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Euler's Number via Difference Equations

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Abstract

We use two second-order linear homogeneous difference equations with variable coefficients as well as one second-order linear homogeneous difference equation with constant coefficients to obtain Euler's number e. Also, we obtain Euler's number by using two first-order linear difference equations with variable coefficients, one homogeneous and one nonhomogeneous. We conclude the article by inviting the reader to obtain the Euler's number e in connection with some other suitable difference equations.

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1. Introduction

The transcendental number $e = \lim_{n \longrightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}$ is called *Euler's number* in honor of Leonhard Euler (1707 – 1783). Since John Napier (1550 – 1617) used e as the base of natural logarithm it is also known as the *Napier's constant* (or logarithmic constant). John Napier was the first who used e as the base of the natural logarithm, and hence he is considered the inventor of e. However, the symbol e as we know it was introduced by Euler. While Euler proved that e is irrational, Charles Hermite (1822 – 1901) proved that e is transcendental. There seems to be confusion in some literature between *Euler's number* and *Euler's constant* (also known as Euler-Mascheroni constant). We note that *Euler's constant* is $\gamma = 0.57721...$ while *Euler's number* is $e = 2.71828....$

A classic example of a second-order linear homogeneous difference equation with constant coefficients is the difference equation that defines the *Fibonacci numbers.* Namely,

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_0 = 0,\ F_1 = 1, \text{ and } n \geq 2.$$

Another well-known example of a second-order linear homogeneous difference equation with constant coefficients is the difference equation

$$L_n = L_{n-1} + L_{n-2}, \text{ where } L_0 = 2,\ L_1 = 1, \text{ and } n \geq 2,$$

that defines the Lucas numbers. Fibonacci numbers and Lucas numbers are named after Leonardo Fibonacci (c. 1170 – c. 1250) and Francois Lucas (1842–1891), respectively. Throughout this article $\mathbb{Z}^+$ represents the set of all positive integers.

Our goal in this article is to find e in connection with particular solutions of difference equations (recurrence relations). We use two second-order linear homogeneous difference equations with variable coefficients as well as one second-order linear homogeneous difference equation with constant coefficients to obtain Euler's number e. Also, we obtain Euler's number by using two first-order linear difference equations with variable coefficients, one homogeneous and one nonhomogeneous. We conclude the article by inviting the reader to obtain the Euler's number e in connection with some other suitable difference equations.

2. Using Difference Equations of Order Two

In this section we find Euler's number by using three different second-order linear homogeneous difference equations.

Problem 2.1. *Let* $x_n - n(x_{n-1} + x_{n-2}) + (x_{n-1} + x_{n-2}) = 0$, *where* $x_1 = 0$, $x_2 = 1$, *and* $n \geq 3$. *Then*

$$1 + \left(\lim_{n \longrightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{x_n}{n!} \right)^k \right)^{-1} = e.$$

Proof. The given difference equation can be rewritten as

$$x_n - nx_{n-1} = -x_{n-1} + (n-1)x_{n-2}. \tag{1}$$

Next, using the initial conditions we can easily see that the right-hand side of (1) is always 1 or -1. Hence, for $i \geq 2$, we conjecture that

$$x_i - i x_{i-1} = (-1)^i. \tag{2}$$

We can easily show that (2) is true for all $i \geq 2$, by using (1) and mathematical induction. Now, we divide both sides of (2) by $i!$ to obtain

$$\frac{x_i}{i!} - \frac{x_{i-1}}{(i-1)!} = \frac{(-1)^i}{i!}. \tag{3}$$

Thus, from (3) we get

$$\sum_{i=2}^{n} \left(\frac{xi}{i!} - \frac{x_{i-1}}{(i-1)!} \right) = \sum_{i=2}^{n} \frac{(-1)^i}{i!}. \tag{4}$$

If we simplify the left-hand side of (4), which is a telescoping series, we get

$$\frac{x_n}{n!} = \sum_{i=2}^{n} \frac{(-1)^i}{i!} = \sum_{i=0}^{n} \frac{(-1)^i}{i!}. \tag{5}$$

Now, from (5) we deduce that

$$\sum_{k=1}^{\infty} \left(\frac{x_n}{n!} \right)^k = \sum_{k=1}^{\infty} \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!} \right)^k.$$

Therefore,

$$\begin{aligned} \lim_{n \longrightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{x_n}{n!} \right)^k &= \lim_{n \longrightarrow \infty} \sum_{k=1}^{\infty} \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!} \right)^k = \sum_{k=1}^{\infty} \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \right)^k \\ &= \sum_{k=1}^{\infty} \left(e^{-1} \right)^k = \frac{1}{e-1}. \end{aligned}$$

Consequently,

$$1 + \left(\lim_{n \longrightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{x_n}{n!} \right)^k \right)^{-1} = e.$$

Note 2.2. Problem 2.1 was a problem proposal with a solution by the author in the *College Mathematics Journal* [2]. However, the solution that was provided by the author (namely proof of Problem 2.1) was not published. Aside from the author, this problem was also solved by thirty other individual mathematicians or a group of mathematicians The solution that was provided by Li Zhou was published as the solution to this problem. Li Zhou used mathematical induction as well as Lebesgue's dominated convergence theorem to arrive at his conclusion. For an alternative proof of this problem the reader is referred to [2].

Problem 2.3. *Let F_n and F_{2n} be Fibonacci numbers and let*

$$2nx_{n+1} + (1-2n)x_n - x_{n-1} = 0, x_0 = F_n \neq x_1 = F_{2n}, n \in \mathbb{Z}^+.$$

Then

$$(i) \left[\frac{F_n - F_{2n}}{F_n - \lim_{n\to\infty} x_n}\right]^2 = e \quad \text{and} \quad (ii) \left[\frac{(1-L_n)(L_{n+1}+L_{n-1})}{L_{n+1}+L_{n-1}-5\lim_{n\to\infty} x_n}\right]^2 = e,$$

where L_n, L_{n+1}, and L_{n-1} are Lucas numbers ($n \in \mathbb{Z}^+$).

Proof (i). First we note that for $n \in \mathbb{Z}^+$ the given difference equation can be rewritten as

$$x_{n+1} - x_n = -\frac{1}{2n}(x_n - x_{n-1}).$$

From this difference equation we can easily deduce that

$$x_{n+1} - x_n = (-\frac{1}{2})^n \frac{1}{n!}(x_1 - x_0), \quad n \in \mathbb{Z}^+. \tag{6}$$

Now, we note that

$$\begin{aligned} x_{n+1} &= x_0 + (x_1 - x_0) + (x_2 - x_1) + ... + (x_{n+1} - x_n) \\ &= x_0 + \sum_{i=0}^{n}(x_{i+1} - x_i). \end{aligned} \tag{7}$$

Next, using (6) we can rewrite (7) as

$$x_{n+1} = x_0 + (x_1 - x_0)\sum_{i=0}^{n}(-\frac{1}{2})^i \frac{1}{i!}.$$

Therefore,

$$\lim_{n\to\infty} x_n = x_0 + (x_1 - x_0)e^{-\frac{1}{2}} = F_n + (F_{2n} - F_n)e^{-\frac{1}{2}}. \tag{8}$$

Finally, we obtain the desired equality by solving (8) for e.

Proof *(ii)*. Since $F_{2n} = F_n L_n$ and $F_n = \frac{1}{5}(L_{n+1} + L_{n-1})$ (for example see, [4, *p*.2]), we can rewrite (8) as

$$\lim_{n\to\infty} x_n = \frac{1}{5}(L_{n+1} + L_{n-1})[1 + (L_n - 1)e^{-\frac{1}{2}}]. \tag{9}$$

Now, we get the desired equality by simply solving (9) for e.

Note 2.4. Problem 2.3 was a problem proposal with a solution by the author in the *Fibonacci Quarterly* [1]. However, the solution provided by the author (namely proof of Problem 2.3) was not published. Aside from the author, this problem was also solved by three other mathematicians and the solution that was provided by Francisco Perdomo and Angel Plaza (jointly) was published as the solution of the problem. An alternative proof of this problem can be found in [1].

Problem 2.5. *Let m and n be positive integers greater than 1, and let $F = \dfrac{F_{mn}}{F_m F_n}$, where F_m, F_n, and F_{mn} are Fibonacci numbers. Then*

$$\left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)(F^{-i} - F^{-i-1})\right]^F = e.$$

Proof. First we note that for $|x| < 1$ we have

$$\frac{e^x}{1-x} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)x^i.$$

Also, it is well-known that $F > 1$ (for example, see [3]). Next, if we substitute $\dfrac{1}{F}$ for x in the above expression, we obtain

$$\frac{Fe^{\frac{1}{F}}}{F-1} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)F^{-i}.$$

Next, if we multiply both sides of the above expression by $\dfrac{F-1}{F}$ we get

$$e^{\frac{1}{F}} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)(F^{-i} - F^{-i-1}).$$

Now, we achieve the desired result by reaching both sides of this expansion to the power F.

Note 2.6. Problem 2.5 was a problem proposal with a solution by the author in the *Fibonacci Quarterly* [3]. However, the solution that was provided by the author (namely proof of Problem 2.5) was not published in the *Fibonacci Quarterly*. Aside from the author, this problem was also solved by ten other mathematicians and the solution that was provided by Norbert Jensen was the solution that was published. For another proof of this problem see [3].

3. Using Difference Equations of Order One

In this section we find Euler's number by using two different first-order linear difference equations with variable coefficients.

Problem 3.1. *Let* $n!(n+2)(x_{n+1}-x_n)=1$, *where* $x_1=\frac{1}{2}$, *and* $n\in\mathbb{Z}^+$. *Then*

$$\left(\sum_{n=1}^{\infty}(-1)^n x_n\right)^{-1}=e.$$

Proof. First we note that the given difference equation can be rewritten as

$$x_{n+1}=x_n+\frac{1}{n!(n+2)}.$$

Hence, $x_1=\dfrac{1}{2}$, $x_2=\dfrac{5}{6}$, $x_3=\dfrac{23}{24}$, $x_4=\dfrac{119}{120}$, and we conjecture that

$$x_n=\frac{(n+1)!-1}{(n+1)!}.$$

This conjecture can be proven by mathematical induction. Thus,

$$\sum_{n=1}^{\infty}(-1)^n x_n=\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)!-1}{(n+1)!}=e^{-1}.$$

Now, we obtain the desired result form the above equality.

Problem 3.2. *Let* $nx_n-x_{n-1}=0$, *where* $x_0=1$. *Then*

$$\left(\sum_{n=0}^{\infty}x_n\right)^{-1}=e.$$

Proof follows from mathematical induction and the fact that $e=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$.

4. Questions

We used two second-order linear homogeneous difference equations with variable coefficients as well as one second-order linear homogeneous difference equation with constant coefficients to obtain e. Also, we obtained e by using two first-order linear difference equations with variable coefficients, one homogeneous and one nonhomogeneous. We invite the reader to obtain the Euler's number e in connection with some other suitable difference equations.

Question 4.1. *Can you think of other linear difference equations of order one or two that would produce Euler's number e?*

Question 4.2. *Can you think of linear difference equations of order larger than two that would produce Euler's number e?*

Question 4.3. *Can you think of nonlinear difference equations of any order that would produce Euler's number e?*

We note that there are some rather obvious difference equations that would produce e. For example, the particular solution to the difference equation $x_{n+1} - x_n = 0$ will be e provided $x_0 = e$.

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