AN AMAZING PRIME HEURISTIC

CHRIS K. CALDWELL

1. Introduction

The record for the largest known twin prime is constantly changing. For example, in October of 2000, David Underbakke found the record primes:

$$83475759 \cdot 2^{64955} \pm 1.$$

The very next day Giovanni La Barbera found the new record primes:

$$1693965 \cdot 2^{66443} \pm 1.$$

The fact that the size of these records are close is no coincidence! Before we seek a record like this, we usually try to estimate how long the search might take, and use this information to determine our search parameters. To do this we need to know how common twin primes are.

It has been conjectured that the number of twin primes less than or equal to N is asymptotic to

$$2C_2 \int_2^N \frac{dx}{(\log x)^2} \sim \frac{2C_2 N}{(\log N)^2}$$

where C_2, called the twin prime constant, is approximately 0.6601618. Using this we can estimate how many numbers we will need to try before we find a prime. In the case of Underbakke and La Barbera, they were both using the same sieving software (NewPGen[1] by Paul Jobling) and the same primality proving software (Proth.exe[2] by Yves Gallot) on similar hardware–so of course they choose similar ranges to search. But where does this conjecture come from?

In this chapter we will discuss a general method to form conjectures similar to the twin prime conjecture above. We will then apply it to a number of different forms of primes such as Sophie Germain primes, primes in arithmetic progressions, primorial primes and even the Goldbach conjecture. In each case we will compute the relevant constants (e.g., the twin prime constant), then compare the conjectures to the results of computer searches. A few of these results are new–but our main goal is to illustrate an important technique in heuristic prime number theory and apply it in a consistent way to a wide variety of problems.

1.1. The key heuristic.

A heuristic is an educated guess. We often use them to give a rough idea of how long a program might run–to estimate how long we might need to wait in the brush before a large prime comes wandering by. The key to all the results in this chapter is the following:

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[1]Available from http://www.utm.edu/research/primes/programs/NewPGen/

[2]Available from http://www.utm.edu/research/primes/programs/gallot/

> The prime number theorem states that the number of primes less than n is asymptotic to $1/\log n$. So if we choose a random integer m from the interval $[1, n]$, then the probability that m is prime is asymptotic to $1/\log n$.

Let us begin by applting this to a few simple examples.

First, as n increases, $1/\log n$ decreases, so it seemed reasonable to Hardy and Littlewood to conjecture that there are more primes in the set $\{1, 2, 3, \ldots, k\}$ than in $\{n+1, n+2, n+3, \ldots, n+k\}$. In other words, Hardy and Littlewood [21] conjectured.

Conjecture 1.1. *For sufficiently large integers n, $\pi(k) \geq \pi(n+k) - \pi(n)$.*

They made this conjecture on the basis of very little numerical evidence saying "An examination of the primes less than 200 suggests forcibly that $\rho(x) \leq \pi(x) (x \geq 2)$" (page 54). (Here $\rho(x) = \limsup_{n\to\infty} \pi(n+x) - \pi(x)$.) By 1961 Schinzel [37] had verified this to $k = 146$ and by 1974 Selfridge *et. al.* [19] had verified it to 500. As reasonable sounding as this conjecture is, we will give a strong argument against it in just a moment.

Second, suppose the Fermat numbers $F_n = 2^{2^n} + 1$ behaved as random numbers.[3] Then the probability that F_n is prime should be about $1/\log(F_n) \sim 1/(2^n \log 2)$. So the expected number of such primes would be on the order of $\sum_{n=0}^{\infty} 1/(2^n \log 2) = 2/\log 2$. This is the argument Hardy and Wright used when they presented the following conjecture [22, pp. 15, 19]:

Conjecture 1.2. *There are finitely many Fermat primes.*

The same argument, when applied to the Mersenne numbers, Woodall numbers, or Cullen numbers suggest that there are infinitely many primes of each of these forms. But it would also imply there are infinitely many primes of the form $3^n - 1$, even though all but one of these are composite. So we must be a more careful than just adding up the terms $1/\log n$. We will illustrate how this might be done in the case of polynomials in the next section.

As a final example we point out that in 1904, Dickson conjectured the following:

Conjecture 1.3. *Suppose a_i and b_i are integers with $a_i > 1$. If there is no prime which divides each of*

$$\{b_1 x + a_1, b_2 x + a_2, \ldots, b_n x + a_n\}$$

for every x, then there are infinitely many integers values of x for which these terms are simultaneously prime.

How do we arrive at this conclusion? By our heuristic, for each x the number $b_i x + a_i$ should be prime with a probability $1/\log N$. If the probabilities that each term is prime are independent, then the whole set should be prime with probability $1/(\log N)^n$. They are not likely to be independent, so we expect something on the order of $C/(\log N)^n$ for some constant C (a function of the coefficients a_i and b_i).

In the following section we will sharpen Dickson's conjecture in to a precise form like that of the twin prime conjecture above.

[3]There are reasons not to assume this such as the Fermat numbers are pairwise relatively prime.

1.2. **A warning about heuristics.** What (if any) value do such estimates have?

They may have a great deal of value for those searching for records and predicting computer run times, but mathematically they have very little value. They are just (educated) guesses, not proven statements, so not "real mathematics." Hardy and Littlewood wrote: "*Probability* is not a notion of pure mathematics, but of philosophy or physics" [21, pg 37 footnote 4]. They even felt it necessary to apologize for their heuristic work:

> Here we are unable (with or without Hypothesis R) to offer anything approaching a rigorous proof. What our method yields is a *formula*, and one which seems to stand the test of comparison with the facts. In this concluding section we propose to state a number of further formulae of the same kind. Our apology for doing so must be (1) that no similar formulae have been suggested before, and that the process by which they are deduced has at least a certain algebraic interest, and (2) that it seems to us very desirable that (in default of proof) the formula should be checked, and that we hope that some of the many mathematicians interested in the computative side of the theory of numbers may find them worthy of their attention. ([21, pg 40])

When commenting on this Bach and Shallit wrote:

> Clearly, no one can mistake these probabilistic arguments for rigorous mathematics and remain in a state of grace.[4] Nevertheless, they are useful in making educated guesses as to how number-theoretic functions should "behave." ([2, p. 248])

Why this negative attitude? Mathematics seeks proof. It requires results without doubt or dependence on unnecessary assumptions. And to be blunt, sometimes heuristics fail! Not only that, they sometimes fail for even the most cautious of users. In fact we have already given an example (perhaps you noticed?)

Hardy and Littlewood, like Dickson, conjectured that if there is no prime which divides each of the following terms for every x, then they are simultaneously primes infinitely often:

$$\{x + a_1, x + a_2, x + a_3, x + a_4, x + a_5, \ldots, x + a_k\}$$

[21, Conjecture X]. This is a special case of Dickson's Conjecture is sometimes called **the prime k-tuple conjecture**. We have also seen that they conjectured $\pi(k) \geq \pi(n + k) - \pi(n)$ (conjecture 1.1). But in 1972, Douglas Hensley and Ian Richards proved that one of these two conjectures is false [24, 25, 35]!

Perhaps the easiest way to see the conflict between these conjectures is to consider the following set of polynomials found by Tony Forbes [16]:

$$\begin{aligned}\{&n - p_{24049}, n - p_{24043}, \ldots, n - p_{1223}, n - p_{1217},\\ &n + p_{1217}, n + p_{1223}, \ldots, n + p_{24043}, n + p_{24049}\}\end{aligned}$$

where p_n is the n-th prime. By Hardy and Littlewood's first conjecture there are infinitely many integers n so that each of these 4954 terms are prime. But the

[4]Compare this quote to John von Neumann's remark in 1951 "Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin." [27, p. 1]

width of this interval is just 48098 and $\pi(48098) < 4954$. So this contradicts the second conjecture.

If one of these conjectures is wrong, which is it? Most mathematicians feel it is the second conjecture that $\pi(k) \geq \pi(n+k) - \pi(n)$ which is wrong. The prime k-tuple conjecture receives wide support (and use!) Hensley and Richards predicted however

> Now we come to the second problem mentioned at the beginning of this section: namely the smallest number $x_1 + y_1$ for which $\pi(x_1 + y_1) > \pi(x_1) + \pi(y_1)$. We suspect, even assuming the k-tuples hypothesis (B) is eventually proved constructively, that the value of $x_1 + y_1$ will never be found; and moreover that no pair x, y satisfying $\pi(x+y) > \pi(x) + \pi(y)$ will ever be computed. ([19, p. 385])

What can we conclude from this example of clashing conjectures? First that heuristics should be applied only with care. Next they should then be carefully tested. Even after great care and testing you should not stake to much on their predictions, so read this chapter with the usual bargain hunter's mottoes in mind: "buyer beware" and "your mileage may vary."

1.3. **Read the masters.** [[Chris, write something here. Here is a start.]]

The great mathematician Abel once wrote "It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils." Good advice, but this is an area short of masters.

Hardy and Littlewood's third paper on their circle method [21] is one of the first key papers in this area. In this paper they made the first real step toward the proving the Goldbach conjecture, then gave more than a dozen conjectures on the distribution of primes. Their method is far more complicated than what we present here–but it laid the basis for actual proofs of some related results.

The approach we take here may have first been laid out by Cherwell and Wright [11, section 3] (building on earlier work by Cherwell [10], Stäckel [43], and of course Hardy and Littlewood). The same approach was taken by Bateman and Horn [3] (see also [4]).

Many authors give similar arguments including Brent, Shanks [40, 42, 41] and P'olya [32].

There are also a couple excellent "students" we should mention. Ribenboim included an entire chapter on heuristics in his text "the new book of prime number records" [34]. Riesel also develops much of this material in his fine book [36]. See also Schroeder [39, 38].

And as enthusiastic students we also add our little mark. Again, most of what we present here was first done by others. Our only claim to fame is a persistent unrelenting application of one simple idea to a wide variety of problems. Enough talk, let's get started!

2. The prototypical example: sets of polynomials

2.1. **Sets of polynomials.** We regularly look for integers that make a set of (one or more) polynomials simultaneously prime. For example, simultaneous prime values of $\{n, n+2\}$ are twin primes, of $\{n, 2n+1\}$ are Sophie Germain primes, and of

$\{n, 2n+1, 4n+3\}$ are Cunningham chains of length three. So this is an interesting test case for our heuristic.

What might stop a set of integer valued polynomials from being simultaneously prime? The same things that keep a single polynomial from being prime: It might factor like $9x^2 - 1$, or there might be a prime which divides every value of the polynomial such as 3 and $x^3 - x + 9$. So before we go much further we need a few restrictions on our polynomials $f_1(x)$, $f_2(x)$, ..., $f_k(x)$. We require that

- the polynomials $f_i(x)$ are irreducible, integer valued, and have positive leading terms, and
- the degree d_i of $f_i(x)$ is greater than zero $(i = 1, 2, ..., k)$.

If we could treat the values of these polynomials at n as independent random variables, then by our key heuristic, the probability that they would be simultaneously prime at n would be

$$\prod_{i=1}^{k} \frac{1}{\log\ f_i(n)} \sim \frac{1}{d_1 d_2 ... d_k (\log n)^k}. \tag{2.1}$$

So the number of values of n with $0 < n \leq N$ which yield primes would be primes approximately

$$\frac{1}{d_1 d_2 ... d_k} \int_2^N \frac{dx}{(\log x)^k} \sim \frac{N}{d_1 d_2 ... d_k (\log N)^k}.$$

However, the polynomials are unlikely to behave both randomly and independently. For example, $\{n, n+2\}$ are either both odd or both even; and the second of $\{n, 2n+1\}$ is never divisible by two. To attempt to adjust for this, for each prime p, we will multiply by the ratio of the probability that p does not divide the product of the polynomials at n, to the probability that p does not divides one of k random integers. In other words, we will adjust by multiplying by a measure of how far from independently random the values are.

To find this *adjustment factor*, we start with the following definition:

Definition 2.1. $w(p)$ is the number of solutions to $f_1(x)f_2(x) \cdot ... \cdot f_k(x) \equiv 0 \pmod{p}$ with x in $\{0, 1, 2, ..., p-1\}$.

For each prime then, we need to multiply by

$$\frac{\frac{p-w(p)}{p}}{(\frac{p-1}{p})^k} = \frac{1 - w(p)/p}{(1 - 1/p)^k}, \tag{2.2}$$

and our complete adjustment factor is found by taking the product over all primes p:

$$\prod_p \frac{1 - w(p)/p}{(1 - 1/p)^k}. \tag{2.3}$$

This gives us the following conjecture (see [3, 13]).

Conjecture 2.2 (Dickson's Conjecture)**.** *Let the irreducible polynomials $f_1(x)$, $f_2(x)$, ..., $f_k(x)$ be integer valued, have a positive leading term, and suppose $f_i(x)$ has degree $d_i > 0$ $(i = 1, 2, \ldots, k)$. The number of values of n with $0 < n \leq N$ which yield simultaneous primes is approximately*

$$\frac{1}{d_1 d_2 ... d_k} \prod_p \frac{1-w(p)/p}{(1-1/p)^k} \int_2^N \frac{dx}{(\log x)^k} \sim \frac{N}{d_1 d_2 ... d_k (\log N)^k} \prod_p \frac{1-w(p)/p}{(1-1/p)^k}. \tag{2.4}$$

The ratio on the right is sufficient if N is very large or we just need a rough estimate, but the integral usually gives a better estimate for small N. Sometimes we wish an even tighter estimate for relatively small N. Then we use the right side of equation 2.1 and write the integral in the conjecture above as

$$\prod_p \frac{1-w(p)/p}{(1-1/p)^k} \int_2^N \frac{dx}{\log f_1(x) \log f_2(x) \cdot \ldots \cdot \log f_k(x)} \tag{2.5}$$

3. Sequences of linear polynomials

Conjecture 2.2 gives us an approach to estimating the number of primes of several forms. In this section we will apply conjecture it to twin primes, Sophie Germain primes, primes of the form $n^2 + 1$, and several other forms of primes. In each case, we will compare the estimates in the conjecture to the actual numbers of such primes.

3.1. **Twin primes.** To find twin primes we can use the polynomials n and $n + 2$. Note that $w(2) = 1$, and $w(p) = 2$ for all odd primes p. With a little algebra, we see our adjustment factor 2.3 is

$$2 \prod_{p>2} 1 - \frac{1}{(p-1)^2} = 2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2}. \tag{3.1}$$

This product over odd primes is called the twin primes constant:

$$C_2 = 0.66016\ 18158\ 46869\ 57392\ 78121\ 10014\ 55577\ 84326\ ...$$

Gerhard Niklasch has computed C_2 to over 1000 decimal places using the methods of Moree [30].

In this case, conjecture 2.2 becomes:

Conjecture 3.1 (Twin prime conjecture)**.** *The expected number of twin primes* $\{p, p+2\}$ *with* $p \le N$ *is*

$$2C_2 \int_2^N \frac{dx}{(\log x)^2} \sim \frac{2C_2 N}{(\log N)^2}. \tag{3.2}$$

(This is [20, Conjecture ??].) For a different heuristic argument for the same result see [22, section 22.20].

In practice this seems to be a exceptionally good estimate (even for small N)–see Table 1. (The last few values in Table 1 were calculated by T. Nicely [31].[5].)

It has been proven by sieve methods, that if you replace the 2 in our estimate (3.2) for the number of twin primes with a 7, then you have a provable upper bound for N sufficiently large. Brun first took this approach in 1919 when he showed we could replace the 2 with a 100 and get an upper bound from some point N_0 onward [8]. There has been steady progress reducing the constant since Brun's article (and

[5]See also http://www.trnicely.net/counts.html

TABLE 1. Twin primes less than N

N	**actual number**	**predicted integral**	**ratio**
10^3	**35**	**46**	28
10^4	**205**	**214**	155
10^5	**1224**	**1249**	996
10^6	**8169**	**8248**	6917
10^7	**58980**	**58754**	50822
10^8	**440312**	**440368**	389107
10^9	**3424506**	**3425308**	3074426
10^{10}	**27412679**	**27411417**	24902848
10^{11}	**224376048**	**224368865**	205808661
10^{12}	**1870585220**	**1870559867**	1729364449
10^{13}	**15834664872**	**15834598305**	14735413063
10^{14}	**135780321665**	**135780264894**	127055347335
10^{15}	**1177209242304**	**1177208491861**	1106793247903

7 is not the current best possible value). Unfortunately there is no known way of changing this into a lower bound–as it has not yet been proven there are infinitely many twin primes.

3.2. **Prime pairs $\{n, n+2k\}$ and the Goldbach conjecture.** What if we replace the polynomials $\{n, n+2\}$ with $\{n, n+2k\}$? In this case $w(p) = 1$ if $p|2k$ and $w(p) = 2$ otherwise, so the adjustment factor 3.1 becomes

$$C_{2,k} = C_2 \prod_{p|k, p>2} \frac{p-1}{p-2}. \tag{3.3}$$

With this slight change, conjecture 2.2 now is

Conjecture 3.2 (Prime pairs conjecture)**.** *The expected number of prime pairs $\{p, p+2k\}$ with $p \le N$ is*

$$2C_{2,k} \int_2^N \frac{dx}{(\log x)^2} \sim \frac{2C_{2,k}N}{(\log N)^2}. \tag{3.4}$$

(This is [21, Conjecture B].)

For example, when searching for primes $\{n, n+210\}$ we expect to find (asymptotically) $\frac{2}{1}\frac{4}{3}\frac{6}{5} = 3.2$ times as many primes as we find twins. Table 2 shows that this is indeed the case.

Note that asymptotically equation 3.4 must also give the expected number of consecutive primes whose difference is k. This can be shown (and the values estimated more accurately for small N) using the inclusion-exclusion principle [7, 29]. From this it is conjectured that the most common gaps between primes $\le N$ is always either 4 or a primorial number $(2, 6, 30, 210, 2310, \ldots)$ [23].

"But wait–there is more" the old infomercial exclaimed "it dices, it slices..." Look at the prime pairs set this way: $\{n, 2k-n\}$. Now when both terms are prime we have found two primes which add to $2k$. Our adjustment factor is unchanged,

TABLE 2. Prime pairs $\{n, n+2k\}$ with $n \le N$

	$k=6$		$k=30$		$k=210$	
N	**actual**	**predicted**	**actual**	**predicted**	**actual**	**predicted**
10^3	**74**	**86**	**99**	**109**	**107**	**118**
10^4	**411**	**423**	**536**	**558**	**641**	**653**
10^5	**2447**	**2493**	**3329**	**3316**	**3928**	**3962**
10^6	**16386**	**16491**	**21990**	**21981**	**26178**	**26358**
10^7	**117207**	**117502**	**156517**	**156663**	**187731**	**187976**
10^8	**879980**	**880730**	**1173934**	**1174300**	**1409150**	**1409141**
10^9	**6849047**	**6850611**	**9136632**	**9134141**	**10958370**	**10960950**

so the number of ways of writing $2k$ as a sum of two primes, often denoted $G(2k)$, is approximately:

$$G(2k) \sim 2C_{2,k}N \int_2^N \frac{dx}{\log x \log(2k-x)}. \tag{3.5}$$

This is equivalent to the conjecture as given by Hardy and Littlewood [21, Conjecture A]:

Conjecture 3.3 (Extended Goldbach conjecture)**.** *The number of ways of writing $2k$ as a sum of two primes is asymptotic to*

$$2C_{2,k} \int_2^N \frac{dx}{(\log x)^2} \sim \frac{2C_{2,k}N}{(\log N)^2}. \tag{3.6}$$

Hardy and Littlewood suggest that for testing this against the actual results for small numbers, we follow Shah and Wilson and use $1/((\log N)^2 - \log N)$ instead of $1/(\log N)^2$.

3.3. **Primes in Arithmetic Progression.** The same reasoning could be applied to estimate the number of arithmetic progressions of primes with length k by seeking integers n and k such that each term of

$$\{n, n+d, n+2d, \ldots, n+(k-1)d\}$$

is prime. In this case $w(p) = 1$ if p divides d, and $w(p) = \min(p, k)$ otherwise. In particular, if we wish all of the terms to be primes we must have $p|d$ for all primes $p \le k$. When this is the case, for a fixed d we have

$$A_{k,d} = \prod_{p|d} \frac{1}{(1-1/p)^{k-1}} \prod_{p\nmid d} \frac{1-k/p}{(1-1/p)^k}. \tag{3.7}$$

We can rewrite these in terms of the Hardy-Littlewood constants

$$c_k = \prod_{p>k} \frac{1-k/p}{(1-1/p)^{k-1}} \tag{3.8}$$

as follows

$$A_{k,d} = c_k \prod_{p\le k} \frac{1}{(1-1/p)^{k-1}} \prod_{p>k, p|d} \frac{p-1}{p-k}.$$

Of course $A_{k,d} = 0$ if $k\#$ does not divide d.

It is possible to estimate c_k and $A_{k,k\#}$ to a half dozen significant digits using product above over the first several hundred million primes–but at the end of this section we will show a much better method. Table 3 contains approximations of the first of these constants.

TABLE 3. Adjustment factors $A_{k,k\#}$ for arithmetic sequences

k	$k\#$	$A_{k,k\#}$	k	$k\#$	$A_{k,k\#}$
2	2	1.32032363169374	11	2310	629913.461423349
3	6	5.71649719143844	12	2310	1135007.50238685
4	6	8.30236172647483	13	30030	45046656.1742087
5	30	81.0543595999686	13	30030	132128113.722194
6	30	138.388898492679	15	30030	320552424.308155
7	210	2590.65351840622	16	30030	527357440.662591
8	210	7130.47817586170	17	510510	23636723084.1607
9	210	16129.6476839631	18	510510	47093023670.0967
10	210	24548.2695388318	19	9699690	3153485401596.08

Once again we reformulate conjecture 2.2 for our specific case and this time find the following.

Conjecture 3.4. *The number of arithmetic progressions of primes with length k, common difference d, and beginning with a prime $p \leq N$ is*

$$A_{k,d} \int_2^N \frac{dx}{(\log x)^k} \sim \frac{A_{k,d} N}{(\log N)^k}. \tag{3.9}$$

To check this conjecture we counted the number of such arithmetic progressions with common differences 6, 30, 210 and 2310. The results (table 4) seem to support this estimate well.

TABLE 4. Primes in arithmetic progression, starting before 10^9

common	length $k=3$		length $k=4$		length $k=5$	
difference	actual	predicted	actual	predicted	actual	predicted
6	**758163**	**759591**	**56643**	**56772**	**0**	**0**
30	**1519360**	**1519170**	**227620**	**227074**	**28917**	**28687**
210	**2276278**	**2278725**	**452784**	**454118**	**85425**	**86037**
2310	**2847408**	**2848284**	**648337**	**648640**	**142698**	**143326**
	length $k=6$		length $k=7$		length $k=8$	
30	**2519**	**2555**	**0**	**0**	**0**	**0**
210	**15146**	**15315**	**2482**	**2515**	**353**	**370**
2310	**30339**	**30588**	**6154**	**6266**	**1149**	**1221**

This conjecture also includes some of the previous results as special cases. For example, when k is one, we are just counting primes, and as expected, $A_{1,d} = 1$. It is also easy to show that $A_{2,d} = 2C_{2,d}$ and $A_{2,2} = 2C_2$, so $A_{2,d}$ matches the values from the Prime Pairs Conjecture 3.2.

What if we do not fix the common difference? Instead we might ask how many arithmetic progressions of primes(with any common difference) there are all of

whose terms are less than x. Call this number $N_k(x)$. Grosswald [17] modified Hardy & Littlewoods' Conjecture X to conjecture:

Conjecture 3.5. *The number of arithmetic progressions of primes $N_k(N)$ with length k all of whose terms are less than N is*

$$N_k(x) \sim \frac{D_k N^2}{2(k-1)(\log N)^k} \tag{3.10}$$

where

$$D_k = \prod_{p \le k} \frac{1}{p} \left(\frac{p}{p-1}\right)^{k-1} \prod_{p > k} \left(\frac{p}{p-1}\right)^{k-1} \frac{p-k+1}{p}. \tag{3.11}$$

Grosswald was able to prove this result in the case $k = 3$ [18]. His article also included approximations of these constants D_k with five significant digits. Writing these in terms of the Hardy-Littlewood constants

$$D_k = c_{k-1} \prod_{p < k} \frac{1}{p} \left(\frac{p}{p-1}\right)^{k-1}$$

we have calculated these with 13 significant digits in Table 5.

TABLE 5. Adjustment factors D_k for arithmetic sequences

k	D_k	k	D_k
3	1.320323631694	12	1312.319711299
4	2.858248595719	13	2364.598963306
5	4.151180863237	14	7820.600030245
6	10.13179495000	15	22938.90863233
7	17.29861231159	16	55651.46255350
8	53.97194830013	17	91555.11122614
9	148.5516286638	18	256474.8598544
10	336.0343267492	19	510992.0103092
11	511.4222820590	20	1900972.584874

The longest known sequence of arithmetic primes (at the time this was written) was found in 1993 [33]: it begins with the prime 11410337850553 and continues with common difference 4609098694200. Ribenboim [34, p. 287] has a table of the first known occurrence of arithmetic sequences of primes of length k for $12 \le k \le 22$.

3.4. **Evaluating the adjustment factors.** In 1961 Wrench [45] evaluated the the twin prime constant with just forty-two decimal place accuracy. He clearly did not do this with the product from equation (3.1)! Just how do we calculate these adjustment factors (also called Hardy Littlewood constants and Artin type constants) with any desired accuracy? The key is to rewrite them in terms of the zeta-functions which are easy to evaluate [1, 6].

Let $P(s)$ be the prime zeta-function:

$$P(s) = \sum_p \frac{1}{p^s}.$$

We can rewrite this using the usual zeta-function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and the Möbius function $\mu(k)$ as follows (see [36, pg. 65]):

$$P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \zeta(ks).$$

To evaluate c_k we take the logarithm of equation 3.8 and find

$$\log\left(\prod_{p>k} \frac{1-k/p}{(1-1/p)^k}\right) = \sum_{p>k} \left(\log(1-k/p) - k\log(1-1/p)\right).$$

Using the McClaurin expansion for the log this is

$$-\sum_{p>k}\sum_{j=1}^{\infty} \frac{k^j-k}{jp^j} = -\sum_{j=2}^{\infty} \frac{k^j-k}{j} \sum_{p>k} p^j = -\sum_{j=2}^{\infty} \frac{k^j-k}{j}\left(P(j) - \sum_{p\le k} p^{-j}\right).$$

It is relatively easy to calculate the zeta-function (see [6, 36]), so we now have a relatively easy way to calculate c_k and $A_{k,d}$. This approach will easily get us the fifteen significant decimal place accuracy shown in table 3.

If we need more accuracy, then we could use the techniques found in Moree [30] which Niklasch[6] used to calculate many such constants with 1000 decimal places of accuracy. Moree's key result is that the product

$$C_{f,g}(n) = \prod_{p>p_n} \left(1 - \frac{f(p)}{g(p)}\right)$$

(where f and g are monic polynomials with integer coefficients satisfying $\deg(f) + 2 \le \deg(g)$ and p_n is the nth prime) can be written as

$$C_{f,g}(n) = \sum_{k=2}^{\infty} \zeta_n(k)^{-e_k}$$

where the exponents $-e_k$ are integers and $\zeta_n(s) = \zeta(s)\prod_{p\le p_n}(1-p_n^{-s})$ is the partial zeta function.

3.5. **Sophie Germain primes.** Recall that p is a Sophie Germain prime if $2p+1$ is also prime [46]. Therefore, we will use the polynomials n and $2n+1$. Again, $w(2) = 1$ and $w(p) = 2$ for all odd primes p; so again our adjustment factor is the twin primes constant C_2. This gives us exactly the same estimated number of primes as in (3.2). We can improve this estimate by not replacing $\log(2n+1)$ with $\log n$ (as we did in (2.1)). This gives us the following,

Conjecture 3.6. *The number of Sophie Germain primes p with $p \le N$ is approximately*

$$2C_2 \int_2^N \frac{dx}{\log x \log 2x} \sim \frac{2C_2 N}{(\log N)^2}$$

Again, this estimate (at least the integral) is surprisingly accurate for small values of N, see Table[7] 6.

[6] http://www.gn-50uma.de/alula/essays/Moree/Moree.en.shtml

[7] Chip Kerchner provided the last two entries in table 6. (Personal e-mail 25 May 1999.)

TABLE 6. Sophie Germain primes less than N

N	**actual number**	**predicted integral**	**ratio**
1,000	**37**	**39**	28
10,000	**190**	**195**	156
100,000	**1171**	**1166**	996
1,000,000	**7746**	**7811**	6917
10,000,000	**56032**	**56128**	50822
100,000,000	**423140**	**423295**	389107
1,000,000,000	**3308859**	**3307888**	3074425
10,000,000,000	**26569515**	**26568824**	24902848
100,000,000,000	**218116524**	**218116102**	205808662

3.6. **Cunningham chains.** Cunningham chains can be thought of as a generalization of Sophie Germain primes. If the terms of the sequence

$$\{p, 2p+1, 4p+3, \ldots, 2^{k-1}p + 2^{k-1} - 1\}$$

are all prime, then this sequence is called a Cunningham chain of length k. (Sophie Germain primes yield Cunningham chains of length two.) There is a second type of these chains, called Cunningham chains of the second kind, which are prime sequences of the form

$$\{p, 2p-1, 4p-3, \ldots, 2^{k-1}p - 2^{k-1} + 1\}.$$

For either of these forms it is easy to show that $w(2) = 1$, and that for odd primes p, $w(p) = \min(k, ord_p(2))$. So the resulting estimate is as follows.

Conjecture 3.7. *The number of Cunningham chains of length k beginning with primes p with $p \le N$ is approximately*

$$B_k \int_2^N \frac{dx}{\log x \log(2x) \ldots \log(2^{k-1}x)} \sim \frac{B_k N}{(\log N)^k}$$

where B_k is the product

$$B_k = 2^{k-1} \prod_{p>2} \frac{p^k - p^{k-1} min(k, ord_p(2))}{(p-1)^k}.$$

(This conjecture for $k = 2, 3$ and 4, can be found in [28].)

Note that $min(k, ord_p(2))$ is just k when $p > 2^k$, so we can again write these adjustment factors in terms of the Hardy-Littlewood constants:

$$B_k = 2^{k-1} c_k \prod_{k<p<2^k} \frac{p - min(k, ord_p(2))}{p-k} \prod_{2<p\le k} \frac{1 - min(k, ord_p(2))/p}{(1-1/p)^k}.$$

We then count the Cunningham Chains less than 10^9 as an example to test our conjecture. As one would expect the agreement is better for the lower k because these forms yield many more primes for this small choice of N.

[[[Chris: Something wrong with this table! Constants are off]]]

TABLE 7. Cunningham chains of length k starting before 10^9

length k	adjustment factor B_k	**actual number** **first kind**	**second kind**	**predicted** **integral**	**ratio**
2	1.320323631694	**3308859**	**3306171**	**3307888**	3074426
3	2.858248595719	**342414**	**341551**	**342313**	321163
4	5.534907817650	**30735**	**30962**	**30784**	30011
5	20.26358989999	**5072**	**5105**	**5092**	5302
6	71.96222721619	**531**	**494**	**797**	909
7	233.8784426339	**47**	**46**	**112**	142

3.7. **Primes of the form n^2+1.** If we use the single polynomial n^2+1, then $w(2)=1$, and $w(p)=1+(-1|p)$ for odd primes p. Here $(-1|p)$ is the Legendre symbol, so it is 1 if there is a solution to $n^2\equiv -1 \pmod p$, and -1 otherwise. Now the adjustment factor (after a little algebra) is

$$2\prod_{p>2} 1-\frac{(-1|p)}{(p-1)^2} = 1.3728134628...$$

Calling this constant C_+ we conjecture that the expected number of values of $n\le N$ yielding primes n^2+1 is

$$\frac{C_+}{2}\int_2^N \frac{dx}{\log x} \sim \frac{C_+}{2}\frac{N}{\log N}.$$

But this is not how we usually word our estimates. Often, we would desire instead the number of primes n^2+1 that are at most N (the resulting prime is at most N, rather than the variable n is at most N). So we need to replace N by $\sqrt{N}$ in the integrals' limit, to get:

Conjecture 3.8. *The expected number of primes of the form n^2+1 less than or equal to N is*

$$\frac{C_+}{2}\int_2^{\sqrt{N}} \frac{dx}{\log x} \sim C_+\frac{\sqrt{N}}{\log N}. \tag{3.12}$$

(This is [21, Conjecture E].)

Again these estimates are quite close, see Table 8.

TABLE 8. Primes n^2+1 less than N

N	**actual number**	**predicted integral**	**ratio**
1,000,000	**112**	**121**	99
100,000,000	**841**	**855**	745
10,000,000,000	**6656**	**6609**	5962
1,000,000,000,000	**54110**	**53970**	49684
100,000,000,000,000	**456362**	**456404**	425861

In 1978 Iwaniec showed [26] that there are infinitely many P_2's (products of two primes) among the numbers of the form n^2+1.[8] It has also be shown that there are infinitely many of the form n^2+m^4, but both of these results are far from proving there are infinitely many primes of the form n^2+1.

4. Non-polynomial forms

In this section we attempt to apply similar reasoning to non-polynomial forms. There are quite a few examples of this in the literature: Mersenne [39, 44], Wieferich [12], generalized Fermat [14][9], primorial and factorial [9], and primes of the form $k \cdot 2^n + 1$ [5]. We will look at several of these cases below including the Cullen and Woodall primes (perhaps for the first time).

In the previous sections we took advantage of the fact that for a polynomial $f(x)$ with integer coefficients, $f(x+p) \equiv f(x+p) \pmod p$. This is rarely the case when $f(x)$ has a more general form, and is definitely not true for the form $2^n - 1$. So rather than use Dickson's Conjecture 2.2 as we did in all of the previous sections, we will proceed directly from our key heuristic: associating with the random number n the probability $1/\log n$ of being prime–then trying to adjust for 'non-randomness' in each case.

A second common problem we will have with these examples is that very few primes of each form are known, usually only a couple dozen at best. When we look back at the numerical evidence for the polynomial examples, we can not help but notice the spectacular agreement between the heuristic estimate and the actual count just begins to show itself after we have many hundreds, or thousands, of examples. For that reason it will be difficult to draw conclusion below from simply counting. We will also look at the distribution of the know examples and in some cases the gaps between these examples.

Why the gaps? [Chris: do we want to answer this here?]

4.1. Mersenne primes and the Generalized Repunits.

A repunit is an integer all of whose digits are all one such as the primes 11 and 1111111111111111111. The generalized repunits (repunits in radix a) are the numbers $R_k(a) = (a^k-1)/(a-1)$. When a is 2 these are the Mersenne numbers. When a is 10, they are the usual repunits.

Before we estimate the number of generalized repunit primes, we must first consider their divisibility properties. For example, if k is composite, then the polynomial $x^k - 1$ factors, so for $R_k(a)$ to be prime, k must be a prime p. As a first estimate we might guess the probability that $R_k(a)$ is prime is roughly $(1/\log R_k(a))(1/\log k) \sim 1/((k-1)\log k \log a)$.

Next suppose that the prime q divides $R_p(a)$ with p prime. Then the order of $a \pmod q$ divides p, so is 1 or p. If the order is 1, then $a \equiv 1 \pmod q$, $R_p(a) \equiv p$ and therefore $p = q$. If the order is p, then since the order divides $q-1$, we know p divides $q-1$. We have shown that every prime divisor q of $R_p(a)$ is either p (and divides $a-1$) or has the form $kp+1$ for some integer k.

[8] He proved that if we divide C_+ by 77 in equation 3.8, then we get a lower bound for the number of P_2's represented.

[9] The authors treated these as polynomials by fixing the exponent and varying the base.

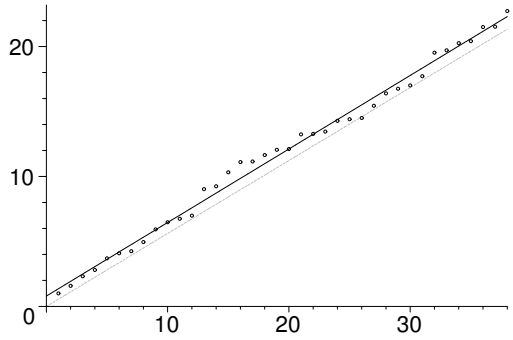
Among other things, this means that for most primes p, $R_p(a)$ in not divisible by any prime $q < p$, so we can adjust our estimate that $R_k(a)$ is prime by multiplying by $1/(1-1/q)$ for each of these primes. Here we need to recall an important tool:

Theorem 4.1 (Merten's Theorem)**.**

$$\prod_{\substack{q \le x \\ q \ prime}} \left(1 - \frac{1}{q}\right) = \frac{e^{-\gamma}}{\log x} + O(1),$$

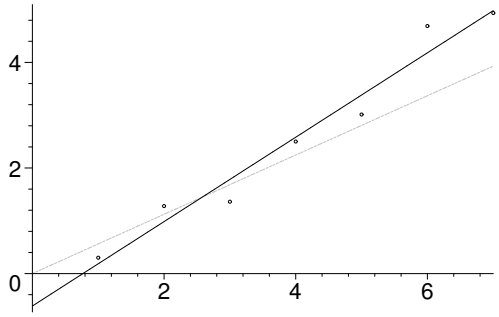
(For a proof see [22, p. 351].) So our second estimate of the probability that $R_k(a)$ is prime is $e^{\gamma}/k \log a$.

FIGURE 1. $\log_2 \log_2 n$th Mersenne prime verses n



http://www.utm.edu/research/primes/mersenne/heuristic.html

FIGURE 2. $\log_{10} \log_{10} n$th repunit prime verses n



4.2. **Cullen and Woodall primes.** The Cullen and Woodall primes are $C(n) = n2^n + 1$, and $W(n) = n2^n - 1$. In this case we have

$$C(n + p(p-1)) \equiv C(n) \pmod{p(p-1)}$$

and

$$W(n + p(p-1)) \equiv W(n) \pmod{p(p-1)}.$$

By the Chinese remainder theorem, both of these have $ord_p(2)$ solutions in

$$\{0, 1, 2, ..., p \cdot ord_p(2)\},$$

so we might again assume that the probabilities that p divides $C(n)$ and $W(n)$ are both $1/p$ for odd primes p–the same as for an arbitrary random integers. But are these probabilities independent for different primes p and q? We must ask this because $p(p-1)$ and $q(q-1)$ are not relatively prime. We verify this independence as follows:

Theorem 4.2. *Let p and q be distinct odd primes and let a and b be any integers. The the system of congruences*

$$\begin{cases} n2^n \equiv a \pmod{p} \\ n2^n \equiv b \pmod{q} \end{cases}$$

has $lcm(pq, ord_p(2), ord_q(2))/pq$ solutions in $\{0, 1, 2, ..., lcm(pq, ord_p(2), ord_q(2))\}$.

Proof. For each r modulo $d = lcm(ord_p(2), ord_q(2))$ write $n = r + sd$. Then the system above is

$$\begin{cases} sd \equiv a/2^r - r \pmod{p} \\ sd \equiv b/2^r - r \pmod{q} \end{cases}$$

Assume that q is the larger of the two primes, then we know $q \nmid ord_p(2)$, so the second of these congruences has a unique solution (modulo q). If $p \nmid d$, then the first congruence also has a unique solution, giving a total of d solution to the original system (one for each r). In this case d is $lcm(pq, ord_p(2), ord_q(2))/pq$. On the other hand, if $p \mid d$, then the only acceptable choices of r are those for which $r2^r \equiv a \pmod{p}$. There are d/p of these–which again is $lcm(pq, ord_p(2), ord_q(2))/pq$. □

For each odd prime the analog of the adjustment factor (2.2) is therefore one, and the complete adjustment factor (2.3) is 2 in both cases (Cullen and Woodall). This gives us the following.

Conjecture 4.3. *The expected number of Cullen and Woodall primes with $n \le N$ are each*

$$2\int_2^N \frac{dx}{\log x 2^x} \backsim 2\frac{\log N - \log 2}{\log 2} \tag{4.1}$$

Table 9 shows us that what little evidence we have does not support (4.1) well for the Cullen numbers, though it does appear reasonable for Woodall numbers.

TABLE 9. Cullen $C(n)$ and Woodall $W(n)$ primes

	actual (with $n < N$)		**predicted**	
N	**Woodall**	**Cullen**	**integral**	**ratio**
1000	**15**	**2**	**15**	18
10,000	**18**	**5**	**22**	25
100,000	**24**	**10**	**29**	31
500,000	≥**26**	≥**13**	**33**	36
1,000,000			**35**	38

[[Chris: Why is this so bad? Keller's article lists the main divisibility properties of these numbers, perhaps we should work it in!]]

Since we have so few data points it might offer some insight to graph the expected number of Cullen and Woodall primes below each of the known primes of these forms (see graph ?removed?). If our estimate holds, then this graph would remain "near" the diagonal.

4.3. **Primorial primes.** Primes of the form $p\# \pm 1$ are sometimes called the primorial primes (a term introduced by H. Dubner as a play on the words prime and factorial). Since $\log p\#$ is the Chebyshev theta function, it is well known that asymptotically $\theta(p) = \log p\#$ is approximately p. In fact Dusart [15] has shown

$$|\theta(x) - x| \leq 0.006788 \frac{x}{\log x} \qquad \text{for } x \geq 2.89 \times 10^7.$$

We begin (as usual) noting that by the prime number theorem the probability of a "random" number the size of $p\# \pm 1$ being prime is asymptotically $\frac{1}{p}$. However, $p\# \pm 1$ does not behave like a random variable because primes q less than p divide $1/q^{\text{th}}$ of a random set of integers, but can not divide $p\# \pm 1$. So we adjust our estimate by dividing by $1 - \frac{1}{q}$ for each of these primes q. By Mertens' theorem 4.1 our final estimate of the probability that $p\# \pm 1$ is prime is $\frac{e^\gamma \log p}{p}$.

By this simple model, the expected number of primes $p\# \pm 1$ with $p \leq N$ would then be

$$\sum_{p \leq N} \frac{e^\gamma \log p}{p} \sim e^\gamma \log N$$

Conjecture 4.4. *The expected number of primorial primes of each of the forms $p\# \pm 1$ with $p \leq N$ are both approximately $e^\gamma \log N$.*

The known, albeit limited, data supports this conjecture. What is known is summarized in Table 10.

TABLE 10. The number of primorial primes $p\# \pm 1$ with $p \leq N$

	actual		**predicted**
N	$p\# + 1$	$p\# - 1$	(of each form)
10	**4**	**2**	**4**
100	**6**	**6**	**8**
1000	**7**	**9**	**12**
10000	**13**	**16**	**16**
100000	**19**	**18**	**20**

Remark 4.5. By the above estimate, the n^{th} primorial prime should be about e^{n/e^γ}.

4.4. **Factorial primes.** The primes of the forms $n! \pm 1$ are regularly called the factorial primes, and like the "primorial primes" $p\# \pm 1$, they may owe their appeal to Euclid's proof and their simple form. Even though they have now been tested up to $n = 10000$ (approximately 36000 digits), there are only 39 such primes known. To develop a heuristical estimate we begin with Stirling's formula:

$$\log n! = (n + \frac{1}{2}) \log n - n + \frac{1}{2} \log 2\pi + O\left(\frac{1}{n}\right)$$

or more simply: $\log n! \sim n(\log n - 1)$. So by the prime number theorem the probability a random number the size of $n! \pm 1$ is prime is asymptotically $\frac{1}{n(\log n - 1)}$.

Once again our form, $n! \pm 1$ does not behave like a random variable–this time for several reasons. First, primes q less than n divide $1/q$–th of a set of random integers, but can not divide $n! \pm 1$. So we again divide our estimate by $1 - \frac{1}{q}$ for each of these primes q and by Mertens' theorem we estimate the probability that $n! \pm 1$ is prime to be

$$\frac{e^\gamma \log n}{n(\log n - 1)}. \tag{4.2}$$

To estimate the number of such primes with n less than N, we may integrate this last estimate to get:

Conjecture 4.6. *The expected number of factorial primes of each of the forms $n! \pm 1$ with $n \le N$ are both asymptotic to $e^\gamma \log N$*

Table 11 shows a comparison of this estimate to the known results.

TABLE 11. The number of factorial primes $n! \pm 1$ with $n \le N$

	actual		**predicted**
N	$n! + 1$	$n! - 1$	(of each form)
10	**3**	**4**	**4**
100	**9**	**11**	**8**
1000	**16**	**17**	**12**
10000	**18**	**21**	**16**

As an alternate check on this heuristic model, notice that it also applies to the forms $k \cdot n! \pm 1$ (k small). For $1 \le k \le 500$ and $1 \le n \le 100$ the form $k \cdot n! + 1$ is a prime 4275 times, and the form $k \cdot n! - 1$, 4122 times. This yields an average of 8.55 and 8.24 primes for each k, relatively close to the predicted 8.20.

But what of the other obstacles to $n! \pm 1$ behaving randomly? Most importantly, what effect does accounting for Wilson's theorem have? These turn out not to significantly alter our estimate above. To see this we first we summarize these divisibility properties as follows.

Theorem 4.7. *Let n be a positive integer.*

i) *n divides $1! - 1$ and $0! - 1$.*
ii) *If n is prime, then n divides both $(n-1)! + 1$ and $(n-2)! - 1$.*
iii) *If n is odd and $2n + 1$ is prime, then $2n + 1$ divides exactly one of $n! \pm 1$.*
iv) *If the prime p divides $n! \pm 1$, then $p - n - 1$ divides one of $n! \pm 1$.*

Proof. (ii) is Wilson's theorem. For (iii), note that if $2n+1$ is prime, then Wilson's theorem implies

$$-1 \equiv 1 \cdot 2 \cdot \ldots \cdot n \cdot (-n) \cdot \ldots \cdot (-1) \equiv (-1)^n (n!)^2 \pmod{2n+1}.$$

When n is odd this is $(n!)^2 \equiv 1$, so $n! \equiv \pm 1 \pmod{2n+1}$. Finally, to see (iv), suppose $n! \equiv \pm 1 \pmod{p}$. Since $(p-1)! \equiv -1$, this is

$$(p-1)(p-2) \cdot \ldots \cdot (n+1) \equiv (-1)^{p-n-1}(p-n-1)! \equiv \mp 1 \pmod{p}.$$

This shows p divides exactly one of $(p-n-1)! \pm 1$. □

To adjust for the divisibility properties (ii) and (iii), we should multiply our estimate 4.2 by $1 - \frac{1}{\log n}$, which is roughly the probability that $n+1$ or $n+2$ is composite; and then by $1 - \frac{1}{4 \log 2n}$, which is the probability n is odd and $2n+1$ is prime. The other two cases of Theorem 4.7 require no adjustment. This gives us the following estimate of the primality of $n! \pm 1$.

$$\left(1 - \frac{1}{4 \log 2n}\right) \frac{e^\gamma}{n} \tag{4.3}$$

Integrating as above suggests there should be $e^\gamma(\log N - \frac{1}{4} \log \log 2N)$ primes of the forms $n! \pm 1$ with $n \le N$. Since we are using an integral of probabilities in our argument, we can not hope to do much better than an error of $o(\log N)$, so this new estimate is essentially the same as our conjecture above.

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Department of Mathematics and Computer Science, University of Tennessee at Martin

E-mail address: caldwell@utm.edu

URL: http://www.utm.edu/~caldwell/