\mu_P(a, b) &= -\mu_P(a, z) - \mu_P(a, x) - \mu_P(a, y) - \sum_{a \leq p < b : p \neq x, y, z} \mu_P(a, p) \\
&= \sum_{a \leq p < x} \mu_P(a, p) - \sum_{a \leq p < b : p \neq x, y, z} \mu_P(a, p) \\
&= -\mu_{P'}(a, xy) - \sum_{a \preceq p \prec b : p \neq xy} \mu_{P'}(a, p) \\
&= \mu_{P'}(a, b).
\end{aligned}$$

Here the second line is obtained by properties (i) and (iii). If b does not cover z, use the same calculation, employing induction to go from the second line to the third line. $\square$

Proposition 3.2.3. *If $xy \preceq b \in P'$, then $\mu_{P'}(xy, b) = \mu_P(x, b) + \mu_P(y, b) + \mu_P(z, b)$.*

Proof. In light of Proposition 3.2.2, one can write:

$$\begin{aligned}
\mu_{P'}(xy,b) &= -\sum_{xy\prec p\preceq b}\mu_{P'}(p,b)\\
&= \left(-\sum_{\substack{p: x<p\le b\\ \text{or}\\ y<p\le b}}\mu_P(p,b)\right)+\mu_P(z,b)\\
&= \left(-\sum_{x<p\le b}\mu_P(p,b)-\sum_{y<p\le b}\mu_P(p,b)+\sum_{z\le p\le b}\mu_P(p,b)\right)+\mu_P(z,b)\\
&= \mu_P(x,b)+\mu_P(y,b)+\mu_P(z,b).
\end{aligned}$$

□

The following two corollaries follow trivially from Propositions 3.2.2 and 3.2.3 and the observation that if P is ranked, then P' inherits a rank function.

Corollary 3.2.4. *If P is thin, then so is P'* □

Corollary 3.2.5. *If P is Eulerian, then so is P'.* □

Theorem 3.2.6. *P has a cd-index if and only if P' has cd-index. The cd-indices are related by:*

$$\Psi_{P'} = \Psi_P - \Psi_{[\hat{0},x]_{\le}}\cdot d\cdot \Psi_{[z,\hat{1}]_{\le}}.$$

Proof. We subtract from Υ_P the chains which disappear under the zipping. First subtract the terms which came from chains through x and z. Any such chain is a chain in $[\hat{0},x]_P$ concatenated with a chain in $[z,\hat{1}]_P$. So the terms subtracted off are $\Upsilon_{[\hat{0},x]_P}\cdot b\cdot b\cdot\Upsilon_{[z,\hat{1}]}$. Then subtract a similar term for chains through y and z. In fact, by condition (iii) of the definition of a zipper, the term for chains through y and z is identical to the term for chains through x and z. Subtract $\Upsilon_{[\hat{0},x]_P}\cdot a\cdot b\cdot\Upsilon_{[z,\hat{1}]}$ for the chains which go through z but skip the rank below z. Finally, x is identified with y, so there is a double-count which must be subtracted off. If two chains are identical except that one goes through x and the other goes through y, then they are counted twice in P but only once in P'. Because $x\vee y = z$, if such a pair of chains include an element whose rank is $\mathrm{rank}(z)$, then that element is z. But the chains through z have already been subtracted, so we need to subtract off $\Upsilon_{[\hat{0},x]_P}b\cdot a\cdot\Upsilon_{[z,\hat{1}]}$. We have again used condition (iii) here. Thus:

$$\Upsilon_{P'} = \Upsilon_P - \Upsilon_{[\hat{0},x]_P}(2bb+ab+ba)\cdot\Upsilon_{[z,\hat{1}]_P}. \tag{3.1}$$

Replacing a by $a-b$ one obtains:

$$\begin{aligned}
\Psi_{P'} &= \Psi_P - \Psi_{[\hat{0},x]_P}\cdot(ab+ba)\cdot\Psi_{[z,\hat{1}]_P} &(3.2)\\
&= \Psi_P - \Psi_{[\hat{0},x]_P}\cdot d\cdot\Psi_{[z,\hat{1}]_P}. &(3.3)
\end{aligned}$$

□

Theorem 3.2.7. *If P is a PL sphere, then so is P'.*

Proof. To avoid tedious repetition, we will omit "PL" throughout the proof. All spheres and balls are assumed to be PL.

Suppose P is a k-sphere. Let $\Delta_{xyz} \subset \Delta(\hat{0}, \hat{1})$ be the simplicial complex whose facets are maximal chains in $P - \{\hat{0}, \hat{1}\}$ passing through x, y or z. Our first goal is to prove that Δ_{xyz} is a ball. Let $\Delta_x \subset \Delta(\hat{0}, \hat{1})$ be the simplicial complex whose facets are maximal chains in $(\hat{0}, \hat{1})$ through x. Similarly Δ_y. One can think of Δ_x as $\Delta(\hat{0}, x) * x * \Delta(x, \hat{1})$. Thus, by Proposition 2.4.4, Δ_x is a k-ball, and similarly, Δ_y. Let $\Gamma = \Delta_x \cap \Delta_y$. Then Γ is the complex whose facets are almost-maximal chains that can be completed to maximal chains either by adding x or y. These are the chains through z which have elements at every rank except at the rank of x. Thus Γ is $\Delta(\hat{0}, x) * z * \Delta(z, \hat{1})$, a $(k-1)$-ball, and Γ lies in the boundary of Δ_x, because there is exactly one way to complete a facet of Γ to a facet of Δ_x, namely by adjoining x. Similarly, Γ lies in the boundary of Δ_y. So by Proposition 2.4.1, $\Delta_{xyz} = \Delta_x \cup \Delta_y$ is a k-ball.

Consider $\Delta((\hat{0}, \hat{1}) - \{x, y, z\})$, which is the closure of $\Delta(\hat{0}, \hat{1}) - \Delta_{xyz}$. By Proposition 2.4.3, $\Delta((\hat{0}, \hat{1}) - \{x, y, z\})$ is also a k-ball. Also consider $\Delta((\hat{0}, \hat{1})_{\preceq} - \{xy\})$, which is isomorphic to $\Delta((\hat{0}, \hat{1}) - \{x, y, z\})$. The boundary of $\Delta((\hat{0}, \hat{1})_{\preceq} - \{xy\})$ is a complex whose facets are chains c with the property that for each c there is a unique element of $(\hat{0}, \hat{1})_{\preceq} - \{xy\}$ that completes c to a maximal chain. However, since $(\hat{0}, \hat{1})_{\preceq}$ is thin, it has the property that any chain of length $k-1$ can be completed to a maximal chain in $(\hat{0}, \hat{1})_{\preceq}$ in exactly two ways. Therefore every facet of the boundary of $\Delta((\hat{0}, \hat{1})_{\preceq} - \{xy\})$ is contained in a chain through xy. So $\Delta((\hat{0}, \hat{1})_{\preceq})$ is a k-ball $\Delta((\hat{0}, \hat{1})_{\preceq} - \{xy\})$ union the pyramid over the boundary of $\Delta((\hat{0}, \hat{1})_{\preceq} - \{xy\})$. Thus by Proposition 2.4.2, $\Delta((\hat{0}, \hat{1})_{\preceq})$ is a k-sphere. $\square$

In the case where P is thin, the conditions for a zipper can be simplified.

Proposition 3.2.8. *If P is thin, then (i) implies (iii). Thus (x, y, z) is a zipper if and only if it satisfies conditions (i) and (ii).*

Proof. Suppose condition (i) but suppose that $[\hat{0}, x) \neq [\hat{0}, y)$. Then without loss of generality x covers some a which y does not cover. Since z covers no element besides x and y, $[a, z]$ is a chain of length 2, contradicting thinness. $\square$

There is an alternate simplification when P is a CW poset.

Proposition 3.2.9. *If P is a CW poset, z covers x and y and (x, y, z) satisfies (iii), then (x, y, z) satisfies (i). Thus (x, y, z) is a zipper if and only z covers x and y and (x, y, z) satisfies conditions (ii) and (iii).*

Proof. Both of the intervals $(\hat{0}, x) = (\hat{0}, y)$ and $(\hat{0}, z)$ are homeomorphic to spheres. The induced subposet $(\hat{0}, x) \cup \{x, y\} \subseteq (\hat{0}, z)$ is a suspension of a sphere, so it is already homeomorphic to a sphere of the same dimension as $(\hat{0}, z)$. Thus $(\hat{0}, x) \cup \{x, y\} = (\hat{0}, z)$ $\square$

3.3 Order-projections

In this section, we define order-projections and fiber posets, which will be convenient for proving the structural recursion on Bruhat intervals. Let P and Q be posets, with some map $\eta : P \to Q$. Consider the set $\bar{P} := \{\eta^{-1}(q) : q \in Q\}$ of fibers of η, and define a relation $\leq_{\bar{P}}$ on $\bar{P}$ by $F_1 \leq_{\bar{P}} F_2$ if there exist $a \in F_1$ and $b \in F_2$ such that $a \leq_P b$. If $\leq_{\bar{P}}$ is a partial order, $\bar{P}$ is called the *fiber poset* of P with respect to η. In this case, there is a surjective order-preserving map $\nu : P \to \bar{P}$ given by $\nu : a \mapsto \eta^{-1}(\eta(a))$, and an injective order-preserving map $\bar{\eta} : \bar{P} \to Q$ such that $\eta = \bar{\eta} \circ \nu$.

Call η an *order-projection* if it is order-preserving and has the following property: For all $q \leq r$ in Q, there exist $a \leq b \in P$ with $\eta(a) = q$ and $\eta(b) = r$. In particular, an order projection is surjective.

Proposition 3.3.1. *Let $\eta : P \to Q$ be an order-projection. Then*

(i) *$\bar{P}$ is a fiber poset.*

(ii) *$\bar{\eta}$ is an order-isomorphism.*

Proof. Assertion (i) is just the statement that $\leq_{\bar{P}}$ is a partial order. The reflexive property is trivial. Let $A = \eta^{-1}(q)$ and $B = \eta^{-1}(r)$ for $q, r \in Q$. If $A \leq_{\bar{P}} B$ and $B \leq_{\bar{P}} A$, we can find a_1, a_2, b_1, b_2 with $\eta(a_1) = \eta(a_2) = q$, $\eta(b_1) = \eta(b_2) = r$, $a_1 \leq b_1$ and $b_2 \leq a_2$. Because η is order-preserving, $q \leq r$ and $r \leq q$, so $q = r$ and therefore $A = B$. Thus the relation is anti-symmetric.

To show that $\leq_{\bar{P}}$ is transitive, suppose $A \leq_{\bar{P}} B$ and $B \leq_{\bar{P}} C$. Then there exist $a \in A$, $b_1, b_2 \in B$ and $c \in C$ with $\eta(a) = q$, $\eta(b_1) = \eta(b_2) = r$, $\eta(c) = s$ $a \leq b_1$ and $b_2 \leq c$. Because η is order-preserving, $q \leq r \leq s$. By hypothesis, one can find $a' \leq c' \in P$ with $\eta(a') = q$ and $\eta(c') = s$. So $A \leq_{\bar{P}} C$.

Since η is surjective, $\bar{\eta}$ is an order-preserving bijection. Let $q \leq r$ in Q. Then, because η is an order-projection, there exist $a \leq b \in P$ with $\eta(a) = q$ and $\eta(b) = r$. So $\bar{\eta}^{-1}(q) = \eta^{-1}(q) \leq \eta^{-1}(r) = \bar{\eta}^{-1}(r)$ in $\bar{P}$. Thus $\bar{\eta}^{-1}$ is order-preserving. ☐

3.4 Building intervals in Bruhat order

This section states and proves the structural recursion for Bruhat intervals. When $s \in S$, $u < us$ and $w < ws$, define a map $\eta : [u, w] \times [1, s] \to [u, ws]$, as follows:

$$\begin{aligned} \eta(v, 1) &= v \\ \eta(v, s) &= \begin{cases} vs & \text{if } vs > v \\ v & \text{if } vs < v. \end{cases} \end{aligned}$$

To show that η is well-defined, let $v \in [u, w]$. Then $\eta(v, 1) = v \in [u, ws]$ because $ws > w \geq v \geq u$. Either $\eta(v, s) = v \in [u, ws]$ or $\eta(v, s) = vs$. In the latter case, $vs > v$, so $u < vs \leq ws$ by the lifting property.

Proposition 3.4.1. *If $u < us$ and $w < ws$, then $\eta : [u, w] \times [1, s] \to [u, ws]$ is an order-projection.*

Proof. To check that η is order-preserving, suppose $(v_1, a_1) \leq (v_2, a_2)$ in $[u, w] \times [1, s]$. We have to break up into cases to check that $\eta(v_1, a_1) \leq \eta(v_2, a_2)$.

Case 1: $a_1 = 1$.

If $a_2 = 1$ as well, $\eta(v_1, a_1) = v_1 \leq v_2 = \eta(v_2, a_2)$. If $a_2 = s$, then $\eta(v_2, a_2)$ is either v_2 with $v_2 \geq v$ or it is $v_2 s$ with $v_2 s > v_2 \geq v_1$.

Case 2: $a_1 = s$.

So $\eta(v_1, a_1)$ is either v_1, with $v_1 > v_1 s$ or it is $v_1 s$ with $v_1 s > v_1$. We must also have $a_2 = s$, so $\eta(v_2, a_2)$ is either v_2 with $v_2 > v_2 s$ or it is $v_2 s$ with $v_2 s > v_2$. If $\eta(v_1, a_1) = v_1$ then $\eta(v_1, a_1) \leq v_2 \leq \eta(v_2, a_2)$. If $\eta(v_1, a_1) = v_1 s$ and $\eta(v_2, a_2) = v_2$, we have $v_1 s > v_1$ and $v_2 > v_2 s$. By hypothesis, $v_1 < v_2$, so by the lifting property $\eta(v_1, a_1) = v_1 s < v_2 = \eta(v_1, a_1)$. If $\eta(v_1, a_1) = v_1 s$ and $\eta(v_2, a_2) = v_2 s$ then $v_1 s > v_1$ and $v_2 s > v_2$, so by the lifting property $v_s \leq v_2 s$.

It will be useful to identify the inverse image of an element $v \in [u, ws]$. The inverse image is:

$$\eta^{-1}(v) = \begin{cases} \{(v, 1)\} & \text{if } v < vs \\ \{(v, 1), (vs, s), (v, s)\} & \text{if } v > vs, \end{cases}$$

provided that these elements are actually in $[u, w] \times [1, s]$. In the case where $v < vs$, we have $u \leq v \leq w$, where the second inequality is by the lifting property. So $(v, 1)$ is indeed an element of $[u, w] \times [1, s]$. In the case where $vs < v$, we have by hypothesis $us > u$, so by lifting, $vs \geq u$. Also by lifting, since $ws > w$ and $v > vs$, we have $w > vs$. So $(vs, s) \in [u, w] \times [1, s]$.

Now, suppose $x_1 \leq x_2 \in [u, ws]$. To finish the proof that η is an order-projection, we must find elements $(v_1, a_1) \leq (v_2, a_2) \in [u, w] \times [1, s]$ with $\eta(v_1, a_1) = x_1$ and $\eta(v_2, a_2) = x_2$. Consider 4 cases:

Case 1: $x_1 < x_1 s$ and $x_2 < x_2 s$.

By the inverse-image argument of the previous paragraph, $x_1, x_2 \in [u, w]$, so $\eta(x_1, 1) = x_1$, $\eta(x_2, 1) = x_2$ and $(x_1, 1) \leq (x_2, 1)$.

Case 2: $x_1 < x_1 s$ and $x_2 > x_2 s$.

By the previous paragraph, $x_1, x_2 s \in [u, w]$. Again, $\eta(x_1, 1) = x_1$, and $\eta(x_2 s, s) = x_2$. By lifting, $x_1 \leq x_2 s$, so $(x_1, 1) \leq (x_2 s, s)$.

Case 3: $x_1 > x_1 s$ and $x_2 > x_2 s$.

We have $x_1 s, x_2 s \in [u, w]$, $\eta(x_1 s, s) = x_1$ and $\eta(x_2 s, s) = x_2$. By lifting, $x_1 s \leq x_2 s$, so $(x_1 s, s) \leq (x_2 s, s)$.

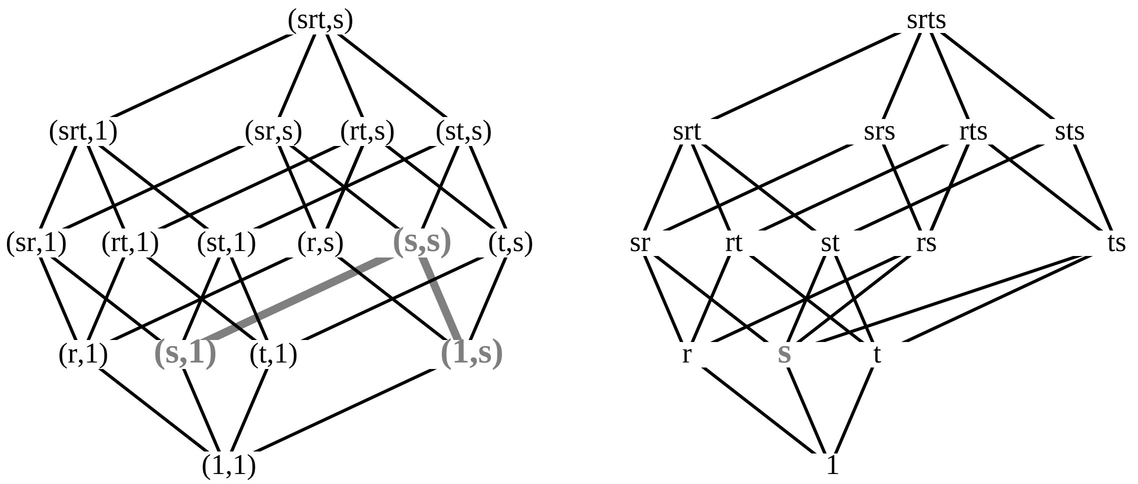
Case 4: $x_1 > x_1 s$ and $x_2 < x_2 s$.

We have $x_2 \in [u, w]$. Since $u \leq x_1 \leq x_2$, $x_1 \in [u, w]$ as well. So $\eta(x_1, 1) = x_1$, $\eta(x_2, 1) = x_2$ and $(x_1, 1) \leq (x_2, 1)$.

$\square$

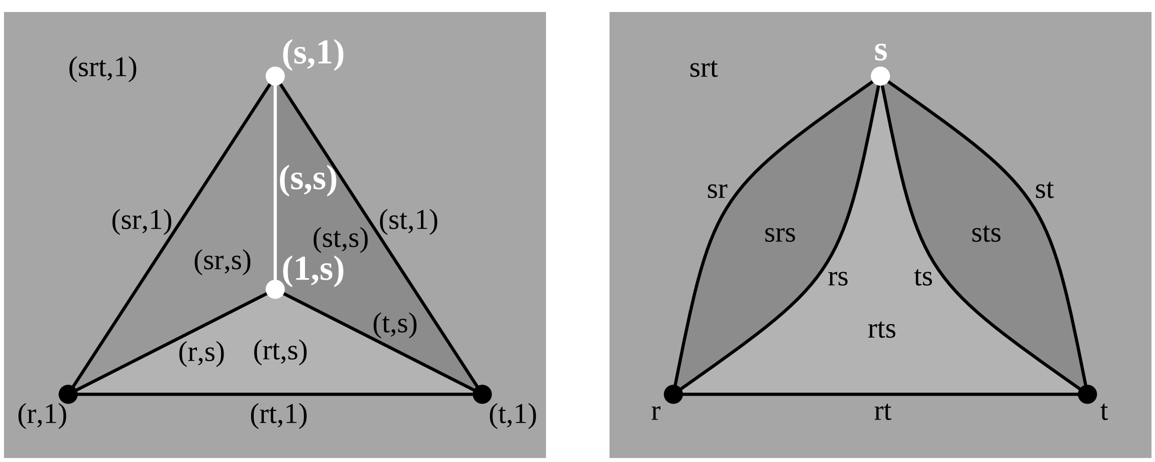
In light of the previous section, η induces an isomorphism $\bar{\eta}$ between $[u, ws]$ and a poset derived from $[u, w] \times [1, s]$, as follows: For every $v \in [u, w]$ with $vs < v$, "identify" $(v, 1)$, (vs, s) and (v, s) to make a single element. Since (v, s) covers $(v, 1)$ and (vs, s) we can also think of η as deleting (v, s) and identifying $(v, 1)$ with (vs, s).

Figure 3.1: The map $\eta : [1, srt] \times [1, s] \to [1, srts]$, where $[1, srt]$ and $[1, srts]$ are intervals in $(S_4, \{r, s, t\})$. All elements (u, v) map to uv except (s, s), which maps to s.



The map η induces a map (also called η) on the CW-spheres associated to Bruhat intervals, as illustrated in Figure 3.2.

Figure 3.2: $\eta : \Omega([1, srt] \times [1, s]) \to \Omega[1, srts]$.



Proposition 3.4.2. *Let $u < us$, $w < ws$ and $us \not\leq w$. Then $vs > v$ for all $v \in [u, v]$, and η is an isomorphism.*

Proof. Suppose for the sake of contradiction that there is a $v \in [u, w]$ with $vs < v$. Since $us > u$ and $u \leq v$, by lifting, $v \geq us$. By transitivity, $w \geq us$. This contradiction shows that that $vs > v$ for all $v \in [u, v]$.

Now, looking back at the proof of Proposition 3.4.1, we see that

$$\eta^{-1}(v) = \begin{cases} \{(v,1)\} & \text{if } v < vs \\ \{(vs,s)\} & \text{if } v > vs, \end{cases}$$

because in the $v > vs$ case, the other two possible elements of $\eta^{-1}(v)$ don't exist. Thus the map ν is an order-isomorphism and therefore $\eta = \bar{\eta} \circ \nu$ is also an order-isomorphism. $\square$

The following corollary is easy.

Corollary 3.4.3. *If $u < us$, $w < ws$ and $us \not\leq w$ the map $\zeta : [u,w] \to [us,ws]$ with $\zeta(v) = vs$ is an isomorphism.* $\square$

We would also like to relate the interval $[us,ws]$ to $[u,w]$ in the case where $us \leq w$. To do this, we need an operator on posets corresponding to vertex-shaving on polytopes or CW spheres. Let P be a poset with $\hat{0}$ and $\hat{1}$, and let a be an atom of P. The *shaving* of P at a is an induced subposet of $P \times [\hat{0}, a]$ given by:

$$\mathcal{S}_a(P) = \big(\big(P - \{\hat{0}, a\}\big) \times \{a\}\big) \cup \big((a, \hat{1}] \times \{\hat{0}\}\big) \cup \{(\hat{0}, \hat{0})\}.$$

We can also describe $\mathcal{S}_v(P)$ as follows: Let P' be obtained from $P \times [\hat{0}, a]$ by zipping the zipper $((a,\hat{0}), (\hat{0}, a), (a,a))$. Denote by a the element created by the zipping. Then $\mathcal{S}_v(P)$ is the interval $[a, (\hat{1}, a)]$ in P'. Figures 3.3 and 3.4 illustrate the operation of shaving.

Let $s \in S$, $u < us$, $w < ws$ and $us \leq w$. Define a map $\theta : \mathcal{S}_{us}[u,w] \to [us,ws]$ as follows. Starting with $[u,w] \times [1,s]$, zip $((us,1),(u,s),(us,s))$, call the new element us, and identify $S_{us}[u,w]$ with the interval $[us,(w,s)]$ in the zipped poset. Now define:

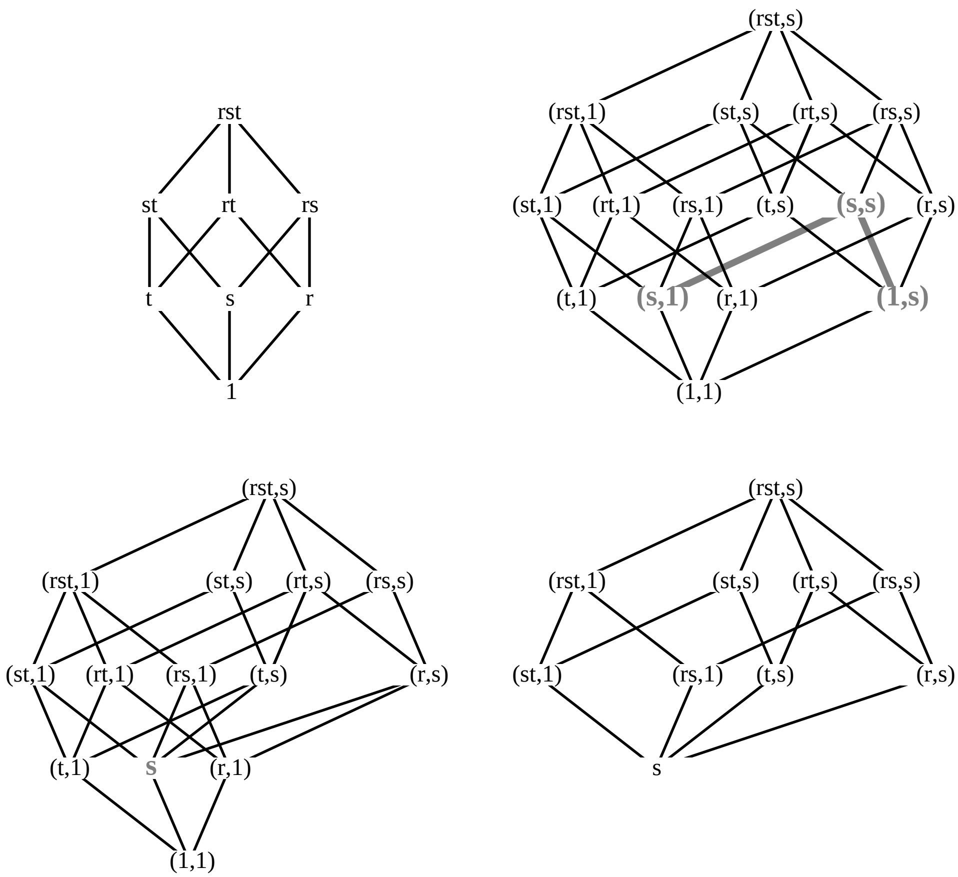
$$\begin{aligned} \theta(us) &= us \\ \theta(v,1) &= v \text{ if } v \in (us,w] \\ \theta(v,s) &= \begin{cases} vs & \text{if } vs > v \\ v & \text{if } vs < v. \end{cases} \end{aligned}$$

To check that θ is well-defined, begin by noting that $\theta(us) \in [us,ws]$, and if $v \in (us,w]$, then $\theta(v,1) = v \in [us,ws]$. If $v \in [u,w] - \{u, us\}$, there are two possibilities, $\theta(v,s) = vs > v$ or $\theta(v,s) = v > vs$. In either case, $us < \theta(vs) < ws$ by lifting. So θ is well-defined.

Proposition 3.4.4. *The map $\theta : \mathcal{S}_{us}[u,w] \to [us,ws]$ is an order-projection.*

Proof. Notice that θ, restricted to $\mathcal{S}_{us}[u,w] - \{us\}$ is just η restricted to an induced subposet. Recall that in the proof of Proposition 3.4.1, it was shown that for $v \in [u,ws]$, if $v < vs$ then $(v,1) \in \eta^{-1}(v)$ and if $v > vs$ then $(vs,s) \in \eta^{-1}(v)$. The existence of these elements of $\eta^{-1}(v)$ was used to check that η is an order-projection. The same argument accomplishes most of the present proof. For $us < x_1 \leq x_2 \leq w$, we are done, and it remains to check that for $us \leq x \leq w$, there exist elements $a \leq b$ in $\mathcal{S}_{us}[u,w]$ with $\theta(a) = us$ and $\theta(b) = x$. This is easily accomplished by setting $a = us \in \mathcal{S}_{us}[u,w]$ and letting b be an element of $\eta^{-1}(x) = \theta^{-1}(x)$. If $x = us$, then set $b = us \in \mathcal{S}_{us}[u,w]$. $\square$

Figure 3.3: The construction of $\mathcal{S}_s([1, rst])$ from $[1, rst] \times [1, s]$, where $[1, srt]$ is an interval in $(S_4, \{r, s, t\})$. The posets are $[1, rst]$, $[1, rst] \times [1, s]$, the same poset with $(s, 1), (1, s), (rs, s)$ zipped, and $\mathcal{S}_s([1, rst])$.

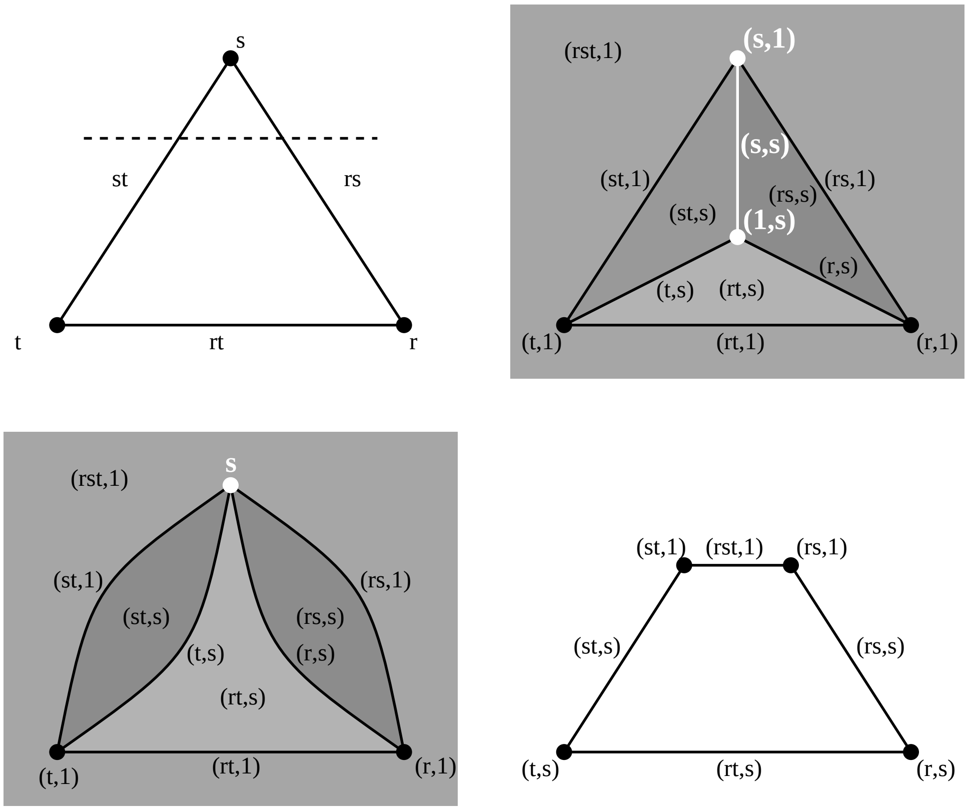


Figures 3.5 and 3.6 illustrate the map θ and the corresponding map on CW spheres.

Since η is an order-projection, $[u, ws]$ is isomorphic to the fiber poset of $[u, w] \times [1, s]$ with respect to η. Similarly, $[us, ws]$ is isomorphic to the fiber poset of $\mathcal{S}_{us}[u, w]$ with respect to θ. We will show that one can pass to the fiber poset in both cases by a sequence of zippings.

Order the set $\{v \in (u, w) : vs < v\}$ linearly by an extension of the partial order from $[u, w]$, such that the elements of rank i in $[u, w]$ precede the elements of rank $i + 1$ for all i. Write this order as $v_1, v_2, \ldots, v_k$. Define $P_0 = [u, w] \times [1, s]$ and inductively define P_i to be the poset obtained by zipping $((v_i, 1), (v_i s, s), (v_i, s))$ in P_{i-1}. We show inductively that this is indeed a proper zipping. First, notice that $(v_i, 1)$, $(v_i s, s)$ and (v_i, s) are indeed elements of P_{i-1}. The element (v_i, s) has not been deleted yet, and we have not identified (v_i, s) with any element because it is at a rank higher than we have yet made identifications. The only elements ever deleted are of the form (x, s) where $x > xs$, so $(v_i, 1)$ and $(v_i s, s)$ have not been deleted. The only identification one could make involving $(v_i, 1)$ and $(v_i s, s)$ is to identify them to each other, and that has not happened yet. We

Figure 3.4: The CW-spheres associated to the posets of Figure 3.3.



check the properties in the definition of a zipper: Properties (i) and (ii) hold in P_0 and therefore in P_{i-1} because we have made no identifications involving $(v_i, 1)$, $(v_i s, s)$ and (v_i, s) or higher-ranked elements. To check property (iii), keep in mind that $v_i > v_i s$.

Suppose $x < (v_i, 1)$ in P_{i-1}. The following cases are possible for x:

Case 1: The element x corresponds to a single (uncombined) element of P_0.

We must have $x = (v, 1)$ with $v < v_i$ and $v < vs$. By lifting, $v \leq v_1 s$, so $x < (v_i s, s)$.

Case 2: The element x corresponds to a pair of identified elements.

These elements must be $(v, 1)$ and (vs, s) for some v with $v > vs$. Thus $v < v_i$, so by lifting, $vs < v_i s$ and thus $x < (v_i s, s)$.

We have shown that $[\hat{0}, (v_i, 1)) \subseteq [\hat{0}, (v_i s, s))$.

Suppose $x < (v_i s, s)$ in P_{i-1}. Again, break up into cases based on the identity of x.

Case 1: x corresponds to a single element $(v, 1)$ of P_0, with $v \leq v_i s$.

By transitivity, $v \leq v_i$ and thus $x < (v_i, 1)$.

Case 2: x is (v, s) in P_0 with $v < v_i s$.

Figure 3.5: The map $\theta : \mathcal{S}_s([1, rst]) \to [s, rsts]$, where $[1, srt]$ and $[s, rsts]$ are intervals in $(S_4, \{r, s, t\})$. All elements (u, v) map to uv except (rs, s), which maps to rs.

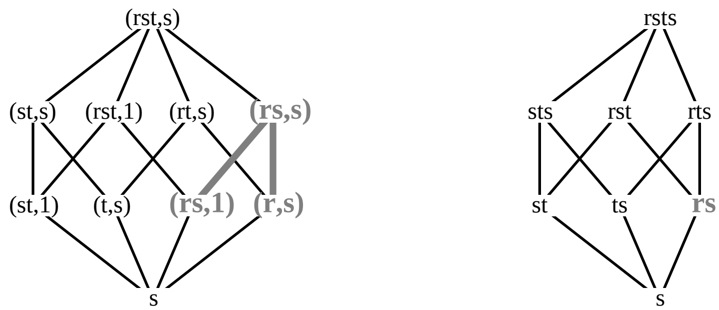
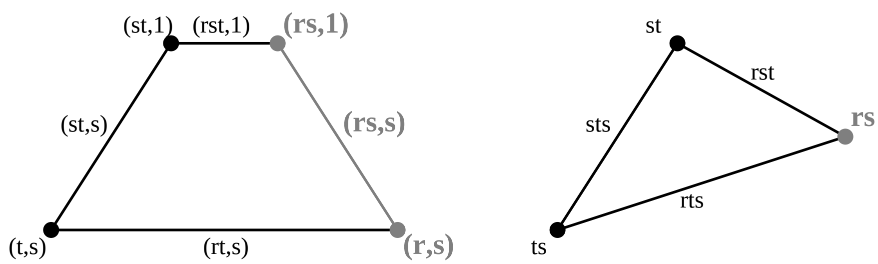


Figure 3.6: $\theta : \Omega(\mathcal{S}_s([1, rst])) \to \Omega[s, rsts]$.



Since x was not deleted previously, $v < vs$. Since x was not identified with $(vs, 1)$, we know that $vs \notin [u, w]$, and specifically that $vs \not\leq w$. However, by lifting, $vs \leq v_i$ and by hypothesis $v_i \leq w$. This contradiction shows that Case 2 cannot occur.

Case 3: x is a pair of identified elements.

The two elements are $(v, 1)$ and (vs, s) with $v > vs$. Since $x < (v_i s, s)$, either $v_i s > vs$ or $v_i s \geq v$. If $v_i s > vs$ then by lifting $v_i > v$, If $v_i s \geq v$, then by transitivity, $v_i > v$. So $x < (v_i, 1)$.

Thus $[\hat{0}, (v_i, 1)) = [\hat{0}, (v_i s, s))$. The zipper is proper because $(v_i, s) < (w, s)$ in P_0 and thus in P_{i-1}.

By definition, $\mathcal{S}_{us}[u, w]$ is an interval in the poset P_1 defined above. Specifically, P_1 was obtained from $[u, w] \times [1, s]$ by zipping $((us, 1), (u, s), (us, s))$. Let us be the element of P_1 resulting from identifying $(us, 1)$ with (u, s). Then $\mathcal{S}_{us}[u, w]$ is isomorphic to the interval $[us, (w, s)]$ in P_1. The remaining deletions and identifications in the map θ are really zippings in the P_i. Therefore they are zippings in the P_i restricted to $[us, (w, s)]$.

We have proven the following:

Theorem 3.4.5. *Let $ws > w$, $us > u$ and $u \leq w$. If $us \notin [u, w]$ then $[u, ws] \cong [u, w] \times [1, s]$ and $[us, ws] \cong [u, w]$. If $us \in [u, w]$, then $[u, ws]$ can be obtained from $[u, w] \times [1, s]$ by a sequence of zippings, and $[us, ws]$ can be obtained from $\mathcal{S}_{us}[u, w]$ by a sequence of zippings.*

Corollary 3.4.6. *Bruhat intervals are PL spheres.*

Proof. One only needs to prove the corollary for lower intervals, because by Proposition 2.4.5 it will then hold for all intervals. Intervals of rank 1 are empty spheres. It is easy to check that a lower interval under an element of rank 2 is a PL 0-sphere. Given the interval $[1, w]$ with $l(w) \geq 3$, there exists $s \in S$ such that $ws < w$. Then $[1, w]$ can be obtained from $[1, ws] \times [1, s]$ by a sequence of zippings. By induction $[1, ws]$ is a PL sphere, and thus $[1, ws] \times [1, s]$ is as well. By repeated applications of Theorem 3.2.7, $[1, w]$ is a PL sphere. $\square$

The following observation will be helpful in Section 3.6, when Theorem 3.4.5 is combined with Theorem 3.2.6.

Proposition 3.4.7. *For $1 \leq i \leq k$,*

$$[(u, 1), (v_i, 1)]_{P_{i-1}} \cong [(u, 1), (v_i, 1)]_{P_0}$$

$$[(v_i, s), (w, s)]_{P_{i-1}} \cong [(v_i, s), (w_s)]_{P_0}$$

Proof. The second statement is obvious because of the way the v_j were ordered. For the first statement, there is the obvious order-preserving bijection between the two intervals. The only question is whether the right-side has any extra order relations. Extra order relations will occur if for some v with $v > vs$, there exists $(x, 1)$ with $(x, 1) \leq (vs, s)$ but $(x, 1) \not\leq (v, 1)$. This is ruled out by transitivity. $\square$

3.5 Polytopal Bruhat intervals

In this section we explore the problem of finding Bruhat intervals which are isomorphic to the face lattices of convex polytopes. For convenience, we will say that such intervals "are" polytopes. Specifically we construct, in the universal Coxeter groups, Bruhat intervals which are dual stacked polytopes. Also, we consider the question of finding large simplices (Boolean algebras) as intervals in a given finite Coxeter group.

A polytope is said to be *dual stacked* if it can be obtained from a simplex by a series of vertex-shavings. As the name would indicate, these polytopes are dual to the *stacked polytopes* [37] which we will not define here. There are Bruhat intervals which are dual stacked polytopes. Let W be a universal Coxeter group with Coxeter generators $S := \{s_1, s_2, \ldots s_{d+1}\}$. Define C_k (for "cyclic word") to be the word $s_1 s_2 \cdots s_k$, where the subscript k is understood to mean $k \pmod{d+1}$. So for example, if $d = 2$, then $C_7 = s_1 s_2 s_3 s_1 s_2 s_3 s_1$. In a universal Coxeter group, every group element corresponds to a unique reduced word. Thus we will use these words interchangeably with group elements.

Proposition 3.5.1. *The interval $[C_k, C_{d+k+1}]$ in W is a dual stacked polytope of dimension d with $d + k + 1$ facets.*

Proof. The proof is by induction on k. For $k = 0$, the interval is $[1, s_1 s_2 \cdots s_{d+1}]$, and because all subwords of $s_1 s_2 \cdots s_{d+1}$ are distinct group elements, this interval is a Boolean algebra—the face poset of a d dimensional simplex. For $k > 0$, the interval $[C_k, C_{d+k+1}]$ is obtained from the interval $[C_{k-1}, C_{d+k}]$ by shaving the vertex C_k and then possibly by performing a sequence of zippings. By induction, $[C_{k-1}, C_{d+k}]$ is a dual stacked polytope of dimension d with $d + k$ facets. The zippings correspond to elements of (C_k, C_{d+k}) which are shortened on the right by s_{d+k+1}. Let $v \in (C_k, C_{d+k})$, and let v also stand for the unique reduced word for the element v. Since every element of W has a unique reduced word, the fact that $C_k < v$ means that v contains C_k as a subword. But the only subword C_k of C_{d+k} is the first k letters of C_{d+k}. Thus v is a subword of C_{d+k} consisting of the first k letters and at least one other letter. Now, $s_k \notin \{s_{k+1}, s_{k+2}, \ldots, s_{d+k}\}$, so v ends in some generator other than s_k, and therefore, v is not shortened on the right by $s_k = s_{d+k+1}$. Thus there are no zippings following the shaving, and so $[C_k, C_{d+k+1}]$ is a dual stacked polytope of dimension d with $d + k + 1$ facets. □

Next, consider the question: What is the largest rank of Boolean algebra which occurs as an interval in a given finite Coxeter group? For a poset P, let $\mathrm{Bool}(P)$ be the largest rank of Boolean algebra which occurs as an interval in P.

Theorem 3.5.2.

$$\begin{array}{rcl}
\mathrm{Bool}(A_n) & \geq & n + \left\lfloor \frac{n-1}{2} \right\rfloor \\
\mathrm{Bool}(B_n) & \geq & n + \left\lfloor \frac{n-1}{2} \right\rfloor \\
\mathrm{Bool}(D_n) & \geq & n + \left\lfloor \frac{n}{2} \right\rfloor \\
\mathrm{Bool}(E_6) & \geq & 8 \\
\mathrm{Bool}(E_7) & \geq & 10 \\
\mathrm{Bool}(E_8) & \geq & 11 \\
\mathrm{Bool}(F_4) & \geq & 5 \\
\mathrm{Bool}(H_3) & \geq & 4 \\
\mathrm{Bool}(H_4) & \geq & 5
\end{array}$$

We would guess that these are the largest possible, but we do not have a proof. Theorem 3.5.2 provides lower bounds for the order dimensions of these finite Coxeter groups, but it will be seen in Chapter 4 that these bounds are very low.

The proof of Theorem 3.5.2 occupies the remainder of this section. Consider $w \in W$ represented by a word

$$a = d_1 t_1 d_2 t_2 \cdots d_k t_k d_{k+1}.$$

where the d_i are words, the t_i are single generators, and $b = t_1 t_2 \cdots t_k$ is a word for some element u. Call b a *Boolean* subword of a if:

(i) For each $i \in [k]$, the letter t_i appears exactly once in a.

(ii) If for $i \leq j$ some letter s occurs in both d_i and d_j, then there is an n with $i \leq n < j$ such that $m(s, t_n) > 2$. In particular, for each $i \in [k+1]$, no letter appears in d_i more than once.

Condition (i) implies that b is reduced, and condition (ii) implies that each d_i is reduced. Write w_i for the element represented by d_i.

Proposition 3.5.3. *If b is a Boolean subword of a, then $[u, w]$ is a Boolean algebra of rank $l(a)-l(b)$ and in particular a is a reduced word.*

Proof. If $[u, w]$ is a Boolean algebra of rank $l(a) - l(b)$, then since b is reduced, $l(w) = l(a)$, so a is reduced. The claim that $[u, w]$ is a Boolean algebra is proven by induction on k and $l(w_{k+1})$.

If $k = 0$, $[u, w] = [1, w_1]$, and by (ii) the letters in d_1 are all distinct, so $[u, w]$ is a Boolean algebra of the correct rank. If $k > 0$ and $l(w_{k+1}) = 0$, let a_- be the word obtained by deleting the last letter, t_k, from a, and let w_- be the element corresponding to a_-. So $[u, w] = [t_1 t_2 \cdots t_k, w_- t_k]$. By (i) and (ii), t_k is distinct from all generators in a_-, so $t_1 t_2 \cdots t_k \not\leq w_-$. For the same reason, $t_1 t_2 \cdots t_{k-1} < t_1 t_2 \cdots t_k$ and $w_- < w_- t_k$. By Corollary 3.4.3, $[t_1 t_2 \cdots t_k, w_- t_k] \cong [t_1 t_2 \cdots t_{k-1}, w_-]$. By induction on k, $[t_1 t_2 \cdots t_{k-1}, w_-]$ is a Boolean algebra of rank $l(a_-) - (k-1)$, so $[u, w]$ is a Boolean algebra of the correct rank.

Now consider the case where $k > 0$ and $l(w_{k+1}) > 0$. Write s for the last letter of a, let a_- be the word obtained by deleting the last letter, s, from a, and let w_- be the element corresponding to a_-. If the hypotheses of Proposition 3.4.2 hold, then $[u, w] \cong [u, w_-] \times [1, s]$, and by induction on $l(w_{k+1})$, $[u, w_-]$ is a Boolean algebra of rank $l(a_-) - l(b)$, so $[u, w]$ is a Boolean algebra of the correct rank.

To verify the hypotheses of Proposition 3.4.2, first notice that $us > u$ because the generators in b are distinct from s. Next, suppose for the sake of contradiction that $us \leq w_-$. Then some subword of a_- is a reduced word for us. But $t_1 t_2 \cdots t_k s$ is a reduced word for us, and the $k+1$ generators in this reduced word are distinct by (i). So any reduced word for us must contain exactly these generators. But condition (i) states that each t_j occurs exactly once in a, so reduced subwords for us must have the form $t_1 t_2 \cdots t_{i-1} s t_i t_{i+1} \cdots t_k$. By (ii) there is an n with $i \leq n \leq k$ so that $m(s, t_n) > 2$. We have reduced words $t_1 t_2 \cdots t_k s$ and $t_1 t_2 \cdots t_{j-1} s t_j t_{j+1} \cdots t_k$ both standing for the same element. We can change one of these words into the other by a series of braid moves. But, since each generator only occurs once, these braid moves are all pairwise commutations. Since s and t_n don't commute, s cannot be moved from the right of t_n to the left of t_n. This contradiction shows that $us \not\leq w_-$. Finally, suppose $w_- s < w_-$. By induction, $u \leq w_-$ and because $us > u$, by lifting, $us \leq w_-$, which contradicts what was just proven. So $w_- s > s$. The hypotheses of Proposition 3.4.2 hold, and the proof is finished. □

Theorem 3.5.2 is proven by finding words and Boolean subwords of the appropriate sizes. Figures 3.7, 3.8 and 3.9 show, for several finite Coxeter groups, a convenient way of naming the generators for A_n, D_n and E_8. For readers not familiar with representing Coxeter groups by a graph, the edges represent pairs s, t of generators for which $m(s,t) = 3$. Pairs s, t of generators not connected by an edge have $m(s,t) = 2$. A labeling for E_6 or E_7 is obtained by deleting the vertices a_3 and/or b_4.

Figure 3.7: A labeling of the generators of A_n.

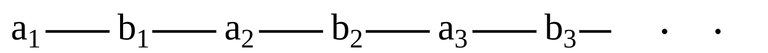


Figure 3.8: A labeling of the generators of D_n.

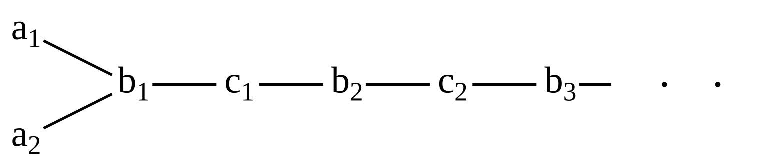


Figure 3.9: A labeling the generators of E_8.

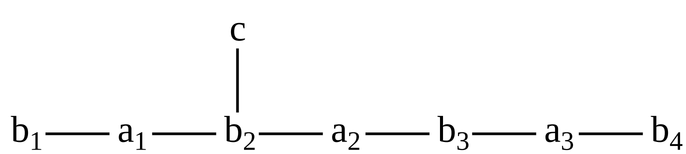


Figure 3.10: Words and Boolean subwords for types A, D and E.

Group	k	General d_i	Exceptions	Rank
A_{2n}	n	$a_{i-1}a_ia_{i+1}$	$w_1 = a_1a_2$, $w_n = a_{n-1}a_n$, $w_{n+1} = a_n$	$3n-1$
A_{2n+1}	n	$a_{i-1}a_ia_{i+1}$	$w_1 = a_1a_2$, $w_{n+1} = a_na_{n+1}$	$3n+1$
D_{2n}	$n-1$	$c_{i-2}c_{i-1}c_i$	$w_1 = a_1a_2c_1$, $w_2 = a_1a_2c_1c_2$, $w_n = c_{n-2}c_{n-1}$	$3n$
D_{2n+1}	n	$c_{i-2}c_{i-1}c_i$	$w_1 = a_1a_2c_1$, $w_2 = a_1a_2c_1c_2$, $w_n = c_{n-2}c_{n-1}$, $w_{n+1} = c_{n-1}$	$3n+1$

Group	Subword	Word	Rank
E_6	$b_1b_2b_3$	$a_1b_1a_1a_2cb_2a_1a_2cb_3a_2$	8
E_7	$b_1b_2b_3$	$a_1b_1a_1a_2cb_2a_1a_2a_3cb_3a_2a_3$	10
E_8	$b_1b_2b_3b_4$	$a_1b_1a_1a_2cb_2a_1a_2a_3cb_3a_2a_3b_4a_3$	11

Figure 3.10 shows how to construct words and Boolean subwords for groups of types A, D and E, and gives the rank of the Boolean algebra produced. The Boolean subword is always $b_1b_2\cdots b_k$ for the k indicated in the table. The other groups are treated like type A. The reader can easily verify that the hypotheses of Proposition 3.5.3 are satisfied.

3.6 A Recursion for the cd-index of Bruhat intervals

Theorems 3.2.6 and 3.4.5 yield Theorem 3.1.1, a set of recursions for the cd-indices of Bruhat intervals. In this section we prove Theorem 3.1.1. For $v \in W$ and $s \in S$, define $\sigma_s(v) := l(vs) - l(v)$. Thus $\sigma_s(v)$ is 1 if v is lengthened by s on the right and -1 if v is shortened by s on the right. We have the following:

Theorem 3.1.1. *Let $u < us$, $w < ws$ and $u \leq w$.*
If $us \notin [u,w]$, then $\Psi_{[u,ws]} = \mathrm{Pyr}\Psi_{[u,w]}$, and $\Psi_{[us,ws]} = \Psi_{[u,w]}$.
If $us \in [u,w]$, then

$$\Psi_{[u,ws]} = \mathrm{Pyr}\Psi_{[u,w]} - \sum_{v \in (u,w): vs<v} \Psi_{[u,v]} \cdot d \cdot \Psi_{[v,w]} \tag{3.4}$$

$$= \frac{1}{2}\left(\Psi_{[u,w]} \cdot c + c \cdot \Psi_{[u,w]} + \sum_{v \in (u,w)} \sigma_s(v)\Psi_{[u,v]} \cdot d \cdot \Psi_{[v,w]}\right) \tag{3.5}$$

$$\Psi_{[us,ws]} = \mathcal{S}_{us}\Psi_{[u,w]} - \sum_{v \in (us,w): vs<v} \Psi_{[us,v]} \cdot d \cdot \Psi_{[v,w]} \tag{3.6}$$

$$= \Psi_{[u,w]} + \frac{1}{2}\left(\Psi_{[us,w]} \cdot c - c \cdot \Psi_{[us,w]} + \sum_{v \in (us,w)} \sigma_s(v)\Psi_{[us,v]} \cdot d \cdot \Psi_{[v,w]}\right). \tag{3.7}$$

Lines (3.5) and (3.7) of these formulas look like a augmented coproducts [23] on Bruhat intervals, with an added sign. Lines (3.4) and (3.6) are more efficient for computation, because the formulas in Proposition 2.5.3 are more efficient than the forms quoted in Proposition 2.5.2.

Proof of Theorem 3.1.1. The statement for $us \notin [u,w]$ follows immediately from Propositions 3.4.2 and 3.4.3. Define the P_i as in Section 3.4. Thus by Theorem 3.2.6,

$$\Psi_{P_{i-1}} - \Psi_{P_i} = \Psi_{[(u,1),(v_i,1)]_{P_{i-1}}} \cdot d \cdot \Psi_{[(v_i,s),(w,s)]_{P_{i-1}}}.$$

Since $P_k = [u,ws]$, sum from $i=1$ to $i=k$ to obtain

$$\Psi_{[u,ws]} = \Psi_{P_0} - \sum_{j=1}^{k} \Psi_{[(u,1),(v_j,1)]_{P_{j-1}}} \cdot d \cdot \Psi_{[(v_j,s),(w,s)]_{P_{j-1}}}.$$

By Proposition 3.4.7, $[(u,1),(v_j,1)]_{P_{j-1}} \cong [(u,1),(v_j,1)]_{P_0}$. This in turn is isomorphic to $[u,v_j]$. Similarly, by Proposition 3.4.7, $[(v_j,s),(w,s)]_{P_{j-1}} \cong [(v_j,s),(w,s)]_{P_0}$ which is isomorphic to $[v_j,w]$. Thus we have established the first line of the first formula in Theorem 3.1.1. The second line follows from the first by Proposition 2.5.2.

A similar proof goes through for the second formula. The isomorphisms from Proposition 3.4.7 restrict to the appropriate isomorphisms for the $[us,ws]$ case. The second line of the formula follows by Proposition 2.5.2. $\square$

3.7 The affine span of the cd-indices of Bruhat intervals

In [2], Bayer and Billera show that the affine span of the cd-indices of polytopes is the entire affine space of monic cd-polynomials. In this section, as an application of Theorem 3.1.1, we prove that the cd-indices of Bruhat intervals have the same affine span. The space of cd-polynomials of degree $n-1$ has dimension F_n, the Fibonacci number, with $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. For each n we will produce a set $\mathcal{F}_n$ consisting of F_n reduced words, corresponding to group elements whose lower Bruhat intervals have linearly independent cd-indices.

Let $(W, S := \{s_1, s_2, \ldots\})$ have a complete Coxeter graph with each edge labeled 3. Each $\mathcal{F}_n$ is a set of reduced words of length n in W, with $\mathcal{F}_1 = \{s_1\}$, $\mathcal{F}_2 = \{s_1 s_2\}$ and

$$\mathcal{F}_n = \mathcal{F}_{n-1} s_n \uplus s_n \mathcal{F}_{n-2} s_n,$$

where $\uplus$ means disjoint union. Given a word $w \in \mathcal{F}_{n-1}$, by Proposition 3.4.2, $[1, ws_n] \cong \mathrm{Pyr}[1, w]$, so $\Psi_{[1,ws_n]} = \mathrm{Pyr}(\Psi_{[1,w]})$. Similarly, given a word $w' \in \mathcal{F}_{n-2}$, $[1, s_n w] \cong \mathrm{Pyr}[1, w]$. Since s_n does not commute with any other generator, and since s_n is not a letter in w, by Proposition 3.4.1, $[1, s_n w s_n]$ is obtained from $[1, s_n w]$ by a single zipping. In particular, by Theorem 3.1.1, $\Psi_{[1,s_n w s_n]} = \mathrm{Pyr}^2(\Psi_{[1,w]}) - d \cdot \Psi_{[s_n, s_n w]}$. By Corollary 3.4.3, $[s_n, s_n w] \cong [1, w]$, so $\Psi_{[1,s_n w s_n]} = \mathrm{Pyr}^2(\Psi_{[1,w]}) - d \cdot \Psi_{[1,w]}$. Let $\Psi(\mathcal{F}_n)$ be the set of cd-indices of lower intervals under words in $\mathcal{F}_n$.

Proposition 3.7.1. *For each $n \geq 1$, the F_n cd-polynomials in $\Psi(\mathcal{F}_n)$ are linearly independent.*

Proof. As a base for induction, the statement is trivial for $n = 1, 2$. For general n, form the matrix M whose rows are the coefficients of the cd-indices in $\Psi(\mathcal{F}_n)$. Order the columns by the lexicographic order on cd-monomials. Order the rows so that the cd-indices in $\Psi(\mathcal{F}_{n-1} s_n)$ appear first. We will show that there are row operations which convert M to an upper-unitriangular matrix. Notice that for each $w \in \mathcal{F}_{n-2}$, $\mathrm{Pyr}^2 \Psi_{[1,w]}$ occurs in $\Psi(\mathcal{F}_{n-1} s_n)$. Also, $\mathrm{Pyr}^2 \Psi_{[1,w]} - d \cdot \Psi_{[1,w]}$ occurs in $\Psi(s_n \mathcal{F}_{n-2} s_n)$. Thus by row operations one obtains a matrix M' whose rows are first $\mathrm{Pyr}(\Psi(\mathcal{F}_{n-1}))$, then $d \cdot \Psi(\mathcal{F}_{n-2})$. By induction, there are row operations which convert the matrix with rows $\Psi(\mathcal{F}_{n-1})$ to an upper-unitriangular matrix. By Proposition 3.7.2, these yield row operations which give the first F_{n-1} rows of M' an upper-unitriangular form. Also by induction, there are row operations which convert the matrix with rows $\Psi(\mathcal{F}_{n-2})$ to an upper-unitriangular matrix. Corresponding operations applied to the rows $d \cdot \Psi(\mathcal{F}_{n-2})$ of M' complete the reduction of M' to upper-unitriangular form. ☐

Proposition 3.7.2. *Let P be a homogeneous cd-polynomial whose lexicographically first term is T. Then the lexicographically first term of $\mathrm{Pyr}(P)$ is $c \cdot T$. In particular, the kernel of the pyramid operation is the zero polynomial.*

Proof. This follows immediately from the second formula in Proposition 2.5.3. ☐

We have shown that the cd-indices of arbitrary Bruhat intervals span the space of cd-polynomials. It would be interesting to know whether the cd-indices of Bruhat intervals in finite Coxeter groups also span, and whether a spanning set of intervals could be found in the finite Coxeter groups of type A.

3.8 Bounds on the cd-index of Bruhat intervals

In this section we discuss lower and upper bounds on the coefficients of the cd-index of a Bruhat interval. The conjectured lower bound is a special case of a conjecture of Stanley [56].

Conjecture 3.8.1. *For any $u \leq v$ in W, the coefficients of $\Phi_{[u,v]}$ are non-negative.*

The coefficient of c^n is always 1, and for the other coefficients the bound is sharp because the dihedral group $I_2(m)$ has cd-index c^m. Computer studies have confirmed the conjecture in S_n with $n \leq 6$.

The most interesting conjectural upper bound was mentioned previously, and is repeated below.

Conjecture 3.1.2. *The coefficientwise maximum of all cd-indices $\Phi_{[u,v]}$ with $l(u) = k$ and $l(v) = d+k+1$ is attained on a Bruhat interval which is isomorphic to a dual stacked polytope of dimension d with $d+k+1$ facets.*

This conjecture is natural in light of Proposition 3.5.1 and Theorem 3.1.1. There are two issues which complicate the conjecture. First, any proof using Theorem 3.1.1 requires non-negativity (Conjecture 3.8.1). Second, and perhaps even more serious, there is the issue of commutation of operators.

Given $p \in P$ denote the corresponding "downstairs" element in $\mathrm{Pyr}(P)$ by p and the "upstairs" element by p'. Denote the operation of zipping a zipper (x, y, z) by $\mathcal{Z}_z$. Then $\mathrm{Pyr}\mathcal{Z}_z P \cong \mathcal{Z}_{z'}\mathcal{Z}_z \mathrm{Pyr} P$. The triple (x', y', z') becomes a zipper only after $\mathcal{Z}_z$ is applied. Pyramid and shaving also commute reasonably well: $\mathrm{Pyr}\mathcal{S}_a P \cong \mathcal{Z}_{a'}\mathcal{S}_a \mathrm{Pyr} P$.

However, zipping does not in general commute nicely with the operation of shaving off a vertex a. Given $p \neq a \in P$ denote the corresponding element of $\mathcal{S}_a P$ again by p, and if in addition $p > a$, write $\bar{p}$ for the new element created by shaving. If $z > a$ and $a \notin \{x, y\}$ then $\mathcal{S}_a\mathcal{Z}_z P \cong \mathcal{Z}_z\mathcal{Z}_{\bar{z}}\mathcal{S}_a P$. If $z \not> a$ then $\mathcal{S}_a\mathcal{Z}_z P \cong \mathcal{Z}_z\mathcal{S}_a P$. However, if x and y are vertices then $\mathcal{S}_{xy}\mathcal{Z}_z P = \mathcal{S}_z P$, where $\mathcal{S}_z$ is the operation of shaving off the edge z.

Since the pyramid operation commutes nicely with zipping, and any lower interval is obtained by pyramid and zipping operations, it is possible to obtain any lower interval by a series of pyramid operations followed by a series of zippings. Thus by Theorem 3.1.1:

Theorem 3.8.2. *Assuming Conjecture 3.8.1, for all $w \in W$,*

$$\Phi_{[1,w]} \leq \Phi_{B_{l(w)}}.$$

Here B_n is the Boolean algebra of rank n. It is not true that the cd-index of general intervals is less than that of the Boolean algebra of appropriate rank. For example, $[1324, 3412]$ is the face lattice of a square, with $\Phi_{[1324,3412]} = c^2 + 2d$. However, $\Phi_{B_3} = c^2 + d$.

Equation (3.2) in the proof of Theorem 3.2.6 is a formula for the change in the ab-index under zipping. Thus Theorem 3.1.1 has a flag h-vector version, and since the flag h-vectors of Bruhat intervals are known to be nonnegative, the following theorem holds.

Theorem 3.8.3. *For any w in an arbitrary Coxeter group,*

$$\Psi_{[1,w]} \leq \Psi_{B_{l(w)}}.$$

Here "$\leq$" means is coefficientwise comparison of the ab-indices, or in other words, comparison of flag h-vectors

3.9 Extracting other invariants from the cd-index

The cd-index contains a large amount of enumerative information about a partially ordered set P. In this section, we will show how to obtain recursions on several invariants of P from the cd-index recursions in the previous section. The key point to remember is that c corresponds to $a+b$ in the ab-index, which corresponds to $a+2b$ in the flag-index. Similarly, d corresponds to $ab+ba+2bb$ in the flag-index. Each of these recursions can also be obtained directly from Propositions 3.4.1 and 3.4.4. Throughout the section, let P be a poset with $\hat{0}$ and $\hat{1}$ which has a cd-index. We will use the notation $l(u,v) := l(v) - l(u)$.

Maximal chains

Let M_P stand for the number of maximal chains in $P - \{\hat{0}, \hat{1}\}$, with $M_P = 0$ when P has only one element and $M_P = 1$ when P is a 2-element chain. To count the number of maximal chains, set $a = 0$ and $b = 1$ in the flag-index. This is accomplished by setting $c = d = 2$ in Φ. One can check that if P has rank n, then $M_{\mathrm{Pyr}(P)} = (n+1)M_P$ and $M_{\mathcal{S}_a(P)} = M_P + (n-2)M_{[a,\hat{1}]}$.

Corollary 3.9.1. *Let $u < us$, $w < ws$ and $u \leq w$.*

If $us \notin [u,w]$, then $M_{[u,ws]} = l(u,ws)M_{[u,w]}$, and $M_{[us,ws]} = M_{[u,w]}$.

If $us \in [u,w]$, then

$$\begin{aligned} M_{[u,ws]} &= l(u,ws)M_{[u,w]} - 2\sum_{v\in(u,w):vs<v} M_{[u,v]}M_{[v,w]} && (3.8)\\ &= 2M_{[u,w]} + \sum_{v\in(u,w)} \sigma_s(v)M_{[u,v]}M_{[v,w]} && (3.9)\\ M_{[us,ws]} &= M_{[u,w]} + (l(us,ws)-2)M_{[us,w]} - 2\sum_{v\in(us,w):vs<v} M_{[us,v]}M_{[v,w]} && (3.10)\\ &= M_{[u,w]} + \sum_{v\in(u,w)} \sigma_s(v)M_{[us,v]}M_{[v,w]}. && (3.11) \end{aligned}$$

Chains

Let C_P be the generating function for chains in $P - \{\hat{0}, \hat{1}\}$, with the coefficient of q^k counting the chains consisting of k elements. If P has one element, $C_P = 0$ and if P is a 2-element chain, $C_P = 1$. The function C_P contains the same information as the zeta-polynomial of P. To obtain C_P, set $a = 1$ and $b = q$ in the flag-index Υ_P. Thus by setting $c = 1 + 2q$ and $d = 2q + 2q^2$ in Theorem 3.1.1, one obtains recursions on C_P for Bruhat intervals.

The Charney-Davis conjecture

One can obtain recursions for the coefficient of d^k in the cd-index of a Bruhat interval by setting $c = 0$ and $d = 1$ in Equations (3.5) and (3.7) of Theorem 3.1.1. This coefficient is interesting because, up to a sign, it is the quantity which appears in the Charney-Davis conjecture [15].

Length generating function

Let $L_P(q)$ be the rank generating function for P. So if P has one element, $L_P(q) = 1$. For convenience we will suppress the explicit q-dependence below. The previous invariants have been obtained by specializing c and d to integers. The length generating function is obtained by applying a $\mathbb{Z}$-linear map which is not a specialization. First, number the factors in each monomial by position, with each d occupying two positions. For example, number $cccddcdcddd$ as $c_1c_2c_3d_4d_6c_8d_9c_{11}d_{12}d_{14}d_{16}$. Map $c_1c_2\cdots c_n$ to $(1+q)(1+q+q^2+\cdots+q^{n-1})$ and map $c_1c_2\cdots c_{i-1}d_ic_{i+2}c_{i+3}\cdots c_n$ to $q^i(1+q)$. All other monomials are mapped to zero. Note that the pyramid operation has the effect of multiplying the length generating function by $(1+q)$, and that

$$L_{Sh_a P} = L_P + L_{[a,\hat{1}]} - (1+q).$$

Corollary 3.9.2. *Let $u < us$, $w < ws$ and $u \leq w$.*

If $us \notin [u,w]$, then $L_{[u,ws]} = (1+q)L_{[u,w]}$, and $L_{[us,ws]} = L_{[u,w]}$.

If $us \in [u,w]$, then

$$L_{[u,ws]} = (1+q)\left(\sum_{v\in[u,w]:vs>v} q^{l(u,v)}\right) \tag{3.12}$$

$$L_{[us,ws]} = \sum_{v\in[u,w]:vs>v} q^{l(u,v)} + \sum_{v\in[us,w]:vs>v} q^{l(us,v)}. \tag{3.13}$$

This formula is particularly interesting because it expresses the length generating function of an interval in terms of length generating functions in a quotient with respect to a parabolic subgroup (see Section 4.7).

3.10 Further questions

Zipping

Let P' be obtained from P by zipping the zipper (x, y, z).

1. If $(a, b)_{\preceq}$ is an interval in P' with $a \neq xy$, is $\Delta((a, b)_{\preceq})$ homotopic to $\Delta((a, b)_{\leq})$? On the other hand, when $a = xy$, the complex $\Delta((xy, b)_{\preceq})$ is obtained by deleting the vertex z from the complex $\Delta((x, b)_{\leq}) \cup \Delta((y, b)_{\leq})$. Thus it is homotopic to the space $\Delta((x, b)_{\leq}) \cup \Delta((y, b)_{\leq})$ with the point z deleted.

2. If every interval of P is a homology sphere, is the same true of P'?

3. If P is Cohen-Macaulay, is P' also?

Polytopal Bruhat intervals

1. Characterize the polytopes which appear as Bruhat intervals. Characterize the polytopes which appear as Bruhat intervals in finite Coxeter groups. Characterize the polytopes which appear as Bruhat intervals in finite Coxeter groups of type A.

2. Are the Boolean algebras constructed in Section 3.5 the largest possible?

Affine span

Given $n \geq 1$, is there a set of intervals in finite Coxeter groups which span the affine space of monic cd-polynomials of degree n? Is there a set of such intervals in the finite Coxeter groups of type A?

Length generating function

Can Corollary 3.9.2 be generalized to a statement about length generating functions for intervals in quotients with respect to other parabolic subgroups?

Chapter 4

Order dimension, Bruhat order and lattice properties for posets

4.1 Main Results

We give here three short summaries of the main results of this chapter, from three points of view. We conclude the introduction by outlining the organization of the chapter.

Bruhat order

From the point of view of Bruhat order, the first main result of this chapter is the following:

Theorem 4.1.1. *The order dimensions of the following Coxeter groups under the Bruhat order are:*

$$\dim(A_n) = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor \tag{4.1}$$

$$\dim(B_n) = \binom{n}{2} + 1 \tag{4.2}$$

$$\dim(H_3) = 6 \tag{4.3}$$

$$\dim(H_4) = 25 \tag{4.4}$$

$$\dim(I_2(m)) = 2. \tag{4.5}$$

The upper bound $\dim(A_n) \leq \frac{(n+1)^2}{4}$ appeared as an exercise in [8], but the proof given here does not rely on the previous bound. The result for type I (dihedral groups) is trivial.

A finite poset is *dissective* if every join-irreducible element generates a principal order filter whose complement is a principal order ideal. Lascoux and Schützenberger [39] show that the Bruhat order on Coxeter groups of types A and B is dissective (or exhibits "clivage"). In types A and B, the dissective property of the strong order is closely related to the *tableau criterion* [8, 9]. Geck and Kim [32] show that Bruhat order on types D, E and F is not dissective. They also cite computer calculations to the effect that the exceptional type H is dissective. Type I is easily seen to be dissective. The following theorem applies:

Theorem 4.1.2. *If P is a dissective poset then* $\dim(P) = \mathrm{width}(\mathrm{Irr}(P))$.

Here, $\mathrm{Irr}(P)$ is the subposet of join-irreducible elements.

The subposet $\mathrm{Irr}(A_n)$ can be realized as a lattice tetrahedron in $\mathbb{R}^4$ [26] or can be characterized by considering certain "rectangular" words in the Coxeter group.

Theorem 4.1.3. $\mathrm{Irr}(A_n)$ *has a symmetric chain decomposition.*

In particular, $\mathrm{width}(\mathrm{Irr}(A_n))$ is the number of chains in the symmetric chain decomposition. The distributive lattice $J(\mathrm{Irr}(A_n))$ is the lattice of monotone triangles [39], which are in bijection with alternating sign matrices. The lattice of monotone triangles is the MacNeille completion of the Bruhat order on A_n and has the same dimension as A_n. (The MacNeille completion of a finite poset P is the smallest lattice containing P as a subposet.) The poset $\mathrm{Irr}(B_n)$ is less well-behaved, but its width can be determined by finding an antichain and a chain-decomposition of the same size. The results for H_3 and H_4 are obtained by computer calculations of the width.

The dissective property is inherited by quotients with respect to parabolic subgroups, so Theorem 4.1.2 can be used to determine the order dimensions of quotients in types A, B, H and I (see Theorems 4.8.3 and 4.9.1). Theorem 4.1.5, below, can in principle be used to compute bounds on the order dimensions of types D, E and F.

The order-dimension calculations reflect a deeper insight into the structure of Bruhat orders and quotients. For a poset, being dissective is, in a very strong sense, analogous to a lattice being distributive—for a precise statement, see Theorem 4.1.6 below. Bruhat orders and quotients of types A, B, H and I are, in some sense, "distributive non-lattices". The fact that Bruhat quotients inherit the dissective property reflects the intimate relationship of Bruhat quotients to lattice quotients. The equivalence relation on the strong order arising from cosets of a parabolic subgroup is an example of a poset congruence, which is in the same strong sense analogous to a lattice congruence (Theorem 4.1.7). Theorem 4.1.7 also shows that given any quotient of Bruhat order on type A, there is a unique corresponding lattice quotient on the lattice of monotone triangles.

The reader who is primarily interested in Theorem 4.1.1 may wish to skip Sections 3 through 6 on the first reading.

Order dimension

From the point of view of order dimension, the main result of this chapter is Theorem 4.1.2, which generalizes the following result of Dilworth:

Theorem 4.1.4. *[18] If L is a distributive lattice, then* $\dim(L) = \mathrm{width}(\mathrm{Irr}(L))$.

Theorem 4.1.2 is a generalization in the sense that a lattice is dissective if and only if it is distributive. The generalization is meaningful because there is an important class of dissective posets, namely the Bruhat orders on finite Coxeter groups of types A, B, H and I.

Theorem 4.1.2 follows from a more general result:

Theorem 4.1.5. *For a finite poset P,* $\mathrm{width}(\mathrm{Dis}(P)) \leq \dim(P) \leq \mathrm{width}(\mathrm{Irr}(P))$.

Here $\mathrm{Dis}(P)$ is a subposet of $\mathrm{Irr}(P)$ consisting of *dissectors* of P, those elements which generate a principal order filter whose complement is a principal order ideal. The upper bound in Theorem 4.1.5 also appears in [52]. The lower bound, in the case where P is a lattice, is implicit in [27]. A poset P is *dissective* if $\mathrm{Dis}(P) = \mathrm{Irr}(P)$. The dissective posets include, for example, distributive lattices and the "standard examples" of order dimension.

Both Theorem 4.1.5 and Theorem 4.1.2 can be expressed geometrically in terms of the *critical complex* $\mathcal{C}(P)$, a simplicial complex such that the dimension of P is the size of a smallest set of faces of $\mathcal{C}(P)$ covering the vertices of $\mathcal{C}(P)$. The critical complex is "dual" to the hypergraph H_P^c of critical pairs in [28], in that H_P^c is a hypergraph on the same vertex set whose edges are the minimal non-faces of $\mathcal{C}(P)$. The critical complex sheds light on the connection between dimension and width: The width of a poset is the size of a smallest covering set of its order complex, while the dimension is the size of a smallest covering set of $\mathcal{C}(P)$. Theorem 4.3.7, stated in detail in Section 3, essentially gives an embedding of the order complex $\Delta(\mathrm{Dis}(P))$ as a subcomplex of $\mathcal{C}(P)$ and a map from $\mathcal{C}(P)$ into $\Delta(\mathrm{Irr}(P))$ which respects the face structure. Theorem 4.4.3, also in Section 3, gives what is essentially an isomorphism between $\Delta(\mathrm{Irr}(P))$ and $\mathcal{C}(P)$, in the case when P is dissective.

The reader interested primarily in order-dimension may wish to skip Sections 4 through 6 on the first reading, and can consider Sections 7 through 9 to be an extended example.

Lattice properties for posets

The third theme of this chapter is taking definitions that apply to finite lattices and finding the "right" generalization to finite posets. We propose that given a lattice property A, the right generalization is the poset property A' such that a poset P has the property A' if and only if the MacNeille completion $L(P)$ has the property A. (The *MacNeille completion* of a finite poset P can be defined as the "smallest" lattice $L(P)$ containing P, in the sense that any lattice containing P as a subposet contains $L(P)$ as a subposet.) For example, the following is [39, Theorem 2.8]. We give a different proof.

Theorem 4.1.6. *For a finite poset P, the following are equivalent:*

(i) P is dissective.

(ii) The MacNeille completion $L(P)$ is a distributive lattice.

(iii) The MacNeille completion $L(P)$ is $J(\mathrm{Irr}(P))$.

The Bruhat orders on finite Coxeter groups of types A, B, H and I provide interesting examples of dissective posets. In Section 4, we explore the extent to which dissective posets have analogous properties to distributive lattices. The most striking case is, of course, Theorem 4.1.2.

Similarly, given a structure on a lattice, we propose that the "right" generalization of the structure to posets should respect the MacNeille completion. For example, in Section 5 we define a notion of poset congruence with the following property:

Theorem 4.1.7. *Let P be a finite poset with MacNeille completion $L(P)$, and let Θ be an equivalence relation on P. Then Θ is a congruence on P if and only if there is a congruence $L(\Theta)$ on $L(P)$ which restricts exactly to Θ, in which case*

(i) *$L(\Theta)$ is the unique congruence on $L(P)$ which restricts exactly to Θ, and*

(ii) *The MacNeille completion $L(P/\Theta)$ is naturally isomorphic to $L(P)/L(\Theta)$.*

The notion of exact restriction is the usual restriction of relations, with an extra condition.

A closely related example is the problem of defining homomorphisms of posets in the right way so as to make them analogous to lattice homomorphisms. Chajda and Snášel [14] give definitions of poset homomorphisms and congruences which correspond to each other in the usual way. The same correspondence holds (by the same proof) between our poset congruences and order-morphisms, which both differ in a trivial way from the definitions in [14]. In light of Theorem 4.1.7, order-morphisms are the right generalization of lattice homomorphisms.

The reader interested primarily in lattice theory may wish to skip Section 3 on the first reading, and can consider Sections 7 through 9 to be an extended example.

Outline

This chapter is structured as follows: Section 4.2 establishes notation, defines join-irreducibles of a non-lattice, dissectors and dissective posets, and concludes with a proof of Theorem 4.1.5. In Section 4.3, the critical complex is defined and Theorem 4.3.7, a geometric version of Theorem 4.1.5, is stated and proved. Dissective posets are characterized in Section 4.4, which also contains a description of the critical complex of a dissective poset, and a comparison of the properties of dissective posets and distributive lattices. Poset congruences and order-quotients are defined in Section 4.5 and shown to behave well with respect to join-irreducibles and dissectors. Section 4.6 is devoted to the MacNeille completion, and the proofs of Theorems 4.1.6 and 4.1.7. Section 4.7 provides a short summary of Bruhat order on a Coxeter group, while Section 4.8 contains the proof of Theorem 4.1.3 and a calculation of the width of $\mathrm{Irr}(A_n)$. Section 4.9 contains the calculation of $\mathrm{width}(\mathrm{Irr}(B_n))$, Section 4.10 is a brief discussion of the other types, and Section 4.11 contains further questions and directions for future research.

4.2 Join-irreducibles and dissectors

In this section, we provide background information about join-irreducibles and dissectors, and finish with a proof of Theorem 4.1.5.

Joins and meets, typically encountered in the context of lattices, can also be defined in general posets. Given x and y, if $U[x] \cap U[y]$ has a unique minimal element, this element is called the *join* of x and y and is written $x \vee^P y$ or simply $x \vee y$. If $D[x] \cap D[y]$ has a unique maximal element, it is called the *meet* of x and y, $x \wedge_P y$ or $x \wedge y$. Given a set $S \subseteq P$, if $\cap_{x \in S} U[x]$ has a unique minimal element, it is called $\vee S$. The join $\vee\emptyset$ is $\hat{0}$ if P has a unique minimal element $\hat{0}$, and otherwise $\vee\emptyset$ does not exist. If $\cap_{x \in S} D[x]$ has a unique maximal element, it is called $\wedge S$. The meet $\wedge\emptyset$ exists if and only if a unique maximal element $\hat{1}$ exists, in which case they coincide. The notation, $x \vee y = a$ means "x and y have a join, which is a," and similarly for other statements about joins and meets.

The description of an element as "join-irreducible" is usually heard in the context of lattices. However, it is useful to apply the definition to general posets, as in [32], [39] and [52]. An element a of a poset P is *join-irreducible* if there is no set $X \subseteq P$ with $a \notin X$ and $a = \vee X$. If P has a unique minimal element $\hat{0}$, then $\hat{0}$ is $\vee\emptyset$ and thus is not join-irreducible. In a lattice, a is join-irreducible if and only if it covers exactly one element. Such elements are also join-irreducible in non-lattices, but an element a which covers distinct elements $\{x_i\}$ is join-irreducible if $\{x_i\}$ has an upper bound incomparable to a. A minimal element of a non-lattice is also join-irreducible, if it is not $\hat{0}$. It is easily checked that if $x \in P$ is not join-irreducible, then $x = \vee D(x)$. The subposet of P induced by the join-irreducible elements is denoted $\mathrm{Irr}(P)$. In [32] and [39], the set $\mathrm{Irr}(P)$ is called the *base* of P. The subposet of meet-irreducibles does not figure strongly in this chapter, and that perhaps excuses the cumbersome notation $\mathrm{MeetIrr}(P)$ for this subposet. For $x \in P$, let I_x denote $D[x] \cap \mathrm{Irr}(P)$, the set of join-irreducibles weakly below x in P.

Proposition 4.2.1. *Let P be a finite poset, and let $x \in P$. Then $x = \vee I_x$.*

Proof. By induction on the cardinality of $D[x]$. The result is trivial if $D[x]$ has one element. If x is join-irreducible, then $x \in I_x$, and every other element of I_x is below x. Thus $x = \vee I_x$. If not, then write $x = \vee D(x)$. By induction, each y in $D(x)$ has $y = \vee I_y$, or in other words $\cap_{i \in I_y} U[i] = U[y]$. Then $U[x] = \cap_{y \in D(x)} U[y] = \cap_{y \in D(x)} \cap_{i \in I_y} U[i] = \cap_{i \in I_x} U[i]$, or in other words, $x = \vee I_x$. ☐

For a finite poset P, define $J(P)$ to be the lattice of order ideals of P, ordered by inclusion. The Fundamental Theorem of Finite Distributive Lattices states that a finite distributive lattice L has $L \cong J(\mathrm{Irr}(L))$, and that for any finite poset P, $J(P)$ is distributive with $\mathrm{Irr}(J(P)) \cong P$.

The proofs of the following two propositions are easy.

Proposition 4.2.2. *[39] An element $x \in P$ is join-irreducible if and only if there exists a $y \in P$ such that x is minimal in $P - D[y]$.* ☐

Proposition 4.2.3. *If x is join-irreducible (or dually meet-irreducible) in a lattice L, then $L - \{x\}$ is a lattice.* ☐

While $L - \{x\}$ is a subposet of L and a lattice, it is not usually a sublattice.

An element $x \in P$ is called a(n) *(upper) dissector* of P if $P - U[x] = D[\beta(x)]$ for some $\beta(x) \in P$. In other words, P can be dissected as a disjoint union of the principal order filter generated by x and the principal order ideal generated by $\beta(x)$. By the same token, call $\beta(x)$ a *lower dissector*.

From now on, however, the term dissector refers to an upper dissector. Thus for each result about dissectors, there is a dual result about lower dissectors which is not stated. The subposet of dissectors of P is called $\mathrm{Dis}(P)$. In the lattice case the definition of dissector coincides with the notion of a prime element. An element x of a lattice L is called *prime* if whenever $x \leq \vee Y$ for some $Y \subseteq L$, then there exists $y \in Y$ with $x \leq y$.

Proposition 4.2.2 implies:

Proposition 4.2.4. *If x is a dissector of P then x is join-irreducible.* □

The converse is not true in general, and the reader can find a 5-element lattice to serve as a counterexample. A poset P in which every join-irreducible is a dissector is called a *dissective* poset. In [39] this property of a poset is called "clivage."

We now give a proof of Theorem 4.1.5. Notice in particular, that this proof actually constructs an embedding of P into $\mathbb{N}^w$, where $w = \mathrm{width}(\mathrm{Irr}(P))$.

First Proof of Theorem 4.1.5. Let $C_1, C_2, \ldots, C_w$ be a chain decomposition of $\mathrm{Irr}(P)$. For each $m \in [w]$, and $x \in P$, let $f_m(x) = |I_x \cap C_m|$. By Proposition 4.2.1, $x \leq y$ if and only if $I_x \subseteq I_y$ if and only if $f_m(x) \leq f_m(y)$ for all $m \in [w]$. Thus $x \mapsto (f_1(x), f_2(x), \ldots, f_w(x))$ is an embedding of P into $\mathbb{N}^w$.

For the lower bound, consider an antichain A in $\mathrm{Dis}(P)$. Each $a \in A$ has $P - U_P[a] = D_P[\beta(a)]$ for some $\beta(a) \in P$. In particular $A - \{a\} \subseteq D_P[\beta(a)]$. So the subposet of P induced by $A \cup \beta(A)$ is a "standard example" of size $|A|$. Thus $\dim(P) \geq \dim(A \cup \beta(A)) = |A|$. □

This proof relies on knowing the order dimensions of the "standard examples." One way to find the order dimension of the standard examples is to notice that they are dissective posets whose join irreducibles form an antichain. However, to avoid this circular reasoning, one can easily compute the order dimension of the standard examples directly, or by the method of the next section.

4.3 The critical complex of a poset

In this section, we give the definition of the critical complex, relate the critical complex to join-irreducibles and dissectors, and give another proof of Theorem 4.1.5. The simple proofs of some propositions are omitted.

A *critical pair* in a poset P is (a, b) with the following properties:

(i) $a \parallel b$,

(ii) $D(a) \subseteq D(b)$, and

(iii) $U(b) \subseteq U(a)$.

As motivation, note that properties (ii) and (iii) hold for a related pair $a \leq b$. If (a, b) is a critical pair, the partial order $\leq$ can be extended to a new partial order $\leq'$ by putting $x \leq' y$ if $x \leq y$ or if

$(x, y) = (a, b)$. So in some sense, a critical pair (a, b) is "almost" a related pair $a \leq b$. The set of critical pairs of P is denoted $\mathrm{Crit}(P)$.

Say an extension E of P *reverses* a critical pair (a, b) if $b < a$ in E. The following fact is due to I. Rabinovitch and I. Rival [46]:

Proposition 4.3.1. *If $L_1, L_2, \dots, L_n$ are linear extensions of P then $P = \cap_{i \in [n]} L_i$ if and only if for each critical pair (a, b), there is some L_i for which $b < a$ in L_i.* □

The *critical digraph* $\mathcal{D}(P)$ of P is the directed graph whose vertices are the critical pairs, with directed edge $(a, b) \to (c, d)$ whenever $b \geq c$. The next proposition follows from Lemma 6.3 of Chapter 1 of [58].

Proposition 4.3.2. *Let S be any set of critical pairs of P. Then there is a linear extension of P reversing every critical pair in S if and only if the subgraph of $\mathcal{D}(P)$ induced by S is acyclic.* □

Proposition 4.3.2 motivates the definition of the *critical complex* $\mathcal{C}(P)$ of P, an abstract simplicial complex whose vertices are the critical pairs of P, and whose faces are the sets of vertices which induce acyclic subgraphs of $\mathcal{D}(P)$. A set of faces $\{F_i\}$ of a simplicial complex $\mathcal{C}$ with vertex set V is a *covering set* if $\cup_i F_i = V$. Propositions 4.3.1 and 4.3.2 imply that (when P is not a total order) the order dimension of P is the size of a smallest covering set of $\mathcal{C}(P)$.

A similar (and in some sense dual) construction to the critical complex is given by Felsner and Trotter [28]. Their *hypergraph H_P^c of critical pairs* is exactly the hypergraph whose vertices are critical pairs and whose edges are minimal non-faces of $\mathcal{C}(P)$. They also define the *graph G_P^c of critical pairs* whose vertices are the critical pairs and whose edges are the edges of cardinality 2 of H_P^c. The size of a smallest covering set of $\mathcal{C}(P)$ is exactly the chromatic number $\chi(H_P^c)$. The following is [28, Lemma 3.3]:

$$\dim(P) = \chi(H_P^c) \geq \chi(G_P^c) \tag{4.6}$$

The easy proofs of the following propositions are omitted. Proposition 4.3.3 was noticed by Rabinovitch and Rival [46] in the context of distributive lattices.

Proposition 4.3.3. *If (a, b) is a critical pair, then a is join-irreducible.* □

Proposition 4.3.4. *Let $(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$ be critical pairs in P with $a_1 \leq a_2 \leq \cdots \leq a_k$. Then $\{(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)\}$ is a face of $\mathcal{C}(P)$.* □

Proposition 4.3.5. *Let $a \in P$ be a non-pivot dissector. Then $(a, \beta(a))$ is a critical pair. Furthermore the only critical pair (a, b) is the pair with $b = \beta(a)$.* □

Proposition 4.3.6. *Let $a, x \in P$ be dissectors with $a \parallel x$. Then $\{(a, \beta(a)), (x, \beta(x))\}$ is not a face in $\mathcal{C}(P)$.* □

Since the width of a poset P is the size of a smallest covering set of the order complex $\Delta(P)$, one might expect that Theorem 4.1.5 follows from some relationships between the order complexes

$\Delta(\mathrm{Dis}(P))$ and $\Delta(\mathrm{Irr}(P))$ and the critical complex $\mathcal{C}(P)$. The following theorem explains such a relationship. Write $\mathrm{Dis}(P)_{nonpiv}$ for the subposet of $\mathrm{Dis}(P)$ consisting of non-pivots and $\mathrm{Dis}(P)_{piv}$ for the subposet of pivots. Similarly $\mathrm{Irr}(P)_{nonpiv}$ and $\mathrm{Irr}(P)_{piv}$.

Given two abstract simplicial complexes A and B, let $A * B$ be the *join* of A and B, a simplicial complex whose vertex set is the disjoint union of the vertices of A and of B, and whose faces are exactly the sets $F \cup G$ for all faces F of A and G of B. It is evident that $\Delta(\mathrm{Dis}(P)_{piv})$ is a simplex and that

$$\Delta(\mathrm{Dis}(P)) \cong \Delta(\mathrm{Dis}(P)_{piv}) * \Delta(\mathrm{Dis}(P)_{nonpiv}).$$

Similarly $\Delta(\mathrm{Irr}(P)_{piv})$ is a simplex and

$$\Delta(\mathrm{Irr}(P)) \cong \Delta(\mathrm{Irr}(P)_{piv}) * \Delta(\mathrm{Irr}(P)_{nonpiv}).$$

In light of Propositions 4.3.3 and 4.3.5, we have well defined set maps:

$$\begin{array}{cc} i : \mathrm{Dis}(P)_{nonpiv} \to \mathrm{Crit}(P), & p : \mathrm{Crit}(P) \to \mathrm{Irr}(P)_{nonpiv} \\ a \overset{i}{\mapsto} (a, \beta(a)) & (a, b) \overset{p}{\mapsto} a \end{array}.$$

Theorem 4.3.7. *The set map i induces a simplicial map $i : \Delta(\mathrm{Dis}(P)_{nonpiv}) \to \mathcal{C}(P)$ which embeds $\Delta(\mathrm{Dis}(P)_{nonpiv})$ as a vertex-induced subcomplex of $\mathcal{C}(P)$. Also, if F is any face in the image of p, then $p^{-1}(F)$ is a face of $\mathcal{C}(P)$.*

Saying that i embeds $\Delta(\mathrm{Dis}(P)_{nonpiv})$ as a vertex-induced subcomplex of $\mathcal{C}(P)$ means that i is one-to-one, maps faces of $\Delta(\mathrm{Dis}(P)_{nonpiv})$ to faces of $\mathcal{C}(P)$ and for any face F of $i(\Delta(\mathrm{Dis}(P)_{nonpiv})$, $i^{-1}(F)$ is a face of $\Delta(\mathrm{Dis}(P)_{nonpiv})$. If F is a face of $\mathcal{C}(P)$, then $p(F)$ need not be a face of $\Delta(\mathrm{Irr}(P)_{nonpiv})$. For example, let P be an antichain $\{a, b, c\}$ and let F be $\{(a, b), (b, c)\}$.

Proof. The statement that i is one-to-one and maps faces of $\Delta(\mathrm{Dis}(P)_{nonpiv})$ to faces of $\mathcal{C}(P)$ follows immediately from Propositions 4.3.4 and 4.3.5. Proposition 4.3.6 is exactly the statement that for any face F of $i(\Delta(\mathrm{Dis}(P)_{nonpiv})$, $i^{-1}(F)$ is a face of $\Delta(\mathrm{Dis}(P)_{nonpiv})$. The last statement of the theorem also follows immediately from Proposition 4.3.4. ☐

Second Proof of Theorem 4.1.5. Since $\Delta(\mathrm{Dis}(P)_{nonpiv})$ is embedded into $\mathcal{C}(P)$, any covering set of $\mathcal{C}(P)$ restricts to a covering set of$\Delta(\mathrm{Dis}(P)_{nonpiv})$. Whenever $\Delta(\mathrm{Dis}(P)_{nonpiv})$ is non-empty, a covering set of $\Delta(\mathrm{Dis}(P)_{nonpiv})$, is easily extended to a set of the same cardinality covering $\Delta(\mathrm{Dis}(P)) \cong \Delta(\mathrm{Dis}(P)_{piv}) * \Delta(\mathrm{Dis}(P)_{nonpiv})$.

Any covering set of $\Delta(\mathrm{Irr}(P))$ restricts to a covering set of $\Delta(\mathrm{Irr}(P)_{nonpiv})$, which maps by p^{-1} to a covering set of $\mathcal{C}(P)$. ☐

4.4 Dissective posets

In this section, we study dissective posets: posets in which every join-irreducible is a dissector. In light of this definition, Theorem 4.1.2 follows trivially from Theorem 4.1.5, and the embedding given in the first proof of Theorem 4.1.5 is an optimal embedding. The dissective property is a generalization of the distributive property, in the following sense:

Proposition 4.4.1. *[27, 42] A finite lattice L is distributive if and only if every join-irreducible is prime.* $\square$

In other words, a lattice is distributive if and only if it is dissective. This statement is strengthened in the next section. In this section we characterize dissective posets, describe the critical complex of a dissective poset and discuss the extent to which dissective posets have properties analogous to distributive lattices.

Proposition 4.4.2. *Let L be a finite distributive lattice and let P be a subposet with*

$$\mathrm{Irr}(L) \cup \mathrm{MeetIrr}(L) \subseteq P.$$

Then P is a dissective poset and $\mathrm{Irr}(P) = \mathrm{Irr}(L)$.

Proof. Suppose $x \in \mathrm{Irr}(L) = \mathrm{Dis}(L)$. Then there is a

$$\beta(x) := \vee^L\{y \in L : y \not\geq x\} = \vee^L\{y \in \mathrm{Irr}(L) : y \not\geq x\}.$$

But $\beta(x) \in \mathrm{MeetIrr}(L)$, so $\beta(x) \in P$. Any upper bound z for $\{y \in P : y \not\geq x\}$ is in particular an upper bound for $\{y \in \mathrm{Irr}(L) : y \not\geq x\}$ so in particular $z \geq \beta(x)$. Thus $\beta(x) = \vee^P\{y \in P : y \not\geq x\}$. Therefore $x \in \mathrm{Dis}(P)$, and so $\mathrm{Irr}(L) \subseteq \mathrm{Dis}(P) \subseteq \mathrm{Irr}(P)$.

Suppose $x \in \mathrm{Irr}(P)$ and write $x = \vee^L(D_L[x] \cap \mathrm{Irr}(L))$. By the previous paragraph, $D_L[x] \cap \mathrm{Irr}(L) \subseteq P$, and since P is a subposet, $x = \vee^P(D_L[x] \cap \mathrm{Irr}(L))$. But $x \in \mathrm{Irr}(P)$, so $x \in D_L[x] \cap \mathrm{Irr}(L)$. Therefore $\mathrm{Irr}(P) \subseteq \mathrm{Irr}(L)$. $\square$

In light of Theorem 7, L is the MacNeille completion of P, and every dissective poset arises as in Proposition 4.4.2. For example, the "standard examples" of n-dimensional posets arise in this manner from the Boolean algebra of rank n.

When P is dissective, Theorem 4.3.7 simplifies greatly. Since every join-irreducible is a dissector and by Proposition 4.3.5, every dissector gives rise to exactly one critical pair, the map i is a bijection with inverse p, and therefore an isomorphism of simplicial complexes.

Theorem 4.4.3. *If P be a dissective poset, then the order complex $\Delta(\mathrm{Irr}(P))$ is isomorphic to $\mathcal{C}(P) * \Delta(\mathrm{Irr}(P)_{piv})$.* $\square$

The statement is even simpler than it looks since $\Delta(\mathrm{Irr}(P)_{piv})$ is a simplex. Similar considerations also show that for a dissective poset, the hypergraph H_P^c of critical pairs is equal to the graph G_P^c of critical pairs. Thus equality holds in Equation (4.6) when P is dissective.

We now list some properties of dissective posets which are analogous to familiar properties of distributive lattices. The proofs are straightforward, and are omitted.

Proposition 4.4.4. *If P is dissective, then so is the dual of P.* $\square$

Proposition 4.4.5. *If P is a dissective poset then $\beta : \mathrm{Irr}(P) \to \mathrm{MeetIrr}(P)$ is an order isomorphism.* $\square$

Proposition 4.4.6. *If a dissective poset P is self-dual then* $\mathrm{Irr}(P)$ *is self-dual.* $\square$

Even when P is not dissective, β is an order isomorphism from $\mathrm{Dis}(P)$ to the subposet of lower dissectors, and if P is self-dual, then $\mathrm{Dis}(P)$ is also self-dual.

Finally, we mention several properties of distributive lattices which appear not to have analogues for dissective posets. The converse of Proposition 4.4.6 holds for distributive lattices, but not for dissective posets. The distributive property in a finite lattice is inherited by intervals, but the analogous property is not true of the dissective property in a finite poset. Finally, distributive lattices can be characterized by the fact that they avoid certain sublattices. No similar characterization for dissective posets is immediately apparent.

4.5 Order-quotients

In this section we define order-quotients and prove that they behave nicely with respect to join-irreducibles and dissectors. The reader familiar with Bruhat order may want to keep in mind quotients with respect to parabolic subgroups as a motivating example. Let P be a finite poset with an equivalence relation Θ defined on the elements of P. Given $a \in P$, let $[a]_\Theta$ denote the equivalence class of a under Θ. The equivalence relation Θ is a *congruence* if:

(i) Every equivalence class is an interval.

(ii) The projection $\pi_\downarrow : P \to P$, mapping each element a of P to the minimal element in $[a]_\Theta$, is order-preserving.

(iii) The projection $\pi^\uparrow : P \to P$, mapping each element a of P to the maximal element in $[a]_\Theta$, is order-preserving.

The definition given here essentially coincides, when P is finite, to the notion of *poset congruence*, as defined in [14]. The difference is that in [14], $P \times P$ is by definition always a congruence. Also in [14], there is the definition of *LU-morphisms*, which we call *order-morphisms*. The definition given here differs from [14], in a way that corresponds to the difference in the definitions of congruence. A map $f : P \to Q$ for finite P and Q is an *order-morphism* if for any $x, y \in P$,

$$f(D_P[x] \cap D_P[y]) = D_{f(P)}[f(x)] \cap D_{f(P)}[f(y)]$$

and if the dual statement also holds. Congruences and order-morphisms are related in the usual way. The proof can be found in [14] and still works with the slightly modified definitions.

A congruence on a lattice L is an equivalence relation which respects joins and meets. Specifically, if $a_1 \equiv a_2$ and $b_1 \equiv b_2$ then $a_1 \vee b_1 \equiv a_2 \vee b_2$ and similarly for meets. For a finite lattice L, the two notions of congruence coincide. So from now on, the term congruence is used without specifying "lattice" or "poset." A connection is made in Section 6 between congruences on a finite poset and congruences on its MacNeille completion.

Define a partial order on the congruence classes by $[a]_\Theta \leq [b]_\Theta$ if and only if there exists $x \in [a]_\Theta$ and $y \in [b]_\Theta$ such that $x \leq_P y$. The set of equivalence classes under this partial order is P/Θ, the *quotient* of P with respect to Θ. It is convenient to identify P/Θ with the induced subposet $Q := \pi_\downarrow(P)$, as is typically done for example with quotients of Bruhat order. Such a subposet Q is called an *order-quotient* of P. It is easily seen that $\pi^\uparrow$ maps Q isomorphically onto $\pi^\uparrow(P)$. The inverse is $\pi_\downarrow$.

We wish to compare $\mathrm{Dis}(P/\Theta)$ and $\mathrm{Irr}(P/\Theta)$ to $\mathrm{Dis}(P)$ and $\mathrm{Irr}(P)$.

Proposition 4.5.1. *Suppose Q is an order-quotient of P. If $x = \vee^Q Y$ for some $Y \subseteq Q$, then $x = \vee^P Y$. If $x = \vee^P Y$ for some $Y \subseteq P$, then $\pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$.*

Proof. Suppose $x = \vee^Q Y$ for $Y \subseteq Q$ and suppose $z \in P$ has $z \geq y$ for every $y \in Y$. Then $\pi_\downarrow(z) \geq \pi_\downarrow(y) = y$ for every $y \in Y$. Therefore $z \geq \pi_\downarrow(z) \geq x$. Thus $x = \vee^P Y$.

Suppose $x = \vee^P Y$ for $Y \subseteq P$, and suppose that for some $z \in Q$, $z \geq \pi_\downarrow(y)$ for every $y \in Y$. Then $\pi^\uparrow(z) \geq \pi^\uparrow(y) \geq y$ for every $y \in Y$, and so $\pi^\uparrow(z) \geq x$. Thus also $\pi_\downarrow(\pi^\uparrow(z)) \geq \pi_\downarrow(x)$, but $\pi_\downarrow(\pi^\uparrow(z)) = z$, and so $\pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$. $\square$

Proposition 4.5.2. *Suppose Q is an order-quotient of P and let $x \in Q$. Then x is join-irreducible in Q if and only if it is join-irreducible in P, and x is a dissector of Q if and only if it is a dissector of P. In other words,*

$$\begin{aligned} \mathrm{Irr}(Q) &= \mathrm{Irr}(P) \cap Q \text{ and,} & (4.7)\\ \mathrm{Dis}(Q) &= \mathrm{Dis}(P) \cap Q. & (4.8) \end{aligned}$$

In particular, if P is dissective, then so is any order-quotient. Also, for any P with order-quotient Q such that $Q \cap \mathrm{Irr}(P) \subseteq \mathrm{Dis}(P)$, Q is dissective.

Proof. Suppose $x \in Q$ is join-irreducible in Q. Then by Proposition 4.2.2, there is some $y \in Q$ so that x is minimal in $Q - D_Q[y]$. Then x is also minimal in $P - D_P[\pi^\uparrow(y)]$, so x is join-irreducible in P. Conversely, suppose $x \in Q$ is join-irreducible in P, and suppose $x = \vee^Q Y$ for some $Y \subseteq Q$. Then by Proposition 4.5.1, $x = \vee^P Y$, so $x \in Y$. Thus x is join-irreducible in Q.

Suppose $x \in Q$ is a dissector of Q. Then there is some $\beta^Q(x) \in Q$ such that $Q - U_Q[x] = D_Q[\beta^Q(x)]$. Then $\pi^\uparrow(\beta^Q(x)) \not\geq x$ because otherwise $\beta^Q(x) \geq \pi_\downarrow(x) = x$. Furthermore, for any $z \not\geq x$, necessarily $\pi_\downarrow(z) \not\geq x$, and therefore $\pi_\downarrow(z) \leq \beta^Q(x)$. So $z \leq \pi^\uparrow(z) \leq \pi^\uparrow(\beta^Q(x))$. Thus x is a dissector of P with $P - U_P[x] = D_P[\pi^\uparrow(\beta^Q(x))]$. Conversely, suppose $x \in Q$ is a dissector of P, or in other words, there is some $\beta^P(x) \in P$ such that $\beta^P(x) = \vee^P(P - U_P[x])$. Then by Proposition 4.5.1, $\pi_\downarrow(\beta^P(x)) = \vee^Q \pi_\downarrow(P - U_P[x]) = \vee^Q(Q - U_Q[x])$, so x is a dissector of Q. $\square$

Quotients of Bruhat order with respect to parabolic subgroups are order-quotients (Proposition 4.7.1). There are also several examples in the literature relating to weak Bruhat order. A. Björner and M. Wachs [13, Section 9] show that the Tamari lattices are quotients of the weak order on A_n.

R. Simion [54, Section 4] defines a congruence on the Coxeter group B_n under the weak order, such that the resulting quotient is the weak order on A_n.

4.6 The MacNeille completion

In this section we define the MacNeille completion of a finite poset and point out that it preserves join-irreducibles, dissectors and critical pairs. We strengthen the assertion that the dissective property generalizes the distributive property. This leads in particular to a different proof of Theorem 4.1.2. We also strengthen the assertion that congruences on posets are a generalization of congruences on lattices.

The *MacNeille completion* (also known as the *MacNeille-Dedekind completion*, *completion by cuts* or *enveloping lattice*) of a poset P generalizes Dedekind's construction of the reals from the rationals. One construction of the completion is due to MacNeille [41] and more information can be found in [58, Section 2.5] and [6, Section V.9]. Here we confine our attention to the MacNeille completion of a finite poset. For a finite poset P, the MacNeille completion $L(P)$ is the "smallest" lattice containing P, in the sense that any lattice containing P as an induced subposet contains $L(P)$ as an induced subposet. One way to obtain $L(P)$ for a finite poset is as the smallest collection of subsets which contains P and $U_P[x]$ for each $x \in P$ and which is closed under intersection [39]. The partial order on $L(P)$ is reverse-inclusion, the join is intersection, and $x \mapsto U_P[x]$ is an embedding of P as a subposet of $L(P)$. Whatever joins exist in P are preserved by MacNeille completion: If $x = \vee^P S$ for some $S \subseteq P$, then $U_P[x] = \cap_{y \in S} U_P[y]$, or in other words, $x = \vee^{L(P)} S$. Conversely, if $x \in P$ is $\vee^{L(P)} S$ for some $S \subseteq P$, then $x = \vee^P S$. This construction also shows that any element of $L(P)$ is a join of elements of P.

The construction is seen to coincide with its dual construction (by order ideals) as follows: For $x \in L(P)$, define P_x to be the elements of P below x in $L(P)$. Then $x = \cap_{y \in P_x} U_P[y]$, the partial order on $L(P)$ is inclusion of the sets P_x and the meet is intersection of sets P_x. It also follows that $x = \vee^{L(P)} P_x$.

Proposition 4.6.1. *Let P be a finite poset and let $L(P)$ be its MacNeille completion. Then*

(i) $\mathrm{Irr}(P) = \mathrm{Irr}(L(P))$

(ii) $\mathrm{Dis}(P) = \mathrm{Dis}(L(P))$*, and*

(iii) $\mathrm{Crit}(P) = \mathrm{Crit}(L(P))$.

Remark 4.6.2. Assertion (i) is implicit in the proof of [39, Théorème 2.8]. Although we were unable to find (iii) in the literature, it is closely related to [58, Exercise 2.5.7].

Proof of Proposition 4.6.1. Suppose that $x \in \mathrm{Irr}(L(P))$. Then by Proposition 4.2.3, $L(P) - \{x\}$ is a lattice. Thus by the definition of $L(P)$, necessarily $x \in P$. Since joins are preserved in $L(P)$, join-irreducibility of x in $L(P)$ implies join-irreducibility in P. Conversely, suppose that $x \in \mathrm{Irr}(P)$ and write $x = \vee^{L(P)} S$ for some $S \subseteq \mathrm{Irr}(L(P))$. Since we have just shown that every join-irreducible in $L(P)$ is join-irreducible in P, $S \subseteq \mathrm{Irr}(P)$. Since joins are preserved, $x = \vee^P S$, and therefore $x \in S$. Thus $x \in \mathrm{Irr}(L(P))$.

Let x be a dissector in P and $\beta_P(x) = \vee^P(P - U_P[x])$. Because joins are preserved, $\beta_P(x) = \vee^{L(P)}(P - U_P[x])$. But each element of the set $(L(P) - U_{L(P)}[x])$ can be written as the join of elements of $P - U_P[x]$, so $\beta_P(x) = \vee^{L(P)}(L(P) - U_{L(P)}[x])$. Thus x is a dissector in $L(P)$ with $\beta_{L(P)}(x) = \beta_P(x)$. Conversely, suppose x is a dissector in $L(P)$ and $\beta_{L(P)}(x) = \vee^{L(P)}(L(P) - U_{L(P)}[x])$. Then $\beta_{L(P)}(x)$ is meet-irreducible, so by the dual of assertion (i), $\beta_{L(P)}(x) \in P$. Since $\beta_{L(P)}(x)$ is an upper bound for $P - U_P[x]$, and $\beta_{L(P)}(x)$ is also contained in $P - U_P[x]$, we have $\beta_{L(P)}(x) = \vee^P(P - U_P[x]) = \beta_P(x)$.

Let (a, b) be a critical pair of P. Then a is join-irreducible in P by Proposition 4.3.3 and thus join-irreducible in $L(P)$ by assertion (i). Let x be the single element of $L(P)$ covered by a. But x is the join of all the join-irreducibles weakly below it, and b is above all these join-irreducibles because they are below a. Thus $x \leq b$ and condition (ii) holds for (a, b) to be critical in $L(P)$. Condition (iii) is dual, and condition (i) holds because P is an induced subposet of $L(P)$. Conversely, let (a, b) be a critical pair of $L(P)$. By Proposition 4.3.3, a is join-irreducible, and so $a \in P$. Dually, $b \in P$. Conditions (i), (ii) and (iii) for (a, b) to be critical in P follow easily because P is an induced subposet of $L(P)$. □

Theorem 4.1.6, due to Lascoux and Schützenberger [39], follows easily from Proposition 4.4.1 and assertions (i) and (ii) of Proposition 4.6.1.

Theorem 4.1.6. *For a finite poset P, the following are equivalent:*

(i) P is dissective.

(ii) The MacNeille completion $L(P)$ is a distributive lattice.

(iii) The MacNeille completion $L(P)$ is $J(\mathrm{Irr}(P))$. □

Assertion (iii) of Proposition 4.6.1 implies the known fact [58, Exercise 2.5.7], [6, Section V.9] that order dimension is preserved by MacNeille completion. Thus Theorem 4.1.6 combines with Theorem 4.1.4 to give a different (more complicated) proof of Theorem 4.1.2.

Given a finite lattice L with a subposet P, a congruence Θ on L *restricts exactly to* P if every congruence class $[x, y]$ of Θ has either $x, y \in P$ or $[x, y] \cap P = \emptyset$. The next proposition follows immediately from the definitions.

Proposition 4.6.3. *If a congruence Θ on L restricts exactly to P, then the restriction (as a relation) $\Theta|_P$ of Θ to P is a congruence, and L/Θ is a lattice containing $P/(\Theta|_P)$ as a subposet.* □

Thus we also say Θ restricts exactly to $\Theta|_P$.

Theorem 4.1.7. *Let P be a finite poset with MacNeille completion $L(P)$, and let Θ be an equivalence relation on P. Then Θ is a congruence on P if and only if there is a congruence $L(\Theta)$ on $L(P)$ which restricts exactly to Θ, in which case* *(i)* $L(\Theta)$ *is the unique congruence on* $L(P)$ *which restricts exactly to* Θ*, and*

(ii) *The MacNeille completion* $L(P/\Theta)$ *is naturally isomorphic to* $L(P)/L(\Theta)$*.*

Proof. The "if" direction is Proposition 4.6.3.

Conversely, suppose Θ is a congruence on P and $x \in L(P)$. Let P_x/Θ be the set of equivalence classes in Θ which have non-empty intersection with P_x. Define $L(\Theta)$ to be the equivalence relation which sets $x \equiv y$ if and only if $P_x/\Theta = P_y/\Theta$. More simply, P_x/Θ is determined by Q_x, the set of elements of Q weakly below x. Here Q is the order-quotient associated to Θ, as in Section 5. Thus $x \equiv y$ if and only if $Q_x = Q_y$. Notice that for any $x, y \in L(P)$, $Q_{x \wedge y} = Q_x \cap Q_y$. Suppose $x_1 \equiv_{L(\Theta)} x_2$ and $y_1 \equiv_{L(\Theta)} y_2$ in $L(P)$. Then

$$Q_{x_1 \wedge y_1} = Q_{x_1} \cap Q_{y_1} = Q_{x_2} \cap Q_{y_2} = Q_{x_2 \wedge y_2},$$

and a dual argument shows that $L(\Theta)$ respects joins. Given any congruence class $[a, b]_P$ in Θ, the element a of $L(P)$ is minimal among elements x of $L(P)$ with $Q_x = Q_a$, and dually, b is maximal. Thus there is a congruence class $[a, b]_{L(P)}$ in $L(\Theta)$. Any element of P is in some Θ-class, and so $L(\Theta)$ restricts exactly to P.

Since $P/\Theta \cong Q$, the natural isomorphism between $L(P/\Theta)$ and $L(P)/L(\Theta)$ is easily seen by identifying the elements of each lattice with order ideals in Q. The lattice $L(Q)$ consists of Q and intersections $\cap_{q \in S} D_Q[q]$ for $S \subseteq Q$, and S may as well be an order filter. Elements of $L(P)/L(\Theta)$ are $X \cap Q$, where $X = P$ or $X = \cap_{y \in T} D_P[y]$, where T is an order filter in P. But then $X \cap Q$ is Q or $\cap_{y \in T}(D_P[y] \cap Q) = \cap_{y \in T \cap Q} D_Q[y]$, and $T \cap Q$ ranges over all order filters in Q.

Let Φ be a congruence on $L(P)$ which restricts exactly to Θ. Proposition 4.6.3 says that $L(P)/\Phi$ is a lattice containing P/Θ as a subposet. Since $L(P)/L(\Theta)$ is the MacNeille completion of P/Θ, $L(P)/\Phi$ contains $L(P)/L(\Theta)$ as a subposet, and therefore $\Phi \subseteq L(\Theta)$ (as relations). Suppose $x \equiv_{L(\Theta)} y$, or in other words $P_x/\Theta = P_y/\Theta$. Write $x = \vee^{L(P)} P_x$ and $y = \vee^{L(P)} P_y$. Since Φ restricts to Θ on P and respects joins, $x \equiv_\Phi y$, and thus $\Phi = L(\Theta)$. $\square$

4.7 Quotients and the tableau criterion

In this section, we give some necessary background information about quotients in Coxeter groups, and about types A and B in particular. The reader should refer to [8] or [35] for proofs and details.

When J is any subset of S, the subgroup of W generated by J is another Coxeter group, called the *parabolic subgroup* W_J. When the generators of a Coxeter group are denoted as s_i, use shorthand notations such as $J = \{1, 2, 4\}$ to denote the subset $\{s_1, s_2, s_4\} \subseteq S$. The following proposition defines and proves the existence of *two-sided quotients* ${}^J W^K$, where $J, K \subseteq S$, and shows that such quotients are order-quotients. The more widely used one-sided quotients can be obtained by letting $J = \emptyset$. Parabolic subgroups W_J with $J = S - \{s\}$ for some s are called *maximal parabolic subgroups*. Quotients with respect to W_J will be denoted W^s and similarly for two-sided quotients. This should not be confused with $W^{\{s\}}$.

Proposition 4.7.1. *For any $w \in W$ and $J, K \subseteq S$, the double coset $W_J w W_K$ has a unique Bruhat minimal element ${}^J w^K$. If W is finite, the subset ${}^J W^K$ consisting of the minimal coset representatives is an order-quotient of W.*

Proof. The proof of the first statement can be found in [30, Proposition 8.3], where it is also shown that w can be factored (non-uniquely) as $w_J \cdot {}^J w^K \cdot w_K$, where $w_J \in W_J$ and $w_K \in W_K$, such that $l(w) = l(w_J) + l({}^J w^K) + l(w_K)$. Let $\pi_\downarrow : W \to {}^J W^K$ be the projection $w \mapsto {}^J w^K$ onto minimal double coset representatives. We must show that $\pi_\downarrow$ is order-preserving: Suppose $v \leq w$, and write $w = w_J \cdot {}^J w^K \cdot w_K$. Choose reduced words a, b and c for w_J, ${}^J w^K$, and w_K respectively. Since $l(w) = l(w_J) + l({}^J w^K) + l(w_K)$, abc is a reduced word for w. By the subword property, there is a subword of abc which is a reduced word for v. This subword breaks into a', b' and c', which are subwords of a, b and c, respectively. Let x, y, and z be the respective elements represented by a', b' and c'. Thus $v = xyz$, and since $x \in W_J$ and $z \in W_K$, we have $y \in W_J v W_K$. In particular, $y \geq {}^J v^K$, and by the subword property, $y \leq {}^J w^K$, so ${}^J v^K \leq {}^J w^K$.

If W is finite, then multiplication on the left by w_0 is an anti-automorphism of W. If $x \in W_J w W_K$, write $x = w_J w w_K$. Then $w_0 x = w_0 w_J w_0 w_0 w w_K$, and $w_0 w_J w_0 \in W_{w_0 J w_0}$, and so $w_0 x \in W_{w_0 J w_0} w_0 w W_K$. Conversely, if $x \in W_{w_0 J w_0} w_0 w W_K$, then $w_0 x \in W_J w W_K$. Thus left-multiplication by w_0 acts as a Bruhat anti-isomorphism $W_J w W_K \overset{w_0 \cdot}{\mapsto} W_{w_0 J w_0} w W_K$. The maximal element of $W_J w W_K$ is $w_0 m$, where m is the minimal element of $W_{w_0 J w_0} w W_K$. The projection $\pi^\uparrow$ onto the maximal element is order-preserving because it is $w \mapsto w_0 \pi_\downarrow(w_0 w)$. $\square$

In [32, 39] it is shown that join-irreducibles in the Bruhat order are always *bigrassmannians*. That is, any join-irreducible x in W is contained in ${}^s W^t$ for some (necessarily unique) choice of $s, t \in S$. Thus Proposition 4.5.2 can be used to simplify the task of finding join-irreducibles and dissectors in W.

Proposition 4.7.2. *For a finite Coxeter group W under the Bruhat order:*

(i) $\mathrm{Irr}(W) = \cup_{s,t \in S} \mathrm{Irr}({}^s W^t)$ *and*

(ii) $\mathrm{Dis}(W) = \cup_{s,t \in S} \mathrm{Dis}({}^s W^t)$.

$\square$

Assertion (i) in Proposition 4.7.2 is due to Geck and Kim [32], who used it find the join-irreducibles for the infinite families of finite Coxeter groups and to write a GAP [53] program to compute the join-irreducibles of the other finite Coxeter groups.

Corollary 4.7.3. *Let W be a finite Coxeter group. The following are equivalent:*

(i) The Bruhat order on W is dissective.

(ii) The Bruhat order on ${}^J W^K$ is dissective for any maximal parabolic subgroups J and K.

(iii) The Bruhat order on ${}^J W^K$ is dissective for any parabolic subgroups J and K.

Proof. This follows from Proposition 4.7.2 and the observation that ${}^{J_1}W^{K_1} \subseteq {}^{J_2}W^{K_2}$ whenever $J_1^c \subseteq J_2^c$ and $K_1^c \subseteq K_2^c$, where $J^c := S - J$. $\square$

We recall the classification of finite irreducible Coxeter groups, traditionally named with letters. There are infinite families A, B and D, indexed by natural numbers n. There are also the exceptional groups E_6, E_7, E_8, F_4, H_3 and H_4, and some groups $I_2(m)$ on two generators. We describe types A, B and H here, and refer the reader to [8] or [35] for the other types. Types A and B are interpreted as permutations. Since many of the permutations we refer to have long strings of values increasing by one, the following notation is helpful: i —— j stands for $i(i+1)(i+2)\cdots j$ if $i \leq j$ or an empty string of entries if $i > j$.

The Coxeter group A_n is isomorphic to the group S_{n+1} of permutations of $[n+1]$. A permutation x can be written in *one-line notation* $x_1 x_2 \cdots x_{n+1}$, meaning $i \mapsto x_i$ for each i. The generators S are the transpositions $s_i := (i \;\; i{+}1)$, which switch the elements i and $i{+}1$ and fix all other elements. It is easy to check that A_n is a Coxeter group with $m(s_i, s_j) = 3$ for $|i-j| = 1$ and $m(s_i, s_j) = 2$ for $|i-j| > 1$. The length of an element is the inversion number $\#\{(i,j) : i < j, x_i > x_j\}$. Multiplying a permutation on the right by a generator s_i has the effect of switching the entry x_i with the entry x_{i+1}. Multiplying on the left by s_i switches the entry i with the entry $i+1$.

Elements of $A_n^{s_i}$ are permutations whose one-line notation increases from left to right except possibly between positions i and $i+1$. Bruhat comparisons in $A_n^{s_i}$ can be made by entrywise comparison of the entries from 1 to i. The *tableau criterion* characterizes Bruhat order on A_n as follows: Let $x = x_1 x_2 \cdots x_{n+1}$, and form a tableau with rows $T_a(x)$ for each $a \in [n]$, such that $T_a(x) = (T_{a,1}, T_{a,2}, \ldots, T_{a,a})$ is the increasing rearrangement of $\{x_i : i \in [a]\}$.

Proposition 4.7.4. *$x \leq y$ if and only if $T_{a,b}(x) \leq T_{a,b}(y)$ for every $1 \leq b \leq a \leq n$.* $\square$

The tableau $T(x)$ is a special case of a *monotone triangle*. A monotone triangle of size n is a tableau of staircase shape (written in the French style), with n rows and n columns, with entries from $[n+1]$, such that rows are strictly increasing, columns are weakly decreasing and elements are weakly increasing in the southeast ($\searrow$) direction. The permutations are exactly the monotone triangles such that for every $1 \leq b \leq a < n$, either $T_{a,b} = T_{a+1,b}$ or $T_{a,b} = T_{a+1,b+1}$. The tableau criterion states that Bruhat order is the restriction to permutations of componentwise order on monotone triangles. There is a simple bijection between monotone triangles and *alternating sign matrices* [49].

Given a permutation $x = x_1 x_2 \cdots x_{n+1}$, form $\alpha(x) = y_1 y_2 \cdots y_{n+1}$ according to $y_i = n + 2 - x_i$. It is easily checked that α is the anti-automorphism $w \mapsto w_0 w$ of the Bruhat order. The operation of α on tableaux is to replace each entry a by $n + 2 - a$, and to reverse the order of entries within the rows.

For the purposes of order dimension there is a much better tableau criterion [8, Exercise 2.13] than Proposition 4.7.4. Given a permutation in $A_n^{s_i}$ the entrywise comparison of the entries from 1 to i is dual to the entrywise comparison of the entries from $i+1$ to $n+1$. Given $x \in A_n$ define a pair (L, R) of tableaux of staircase shape, where L is the increasing rearrangements of the initial segments of x of lengths $\leq \lfloor \frac{n+1}{2} \rfloor$, and R is the increasing rearrangements of the final segments of x of lengths $\leq n - \lfloor \frac{n+1}{2} \rfloor$.

Proposition 4.7.5. *$x \leq y$ if and only if $L(x) \leq L(y)$ and $R(x) \geq R(y)$ componentwise.* □

Note that the existence of this "two-tableau criterion" is related to the existence of the symmetry $w \mapsto w_0 w w_0$ in the Coxeter group A_n.

The total number of entries in (L, R) is

$$\binom{\lfloor \frac{n+1}{2} \rfloor + 1}{2} + \binom{n+1-\lfloor \frac{n+1}{2} \rfloor}{2} = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor.$$

Thus the bound $\dim(A_n) \leq \frac{(n+1)^2}{4}$ was already known. However, the proof of Theorem 4.1.1 does not explicitly use Proposition 4.7.5. In light of Theorem 4.1.1, from the viewpoint of order dimension this two-tableaux criterion is an optimal encoding of Bruhat order. Whether this embedding is actually the fastest way to compute Bruhat order is not quite the same question. Another simplification of the tableau criterion is given in [9]. Here it is shown that for a given x, one need only consider certain rows of $T(x)$ and $T(y)$, depending on the descents of x and y, to compare x to y. This simplification does not affect order dimension, but may speed up computations.

An element $w \in A_n$ is called *321-avoiding* if any of the following equivalent [1, Theorem 2.1] conditions holds:

(i) Let w correspond to a permutation with one-line form $w_1 w_2 \cdots w_{n+1}$. There exist no i, j, k with $1 \leq i < j < k \leq n+1$ such that $w_i > w_j > w_k$.

(ii) Let a be a reduced word for w. For all $i \in [n]$, between any two instances of s_i in a, the letters s_{i-1} and s_{i+1} occur. In particular, s_1 and s_n each occur at most once.

(iii) Any two reduced words for w are related by commutations.

The following is immediate by characterization (iii):

Proposition 4.7.6. *Let w be a 321-avoiding element of A_n, let $s_1 s_2 \cdots s_k$ be a reduced word for w, and let $s_{i_1} s_{i_2} \cdots s_{i_j}$ be a subword with $|i_m - i_{m+1}| = 1$ for every $m \in [j-1]$. Then $s_{i_1} s_{i_2} \cdots s_{i_j}$ occurs as a subword of every reduced word for w.* □

The Coxeter group B_n is the group of *signed permutations*. Signed permutations are permutations x of $\pm[n] := [-n, n] - \{0\}$ subject to the condition that $x(a) = -x(-a)$ for each $a \in [n]$. The *one-line notation* for a signed permutation x is $x_1 x_2 \cdots x_n$, meaning $i \mapsto x_i$ for each i. The generators S are the transpositions $s_i := (i \;\; i{+}1)$ for each $i \in [n-1]$, and the transposition $s_0 := (-1 \;\; 1)$. The group B_n is a Coxeter group with $m(s_i, s_j) = 3$ for $i, j \in [n-1]$ with $|i-j| = 1$, $m(s_0, s_1) = 4$ and $m(s_i, s_j) = 2$ for $|i-j| > 1$. Multiplying a permutation on the right by a generator s_i for $i \in [n-1]$ has the effect of switching the entry x_i with the entry x_{i+1}. Multiplying on the left by s_i switches the entry i with the entry $i+1$. Multiplying on the right by s_0 reverses the sign of x_1 and multiplying on the left by s_0 changes 1 to -1 and vice-versa.

Elements of $B_n^{s_i}$ for $i \in [n-1]$ are signed permutations whose one-line notation is positive and increasing in positions 1 to i, and increasing but not necessarily positive in positions $i+1$ to n. Elements of $B_n^{s_0}$ are signed permutations whose one-line notation is increasing everywhere. Bruhat comparisons in $B_n^{s_i}$ can be made by dual entrywise comparison of the entries from $i+1$ to n. The *signed tableau criterion* [8] characterizes Bruhat order on B_n as follows: Let $x \in B_n$ have one-line notation $x_1 x_2 \cdots x_n$, and form a tableau with rows $\bar{T}_a(x)$ for each $a \in [n]$, such that $\bar{T}_a(x) = (\bar{T}_{a,1}, \bar{T}_{a,2}, \ldots, \bar{T}_{a,a})$ is the increasing rearrangement of $\{x_i : i \in [n+1-a, n]\}$.

Proposition 4.7.7. *$x \leq y$ if and only if $\bar{T}_{a,b}(x) \geq \bar{T}_{a,b}(y)$ for every $1 \leq b \leq a \leq n$.* □

The tableau criterion for B_n associates to each signed permutation in B_n a *signed monotone triangle* of size n: A tableau of staircase shape, with n rows and n columns, with entries from $\pm[n]$, with $+i$ and $-i$ never occurring in the same signed triangle. Also, the rows are required to be strictly increasing, columns weakly decreasing and elements weakly increasing in the Southeast ($\searrow$) direction. The signed permutations are exactly the signed monotone triangles such that for every $1 \leq b \leq a < n$, either $T_{a,b} = T_{a+1,b}$ or $T_{a,b} = T_{a+1,b+1}$. The tableau criterion for B_n states that Bruhat order is dual to the restriction of componentwise order on the signed monotone triangles associated to signed permutations.

The anti-automorphism $w \mapsto w_0 w$ takes a signed permutation $x = x_1 x_2 \cdots x_n$, to $y = y_1 y_2 \cdots y_n$ according to $y_i = -x_i$. The corresponding operation on tableaux is to replace every entry a by $-a$ and to reverse the order of entries within the rows.

The two-tableaux criterion in type A was related to the symmetry $w \mapsto w_0 w w_0$. Since $w \mapsto w_0 w w_0$ is the identity on B_n, one might not expect to find a great improvement over the signed tableau criterion. And indeed, the order dimension of B_n is not much lower than the upper bound given by the signed tableau criterion.

4.8 Order dimension of Bruhat order on type A

In [39], Lascoux and Schützenberger show that Bruhat order on Coxeter groups of type A or B is dissective and identify the join-irreducibles. In this section and the following section, we review their results for types A and B, determine the partial order induced on $\mathrm{Irr}(A_n)$ and determine the widths of $\mathrm{Irr}(A_n)$ and $\mathrm{Irr}(B_n)$. We then apply Theorem 4.1.2 to determine the order dimension of the Bruhat orders on A_n and B_n and of all one-sided quotients.

For any $1 \leq b \leq a \leq n$, and $b \leq c \leq n-a+b+1$, define $J_{a,b,c}$ to be the componentwise smallest monotone triangle such that the a, b entry is $\geq c$. It is easily checked that the permutation

$$1 \text{ —— } (b-1)c \text{ —— } (c+a-b)b \text{ —— } (c-1)(c+a-b+1) \text{ —— } (n+1)$$

gives rise to a tableau which fits the description of $J_{a,b,c}$. If $b = c$, then $J_{a,b,c}$ is the tableau associated to the identity permutation. A monotone triangle T is the join of $\{J_{a,b,T_{a,b}} : 1 \leq b \leq a \leq n\}$. Thus

$$\mathrm{Irr}(A_n) \subseteq \{J_{a,b,c} : 1 \leq b \leq a \leq n, b < c \leq n-a+b+1\}.$$

Given $1 \leq b \leq a \leq n$, and $b < c \leq n - a + b + 1$, define $M_{a,b,c}$ to be the componentwise largest monotone triangle whose a, b entry is $< c$. The tableau $M_{a,b,c}$ can be found by applying the anti-symmetry $w \mapsto w_0 w$ to $J_{a,a-b+1,n+3-c}$. It is the permutation

$$(n+1) \text{ --- } (n-a+b+2)(c-1) \text{ --- } (c-b)(n-a+b+1) \text{ --- } c(c-b-1) \text{ --- } 1.$$

Thus each $J_{a,b,c}$ for $1 \leq b \leq a \leq n$, and $b < c \leq n - a + b + 1$ is a dissector with $\beta(J_{a,b,c}) = M_{a,b,c}$. As a result,

$$\mathrm{Irr}(A_n) = \{J_{a,b,c} : 1 \leq b \leq a \leq n, b < c \leq n - a + b + 1\},$$

A_n is dissective, and by Theorem 4.1.6, the MacNeille completion of A_n is the distributive lattice of monotone triangles [39]. Thus the order-dimension of A_n under the strong order is equal to the order dimension of the lattice of monotone triangles of the same size.

The partial order induced on $\mathrm{Irr}(A_n)$ is studied using the subword definition of Bruhat order. It is convenient to fix a particular word for the maximal element of A_n, and also to write the word as an array:

$$w_0 = \begin{array}{llllll} s_1 & & & & & \\ s_2 & s_1 & & & & \\ s_3 & s_2 & s_1 & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ s_n & s_{n-1} & \cdot & \cdot\ \cdot & s_2 & s_1 \end{array}$$

Reading the array in the standard order for reading English text gives a word $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots$. Elements of A_n are in bijection with left-justified subsets of the array. It is easily seen [32, 39] that $\mathrm{Irr}(A_n)$ consists of left-justified rectangles in the array. That is, an element is join-irreducible if and only if its left-justified form is:

$$\begin{array}{ccccc} s_j & s_{j-1} & s_{j-2} & \cdot\ \cdot\ \cdot & s_{j-i+1} \\ s_{j+1} & s_j & s_{j-1} & \cdot\ \cdot\ \cdot & s_{j-i+2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ s_{j+k-1} & s_{j+k-2} & s_{j+k-3} & \cdot\ \cdot\ \cdot & s_{j+k-i} \end{array}$$

for some $1 \leq i \leq j \leq n$, and $k \leq n - j + 1$. Counting such rectangles shows that there are $\binom{n+2}{3}$ join-irreducibles in A_n. Refer to these rectangles and the corresponding irreducibles by the triples (i, j, k). A triple (i, j, k) corresponds to the tableau $J_{j-i+k,j-i+1,j+1}$.

Example 4.8.1. The monotone triangle $J_{5,3,5}$ in A_7 and the corresponding rectangle $(2, 4, 3)$ are shown below. The corresponding permutation is 12567348.

$$\begin{array}{cccccccc} 1 & & & & & & & \\ 1 & 2 & & & & & & \\ 1 & 2 & 5 & & & & & \\ 1 & 2 & 5 & 6 & & & & \\ 1 & 2 & 5 & 6 & 7 & & & \\ 1 & 2 & 3 & 5 & 6 & 7 & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \qquad \begin{array}{cc} s_4 & s_3 \\ s_5 & s_4 \\ s_6 & s_5 \end{array}$$

A criterion is given in [39] for deciding whether a given permutation is above a given join-irreducible. For now, we are interested in an easy criterion for comparing two join-irreducibles. A subrectangle of (i, j, k) is a rectangle that can be obtained by deleting columns from the left and/or right of (i, j, k) and/or deleting rows from the top and/or bottom of (i, j, k).

Proposition 4.8.2. *Join-irreducibles $u = (i, j, k)$ and $v = (i', j', k')$ in A_n have $u \leq v$ if and only if (i, j, k) is a sub-rectangle of (i', j', k').*

Proof. The "if" direction follows immediately from the subword property.

Suppose $u \leq v$. The subword property requires that some reduced word for the rectangle (i, j, k) be a subword of the rectangle-word for (i', j', k'). It is easily verified that (i, j, k) stands for a 321-avoiding element of A_n. Notice also that the rectangle form for (i, j, k) has a subword

$$s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1} s_{j-i+2} \cdots s_{j-i_k-1} s_{j-i+k}$$

which satisfies the hypotheses of Proposition 4.7.6. Therefore, the subword of (i', j', k') which is a reduced word for (i, j, k) must itself contain the same subword. For the word given by (i', j', k') to contain the letters $s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1}$ in that order, in particular, it must contain the letter s_{j-i+1} somewhere after an occurrence of s_j. Thus there is a row in the rectangle for v containing $s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1}$. For the letters $s_{j-i+2} s_{j-i+1} s_{j-i+2} \cdots s_{j-i_k-1} s_{j-i+k}$ to occur afterwards, there must be at least $k - 1$ more rows. $\square$

There are four types of covers in $\mathrm{Irr}(A_n)$, corresponding to striking the left or right column or the top or bottom row from a rectangle. Of course, a column can only be deleted if there is more than one column present, and similarly for rows. Thus a rectangle (i, j, k) covers the following rectangles:

$$\begin{array}{rl} (i-1, j, k) & \text{if } i > 1, \\ (i-1, j-1, k) & \text{if } i > 1, \\ (i, j+1, k-1) & \text{if } k > 1, \\ (i, j, k-1) & \text{if } k > j. \end{array}$$

The minimal rectangles are $(1, j, 1)$ for $j \in [n]$, and the maximal elements are $(i, i, n-i+1)$ for $i \in [n]$. Also, $\mathrm{Irr}(A_n)$ is ranked by $r(i, j, k) = i + k - 1$, with the lowest rank being 1 and the highest rank being n—a departure from the usual convention that minimal elements have rank zero. A diagram of $\mathrm{Irr}(A_4)$ is given in Figure 4.1.

The rank number $R_r(\mathrm{Irr}(A_n))$ is determined by counting the number of ways to choose i, j and k subject to the constraints:

$$1 \leq i \leq j \leq n,\ 1 \leq k \leq n-j+1 \text{ and } i+k-1=r.$$

Necessarily, $i \in [r]$ (otherwise, $i+k-1 \geq i > r$) and j must be chosen so that $1 \leq i \leq j \leq n$, and $1 \leq r-i+1 \leq n-j+1$, or equivalently, so that $i \leq j \leq n+i-r$. Thus,

$$R_r(\mathrm{Irr}(A_n)) = r(n-r+1). \tag{4.9}$$

The maximum rank number is $R_{\left\lfloor \frac{n+1}{2} \right\rfloor} = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$. Thus, in order to verify the statement about A_n in Theorem 4.1.1, it only remains only to prove Theorem 4.1.3, which asserts that $\mathrm{Irr}(A_n)$ has a symmetric chain decomposition. In particular, Theorem 4.1.3 implies that $\mathrm{Irr}(A_n)$ is Sperner, so its width is equal to its maximum rank number.

Proof of Theorem 4.1.3. Restrict to a weaker order on $\mathrm{Irr}(A_n)$, by allowing a rectangle (i,j,k) to cover only

$$\begin{aligned} &(i-1,j,k) \quad \text{if } i>1 \text{ or,} \\ &(i,j,k-1) \quad \text{if } k>j. \end{aligned}$$

In other words, restrict the covers by only allowing the rightmost column or the bottom row to be deleted. Call this weaker order $\mathrm{Irr}'(A_n)$. Then $\mathrm{Irr}'(A_n)$ consists of n disjoint components, each of which is isomorphic to a product of chains: For each $j \in [n]$, there is a maximal element $(j,j,n-j+1)$ in $\mathrm{Irr}(A_n)$ and the interval below $(j,j,n-j+1)$ in $\mathrm{Irr}'(A_n)$ is isomorphic to the product of chains $[j] \times [n-j+1]$. Thus $\mathrm{Irr}'(A_n)$ has a symmetric chain decomposition. Since $\mathrm{Irr}'(A_n)$ is ranked with the same rank function as $\mathrm{Irr}(A_n)$, the symmetric chain decomposition is inherited by $\mathrm{Irr}(A_n)$. $\square$

Incidentally, the covers in $\mathrm{Irr}'(A_n)$ are exactly the covers in $\mathrm{Irr}(A_n)$ which are order relations in the right weak Bruhat order.

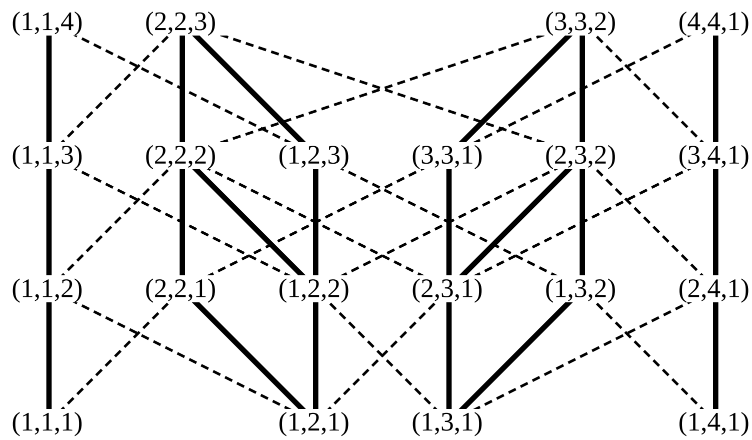
By Corollary 4.7.3, any one-sided or two-sided quotient of A_n is a dissective poset. The same symmetric chain decomposition proves the following:

Theorem 4.8.3. *The order dimension of a one-sided quotient A_n^J of A_n, is:*

$$\dim(A_n^J) = \sum_{s_i \in (S-J)} \min(i, n-i+1).$$

Proof. The symmetric chain decomposition given for $\mathrm{Irr}(A_n)$ arises from symmetric chain decompositions of the components of $\mathrm{Irr}'(A_n)$. Each such component is $\mathrm{Irr}(A_n) \cap {}^{s_j}W$ for some j. Thus the same symmetric chain decomposition can be given to $\mathrm{Irr}({}^JW)$ for any J. The quotients ${}^JW$ and W^J are isomorphic by the map which takes w to w^{-1}. $\square$

Figure 4.1: A diagram of $\mathrm{Irr}(A_4)$. The dotted lines are covers which are not in $\mathrm{Irr}'(A_4)$.



4.9 Order dimension of Bruhat order on type B

In this section, we determine the width of $\mathrm{Irr}(B_n)$, and thus the order dimension of B_n. The calculation of the width of $\mathrm{Irr}(B_n)$ is complicated by the fact that $\mathrm{Irr}(B_n)$ is not graded. For example, in $\mathrm{Irr}(B_3)$ there are maximal chains of lengths 5 and 6. The width is calculated by exhibiting a chain decomposition and an antichain of the same size. We describe the ordering on $\mathrm{Irr}(B_n)$ only as far as is necessary. For convenience, the Coxeter generators $S = \{s_0, s_1, \ldots, s_{n-1}\}$ will be referred to as the integers in $[0, n-1]$, and complementation within S will be denoted by the symbol ^{c}. Specifically, it is proven that

Theorem 4.9.1. *The order dimension of a one-sided quotient B_n^J of B_n, is:*

$$\dim(B_n^J) = \begin{cases} \sum_{j \in J^c} n - j & \text{if } 0 \in J \\ \lfloor \frac{n+1}{2} \rfloor + \sum_{j \in J^*} n - j & \text{if } 0 \notin J, 1 \in J \\ n + \sum_{j \in J^*} n - j & \text{if } \{0,1\} \cap J = \emptyset, \end{cases}$$

where $J^* = J^c \cap [2, n-1]$.

A characterization of the join-irreducibles of B_n as reduced words is in [32, 39]. We will characterize them by tableaux in a manner similar to what was done in type A. Represent the join-irreducibles by tableaux, as follows: For any $1 \leq b \leq a \leq n$, and $-n+b-1 \leq c \leq n-a+b$, define $J_{a,b,c}$ to be the componentwise largest signed monotone triangle such that the (a, b) entry is $\leq c$. Notice that we specify the **largest**, as opposed to the smallest as in type A. If $c = n - a + b$, this is the tableau associated to the identity permutation. One can show that the $J_{a,b,c}$ exist by constructing them. Start by placing an entry c in the (a, b) position, and then move one position to the left and place the largest entry allowed by the definition, and so on to first column. Then move to the right from position (a, b) placing the largest possible entries. Construct row $a-1$ from row a by deleting the smallest entry, and so on to row 1. Construct row $a+1$ from row a by placing the largest possible remaining entry in the appropriate place in the row, and so on to row n. The reader is invited to verify, by inspection of the tableau constructed on the following pages, that the construction indeed produces a tableau answering the description of $J_{a,b,c}$. Notice that some join-irreducibles have more than one representation as a $J_{a,b,c}$.

On the following pages are diagrams of the $J_{a,b,c}$ which arise from this construction. The $J_{a,b,c}$ are represented with the following convention: In the tableaux, lines drawn from one entry i to another entry i represent a line of entries i. Lines from i to j represent a line of entries changing monotonically by ones. Entries not on these lines are filled in in the natural way with adjacent entries differing by at most one. Four cases must be considered, depending on the value of c in relation to a, b and n. The tableau and corresponding signed permutations for each case are shown as Figures 4.2, 4.3, 4.4 and 4.5.

Any signed monotone triangle T is the join of $\{J_{a,b,T_{a,b}} : 1 \leq b \leq a \leq n\}$. The signed monotone triangle $J_{a,b,n-a+b}$ the unique minimal element of B_n, and so is not join-irreducible. Thus the join-irreducibles are contained in the set

$$\{J_{a,b,c} : 1 \leq b \leq a \leq n, -n+b-1 \leq c < n-a+b\}.$$

In fact, each such $J_{a,b,c}$ is join-irreducible, because it is a dissector. Specifically, there exists $M_{a,b,c}$, the componentwise smallest signed monotone triangle whose (a,b) entry is greater than c. It is easily checked that $M_{a,b,c}$ is the image of $J_{a,a-b+1,-c-1}$ under the anti-automorphism $w \mapsto w_0 w$.

The set of signed monotone triangles is not closed under entrywise meet. For example, when $n=4$, The entrywise meet of $J_{3,1,-2}$ and $J_{3,3,3}$ has $-2, 2, 3$ as its third row. In particular, the MacNeille completion of B_n is larger than the set of all signed monotone triangles.

The following proposition is the key to the upper bound on width(Irr(B_n)). It is easily verified by inspecting the signed permutations given in Figures 4.2, 4.3, 4.4 and 4.5. Specifically, one checks that the only decrease in values in the one-line notation occurs between positions $(n-a)$ and $(n-a+1)$. One also checks the inverse signed permutation to verify that the only decrease occurs between positions c and $c-1$ for $c>0$ and between positions $-c-1$ and $-c$ for $c<0$.

Proposition 4.9.2. *If $c \geq 1$, a join-irreducible $J_{a,b,c}$ is in $\mathrm{Irr}(B_n) \cap {}^{c}B_n^{n-a}$. If $c \leq -1$, a join-irreducible $J_{a,b,c}$ is in $\mathrm{Irr}(B_n) \cap {}^{-c-1}B_n^{n-a}$.* □

We now give a decomposition of $\mathrm{Irr}(B_n)$ into chains. For each fixed pair (a,b) with $1 \leq b \leq a \leq n-2$, the set

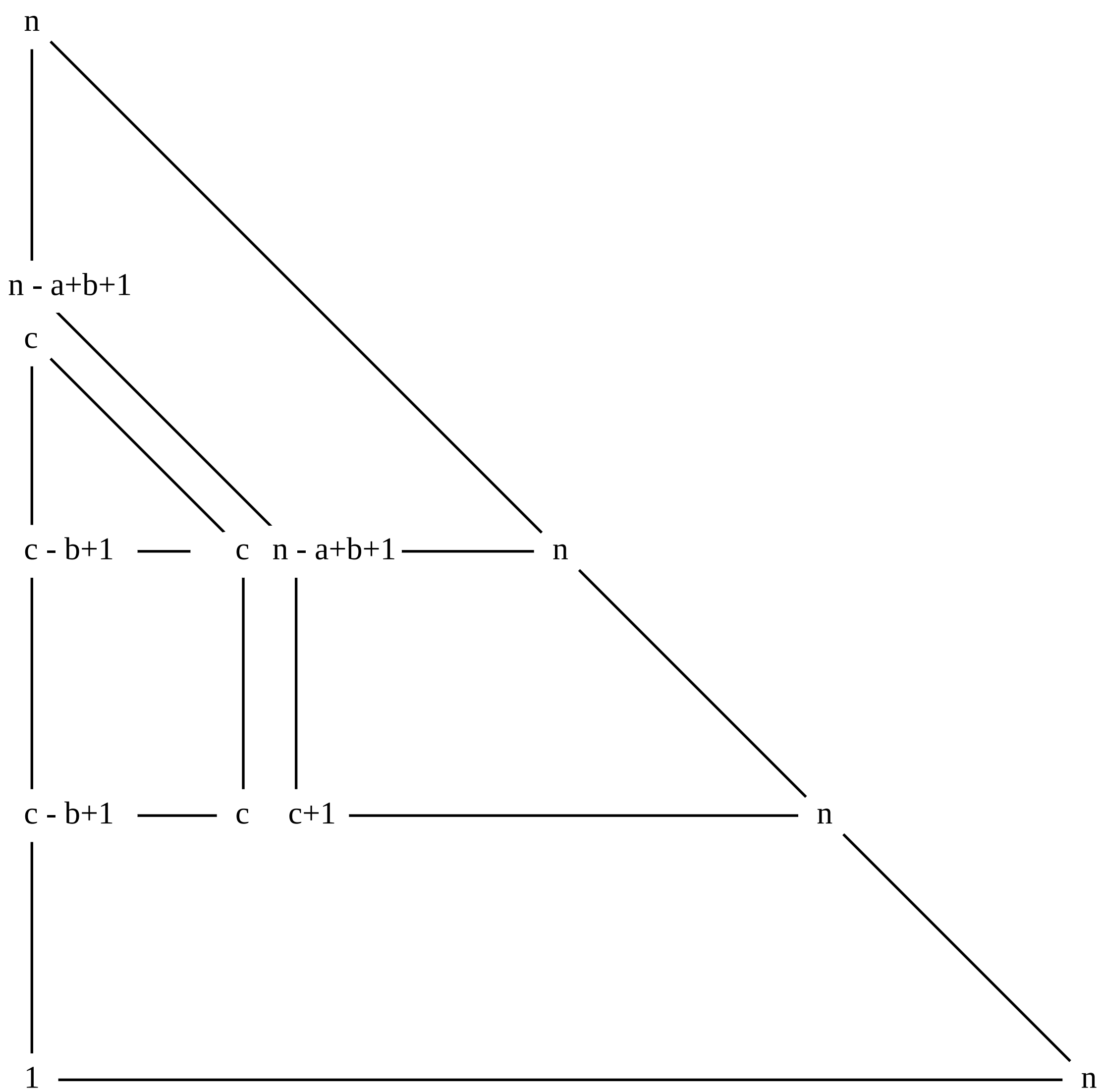
$$\{J_{a,b,c} : -n+b-1 \leq c < n-a+b\}$$

is a chain in $\mathrm{Irr}(B_n)$. By Propositions 4.7.2 and 4.9.2, the remaining join-irreducibles are $\mathrm{Irr}(B_n^{[2,n]})$. However, it is easier to give a chain decomposition of $\mathrm{Irr}({}^{[2,n]}B_n)$, because by Propositions 4.7.2 and 4.9.2, $\mathrm{Irr}({}^{[2,n]}B_n)$ is exactly the set of join-irreducibles with $c \in \{-2,-1,1\}$. Any $J_{a,b,1}$ has at least a "-2" in position $(a, b-1)$, if $(a, b-1)$ is a valid position in the tableau, so $J_{a,b,-2}$ is componentwise greater than or equal to $J_{a,b+1,1}$. Thus for any $d \in [n]$,

$$J_{d,1,1} \geq J_{d,1,-1} \geq J_{d,1,-2} \geq J_{d,2,1} \geq J_{d,2,-1} \geq J_{d,2,-2} \geq J_{d,3,1} \geq \cdots.$$

Figure 4.2: $J_{a,b,c}$ for type B

Case 1: $c \geq b$



The associated signed permutation, where $_*$ marks the unique descent:

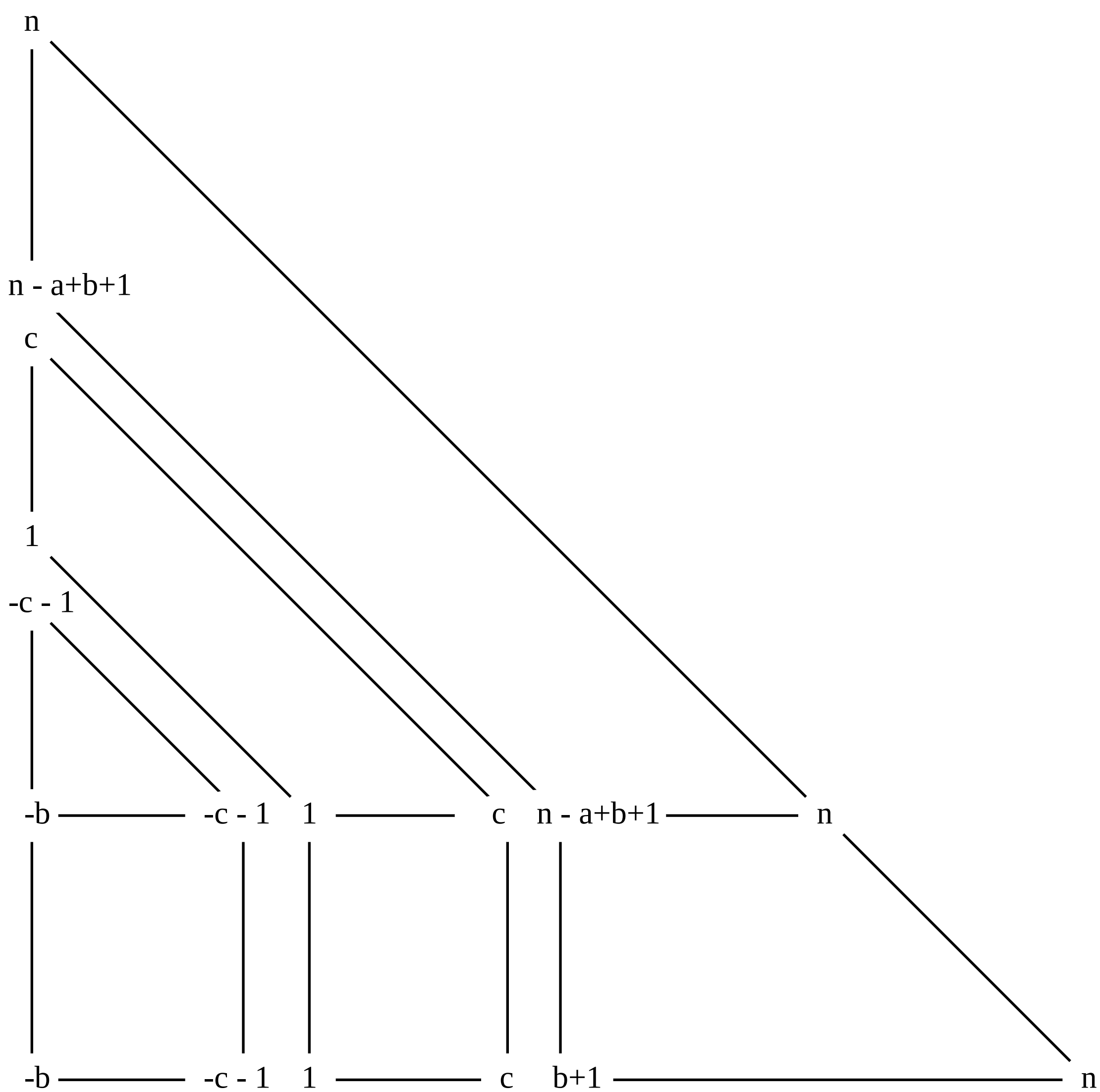
$$1 \text{ —— } (c-b)(c+1) \text{ —— } (n-a+b)_*(c-b+1) \text{ —— } c(n-a+b+1) \text{ —— } n$$

The inverse of this signed permutation:

$$1 \text{ —— } (c-b)(n-a+1) \text{ —— } (n-a+b+1)_*(c-b+1) \text{ —— } (n-a)(n-a+b+1) \text{ —— } n$$

Figure 4.3: $J_{a,b,c}$ for type B

Case 2: $b \geq c \geq 1$



The associated signed permutation, where $_*$ marks the unique descent:

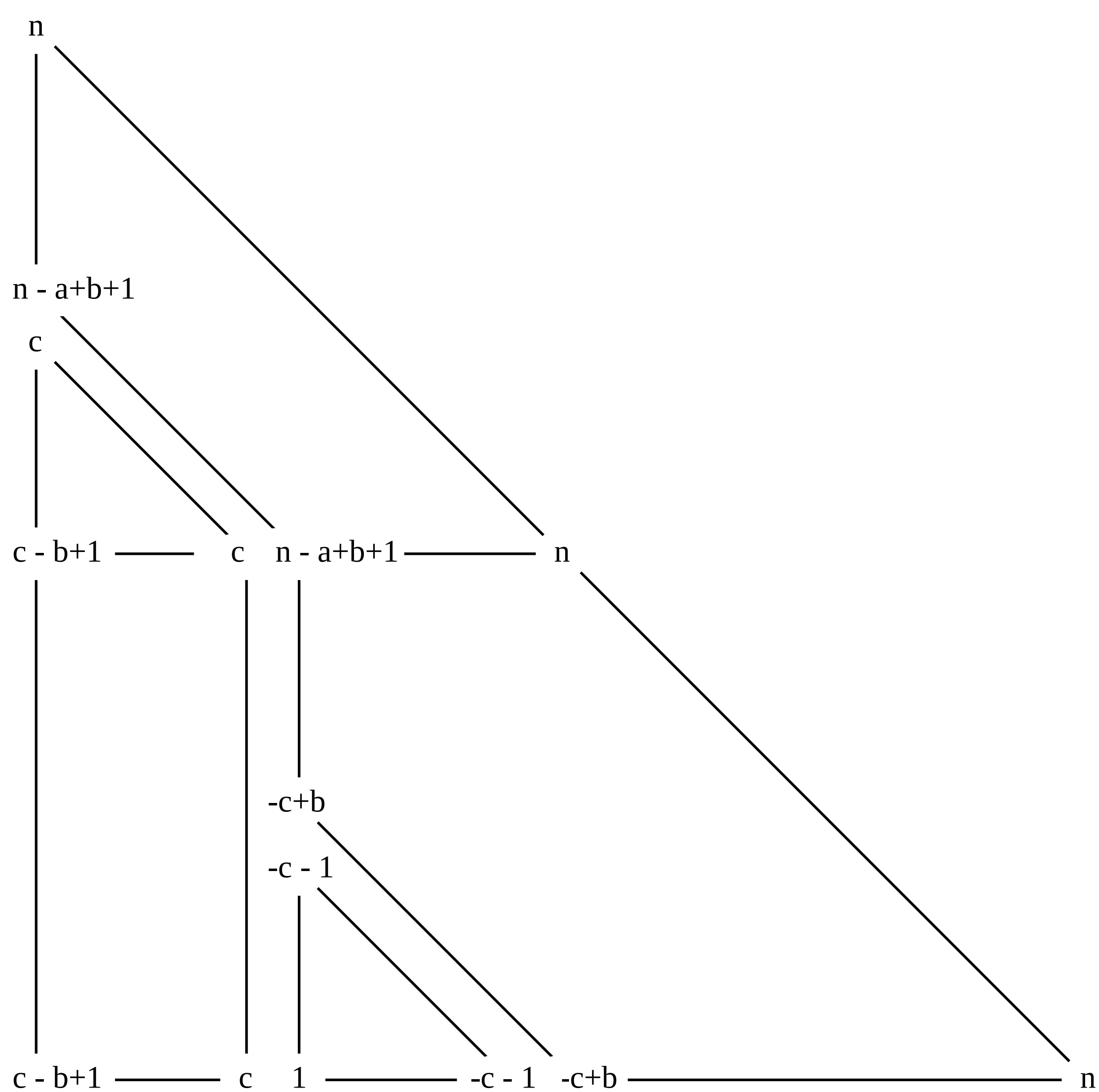
$$(b+1) \text{ —— } (n-a+b)_*(-b) \text{ —— } (-c-1)1 \text{ —— } c(n-a+b+1) \text{ —— } n$$

The inverse of this signed permutation:

$$(n-a+b-c+1) \text{ —— } (n-a+b)_*(-n+a-b+c) \text{ —— } (-n+a-1)1 \text{ —— } (n-a)(n-a+b+1) \text{ —— } n$$

Figure 4.4: $J_{a,b,c}$ for type B

Case 3: $-1 \geq c \geq a-n-1$



The associated signed permutation, where $_*$ marks the unique descent:

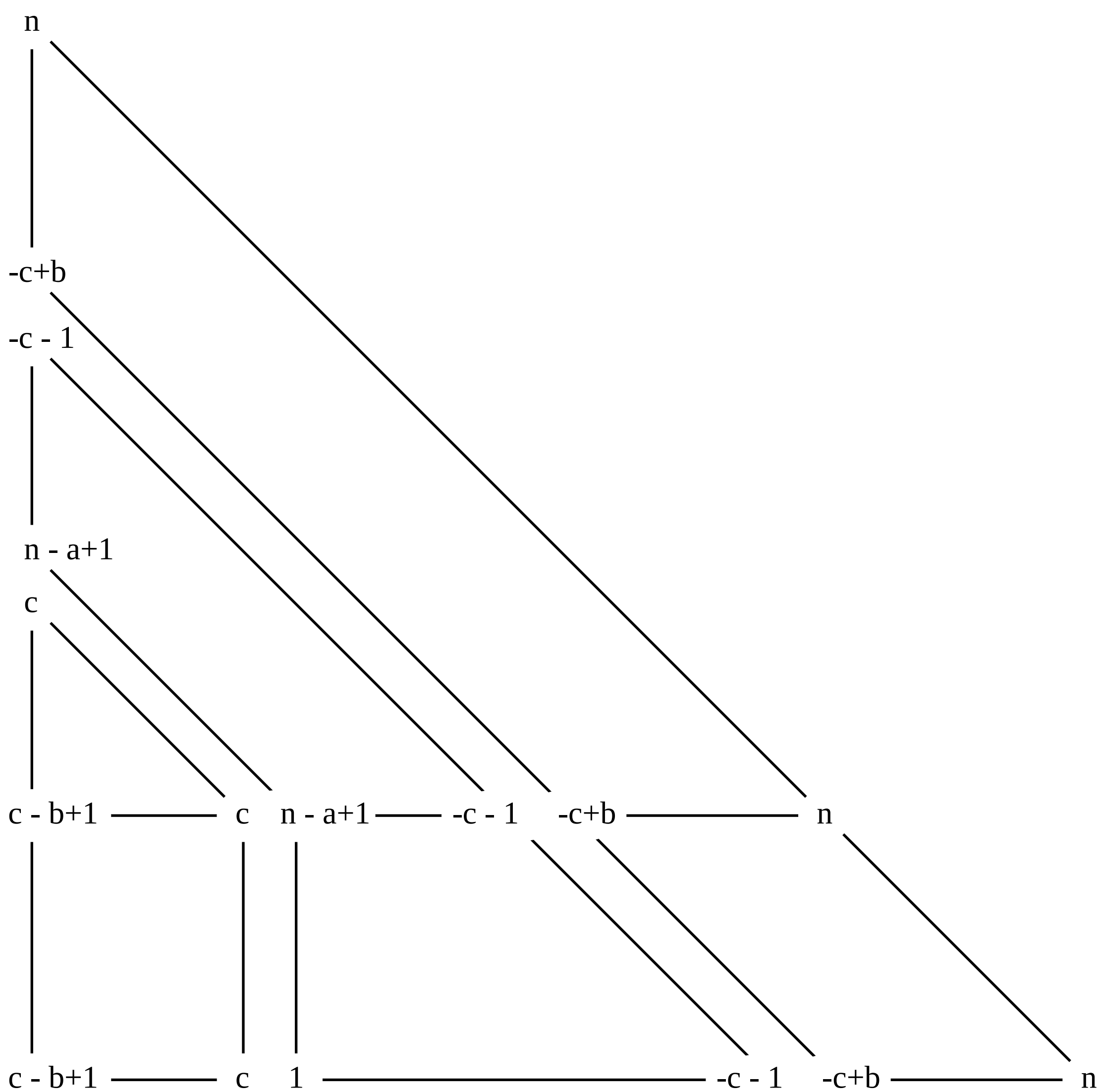
$$1 \text{ —— } (-c-1)(-c+b) \text{ —— } (n-a+b)_*(c-b+1) \text{ —— } c(n-a+b+1) \text{ —— } n$$

The inverse of this signed permutation:

$$1 \text{ —— } (-c-1)_*(-n+a-b) \text{ —— } (-n+a-1)(-c) \text{ —— } (n-a)(n-a+b+1) \text{ —— } n$$

Figure 4.5: $J_{a,b,c}$ for type B

Case 4: $a - n - 2 \geq c$



The associated signed permutation, where $_*$ marks the unique descent:

$$1 \text{ — } (n-a)_*(c-b+1) \text{ — } c(n-a+1) \text{ — } (-c-1)(-c+b) \text{ — } n$$

The inverse of this signed permutation:

$$1 \text{ — } (n-a)(n-a+b+1) \text{ — } (-c+b-1)_*(-n+a-b) \text{ — } (-n+a-1)(-c+b) \text{ — } n$$

with the chain continuing through every $J_{a,b,c}$ with $a = d$ and $c \in \{-2, -1, 1\}$. In this way $\mathrm{Irr}({}^{[2,n]}B_n)$ is decomposed into n chains, and the inverse map $w \mapsto w^{-1}$ gives a decomposition of $\mathrm{Irr}(B_n^{[2,n]})$ into n chains. The total number of chains is $\binom{n-1}{2} + n = \binom{n}{2} + 1$.

The best way to describe an antichain in $\mathrm{Irr}(B_n)$ is as a tableau, as in the examples below. The tableau records the number c in position (a, b) for each $J_{a,b,c}$ in the antichain. Any antichain can be written as such a tableau, but obviously not every tableau is an antichain. The tableau below represents a largest antichain in $\mathrm{Irr}(B_6)$. Some of the elements in the antichain are $J_{1,1,1}$, $J_{2,1,-2}$, $J_{2,2,2}$, $J_{3,1,-3}$, $J_{3,2,1}$, $J_{3,3,3}$, etc.

$$
\begin{array}{cccccc}
1 & & & & & \\
-2 & 2 & & & & \\
-3 & 1 & 3 & & & \\
-4 & -2 & 2 & 4 & & \\
-5 & -3 & 1 & 3 & 5 & \\
\bullet & \bullet & -1 & \bullet & \bullet & \bullet
\end{array}
$$

In general, let A be the tableau with:

$$
\begin{aligned}
A(a,b) &= 2b - a - 2 \text{ for } 1 \le b \le \frac{a}{2} \le \frac{n-1}{2}, \\
A(a,b) &= 2b - a \text{ for } \frac{1}{2} \le \frac{a}{2} < b \le a \le n-1, \\
A(n, \lfloor \frac{n+1}{2} \rfloor) &= -1,
\end{aligned}
$$

and the other $n-1$ entries blank. The tableau A has $\binom{n+1}{2} - (n-1) = \binom{n}{2} + 1$ non-blank entries. Thus the following proposition completes the calculation of the width of $\mathrm{Irr}(B_n)$.

Proposition 4.9.3. *A is an antichain of size $\binom{n}{2} + 1$ in $\mathrm{Irr}(B_n)$.*

Proof. This is equivalent to saying that for any non-blank entry c in position (a, b) in A, every other non-blank entry in A strictly less than the corresponding entry in $J_{a,b,c}$. The proof breaks into cases according to the entries of A, and relies on the specific forms of the $J_{a,b,c}$. The argument is simplified by considering "regions" in the $J_{a,b,c}$.

A region whose entries are constant along northwest–southeast diagonals and decrease by exactly one when moving one position to the left will be called a *diagonal region*. For each diagonal region, it is sufficient to show that the entry in the lowest, rightmost position in the region is greater than the corresponding entry in A. This is because the entries in A decrease in the northwest direction and decrease by at least two when moving one position to the left. A region whose entries are constant along columns and decrease by exactly one when moving one position to the left will be called a *vertical region*. For each vertical region, it is sufficient to check the topmost, rightmost entry, because entries in A decrease down columns and decrease by at least two when moving one position to the left. These special corner of regions will be called *active corners*. To avoid confusion, we apply this simplification only to the part of A with $a < n$ and treat the entry in row n separately.

In each of the four cases of the $J_{a,b,c}$, position (a,b) is the active corner of both a diagonal region and a vertical region. For these regions, the fact that the active corner has entry c is enough to guarantee that the other entries in the region are strictly greater than the corresponding entries in A.

First, consider position $(n, \lfloor \frac{n+1}{2} \rfloor)$ in A, with entry -1. The corresponding monotone triangle falls into Case 3, and there are two regions, both diagonal, that intersect the non-blank entries of A. One has active corner $(n-1, n-1)$ with entry n, as compared to the entry $n-1$ in A, and the other has active corner $(n-1, \lfloor \frac{n+1}{2} \rfloor - 1)$, with entry -1 as opposed to -2 or -3 in A.

The positive entries c in A occur in positions (a,b) with $c \leq b$, so all of these fall into Case 2. The only positions one needs to check are position $(n-1, n-1)$ with entry n greater than the corresponding entry $n-1$ in A, and position $(a, b-c)$ with entry $-c-1$. The entry $A(a,b)$ is c and entries in A decrease by at least 2 when moving one position to the left, with a decrease of 4 when moving left from a positive entry to a negative entry. Thus $A(a, b-c)$ is at most $c - 2c - 2 = -c - 2$. The entry of $J_{a,b,c}$ at position $(n, \lfloor \frac{n+1}{2} \rfloor)$ is greater than -1, because no negative entries occur in $J_{a,b,c}$ except in the first $b-c$ columns, and $b - c = b - (2b - a) = a - b < \frac{a}{2} \leq \frac{n-1}{2}$.

The positions (a,b) with a negative entry c in A fall into Cases 3 and 4. In either Case 3 or Case 4, the argument is the same. The entry of $J_{a,b,c}$ at position $(n, \lfloor \frac{n+1}{2} \rfloor)$ is greater than -1, because no negative entries occur in $J_{a,b,c}$ except weakly to the left of column b, but $b \leq \frac{n-1}{2}$. Position $(n-1, n-1)$ passes as before, and position $(n-1, b-c-2)$ of $J_{a,b,c}$ has entry $-c-1$. But entries in A increase by 2 when moving one position to the right, except that the increase when moving right from a negative entry to a positive entry is 4. So the entry at position $(a, b-c-2)$ in A is $c + 2(-c-2) + 2 = -c - 2$ and the entry at $(n-1, b-c-2)$ is no greater. This argument fails when $c = -2$, because $b - c - 2 = b$ and there is no extra increase due to moving right from a negative entry to a positive entry. However, in this case, the region is a single entry in row n. $\square$

The argument given above also accomplishes much of the work for determining the order dimension of quotients of B_n, and the rest Theorem 4.9.1 is proven by similar methods. It is necessary to prove the following proposition on join-irreducibles $J_{a,b,c}$ with $a = n$.

Proposition 4.9.4. *If $c > 0$, then $J_{n,b,c} = J_{n,b-c,-c-1}$ (and one of these exists and is non-trivial if and only if the other exists and is non-trivial).*

Proof. Existence and non-triviality of $J_{n,b,c}$ is exactly the inequalities $1 \leq b \leq n$ and $-n + b - 1 \leq c \leq b - 1$. Existence and non-triviality of $J_{n,b-c,-c-1}$ is exactly the inequalities $1 \leq b - c \leq n$ and $-n + b - c - 1 \leq -c - 1 \leq b - c - 1$. This second set of inequalities is equivalent to $0 \leq b \leq n$ and $-n + b \leq c \leq b - 1$. However, when $c > 0$, both $-n + b \leq c$ and $-n + b - 1 \leq c$ are redundant inequalities. Also, $0 < c \leq b - 1$ implies $b \geq 1$, and thus the two sets of inequalities are equivalent.

Now we must show that in any signed monotone triangle T, we have $T(n,b) \leq c$ if and only if $T(n, b-c) \leq -c - 1$. Suppose $T(n,b) \leq c$ and suppose for the sake of a contradiction that $T(n, b-c) \geq -c$. Since rows are strictly increasing, the $c+1$ entries in row n columns $b-c$ through b are all between $-c$ and c. But since no entry can occur along with its negative, there are only

c possible entries for the $c+1$ positions. This contradiction shows that $T(n, b-c) \leq -c-1$. Conversely, suppose that $T(n, b-c) \leq -c-1$ and suppose for the sake of a contradiction that $T(n, b) \geq c+1$. Then the $n-c+1$ positions in row n, columns 1 through $b-c$ and columns b through n have have entries in $\pm[c+1, n]$. There are only $n-c$ possible entries for the $n-c+1$ positions, and this contradiction finishes the proof. $\square$

We conclude this section with the:

Proof of Theorem 4.9.1. For $0 \in J$, an antichain of the correct size is obtained by restricting A to the rows $a \in J^c$. A chain decomposition of the same size is obtained by restricting the chains which arise from the tableau criterion. When $\{0,1\} \cap J = \emptyset$, an antichain of the correct size is again obtained by restricting A to the rows a with $a \in J^c$, and a chain decomposition is obtained by restricting the chain decomposition given above for $\mathrm{Irr}(B_n)$.

The case $0 \notin J$, $1 \in J$ requires a new construction. The same chain decomposition of $\mathrm{Irr}(B_n^{\{0,1\}})$ can be used, and a chain decomposition of $\mathrm{Irr}(B_n^{[n-1]})$ is as follows: By Proposition 4.9.2, $\mathrm{Irr}(B_n^{[n-1]})$ consists of irreducibles $J_{a,b,c}$ with $a = n$, and by Proposition 4.9.4 we may as well assume that $c < 0$. For each i with $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$, the following inequalities hold whenever the $J_{a,b,c}$'s mentioned exist:

$$J_{n,1,-2i+1} \geq J_{n,1,-2i} \geq J_{n,2,-2i+1} \geq J_{n,2,-2i} \geq \cdots J_{n,n,-2i+1} \geq J_{n,n,-2i},$$

This is because $J_{n,b,c}$ has at most $c-1$ in the $(n, b-1)$ position. Thus $\mathrm{Irr}(B_n^{[n-1]})$ can be decomposed into $\lfloor \frac{n+1}{2} \rfloor$ chains.

To conclude the proof, we need to exhibit an antichain in $\mathrm{Irr}(B_n^{\{1\}})$ whose intersection with each $\mathrm{Irr}(B_n^j)$ has size $n-j$ for $j \geq 2$ and $\lfloor \frac{n+1}{2} \rfloor$ for $j = 0$. This is done by representing the antichain as a tableau with $\lfloor \frac{n+1}{2} \rfloor$ entries in row n, no entries in row $n-1$ and all other rows full.

Let A' be the tableau with:

$$\begin{aligned} A'(a,b) &= 2b-a-2 \text{ for } 1 \leq b \leq \frac{a}{2} \leq \frac{n-2}{2}, \\ A'(a,b) &= 2b-a \text{ for } \frac{1}{2} \leq \frac{a}{2} < b \leq a \leq n-2, \\ A'(n,b) &= 2b-n-2 \text{ for } 1 \leq b \leq \frac{n+1}{2}, \end{aligned}$$

with row $n-1$ and the other entries in row n blank. For example, if $n = 6$, this is:

$$\begin{array}{cccccc} 1 \\ -2 & 2 \\ -3 & 1 & 3 \\ -4 & -2 & 2 & 4 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ -6 & -4 & -2 & \bullet & \bullet & \bullet \end{array}$$

The first $n-2$ rows of A' are identical to the first $n-2$ rows of A, so we need only check that the signed monotone triangles represented by the entries in row n are incomparable with each other and with the entries in the other rows.

When n is odd the position $(n, \frac{n+1}{2})$ in A', with entry -1, gives a J which has larger entries than rows 1 through $n-2$ of A', as was already checked in the previous proof. Now, consider position (n,b) for $1 \le b < \frac{n+1}{2}$, with entry $2b-n-2$, which corresponds to $J_{n,b,2b-n-2}$ in Case 4. The entries directly to the right of (n,b) in A' are negative, but every entry to the right of (n,b) in $J_{n,b,2b-n-2}$ is positive. We must also compare entries in A' with entries in $J_{n,b,2b-n-2}$ in positions $(n-2,b-2)$, $(n-2,n-b-1)$ and $(n-2,n-2)$. In position $(n-2,b-2)$, $J_{n,b,2b-n-2}$ has $2b-n-2$ while A' has $2(b-2)-(n-2)-2 = 2b-n-4$. In position $(n-2,n-b-1)$, $J_{n,b,2b-n-2}$ has $n+2-2b-1 = n-2b+1$, while A' has at most $2(n-b-1)-(n-2) = n-2b$. In position $(n-2,n-2)$, $J_{n,b,2b-n-2}$ has n, while A' has $n-2$.

The positive entries c in rows 1 through $n-2$ of A' occur in positions (a,b) with $c = 2b-a \le b$, so all of these fall into Case 2. It suffices to check the entries at positions (n,b) and $(n,b-c)$. In position (n,b), $J_{a,b,c}$ has $c>0$, while A' has a negative or blank entry. In position $(n,b-c)$, $J_{a,b,c}$ has $-c-1 = a-2b-1$, while A' is blank or $2(b-c)-n-2 = 2a-2b-n-2$. Since $a < n+1$, $2a-2b-n-2 < a-2b-1$.

The positions (a,b) with a negative entry c in A' fall into Cases 3 and 4. Columns to the right of b in $J_{a,b,c}$ have positive entries, so we need only check that A' has an entry less than c at position (n,b). But $c = 2b-a-2$, while the entry in A' is $2b-n-2$. □

4.10 Order dimension of Bruhat order on other types

Type H

Type H contains two groups H_3 and H_4, the symmetry groups of the icosahedron and the 600-cell respectively. As a Coxeter group, H_3 has generators s_1, s_2, s_3 and with $m(s_1,s_2)=5$, $m(s_2,s_3)=3$ and $m(s_1,s_3)=2$. The Coxeter group H_4 has generators s_1,s_2,s_3,s_4 and with $m(s_1,s_2)=5$, $m(s_2,s_3)=3$, $m(s_3,s_4)=3$ and $m(s,t)=2$ for all other pairs.

Since H_3 and H_4 are dissective, their order dimensions can be calculated as the width of their subposet of irreducibles. We used the `GAP` [53] program `brbase` [32] and the package `CHEVIE` [31] to find $\mathrm{Irr}(H_3)$ and $\mathrm{Irr}(H_4)$. Then we used a program written in Prolog to calculate widths, obtaining the results in Theorem 4.1.1. The width of $\mathrm{Irr}(H_3)$ was easy to calculate, but calculating the width of $\mathrm{Irr}(H_4)$ by brute force proved to be too much even for a very fast computer. (After two weeks, the computer managed to find an antichain of size 24). However, notice that in type B,

$$\mathrm{width}(\mathrm{Irr}(B_n)) = \mathrm{width}(\mathrm{Irr}(B_n^{[2,n-1]})) + \sum_{i\in[2,n-1]} \mathrm{width}(\mathrm{Irr}(B_n^i)).$$

The computer verified that for H_3,

$$\mathrm{width}(\mathrm{Irr}(H_3)) = \mathrm{width}(\mathrm{Irr}(H_3^{\{3\}})) + \mathrm{width}(\mathrm{Irr}(H_3^{\{1,2\}})) = 4+2.$$

Thus one might hope to make the calculation smaller by calculating the analogous quotients for H_4.

The computer found:

$$\begin{aligned}
\mathrm{width}(\mathrm{Irr}(H_4^{\{1,2,3\}})) &= 3 \\
\mathrm{width}(\mathrm{Irr}(H_4^{\{1,2,4\}})) &= 7 \\
\mathrm{width}(\mathrm{Irr}(H_4^{\{3,4\}})) &= 15.
\end{aligned}$$

Since $\mathrm{Irr}(H_4) = \mathrm{Irr}(H_4^{\{1,2,3\}}) \cup \mathrm{Irr}(H_4^{\{1,2,4\}}) \cup \mathrm{Irr}(H_4^{\{3,4\}})$, these calculations give an upper bound $\mathrm{width}(\mathrm{Irr}(H_4)) \leq 25$. Then the computer was able to find an antichain of size 25 in $\mathrm{Irr}(H_4)$ by considering unions with one antichain from each of the three quotients. Thus,

$$\mathrm{width}(\mathrm{Irr}(H_4)) = \mathrm{width}(\mathrm{Irr}(H_4^{\{3,4\}})) + \mathrm{width}(\mathrm{Irr}(H_4^{\{1,2,3\}})) + \mathrm{width}(\mathrm{Irr}(H_4^{\{1,2,4\}})) = 25.$$

Type I

Type I consists of the dihedral groups, each with two generators s and t. The m in $I_2(m)$ is $m(s,t)$. In $I_2(m)$, every element except the identity and w_0 is a dissector. The group $I_2(6)$ is also called G_2. The width of $I_2(m)$ itself is 2, making the order dimension calculation trivial.

Types D, E and F

Theorem 4.1.5 enables the computer to set bounds on the order dimensions of some groups of types D, E and F. We used `brbase` to find the bigrassmannians of several groups, and then used a Prolog program to find Irr and Dis and calculate widths. The results are:

$$\begin{aligned}
6 \leq &\ \dim(D_4) \leq 9 \\
10 \leq &\ \dim(D_5) \leq 14 \\
14 \leq &\ \dim(D_6) \leq 22 \\
18 \leq &\ \dim(D_7)
\end{aligned}$$

$$\begin{aligned}
14 \leq &\ \dim(E_6) \leq 26 \\
18 \leq &\ \dim(E_7)
\end{aligned}$$

$$10 \leq \dim(F_4) \leq 12.$$

Further width calculations were beyond the ability of a fast computer to perform even for run times of about two weeks. Also, it appears that the bounds obtained in this way continue to worsen with increasing numbers of generators, because the number of join-irreducibles appears to grow more rapidly than the number of dissectors.

4.11 Further questions

Bruhat order and alternating sign matrices

1. Determine the order dimension of

   (a) The Bruhat order on the other finite Coxeter groups. Give a uniform treatment, independent of the classification.

   (b) Intervals in the Bruhat order.

   (c) Two-sided quotients of types A, B and H. These are all dissective by Corollary 4.7.3. All of the two-sided quotients by maximal parabolic subgroups in type A are one-dimensional.

   (d) The weak Bruhat order on a finite Coxeter group. The lower bound given by Theorem 4.1.5 is just the number of generators, and the upper bound of Theorem 4.1.5 appears to be much larger than the known upper bound—the number of reflections (cf. [8, Exercise 3.2]). For type A, the order dimension of the weak Bruhat order was determined by Flath [29] in 1993. Recently, using methods from the study of hyperplane arrangements, the author determined the order dimension of the weak Bruhat order for types A and B [47]. For both of these types, the order dimension is equal to the number of generators.

2. Some ideals in $\mathrm{Irr}(A_n)$ correspond to elements of A_n and some do not. Give a purely order-theoretic characterization of the order ideals which are elements of A_n. A necessary but not sufficient condition on an ideal $I \subseteq \mathrm{Irr}(A_n)$ is that $\max(I) \cup \min(I^c)$ is an antichain.

3. What statistic on permutations is $|I_x|$, the number of join-irreducibles below $x \in A_n$? The distributions for $n = 1, 2$ and 3 are:

$$1 + q,$$

$$1 + 2q + 2q^3 + q^4,$$

$$1 + 3q + q^2 + 4q^3 + 2q^4 + 2q^5 + 2q^6 + 4q^7 + q^8 + 3q^9 + q^{10}.$$

4. The MacNeille completion $L(A_n)$ is the componentwise order on monotone triangles, which biject with alternating sign matrices. Does $L(B_n)$ have any connection to alternating sign matrices with symmetry conditions? Since the set of signed monotone triangles is not closed under entrywise meet, $L(B_n)$ is larger than the set of all signed monotone triangles. Okada [44] has type-B and type-C Weyl denominator formulas which are expressed in terms of alternating sign matrices with half-turn symmetry. However, the numbers of such matrices do not agree with the number of order ideals in $\mathrm{Irr}(B_n)$.

5. Study the lattice quotients induced on the componentwise order on monotone triangles by quotients of the Bruhat order on A_n, as in Theorem 4.1.7. This is not as simple as one might guess. For example, the congruence on A_3 obtained from the subgroup $\{1, s_1\}$ induces a lattice

congruence on the lattice of monotone triangle which has 15 congruence classes, rather than the 12 one would expect.

Order dimension

1. Is there any condition weaker than requiring that a poset P be dissective, that would imply $\dim(P) = \mathrm{width}(\mathrm{Irr}(P))$? Is there any condition that would imply $\dim(P) = \mathrm{width}(\mathrm{Dis}(P))$?

2. Develop efficient algorithms for finding the critical complex of a poset. If this can be done, covering sets, and thus order dimension, can in principle be determined or approximated by linear programming [34].

Lattice properties for posets

1. Find other naturally occurring examples of dissective posets. One possibility is the "Bruhat order" on complete matchings on $[2n]$ defined in [17], which is dissective at least for $n \leq 3$.

2. What is the right generalization of modularity to posets [38]? In other words, is there a simple order-theoretic condition on P that is equivalent to requiring that $L(P)$ be modular?

3. Find other naturally occurring examples of congruences and quotients of non-lattices.

4. The lattice $\mathrm{Con}(L)$ of congruences of a lattice L has been studied extensively [33]. Similar questions can be asked about the poset $\mathrm{Con}(P)$ of congruences of a poset. Also, how is $\mathrm{Con}(P)$ related to $\mathrm{Con}(L(P))$?

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