

# Nordhaus-Gaddum Class Theorems for $k$ -Decompositions

Allan Bickle

Department of Mathematics

Western Michigan University

1903 W. Michigan

Kalamazoo, MI 49008

email: allan.e.bickle@wmich.edu

## Abstract

A Nordhaus-Gaddum class theorem provides sharp upper and lower bounds for the sum  $p(G) + p(\overline{G})$  and product  $p(G) \cdot p(\overline{G})$  for some graph parameter  $p(G)$ . This can be generalized to decompositions into  $k$  factors. Let  $p(k; G)$  denote the maximum of  $\sum_{i=1}^k p(G_i)$  over all  $k$ -decompositions of  $G$ . The maximum core number of a graph,  $\widehat{C}(G)$ , is the maximum  $k$  such that  $G$  has a  $k$ -core, the maximal induced subgraph  $H \subseteq G$  such that  $\delta(H) \geq k$ . Furedi, Kostochka, Stiebitz, Skrekovski, and West [2005] determined some bounds and exact values of  $\widehat{C}(k; K_n)$  for  $k \in \{2, 3, 4\}$ . We determine the extremal decompositions for  $k \in \{2, 3, 4\}$  and construct a number of other decompositions that attain large values of this parameter.

Keywords: Nordhaus-Gaddum, decomposition, maximum core number, block design

## 1 Introduction

Consider the following problem. An international round-robin sports tournament is held between teams. The games are split between locations in different countries, which can host multiple games simultaneously. The teams can travel to different locations to play, but it is impractical for the fans to visit more than one location. In this situation, it is reasonable to want teams that play at a given location to play as many games there as possible so that local fans can see them as much as possible. More precisely, we can compute the minimum number of games played by the teams at that location. We then wish to maximize the sum of these minimum numbers over all the locations in the tournament.

Thus we wish to find the maximum sum of the minimum degrees over all possible decompositions of a graph into  $k$  factors. This is a generalization of a Nordhaus-Gaddum Theorem, which finds sharp upper and lower bounds for the sum of a parameter on a graph and its complement. Furedi, Kostochka, Stiebitz, Skrekovski, and West [2005] [3] determined some bounds and exact values of this quantity for  $k \in \{2, 3, 4\}$ . We extend these results by determining all possible decompositions achieving these bounds and considering larger values of  $k$ .

The specific parameter that we will study is the maximum core number, which is equivalent to the degeneracy of a graph. We require a few definitions. (See [2] and [8] for basic terminology.)

**Definition 1.** The  $k$ -core of a graph  $G$ ,  $C_k(G)$ , is the maximal induced subgraph  $H \subseteq G$  such that  $\delta(H) \geq k$ , if it exists.

The maximum core number of a graph,  $\widehat{C}(G)$ , is the maximum  $k$  such that  $G$  has a  $k$ -core. Given  $k = \widehat{C}(G)$ , the maximum core of  $G$  is  $C_k(G)$ .

Cores were introduced by S. B. Seidman [6] and have been studied extensively in [1]. It is easy to show that the  $k$ -core is well-defined and that the cores of a graph are nested. There is a simple algorithm for determining the  $k$ -core of a graph, which we shall call the  $k$ -core algorithm.

**Algorithm 2.** [*k-Core Algorithm*] Iteratively delete vertices of degree less than  $k$  until none remain.

It is straightforward to show that this will produce the  $k$ -core if it exists.

**Definition 3.** [5] A graph is  $k$ -degenerate if its vertices can be successively deleted so that when deleted, each has degree at most  $k$ . The degeneracy of a graph  $G$  is the smallest  $k$  such that it is  $k$ -degenerate.

As a corollary of the  $k$ -core algorithm, we have the following min-max relationship.

**Corollary 4.** For any graph, its maximum core number is equal to its degeneracy.

It is immediate that  $\delta(G) \leq \widehat{C}(G) \leq \Delta(G)$ . We can characterize the extremal graphs for the upper bound. For simplicity, we restrict the statement to connected graphs. (see also [8] p. 199)

**Proposition 5.** Let  $G$  be a connected graph. Then  $\widehat{C}(G) = \Delta(G) \iff G$  is regular.

*Proof.* If  $G$  is regular, then its maximum and minimum degrees are equal, so the result is obvious.

For the converse, let  $\widehat{C}(G) = \Delta(G) = k$ . Then  $G$  has a subgraph  $H$  with  $\delta(H) = \Delta(G) \geq \Delta(H)$ , so  $H$  is  $k$ -regular. If  $H$  were not all of  $G$ , then since  $G$  is connected, some vertex of  $H$  would have a neighbor not in  $H$ , implying that  $\Delta(G) > \Delta(H) = \delta(H) = \Delta(G)$ . But this is not the case, so  $G = H$ , and  $G$  is regular.  $\square$

We can also consider the extremal graphs for the lower bound  $\delta(G) \leq \widehat{C}(G)$ .

**Definition 6.** A graph  $G$  is  $k$ -monocore if  $\widehat{C}(G) = \delta(G) = k$ .

One important application for the maximum core number is the bound  $\chi(G) \leq 1 + \widehat{C}(G)$  for the chromatic number. This bound was first proved by Szekeres and Wilf in 1968 [7], stated in different terms. This bound is useful in proving part of the original Nordhaus-Gaddum Theorem,  $\chi(G) + \chi(\overline{G}) \leq n + 1$ .

One common way to study a graph parameter  $p(G)$  is to examine the sum  $p(G) + p(\overline{G})$  and product  $p(G) \cdot p(\overline{G})$ . A theorem providing sharp upper and lower bounds for this sum and product is known as a theorem of the Nordhaus-Gaddum class. Of the four possible bounds, the sum upper bound has attracted the most attention. We will examine results of this type for maximum core number.

It is convenient to consider a graph and its complement as a decomposition of a complete graph. This makes it possible to generalize the problem to more than two factors.

**Definition 7.** [3] A  $k$ -decomposition of a graph  $G$  is a decomposition of  $G$  into  $k$  factors. For a graph parameter  $p$ , let  $p(k; G)$  denote the maximum of  $\sum_{i=1}^k p(G_i)$  over all  $k$ -decompositions of  $G$ .

It may be that it is possible to delete some edges from one of the factors so that it still has the same maximum core number. Thus we are most interested in the critical subgraphs of the factors of the decomposition. Conversely, given the critical subgraphs, we can distribute the extra edges arbitrarily. This final step is uninteresting, so we will tend to describe a  $k$ -decomposition as  $\{H_1, \dots, H_k\}$ , where each  $H_i$  is a critical subgraph of a factor.

## 2 $k$ -Decompositions and Maximum Core Number for $k \in \{2, 3, 4\}$

Translating our motivating problem into graph theory terms, we wish to find  $\delta(k; K_n)$  over all values of  $k$  and  $n$ . We will investigate  $\widehat{C}(k; K_n)$ , which may be the same thing.

Consider 2-decompositions and maximum core number. The upper bound is implicit in [2].

**Theorem 8.** *We have  $\widehat{C}(G) + \widehat{C}(\overline{G}) \leq n - 1$ . The graphs for which  $\widehat{C}(G) + \widehat{C}(\overline{G}) = n - 1$  are exactly the graphs constructed by starting with a regular graph and iterating the following operation.*

*Given  $k = \widehat{C}(G)$ ,  $H$  a  $k$ -monocore subgraph of  $G$ , add a vertex adjacent to at least  $k + 1$  vertices of  $H$ , and all vertices of degree  $k$  in  $H$  (or similarly for  $\overline{G}$ ).*

*Proof.* Let  $p = \widehat{C}(G)$  and suppose  $\overline{G}$  has an  $n - p$ -core. These cores use at least  $(p + 1) + (n - p + 1) = n + 2$  vertices, and hence share a common vertex  $v$ . But then  $d_G(v) + d_{\overline{G}}(v) \geq p + (n - p) = n$ , a contradiction.

If  $G$  is regular with  $k = \widehat{C}(G)$ , then  $\overline{G}$  is  $n - k - 1$ -regular, so  $\widehat{C}(G) + \widehat{C}(\overline{G}) = n - 1$ . If a vertex  $v$  is added as in the operation, producing a graph  $H$ , a  $k + 1$ -core is produced, so  $\widehat{C}(H) + \widehat{C}(\overline{H}) = (n + 1) - 1$ .

Suppose that for a graph  $G$ ,  $\widehat{C}(G) + \widehat{C}(\overline{G}) = n - 1$ . If  $G$  and  $\overline{G}$  are both monocore, then they must be regular. If  $G$  has a vertex  $v$  that is not contained in the maximum cores of both  $G$  and  $\overline{G}$ , then  $\widehat{C}(G - v) + \widehat{C}(\overline{G} - v) = (n - 1) - 1$ . Then  $v$  is contained in the maximum core of one of them, say  $G$ . Further, given  $k = \widehat{C}(G)$ ,  $v$  is contained in a  $k$ -monocore subgraph  $H$  of  $G$ , and  $H - v$  must be  $k - 1$ -monocore. Then  $v$  must have been adjacent to all vertices of degree  $k - 1$  in  $H - v$ . Thus  $G$  can be constructed as described using the operation. □

This theorem says that in any extremal 2-decomposition into spanning factors, they must be regular. This generalizes to  $k$ -decompositions.

**Corollary 9.** *Let  $D$  be a  $k$ -decomposition of  $K_n$  into factors that are critical with respect to a maximum core number. Then  $\sum_D \left( \widehat{C}(G_i) \right) \leq n - 1$  with equality exactly for decompositions into  $k$  spanning regular graphs.*

*Proof.* Given vertex  $v$ , we have  $\sum_D \left( \widehat{C}(G_i) \right) \leq \sum d_{G_i}(v) \leq n - 1$ . Equality holds exactly when every factor is regular. □

Next we consider  $k$ -decompositions with the restriction that each vertex is contained in exactly two factors. Consider the following construction.

**Algorithm 10.** *Let  $r_1, \dots, r_k$  be nonnegative integers at most one of which is odd. Let  $G_{ij}$ ,  $1 \leq i < j \leq k$  be an  $r_i$ -regular graph of order  $r_i + r_j + 1$ , and let  $G_{ji} = \overline{G_{ij}}$ . Let  $G_i = \bigoplus_{j, j \neq i} G_{ji}$ . Let  $S_k$  be the set of all  $k$ -decompositions of the form  $\{G_1, \dots, G_k\}$  constructed in this fashion.*

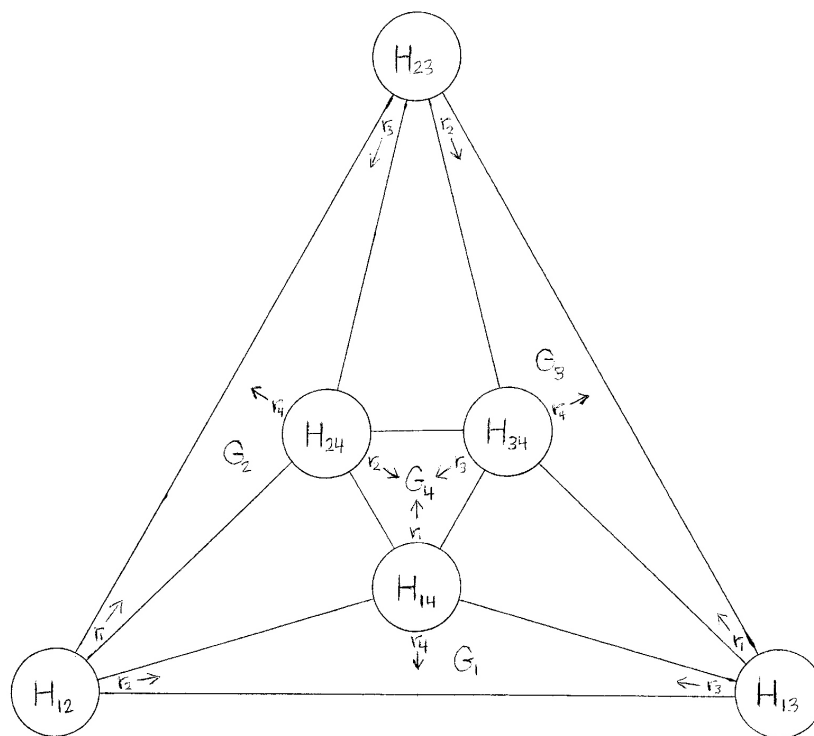


Figure 1: A decomposition produced by Algorithm 10 ( $k = 4$ ).

**Theorem 11.** *A  $k$ -decomposition with order  $n > 1$  and every vertex in exactly two factors has  $\sum_D \widehat{C}(G_i) \leq \binom{2k-3}{k-1} n - \frac{k}{2}$ , and equality holds exactly for those decompositions in the set  $S_k$ .*

*Proof.* Since each vertex is contained in exactly two of the  $k$  factors, so we can partition them into  $\binom{k}{2}$  distinct classes. Let  $H_{ij} = V(G_i) \cap V(G_j)$  and let  $n_{ij} = |H_{ij}|$  for  $i \neq j$ ,  $n_{ii} = 0$ . Hence  $n = \sum_{i,j} n_{ij}$ . For  $v \in H_{ij}$ , we have  $\widehat{C}(G_i) \leq d_{G_i}(v) \leq d_{G_i[H_{ij}]}(v) + \sum_{t=1}^k n_{it}$ . Sum for each of the two factors and each of the  $\binom{k}{2}$  classes. Then  $(k-1) \sum_{i=1}^k \widehat{C}(G_i) \leq 2(k-2) \sum_{i,j} n_{ij} + \sum_{i,j,i \neq j} (n_{ij} - 1) = (2k-3)n - \binom{k}{2}$ , so  $\sum_{i=1}^k \widehat{C}(G_i) \leq \binom{2k-3}{k-1} n - \frac{k}{2}$ .

( $\Rightarrow$ ) If this bound is an equality, then all  $k$  of the factors must be regular. Let  $r_{ij} = d_{G_i[H_{ij}]}(v)$  for  $v \in G_i[H_{ij}]$ . Also, all edges between two classes sharing a common factor must be in that factor, so it is a join of  $k-1$  graphs. A join of graphs is regular only when they are all regular. Now since  $G_i$  is regular, its complement must also be regular. But this implies that all the constants  $r_{ji}$ ,  $j \neq i$  are equal. Let  $r_i$  be this common value. Then  $n_{ij} = r_i + r_j + 1$ , so  $n = (k-1) \sum r_i + \binom{k}{2}$ . This implies that at most one of  $r_i$  and  $r_j$  is odd, so at most one of all the  $r_i$ 's is odd.

( $\Leftarrow$ ) Let  $G_i$  be a factor of a decomposition  $D$  constructed using the algorithm. It is easily seen that  $G_i$  is regular of degree  $(k-3)r_i + (k-2) + \sum_j r_j$ . Summing over all the factors, we see  $\sum_D \widehat{C}(G_i) = \binom{2k-3}{k-1} n - \frac{k}{2}$ .  $\square$

Now consider 3-decompositions. The formula in the following theorem was proven by Furedi et al [3].

**Theorem 12.** *We have  $\widehat{C}(3; K_n) = \lfloor \frac{3}{2}(n-1) \rfloor$ , and the extremal decompositions that achieve  $\sum_{i=1}^3 \widehat{C}(G_i) = \frac{3}{2}(n-1)$  all consist of three  $\frac{n-1}{2}$ -regular graphs. For  $n = 1$ ,  $\{K_1, K_1, K_1\}$  is the only extremal 3-decomposition, and for odd order  $n > 1$  they are exactly those in the set  $S_3$ .*

*Proof.* Let  $G_1, G_2$ , and  $G_3$  be the three factors of an extremal decomposition for  $\widehat{C}(3; K_n)$ . It is obvious that  $\{K_1, K_1, K_1\}$  is the only possibility for  $n = 1$ , so let  $n > 1$ . The previous theorem shows that  $\sum_{i=1}^3 \widehat{C}(G_i) \leq \frac{3}{2}(n-1)$ .

Now any vertex can be contained in at most two of the three factors, since its degrees in the three graphs sum to at most  $n-1$ . Now adding a vertex with adjacencies so that it is contained in exactly one of the three factors increases  $n$  by one and  $\sum_{i=1}^3 \widehat{C}(G_i)$  by at most one, so this cannot violate the bound. Thus deleting a vertex of an extremal decomposition contained in only one of the three factors would decrease  $n$  by one and  $\sum_{i=1}^3 \widehat{C}(G_i)$  by at most one. For  $n$  odd, this is a contradiction and for  $n$  even it can occur only when it is the only such vertex.

If there are only two distinct classes, then add a vertex joined to all the vertices of the two disjoint factors. This increases  $n$  by one and  $\sum_{i=1}^3 \widehat{C}(G_i)$  by

two. Hence if the new decomposition satisfies the bound, so does the original, and if the original decomposition attains the bound, then  $n$  must be even.

Thus by the previous theorem, those decompositions with  $\sum_{i=1}^3 \widehat{C}(G_i) = \frac{3}{2}(n-1)$  are exactly those in  $S_3$ . Further, by the proof of this theorem the factors of such a decomposition are all  $1 + \sum r_j$ -regular. Now  $2(r_1 + r_2 + r_3) = \sum (n_{ij} - 1) = n - 3$ , so  $\sum_j r_j = \frac{n-3}{2}$ . Thus the factors are all  $\frac{n-1}{2}$ -regular.

Finally note that joining a vertex to all vertices of one factor of an extremal decomposition of odd order attains the bound for even order, so  $\widehat{C}(3; K_n) = \lfloor \frac{3}{2}(n-1) \rfloor$  for even orders as well.  $\square$

An extremal decomposition of even order can be formed from one of odd order by either joining a vertex to one of the factors or deleting a vertex contained in two factors. However, the decomposition  $\{K_4, C_4, C_4\}$  shows that not all extremal decompositions of even order can be formed this way.

Furedi et al [3] also proved that  $\widehat{C}(4; K_n) = \lfloor \frac{5}{3}(n-1) \rfloor$ . We employ their proof of this result to show that all extremal decompositions of order  $n = 3r + 1 > 1$  can be constructed using the following algorithm.

**Algorithm 13.** *Let  $n, r, a, b, c$ , and  $s$  be nonnegative integers with  $n = 3r + 1$ ,  $a + b + c = s - 1$  and  $a, b, c$ , even if  $s$  is odd. Let  $G_1, G_2, G_3$  be  $a, b, c$ -regular graphs, respectively, of order  $s$ . Let  $G_4, G_5, G_6$  be  $r - s$ -regular graphs of orders  $r - a, r - b, r - c$ , respectively. Let  $S$  be the set of all decompositions of the form  $\{G_1 + G_4, G_2 + G_5, G_3 + G_6, \overline{G}_3 + \overline{G}_4 + \overline{G}_6\}$ .*

**Theorem 14.** *We have  $\widehat{C}(4; K_n) = \lfloor \frac{5}{3}(n-1) \rfloor$ . For  $n = 1$ , the only extremal 4-decomposition is  $\{K_1, K_1, K_1, K_1\}$ , and the extremal decompositions of order  $n = 3r + 1 > 1$  that achieve  $\sum_{i=1}^4 \widehat{C}(G_i) = \frac{5}{3}(n-1)$  are exactly those in  $S$ .*

*Proof.* It is obvious that  $\{K_1, K_1, K_1, K_1\}$  is the only possibility for  $n = 1$ , so let  $n > 1$ . It is easily checked that the decompositions in  $S$  exist and achieve the stated sum. Joining a vertex to one of the factors achieves the stated bound for  $n = 3r + 2$ , and deleting a vertex contained in two of the factors achieves the bound for  $n = 3r$ .

As in the previous theorem, it is easily shown that no vertex is contained in a single factor or all four factors. If each vertex is contained in exactly two of the four factors, then Theorem 11 says that  $\sum_{i=1}^4 \widehat{C}(G_i) \leq \frac{5}{3}n - 2$ . Hence this decomposition is not extremal for  $n = 3r + 1$ .

Consider an extremal decomposition with a vertex contained in three of the factors. Call these factors 1, 2, and 3 so that  $\widehat{C}(G_1) \leq \widehat{C}(G_2) \leq \widehat{C}(G_3)$ . Let  $H_{123} = V(G_1) \cap V(G_2) \cap V(G_3)$  and  $H_{i4} = V(G_i) \cap V(G_4)$ . Then  $\widehat{C}(G_1) + \widehat{C}(G_2) + \widehat{C}(G_3) \leq n - 1$ , so  $\widehat{C}(G_1) + \widehat{C}(G_2) \leq \frac{2}{3}(n - 1)$ . Now  $\widehat{C}(G_3) + \widehat{C}(G_4) \leq n - 1$ , so  $\sum_{i=1}^4 \widehat{C}(G_i) \leq \frac{5}{3}(n - 1)$ , and  $\widehat{C}(4; K_n) = \lfloor \frac{5}{3}(n - 1) \rfloor$ .

If  $\sum_{i=1}^4 \widehat{C}(G_i) = \frac{5}{3}(n - 1)$ , then  $\widehat{C}(G_1) + \widehat{C}(G_2) = \frac{2}{3}(n - 1)$  and  $\widehat{C}(G_3) + \widehat{C}(G_4) = n - 1$ . The former implies that  $\widehat{C}(G_1) = \widehat{C}(G_2) = \widehat{C}(G_3) =$

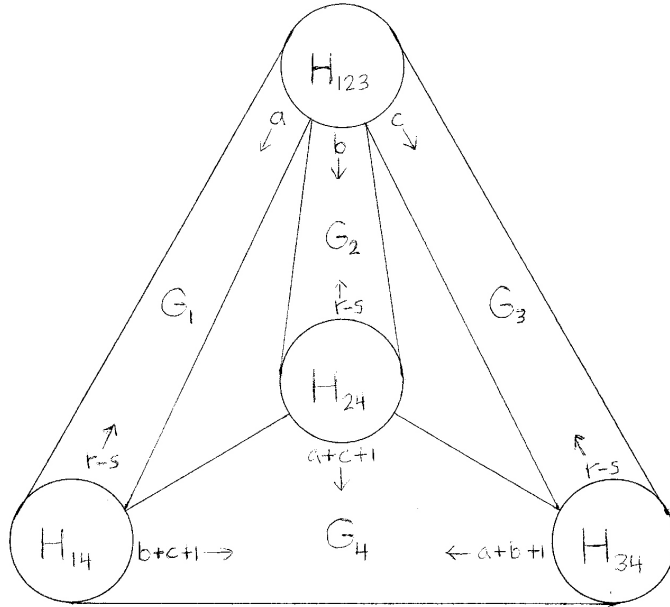


Figure 2: A decomposition produced by Algorithm 13.

$\frac{1}{3}(n-1)$ . The latter and this imply that  $\widehat{C}(G_4) = \frac{2}{3}(n-1)$  and each vertex in  $G_i \cap G_4$ ,  $i \in \{1, 2, 3\}$ , is only adjacent to vertices in these two factors. Hence the vertices partition into  $H_{123}$  and  $H_{i4} = G_i \cap G_4$ ,  $i \in \{1, 2, 3\}$  whose orders we call  $n_{123}$  and  $n_{i4}$ , respectively. Furthermore, each of the factors is regular.

Then  $H_{123}$  is decomposed into three regular spanning factors whose degrees are even if  $n_{123}$  is odd, and the other sets are decomposed into two regular spanning graphs. Let  $r_{i,s} = d_{G_i[H_S]}(v)$  for  $v \in G_i[H_S]$ . Hence  $r_{1,123} + n_{14} = r_{1,14} + n_{123}$ ,  $r_{2,123} + n_{24} = r_{2,24} + n_{123}$ , and  $r_{3,123} + n_{34} = r_{3,34} + n_{123}$ . Now since  $G_4$  is regular, so is  $\overline{G}_4$ . Thus  $r_{1,14} = r_{2,24} = r_{3,34}$ , so each of the factors is regular of the same degree. Let  $r = \frac{1}{3}(n-1)$  be this common value,  $s = n_{123}$ , so  $r-s = r_{1,14} = r_{2,24} = r_{3,34}$ . Let  $a = r_{1,123}$ ,  $b = r_{2,123}$ , and  $c = r_{3,123}$ , so  $a+b+c = s-1$ ,  $n_{14} = r-a$ ,  $n_{24} = r-b$ , and  $n_{34} = r-c$ . There are no parity problems, so the extremal decomposition can be constructed by the algorithm.  $\square$

The decompositions in set  $S_3$  with order  $n = 3r$  are also extremal with  $\widehat{C}(4; K_n) = \lfloor \frac{5}{3}(n-1) \rfloor = \frac{5}{3}n - 2$ . This is also satisfied by those formed by deleting a vertex contained in exactly two factors from a decomposition in  $S$ . However, the decomposition  $\{K_4, C_4, K_3, K_2\}$  shows that not all such extremal decompositions fall into the previous two categories.



### 3 Constructions for Larger Values of $k$

For larger values of  $k$ , we have decompositions that may be extremal but do not know the exact value of  $\widehat{C}(k; K_n)$  except in special cases. To simplify the description of the decompositions, we will use the notation  $r[G]$  to denote the factor  $G$  occurring  $r$  times.

For  $k \in \{2, 3, 4\}$ , we have seen  $\widehat{C}(k; K_n) = \lfloor \frac{2k-3}{k-1}(n-1) \rfloor$ . In fact, Furedi et al [3] produced a simple construction to prove that  $\widehat{C}(k; K_n) \geq \lfloor \frac{2k-3}{k-1}(n-1) \rfloor$ , but this is not an equality for  $k \geq 5$ .

**Algorithm 15.** Let  $S_5^*$  be the set of all decompositions that can be constructed as follows. Take a decomposition  $D$  in  $S_4$  with the additional property that the sum of some two of the four  $r_i$ 's equals the sum of the other two  $r_i$ 's (e.g.  $r_1 + r_2 = r_3 + r_4$ ). Let  $r$  be this common value. Add the factor  $K_{r+1, r+1}$  to the decomposition.

**Proposition 16.** We have  $\widehat{C}(5; K_n) \geq \lfloor \frac{11}{6}n - 2 \rfloor$ .

*Proof.* This construction has  $\sum_{i=1}^5 \widehat{C}(G_i) = \frac{5}{3}n - 2 + \frac{n}{6} = \frac{11}{6}n - 2$  for any order that it can attain. The proof of Theorem 11 shows that a decomposition in  $S_k$  has order  $n = (k-1) \sum r_i + \binom{k}{2}$ . For  $k = 4$ , this gives  $n = 3 \sum r_i + 6$ . To satisfy the property in the construction, all the  $r_i$ 's must be even, and it is obvious that any nonnegative even  $r$  can be attained. Hence for each positive order  $n = 6r$  there is a decomposition in  $S_5^*$  with this order. Successively deleting five vertices contained in exactly two factors from such a decomposition provides decompositions attaining the bound for the other five classes of orders mod 6.  $\square$

**Conjecture 17.** For  $n \geq 2$ ,  $\widehat{C}(5; K_n) = \lfloor \frac{11}{6}n - 2 \rfloor$ .

The decompositions with  $\sum_{i=1}^5 \widehat{C}(G_i) = \frac{11}{6}n - 2$  include those in  $S_5^*$ . However,  $\{K_4, C_4, K_3, K_2, K_2\}$  shows that there are others. The best known upper bound, due to Furedi et al [3] says that  $\widehat{C}(5; K_n) \leq 2n - 3$ .

**Algorithm 18.** Let  $S_6^*$  be the set of all decompositions that can be constructed as follows. Take a decomposition  $D$  in  $S_4$  with the additional property that two pairs of two of the four  $r_i$ 's are equal. (e.g.  $r_1 = r_2$  and  $r_3 = r_4$ ). Let  $r$  be the sum of these two values. Add two copies of the factor  $K_{r+1, r+1}$  to the decomposition.

**Proposition 19.** For  $n \geq 4$ ,  $\widehat{C}(6; K_n) \geq 2n - 2$ .

*Proof.* This construction has  $\sum_{i=1}^6 \widehat{C}(G_i) = \frac{5}{3}n - 2 + 2 \left(\frac{n}{6}\right) = 2n - 2$  for any order that it can attain. The proof of Theorem 11 shows that a decomposition in  $S_k$  has order  $n = (k-1) \sum r_i + \binom{k}{2}$ . For  $k = 4$ , this gives  $n = 3 \sum r_i + 6$ .

To satisfy the property in the construction, all the  $r_i$ 's must be even, and any nonnegative even  $r = 4s$  can be attained. Hence for each positive order  $n = 12s + 6$  there is a decomposition in  $S_6^*$  with this order. Successively deleting vertices contained in exactly two factors from such a decomposition provides decompositions attaining the bound when  $4 \leq n \leq 6$ ,  $12 \leq n \leq 18$ , and  $n \geq 20$ . Joining a vertex to each of the two disjoint factors when  $n = 12s + 6$  works for  $n \in \{7, 19\}$ . Now  $\{2[K_4], 4[C_4]\}$  works for  $n = 8$  and  $\{3[K_4], 3[K_{3,3}]\}$  works for  $n = 10$ . Joining a vertex to disjoint factors in these last two works for  $n \in \{9, 11\}$ .  $\square$

**Conjecture 20.** For  $n \geq 4$ ,  $\widehat{C}(6; K_n) = 2n - 2$ .

The decompositions with  $\sum_{i=1}^6 \widehat{C}(G_i) = 2n - 2$  include those in  $S_6^*$ . There are many others, including  $\{6[K_2]\}$  and  $\{5[K_4], PG\}$  where  $PG$  is the Petersen graph. The best known upper bound, due to Furedi et al [3] says that  $\widehat{C}(6; K_n) \leq \frac{5}{2}n - \frac{7}{2}$ .

The constructions that we have seen so far start with a small decomposition and 'expand' it to a bigger one. In some cases, this process can be generalized.

**Theorem 21.** Suppose there is a  $k$ -decomposition of  $K_n$  into regular subgraphs and  $\sum_{i=1}^k \widehat{C}(G_i) = c(n - 1)$ . Then there are infinitely many other  $k$ -decompositions with order  $n'$  and  $\sum_{i=1}^k \widehat{C}(G_i) = c(n' - 1)$ .

*Proof.* Let  $r = n - 1$ . Let  $D$  be a decomposition of  $K_{rt+1}$  into  $r$   $t$ -regular spanning factors, where  $t$  is even if  $r$  is even. Form a  $k$ -decomposition  $D'$  with order  $n'$  by replacing each vertex of  $K_n$  with a copy of  $D$  so that if vertex  $v$  has degree  $d_i$  in  $G_i$ , then  $d_i$  of the  $r$  factors are merged together. Finally, join the corresponding factors in different copies of  $D$ .

If the factor  $G_i$  has degree  $d_i$  in  $K_n$ , then the corresponding factor has degree  $d_i(rt + 1) + d_it$ . Now since  $\sum_{i=1}^k d_i = c(n - 1)$  and  $n' = n(rt + 1)$ , we have  $\sum_{i=1}^k (d_i(rt + 1) + d_it) = (rt + 1 + t) \sum d_i = (rt + 1 + t)c(n - 1) = c[n(rt + 1) - 1 + t(n - 1 - r)] = c(n' - 1)$ .  $\square$

We now consider a number of decompositions that can be expanded to infinite families via the previous theorem.

Decompose  $K_n$  into  $k = \binom{n}{2}$   $K_2$ 's. Then  $\sum_{i=1}^k \widehat{C}(G_i) = \binom{n}{2} = \frac{n}{2}(n - 1) = \frac{1 + \sqrt{1 + 8k}}{4}(n - 1)$ . Thus this sum can be achieved for any order and infinitely many values of  $k$ .

Decompose  $K_n$  into  $K_3$ 's, which can occur whenever  $n \equiv 1$  or  $3 \pmod{6}$ . Such a decomposition has  $k = \frac{1}{3} \binom{n}{2} = \frac{n(n-1)}{6}$  triangles, so  $\sum_{i=1}^k \widehat{C}(G_i) = 2 \frac{n(n-1)}{6} = \frac{n}{3}(n - 1) = \frac{1 + \sqrt{1 + 24k}}{6}(n - 1)$ .

In particular, consider  $k = 7$ . Let  $H$  be an  $r$ -regular graph of order  $3r + 1$ . Let  $G = H + H + H$ . Then  $G$  is  $7r + 2$ -regular, and 7 copies of  $G$  form a decomposition of order  $n = 7(3r + 1) = 21r + 7$ , so  $\frac{n-1}{3} = 7r + 2$ .

Then  $\sum_{i=1}^7 \widehat{C}(G_i) = 7(7r+2) = \frac{7}{3}(n-1)$ . This construction shows that  $\widehat{C}(7; K_n) \geq \lfloor \frac{7}{3}(n-1) \rfloor$  for  $n = 7(3r+1)$ . It is straightforward to check that  $\widehat{C}(7; K_8) = 15 < \lfloor \frac{7}{3}(8-1) \rfloor$ , so the formulas we consider need not be “floor-linear” in all cases.

Decompose  $K_n$  into  $K_4$ 's, which can occur whenever  $n \equiv 1$  or  $4 \pmod{12}$  [4]. Such a decomposition has  $k = \frac{1}{6} \binom{n}{2} = \frac{n(n-1)}{12}$   $K_4$ 's, so  $\sum_{i=1}^k \widehat{C}(G_i) = 3 \frac{n(n-1)}{12} = \frac{n}{4}(n-1) = \frac{1+\sqrt{1+48k}}{8}(n-1)$ .

Decompose  $K_n$  into  $K_5$ 's, which can occur whenever  $n \equiv 1$  or  $5 \pmod{20}$  [4]. Such a decomposition has  $k = \frac{1}{10} \binom{n}{2} = \frac{n(n-1)}{20}$   $K_5$ 's, so  $\sum_{i=1}^k \widehat{C}(G_i) = 4 \frac{n(n-1)}{20} = \frac{n}{5}(n-1) = \frac{1+\sqrt{1+80k}}{10}(n-1)$ .

Let  $n = p^2 + p + 1$ , where  $p$  is a prime power. Then there is a projective plane with  $n$  points and  $n$  lines, which correspond to vertices and factors of a decomposition. Then  $\sum_{i=1}^k \widehat{C}(G_i) = kp = \frac{kp}{k-1}(n-1) = \frac{p^2+p+1}{p+1}(n-1) = \frac{(-1+\sqrt{4k-3})k}{2(k-1)}(n-1)$ .

$k$	$\sum \widehat{C}(G_i)$	Decomposition
2	$n-1$	$\{2[K_1]\}$
3	$\frac{3}{2}(n-1)$	$\{3[K_2]\}$
4	$\frac{5}{3}(n-1)$	$\{K_3, 3[K_2]\}$
5	$\frac{9}{5}(n-1)$	$\{4[K_3], 3K_2\}$
6	$2(n-1)$	$\{6[K_2]\}$
7	$\frac{7}{3}(n-1)$	$\{7[K_3]\}$
8	$\frac{9}{4}(n-1)$	$\{K_3, 7[K_2]\}$
9	$\frac{12}{5}(n-1)$	$\{3[K_3], 6[K_2]\}$
10	$\frac{5}{2}(n-1)$	$\{10[K_2]\}$
11	$\frac{19}{7}(n-1)$	$\{8[K_3], 2[K_2], 2K_2\}$
12	$3(n-1)$	$\{12[K_3]\}$
13	$\frac{13}{4}(n-1)$	$\{13[K_4]\}$
14	$\frac{25}{8}(n-1)$	$\{11[K_3], 3[K_2]\}$
15	$3(n-1)$	$\{15[K_2]\}$
16	$\frac{17}{5}(n-1)$	$\{K_5, 15[K_3]\}$
20	$4(n-1)$	$\{20[K_4]\}$
21	$\frac{21}{5}(n-1)$	$\{21[K_5]\}$
30	$5(n-1)$	$\{30[K_5]\}$

The table contains the best known constructions for which the previous

theorem is applicable. Note that in the cases  $k \in \{5, 8, 11, 14, 15\}$  these decompositions are not the best possible for their orders.

There is another way to generate decompositions that are better for some orders. If a decomposition has  $\sum_{i=1}^k \widehat{C}(G_i) = c(n-1)$ , then some factor  $G_i$  has  $\widehat{C}(G_i) \leq \frac{c}{k}(n-1)$ . Generalizing this, we have the following.

**Proposition 22.** *If there is a decomposition of  $K_n$  with  $\sum_{i=1}^k \widehat{C}(G_i) = c(n-1)$ , then given  $0 \leq p \leq k-1$ , there is a decomposition of  $K_n$  with  $\sum_{i=1}^{k-p} \widehat{C}(G_i) \geq c \frac{k-p}{k}(n-1)$ .*

Furedi et al [3] also proved the general upper bound that for all positive integers  $n$  and  $k$ ,  $\widehat{C}(k; K_n) \leq \sqrt{k} \cdot n$ . This is not attained for any values of  $n$  and  $k$ . Using essentially the same approach, this can be strengthened to a sharp bound.

**Proposition 23.** *For all positive integers  $n$  and  $k$ , we have  $\widehat{C}(k; K_n) \leq -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn(n-1)}$ . This is an equality exactly when there is a decomposition of  $K_n$  into  $k$  cliques of equal size.*

*Proof.* For a  $k$ -decomposition, let  $d_i = \widehat{C}(G_i)$  and  $D = \sum \widehat{C}(G_i)$ . Then  $m(G_i) \geq \binom{d_i+1}{2}$ . Now

$$\frac{n(n-1)}{2} = \binom{n}{2} \geq \sum_{i=1}^k \binom{d_i+1}{2} = \frac{1}{2} \sum_{i=1}^k (d_i^2 + d_i) \geq \frac{1}{2} \left( \frac{D^2}{k} + D \right).$$

The first inequality is attained exactly when all the factors are cliques, and the second is attained exactly when all the cliques have the same size. Hence  $kn(n-1) \geq D^2 + kD$ , so  $D^2 + kD - kn(n-1) \leq 0$ , and  $D \leq -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn(n-1)}$ .  $\square$

We can obtain the successively simpler but weaker formulas  $\widehat{C}(k; K_n) \leq -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn(n-1)} < \sqrt{kn(n-1)} < \sqrt{k} \left( n - \frac{1}{2} \right) < \sqrt{k} \cdot n$  as corollaries.

A decomposition of  $K_n$  into  $k$  cliques of equal size is a block design. In particular, it is a  $\left( n, k, \frac{k + \sqrt{k^2 + 4kn(n-1)}}{2n}, \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{n(n-1)}{k}}, 1 \right)$ -design. Hence the previous result will attain equality whenever such a design exists.

**Corollary 24.** *We have*

1.  $\widehat{C} \left( \binom{n}{2}; K_n \right) = \binom{n}{2}$  for  $n \geq 2$
2.  $\widehat{C} \left( \frac{n(n-1)}{6}; K_n \right) = \frac{n(n-1)}{3}$  for  $n \equiv 1$  or  $3 \pmod{6}$
3.  $\widehat{C} \left( \frac{n(n-1)}{12}; K_n \right) = \frac{n(n-1)}{4}$  for  $n \equiv 1$  or  $4 \pmod{12}$

4.  $\widehat{C}\left(\frac{n(n-1)}{20}; K_n\right) = \frac{n(n-1)}{5}$  for  $n \equiv 1$  or  $5 \pmod{20}$
5.  $\widehat{C}(n; K_n) = \frac{(-1+\sqrt{4n-3})n}{2}$  for  $n = p^2 + p + 1$ , where  $p$  is a prime power

It is immediate that  $\kappa(k, K_n) \leq \lambda(k, K_n) \leq \delta(k, K_n) \leq \widehat{C}(k; K_n)$ . Furthermore, the decompositions constructed above show that these are all equalities for  $1 \leq k \leq 4$ . Thus the work on maximum core number does apply to the motivating problem. Perhaps these equalities hold for all values of  $n$  and  $k$ .

**Conjecture 25.** *For all positive integers  $n$  and  $k$ , we have*

$$\kappa(k, K_n) = \lambda(k, K_n) = \delta(k, K_n) = \widehat{C}(k; K_n).$$

## References

- [1] Bickle A., *The  $k$ -Cores of a Graph*. Ph.D. Dissertation, Western Michigan University (2010).
- [2] Chartrand G. and L. Lesniak, *Graphs and Digraphs*, (4th ed.). CRC Press, (2005).
- [3] Furedi Z., A. Kostochka, M. Stiebitz, R. Skrekovski, and D. West, *Nordhaus-Gaddum-type theorems for decompositions into many parts*. J. Graph Theory 50 (2005), 273-292.
- [4] Hanani, H. *Balanced incomplete block designs and related designs*. Discrete Math, 11 (1975), 255-369.
- [5] Lick D. R. and A. T. White,  *$k$ -degenerate graphs*. Canad. J. Math. 22 (1970), 1082-1096.
- [6] Seidman, S. B. *Network structure and minimum degree*. Social Networks 5 (1983) 269-287.
- [7] Szekeres G. and H. S. Wilf, *An inequality for the chromatic number of a graph*. J. Comb. Th. 4 (1968), 1-3.
- [8] West D., *Introduction to Graph Theory*, (2nd ed.). Prentice Hall, (2001).