ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-790 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Find the largest constant c such that $F_{n+1}^2 > cF_{2n}$ for all even positive integers n.

B-791 ***Proposed by Andrew Cusumano, Great Neck, NY***

Prove that, for all n, $F_{n+11} + F_{n+7} + 8F_{n+5} + F_{n+3} + 2F_n$ is divisible by 18.

B-792 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let the sequence $\langle a_n \rangle$ be defined by the recurrence $a_{n+1} = a_n^2 - a_n + 1$, $n > 0$, where the initial term, a_1, is an arbitrary real number larger than 1. Express $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$ in terms of a_1.

B-793 ***Proposed by Wray Brady, Jalisco, Mexico***

Show that $2^n L_n \equiv 2 \pmod 5$ for all positive integers n.

B-794 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***

For x a real number and n a positive integer, prove that

$$\left(\frac{F_2}{F_1}\right)^x + \left(\frac{F_3}{F_2}\right)^x + \cdots + \left(\frac{F_{n+1}}{F_n}\right)^x \geq n + x \ln F_{n+1}.$$

B-795 ***Proposed by Wray Brady, Jalisco, Mexico***

Evaluate

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} L_{2n}.$$

SOLUTIONS

A Cauchy Convolution

B-759 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 32, no. 1, February 1994)***

Show that for all positive integers k and all nonnegative integers n,

$$\sum_{j=0}^{n} F_{k(j+1)}P_{k(n-j+1)} = \frac{F_k P_{k(n+2)} - P_k F_{k(n+2)}}{2Q_k - L_k}.$$

Solution by Paul S. Bruckman, Edmonds, WA

Let $S_{n,k}$ denote the given sum. Treating k as fixed, make the following substitutions, for brevity: $s = \alpha^k, t = \beta^k, u = p^k$, and $v = q^k$. Then,

$$\begin{aligned}
S_{n,k}\sqrt{40} &= \sum_{j=0}^{n}(s^{j+1} - t^{j+1})(u^{n-j+1} - v^{n-j+1}) \\
&= \sum_{j=0}^{n}[su^{n+1}(s/u)^j - tu^{n+1}(t/u)^j - sv^{n+1}(s/v)^j + tu^{n+1}(t/v)^j] \\
&= \frac{su}{s-u}(s^{n+1} - u^{n+1}) - \frac{tu}{t-u}(t^{n+1} - u^{n+1}) - \frac{sv}{s-v}(s^{n+1} - v^{n+1}) + \frac{tv}{t-v}(t^{n+1} - v^{n+1}) \\
&= \frac{u}{(s-u)(t-u)}[(s^{n+2} - su^{n+1})(t-u) - (t^{n+2} - tu^{n+1})(s-u)] \\
&\quad - \frac{v}{(s-v)(t-v)}[(s^{n+2} - sv^{n+1})(t-v) - (t^{n+2} - tv^{n+1})(s-v)] \\
&= [(-1)^k p^{-k} + p^k - L_k]^{-1}[s^{n+2}t - stu^{n+1} - s^{n+2}u + su^{n+2} - st^{n+2} + stu^{n+1} + t^{n+2}u - tu^{n+2}] \\
&\quad - [(-1)^k q^{-k} + q^k - L_k]^{-1}[s^{n+2}t - stv^{n+1} - s^{n+2}v + sv^{n+2} - st^{n+2} + stv^{n+1} + t^{n+2}v - tv^{n+2}] \\
&= (2Q_k - L_k)^{-1}[s^{n+2}(t-u) + t^{n+2}(u-s) + u^{n+2}(s-t) - s^{n+2}(t-v) - t^{n+2}(v-s) - v^{n+2}(s-t)] \\
&= (2Q_k - L_k)^{-1}[(u^{n+2} - v^{n+2})(s-t) - (s^{n+2} - t^{n+2})(u-v)].
\end{aligned}$$

The result follows.

Also solved by Glenn Bookhout, C. Georghiou, Pentti Haukkanen, and the proposer.

A Simple Inequality

B-760 ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, MO (Vol. 32, no. 2, May 1994)***

Prove that $F_{n+1}^2 \geq F_{2n}$ for all $n \geq 0$.

Solution by many readers

This inequality follows from Hoggatt's Identity (I_{10}) from [1]:

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2.$$

The inequality is true for all n, not just $n \geq 0$. Equality holds if and only if $F_{n-1} = 0$, i.e., if and only if $n = 1$.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, Calif.: The Fibonacci Association, 1979.

Lord pointed out the stronger inequality, $F_{n+p}^2 \geq F_{2n}F_{2p}$, which follows from Hoggatt's Identity (I_{25}). Prielipp mentions that the result $F_{m+n} \geq F_m F_n$, if $m, n > 0$, which comes from The American Mathematical Monthly, 1960, p. 876, implies that we can extend the given inequality to $F_{n+1}^2 \geq F_{2n} \geq F_n^2$ if $n \geq 0$. Seiffert generalized in another direction, and we present his result as problem B-790 in this issue.

Generalization by Richard André-Jeannin, Longwy, France

Consider the sequences $\langle U_n \rangle$ and $\langle V_n \rangle$ satisfying the recurrence $W_n = PW_{n-1} - QW_{n-2}$, $n \geq 2$, with initial conditions $U_0 = 0, U_1 = 1$ and $V_0 = 2, V_1 = P$. It is known and readily verified that $V_n = U_{n+1} - QU_{n-1}$ and $U_{2n} = U_n V_n$. Thus, $U_{n+1}^2 - Q^2 U_{n-1}^2 = (U_{n+1} + QU_{n-1})(U_{n+1} - QU_{n-1}) = PU_n V_n = PU_{2n}$. Hence, $U_{n+1}^2 = Q^2 U_{n-1}^2 + PU_{2n}$. This gives us the desired generalization: $U_{n+1}^2 \geq PU_{2n}$.

Also solved by Charles Ashbacher, Michel A. Ballieu, Brian D. Beasley, Glenn Bookhout, Paul S. Bruckman, Charles K. Cook., Bill Correll, Jr., M. N. Deshpande, Steve Edwards, Piero Filipponi, Russell Jay Hendel, Norbert Jensen, Joseph J. Kostal, Harris Kwong, Carl Libis, Graham Lord, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David C. Terr, and the proposer.

L Determinants

B-761 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 32, no. 2, May 1994)***

Evaluate the determinants

$$\begin{vmatrix} L_0 & L_1 & L_2 & L_3 & L_4 \\ L_1 & L_0 & L_1 & L_2 & L_3 \\ L_2 & L_1 & L_0 & L_1 & L_2 \\ L_3 & L_2 & L_1 & L_0 & L_1 \\ L_4 & L_3 & L_2 & L_1 & L_0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} L_0^2 & L_1^2 & L_2^2 & L_3^2 & L_4^2 \\ L_1^2 & L_0^2 & L_1^2 & L_2^2 & L_3^2 \\ L_2^2 & L_1^2 & L_0^2 & L_1^2 & L_2^2 \\ L_3^2 & L_2^2 & L_1^2 & L_0^2 & L_1^2 \\ L_4^2 & L_3^2 & L_2^2 & L_1^2 & L_0^2 \end{vmatrix}.$$

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Let $A_n = (a_{ij})$ and $B_n = (b_{ij})$ be $n \times n$ matrices such that $a_{ij} = L_{|i-j|}$ and $b_{ij} = L_{|i-j|}^2$. The values of $\det(A_n)$ and $\det(B_n)$ can be derived as follows.

For small n, we have $\det(A_1) = 2$ and $\det(A_2) = 3$. For $n \geq 3$, subtract the sum of the second and third rows of A_n from the first row, then subtract the difference of the first and third columns from the second column. In the modified matrix, the first row is $(-2, 0, \ldots)$, and the second column is $(0, 2, 0, \ldots)^{\mathrm{T}}$. Hence, for $n \geq 3$, $\det(A_n) = -4\det(A_{n-2})$. Consequently, $\det(A_{2n-1}) = 2(-4)^{n-1}$ and $\det(A_{2n}) = 3(-4)^{n-1}$.

It is straightforward to check that $\det(B_1) = 4$, $\det(B_2) = 15$, and $\det(B_3) = -250$. For $n \geq 4$, add the fourth row of B_n to the first row, then add the third row to the second row. Since

$$L_n^2 + L_{n+3}^2 = (L_{n+2} - L_{n+1})^2 + (L_{n+2} + L_{n+1})^2 = 2(L_{n+2}^2 + L_{n+1}^2),$$

the first and second rows of the modified matrix are proportional, namely, in the ratio of $2:1$. Thus, $\det(B_n) = 0$ for $n \geq 4$.

In particular, the given determinants are $\det(A_5) = 32$ and $\det(B_5) = 0$, respectively.

Also solved by Charles Ashbacher, Paul S. Bruckman, Charles K. Cook, Bill Correll, Jr., Russell Jay Hendel, Norbert Jensen, Carl Libis, H.-J. Seiffert, and the proposer.

Taylor's Series

B-762 ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 32, no. 2, May 1994)

Let n be an integer.

(a) Generalize the numbers (2, 2, 2) to form three three-term arithmetic progressions of integral multiples of Fibonacci and/or Lucas numbers with common differences $3F_n$, $5F_n$, and $3F_n$.

(b) Generalize the numbers (4, 4, 4) to form two such arithmetic progressions with common differences F_n and F_n.

(c) Generalize the numbers (6, 6, 6) to form four such arithmetic progressions with common differences F_n, $5F_n$, $7F_n$, and F_n.

Solution by H.-J. Seiffert, Berlin, Germany

The proofs of the statements presented in the following table are all easy and thus will be omitted.

Arithmetic Progression	Common Difference	Generalizes ($n = 0$)
$(-2F_{n-2}, L_n, 2F_{n+2})$	$3F_n$	$(2, 2, 2)$
$(-2L_{n-1}, L_n, 2L_{n+1})$	$5F_n$	$(2, 2, 2)$
$(2F_{n-1}, F_{n+3}, 2L_{n+1})$	$3F_n$	$(2, 2, 2)$
$(2F_{n+3}, L_{n+3}, 4F_{n+2})$	F_n	$(4, 4, 4)$
$(-4F_{n-2}, -L_{n-3}, 2F_{n-3})$	F_n	$(4, 4, 4)$
$(2L_{n+2}, 3F_{n+3}, 2F_{n+4})$	F_n	$(6, 6, 6)$
$(2L_{n-2}, 3L_n, 2L_{n+2})$	$5F_n$	$(6, 6, 6)$
$(-2F_{n-4}, 3L_n, 2F_{n+4})$	$7F_n$	$(6, 6, 6)$
$(-2F_{n-4}, 3F_{n-3}, 2L_{n-2})$	F_n	$(6, 6, 6)$

Editorial Comment: Larry Taylor asked about three-term arithmetic progressions such that

(1) each term is an integral multiple of either F_{n+a} or L_{n+a} for some integer a;
(2) the common difference is a positive integral multiple of F_n;
(3) the values assumed by the terms when $n = 0$ are positive, equal, and do not exceed 6.

He conjectured that all such arithmetic progressions are given by the solutions of problems B-762 and H-422 (*The Fibonacci Quarterly* **28.3** [1990]:285-87).

Bruckman investigated this problem and came up with the following table of arithmetic progressions of the form $(a_1H_{n+b_1}, a_2J_{n+b_2}, a_3K_{n+b_3})$ with common difference cF_n and where H, J, and K are each either "F" or "L". When $n = 0$, these progressions reduce to (e, e, e).

a_1	H	b_1	a_2	J	b_2	a_3	K	b_3	c	e
−1	L	−1	−1	F	−2	1	F	−1	1	1
−1	F	−2	+1	F	−1	1	F	+1	1	1
+1	F	−1	+1	F	+1	1	F	+2	1	1
+1	F	+1	+1	F	+2	1	L	+1	1	1
−1	L	−1	+1	F	−1	1	F	+2	2	1
−1	F	−2	+1	F	+1	1	L	+1	2	1
−2	F	−2	+1	F	−3	2	F	−1	1	2
+1	F	−3	+2	F	−1	1	L	0	1	2
+2	F	−1	+1	L	0	2	F	+1	1	2
+1	L	0	+2	F	+1	1	F	+3	1	2
+2	F	+1	+1	F	+3	2	F	+2	1	2
+1	F	−3	+1	L	0	1	F	+3	2	2
−2	L	−1	+1	F	−3	2	F	+1	3	2
−2	F	−2	+1	L	0	2	F	+2	3	2
+2	F	−1	+1	F	+3	2	L	+1	3	2
−2	L	−1	+1	L	0	2	L	+1	5	2
−3	F	−2	−1	F	−4	1	L	−2	1	3
−1	F	−4	+1	L	−2	3	F	−1	1	3
+3	F	+1	+1	L	+2	1	F	+4	1	3
+1	L	+2	+1	F	+4	3	F	+2	1	3
−3	L	−1	+1	L	−2	1	L	+2	5	3
+1	L	−2	+1	L	+2	3	L	+1	5	3
−4	F	−2	−1	L	−3	2	F	−3	1	4
+2	F	+3	+1	L	+3	4	F	+2	1	4
−4	L	−1	−1	L	−3	2	L	0	5	4
−1	L	−3	+2	L	0	1	L	+3	5	4
+2	L	0	+1	L	+3	4	L	+1	5	4
−2	F	−4	+3	F	−3	2	L	−2	1	6
+2	L	+2	+3	F	+3	2	F	+4	1	6
+2	L	−2	+3	L	0	2	L	+2	5	6
−2	F	−4	+3	L	0	2	F	+4	7	6

He believes the list is exhaustive [for $e \le 6$ and $\gcd(a_1, a_2, a_3) = 1$] but does not have a proof. The editor did a computer search and did not find any additional examples even if condition (2) is dropped.

Also solved by Paul S. Bruckman and the proposer.

Matrix Power

B-763 ***Proposed by Juan Pla, Paris, France***
(Vol. 32, no. 2, May 1994)

Let

$$A = \begin{pmatrix} e^{i\pi/3} & \sqrt{2} \\ \sqrt{2} & e^{-i\pi/3} \end{pmatrix}.$$

Express A^n in terms of Fibonacci and/or Lucas numbers.

Solution by H.-J. Seiffert, Berlin, Germany

The answer is

$$A^n = F_n A + F_{n-1} I, \tag{1}$$

where I denotes the identity matrix.

We prove this result for any matrix A of the form

$$A = \begin{pmatrix} a & c \\ b & 1-a \end{pmatrix}$$

with determinant -1 (which is true of our present proposal).

By direct calculation, we have $A^2 = A + I$ since $\det A = -1$, so equation (1) is true for $n = 2$. It is also clearly true for $n = 1$. We proceed by induction. Assume that equation (1) holds for some n. Then $A^{n+1} = A^n A = (F_n A + F_{n-1} I)A = F_n A^2 + F_{n-1} A = F_n(A+I) + F_{n-1}A = (F_n + F_{n-1})A + F_n I = F_{n+1}A + F_n I$ and equation (1) holds for $n+1$. Thus it is true for all positive integral n.

Seiffert also showed that equation (1) is true for all negative n as well. We omit the proof.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Steve Edwards, Piero Filipponi, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Murray S. Klamkin, Joseph J. Kostal, Bob Prielipp, M. N. S. Swamy, and the proposer.

Secret Treasures Hidden in Pascal's Triangle

B-764 ***Proposed by Mark Bowron, Tucson, AZ***
(Vol. 32, no. 2, May 1994)

Consider row n of Pascal's triangle, where n is a fixed positive integer. Let S_k denote the sum of every fifth entry, beginning with the k^{th} entry, $\binom{n}{k}$. If $0 \le i < j < 5$, show that $|S_i - S_j|$ is always a Fibonacci number.

Solution by the proposer, Channelview, TX

For $n \ge 1$, let $s_k(n)$ be the sum of every fifth entry in row n (including entries outside Pascal's triangle, which by convention are all zero) that includes $\binom{n}{\lfloor n/2 \rfloor - 2 - k}$ as a summand $(0 \le k < 5)$. The following hold by symmetry of Pascal's triangle:

$$\begin{array}{cc} \underline{n \text{ even}} & \underline{n \text{ odd}} \\ s_0(n) = s_4(n) & s_1(n) = s_4(n) \\ s_1(n) = s_3(n) & s_2(n) = s_3(n) \end{array}$$

Define $D_0(n) = s_1(n) - s_0(n)$, $D_1(n) = s_2(n) - s_1(n)$, and $D_2(n) = s_2(n) - s_0(n)$. By the above, it suffices to show that $D_0(n)$, $D_1(n)$, and $D_2(n)$ are Fibonacci numbers for each $n \ge 1$. Claim:

$$\begin{array}{cc} \underline{n \text{ even}} & \underline{n \text{ odd}} \\ D_0(n) = F_n & D_0(n) = F_{n-1} \\ D_1(n) = F_{n-1} & D_1(n) = F_n \\ D_2(n) = F_{n+1} & D_2(n) = F_{n+1} \end{array}$$

The proof is by induction on n. The claim is easily seen to hold for $n = 1$. Let $n > 1$ and assume that the claim holds for all positive integers less than n.

Suppose n is even. By the recursion that defines Pascal's triangle, we have

$$\begin{aligned}
s_0(n) &= s_0(n-1)+s_1(n-1),\\
s_1(n) &= s_1(n-1)+s_2(n-1),\\
s_2(n) &= s_2(n-1)+s_3(n-1) = 2s_2(n-1).
\end{aligned}$$

Thus, by the induction hypothesis and previous results,

$$\begin{aligned}
D_0(n) &= s_1(n)-s_0(n) = s_1(n-1)+s_2(n-1)-s_0(n-1)-s_1(n-1)\\
&= D_0(n-1)+D_1(n-1) = F_{n-2}+F_{n-1} = F_n;\\
D_1(n) &= s_2(n)-s_1(n) = 2s_2(n-1)-s_1(n-1)-s_2(n-1)\\
&= D_1(n-1) = F_{n-1};\\
D_2(n) &= s_2(n)-s_0(n) = 2s_2(n-1)-s_0(n-1)-s_1(n-1)\\
&= D_2(n-1)+D_1(n-1) = F_n+F_{n-1} = F_{n+1}.
\end{aligned}$$

The equations for odd n are similar. This completes the proof.

Also solved by Paul S. Bruckman, Norbert Jensen, and H.-J. Seiffert.

An Expansion of e

B-765 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 32, no. 2, May 1994)***

Let m and n be positive integers greater than 1, and let $x = F_{mn}/(F_mF_n)$. What famous constant is represented by

$$\left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)\left(\frac{1}{x^i}-\frac{1}{x^{i+1}}\right)\right]^x?$$

Solution by Norbert Jensen, Kiel, Germany

For $m,n>1$, we have $F_{mn}\ge F_{m+n} = F_mF_{n+1}+F_{m-1}F_n > F_mF_n$. Thus $x>1$ and all the terms of the series are positive. We may thus rearrange the order of summation.

We find that for any $x>1$,

$$\begin{aligned}
\left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)\left(\frac{1}{x^i}-\frac{1}{x^{i+1}}\right)\right]^x &= \left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)\frac{1}{x^i}\left(1-\frac{1}{x}\right)\right]^x = \left(\frac{x-1}{x}\right)^x\left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)\frac{1}{x^i}\right]^x\\
&= \left(\frac{x-1}{x}\right)^x\left[\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{i=j}^{\infty}\frac{1}{x^i}\right]^x = \left(\frac{x-1}{x}\right)^x\left[\sum_{j=0}^{\infty}\frac{x^{-j}}{j!}\,\frac{x}{x-1}\right]^x\\
&= \left(\frac{x-1}{x}\right)^x\left[\frac{x}{x-1}e^{1/x}\right]^x = e.
\end{aligned}$$

Thus the magic constant is e.

Also solved by O. Brugia and P. Filipponi (jointly), Paul S. Bruckman, Bill Correll, Jr., Steve Edwards, Russell Jay Hendel, Hans Kappus, Carl Libis, H.-J. Seiffert, and the proposer.

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