Art and Sequences

“If you can’t solve it, make art”

“I can’t get interested in a sequence unless it leads to a beautiful picture”

Neil J. A. Sloane

Math. Dept., Rutgers University, and The OEIS Foundation,
Highland Park, NJ;   njasloane@gmail.com

in collaboration with Scott R. Shannon

(oeis.org/wiki/User:Scott R. Shannon)

Guest lecture, Math 640, Rutgers University

February 6, 2020

Outline:

- Three amazing illustrations of sequences
- Stained-glass windows
  - **Rose windows**
  - **Rectangular windows**
- Points on a line

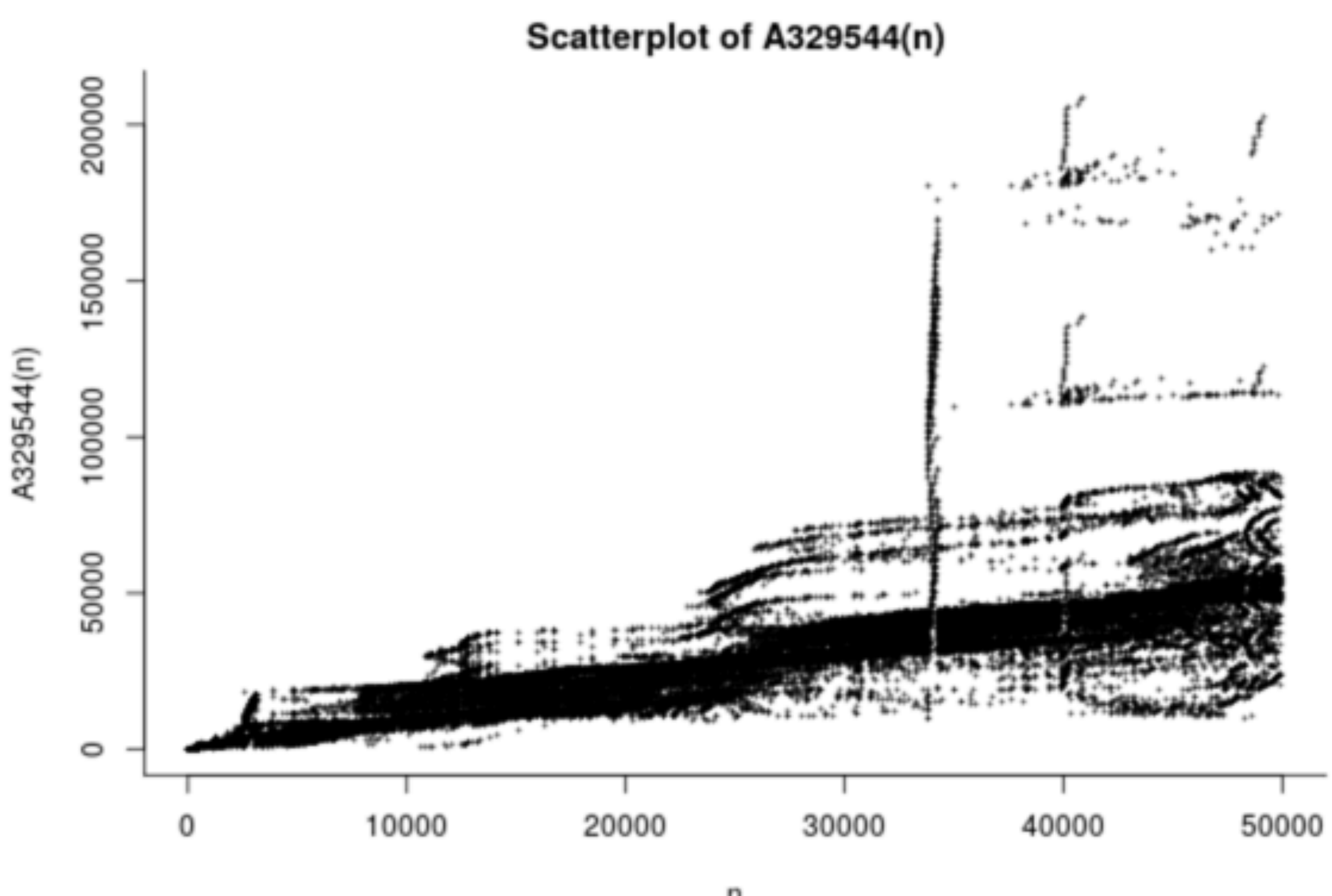
Wild sequence, simple definition, growth is mystery

A329544 (Angelini and Falcoz, Nov. 2019)

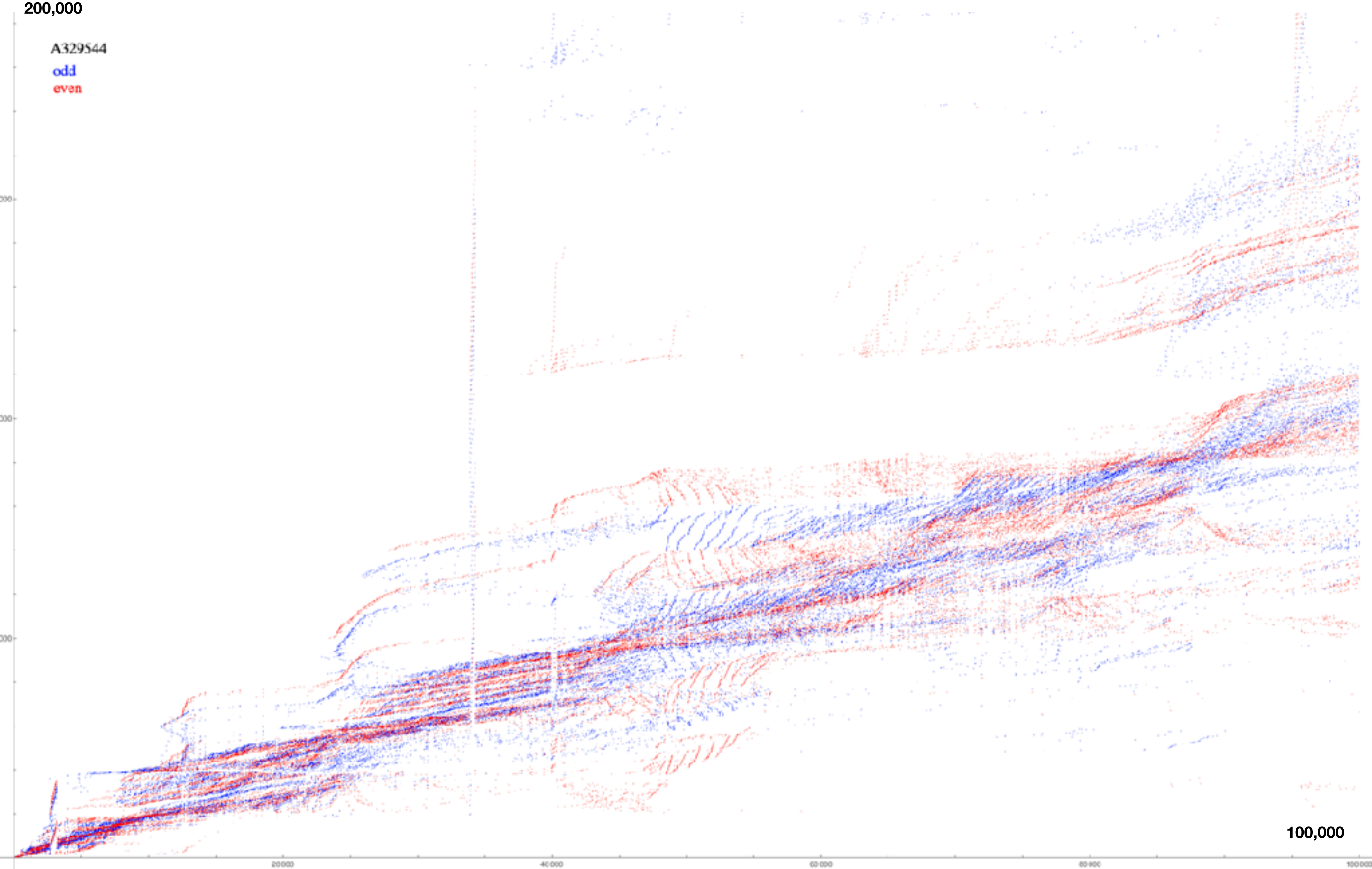
Lex earliest sequence of distinct positive numbers s.t.
if add odd terms, and subtract even terms,
result is always a (positive) palindrome

1 3 2 5 4 ...
1 4 2 7 3 ...

1, 3, 2, 5, 4, 19, 11, 22, 6, 17, 14, 8, 7, 15, 16, 27, 24, 13, 18,

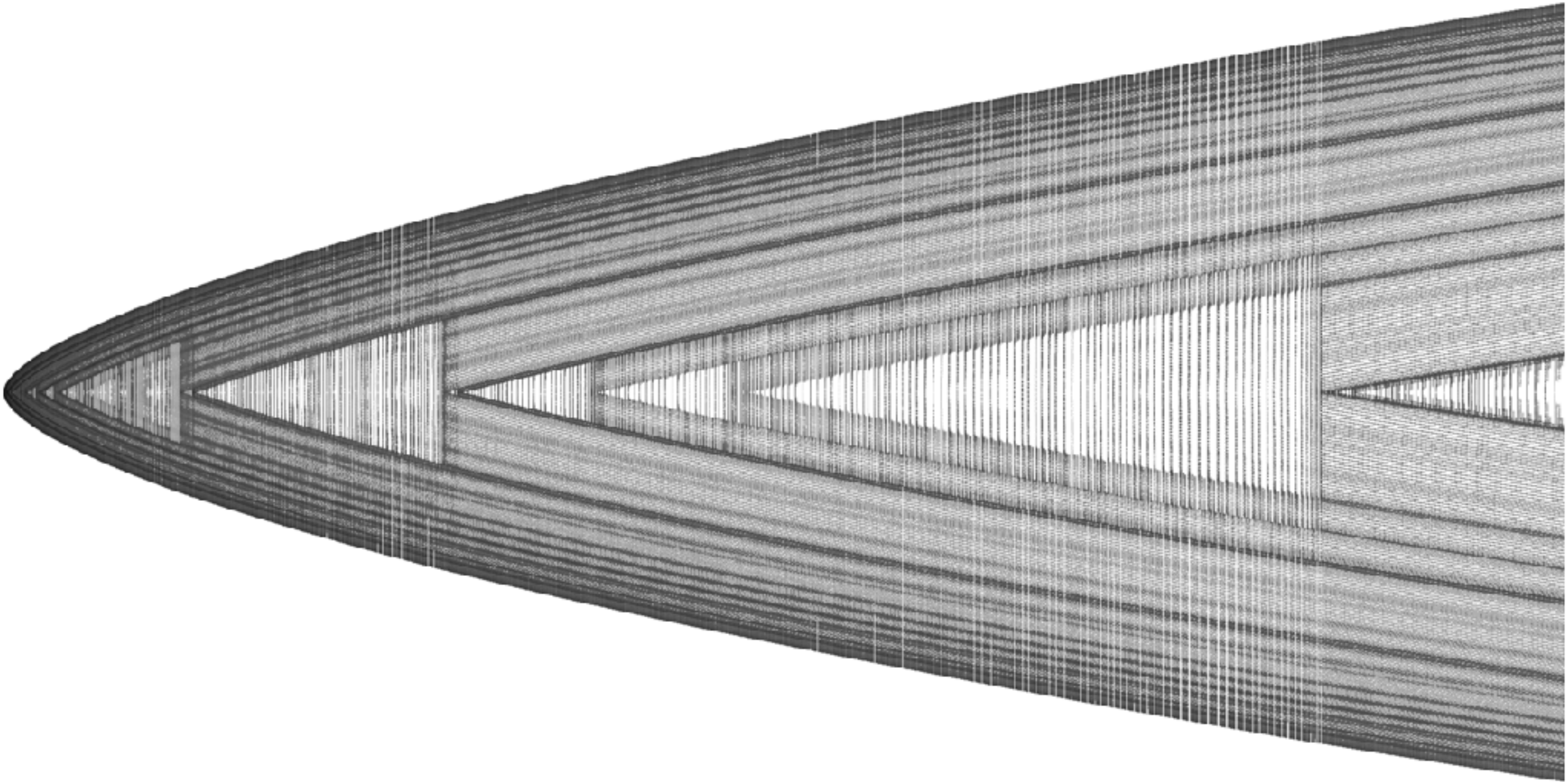


A329544



(Hans Havermann)

A329985 (Rémy Sigrist)



a(n+1) = a(k) - a(n), where k = number of times a(n) appeared in first n terms.

A330189
Scott Shannon

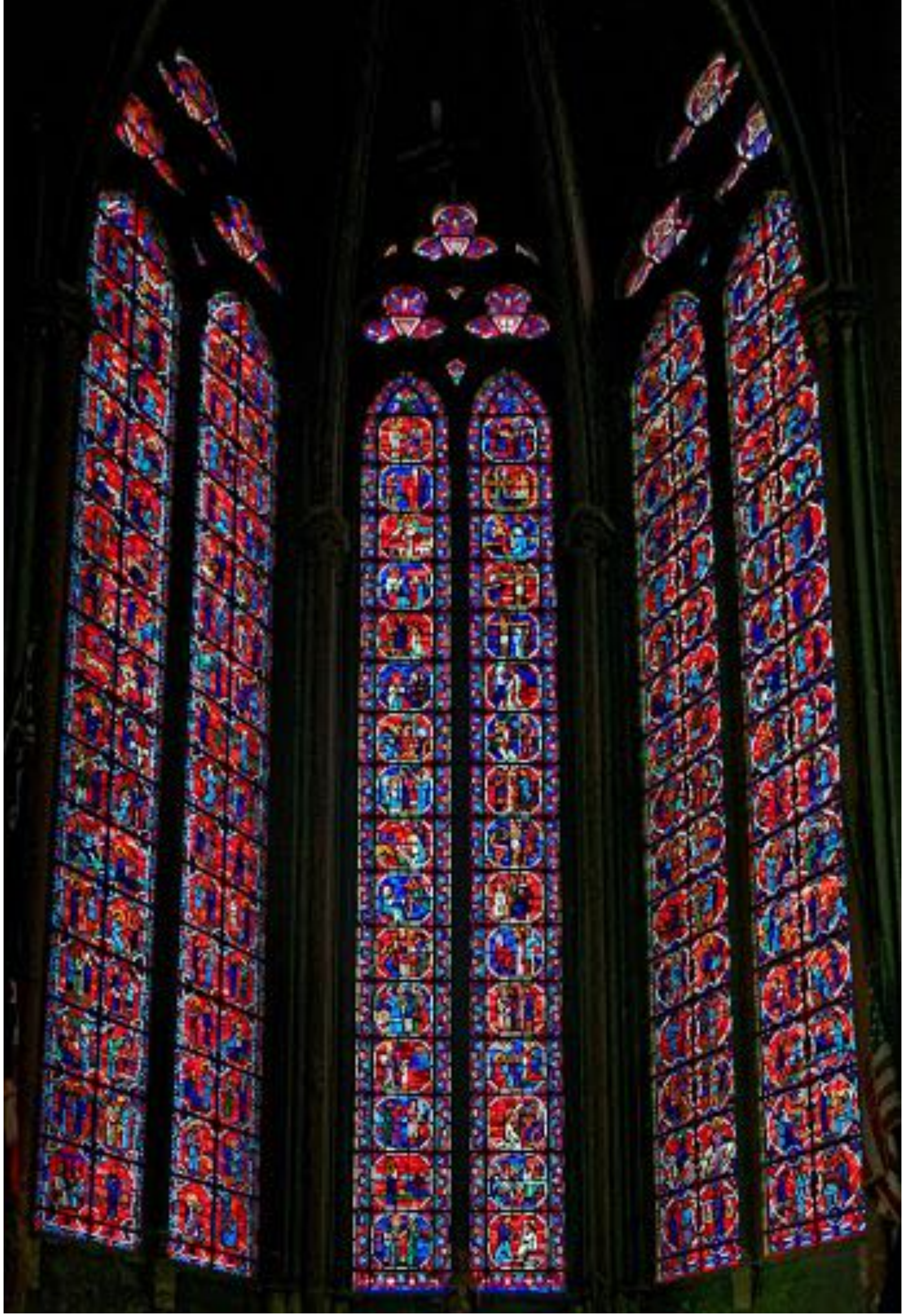
Knight on square spiral, move to unvisited cell with fewest visited nbrs, if a tie choose lowest spiral number

Stained-Glass Windows

Our story begins in France 1967 …

France 1967

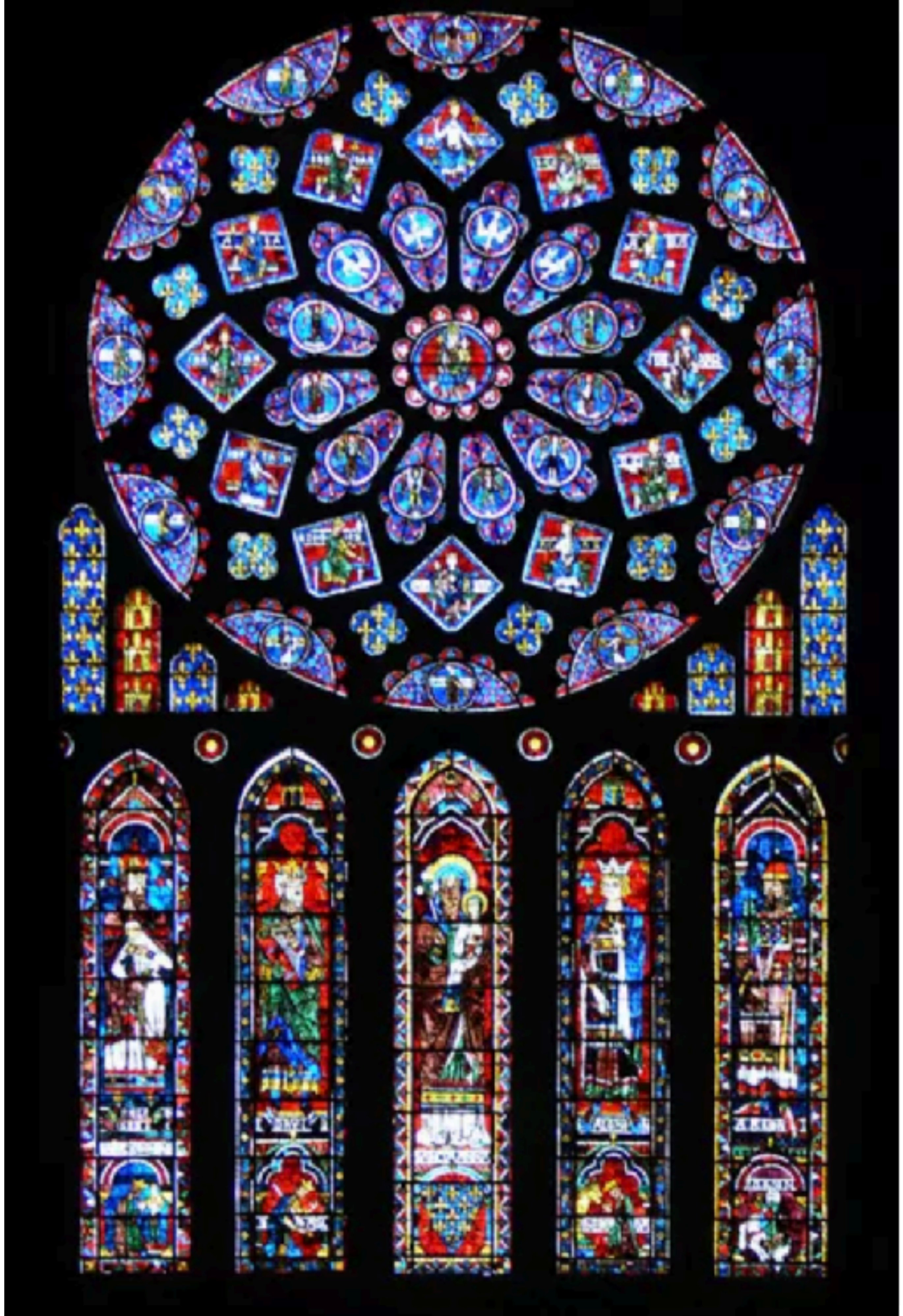
Amiens



Rose window

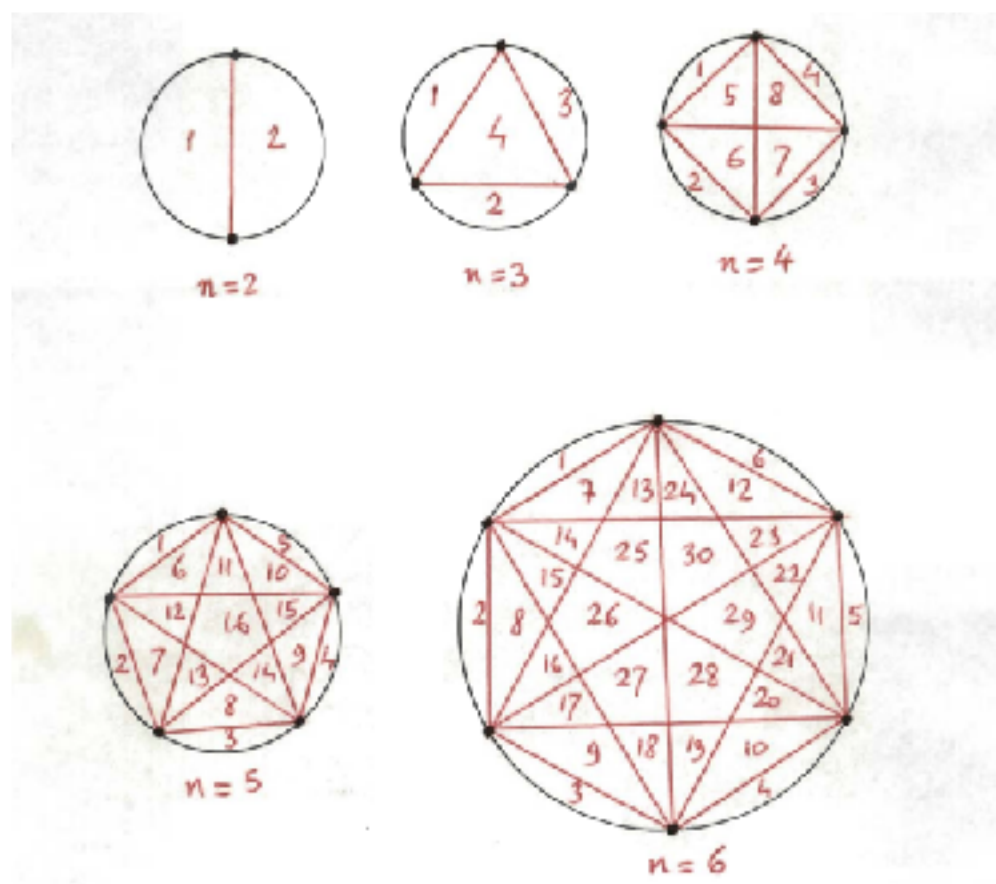
Amiens, France

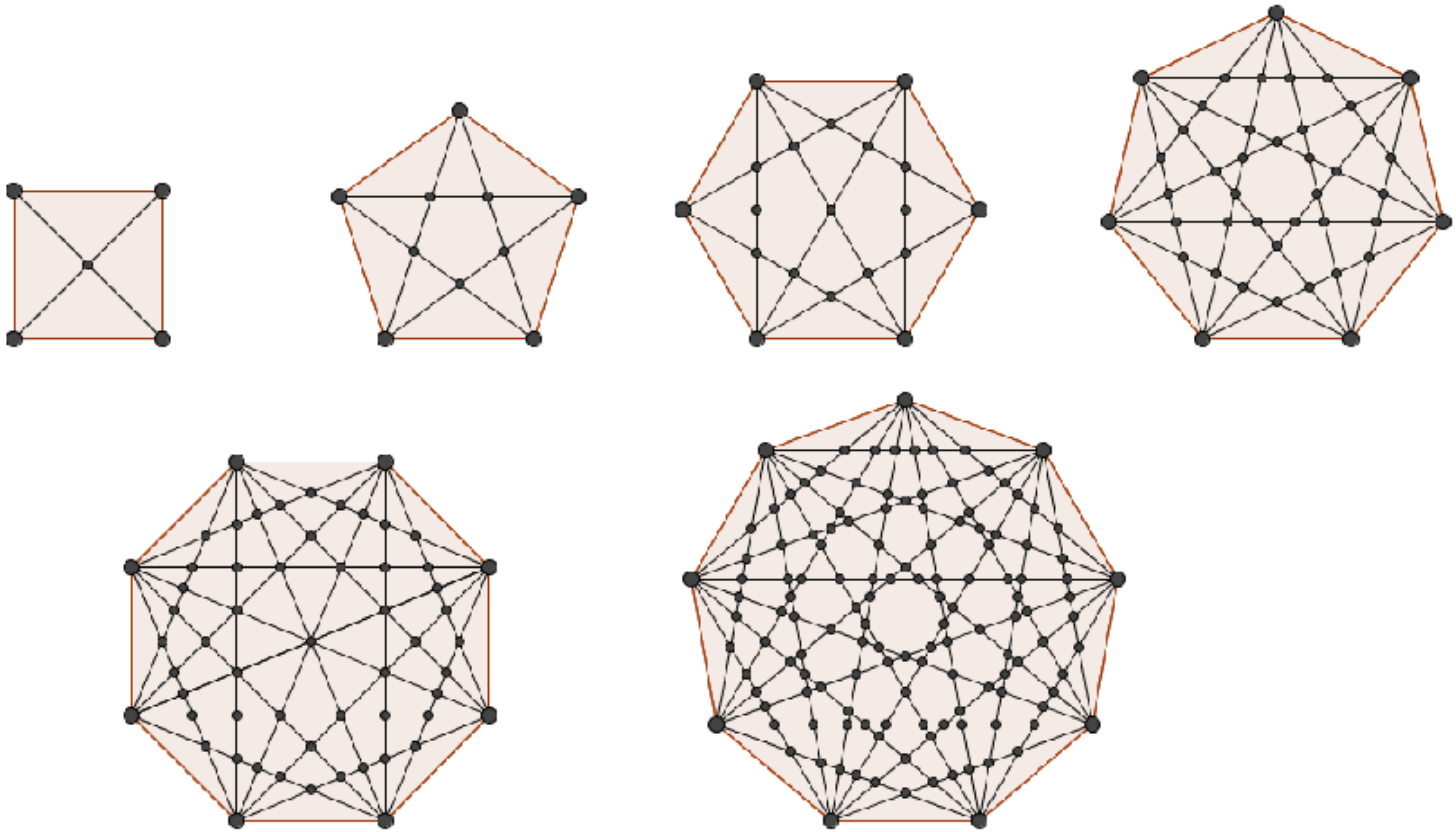
Chartres, France



Counting intersection points of regular polygons with all diagonals drawn

Letter from Jean Meeus in 1974:





A6561: 1, 5, 13, 35, 49, 126, ...

Number of (internal) intersection points of all diagonals

A7569 = total number of points; A7678 = number of regions; A135563 = no. of edges

BELL LABS, 1990’s , I put problem on blackboard in Commons Room

Solved by Bjorn Poonen and Michael Rubinstein, SIAM J Disc. Math., 1998:

Number of interior vertices is

$$\binom{n}{4} + (-5n^3 + 45n^2 - 70n + 24)/24 \cdot \delta_2(n) - (3n/2) \cdot \delta_4(n)$$
$$+ (-45n^2 + 262n)/6 \cdot \delta_6(n) + 42n \cdot \delta_{12}(n) + 60n \cdot \delta_{18}(n)$$
$$+ 35n \cdot \delta_{24}(n) - 38n \cdot \delta_{30}(n) - 82n \cdot \delta_{42}(n) - 330n \cdot \delta_{60}(n)$$
$$- 144n \cdot \delta_{84}(n) - 96n \cdot \delta_{90}(n) - 144n \cdot \delta_{120}(n) - 96n \cdot \delta_{210}(n).$$

where $\delta_4(n) = 1$ iff 4 divides $n, \ldots$

In particular, if n is odd, $a(n) = \binom{n}{4}$

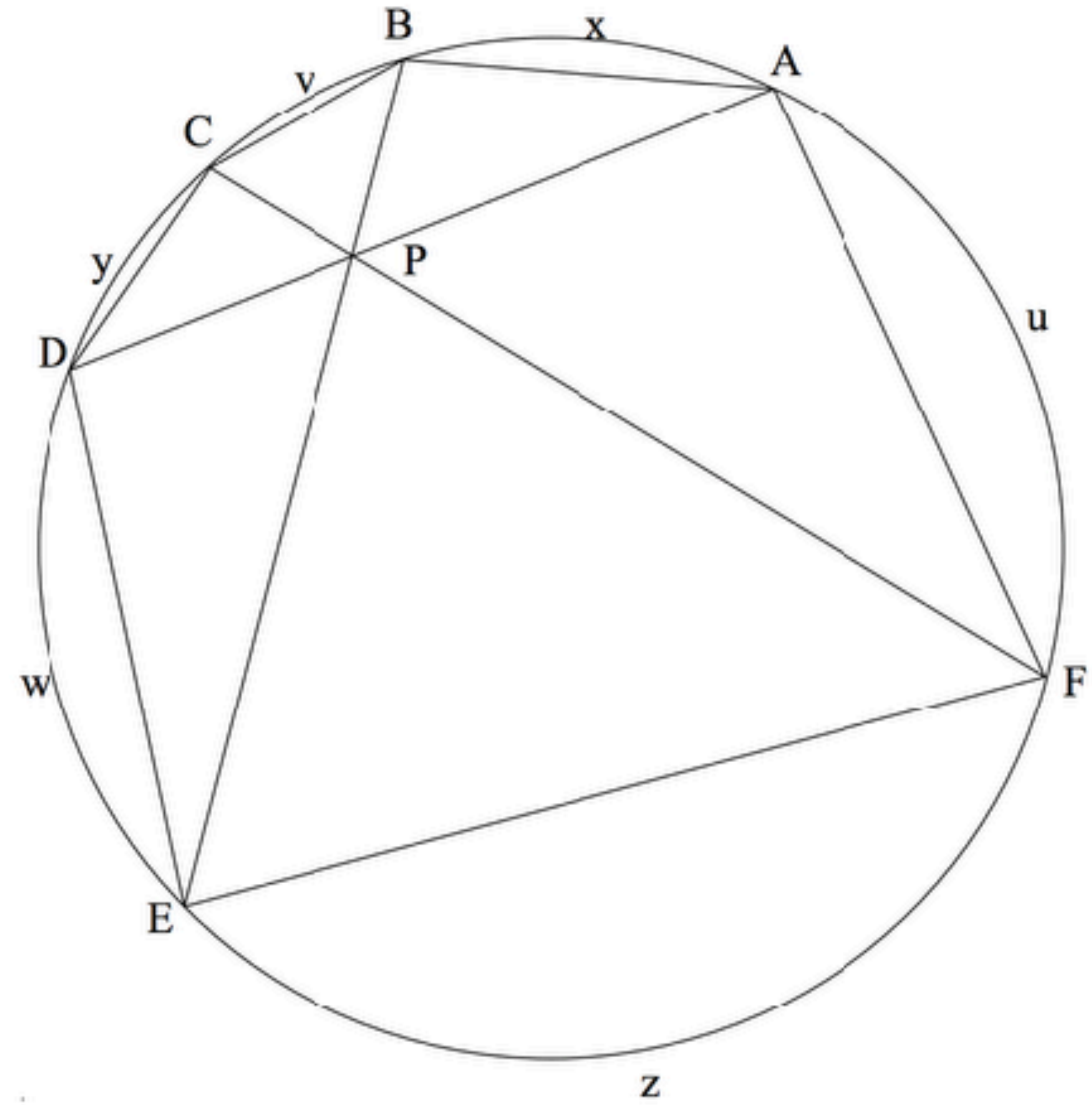
A6561

The triple point lemma:

NASC for 3 diagonals to meet at a point:

$$\sin\pi U \sin\pi V \sin\pi W \ = \ \sin\pi X \sin\pi Y \sin\pi Z$$

$$U+V+W+X+Y+Z \ = \ 1$$



$$U \ = \ \frac{u}{2\pi}, \ \text{etc.}$$

Equivalently:

$\exists$ rationals $\alpha_1, \ldots, \alpha_6$ such that

$$\sum_{j=1..6} \left(e^{i\pi\alpha_j} \ + \ e^{-i\pi\alpha_j}\right) \ = \ 1$$

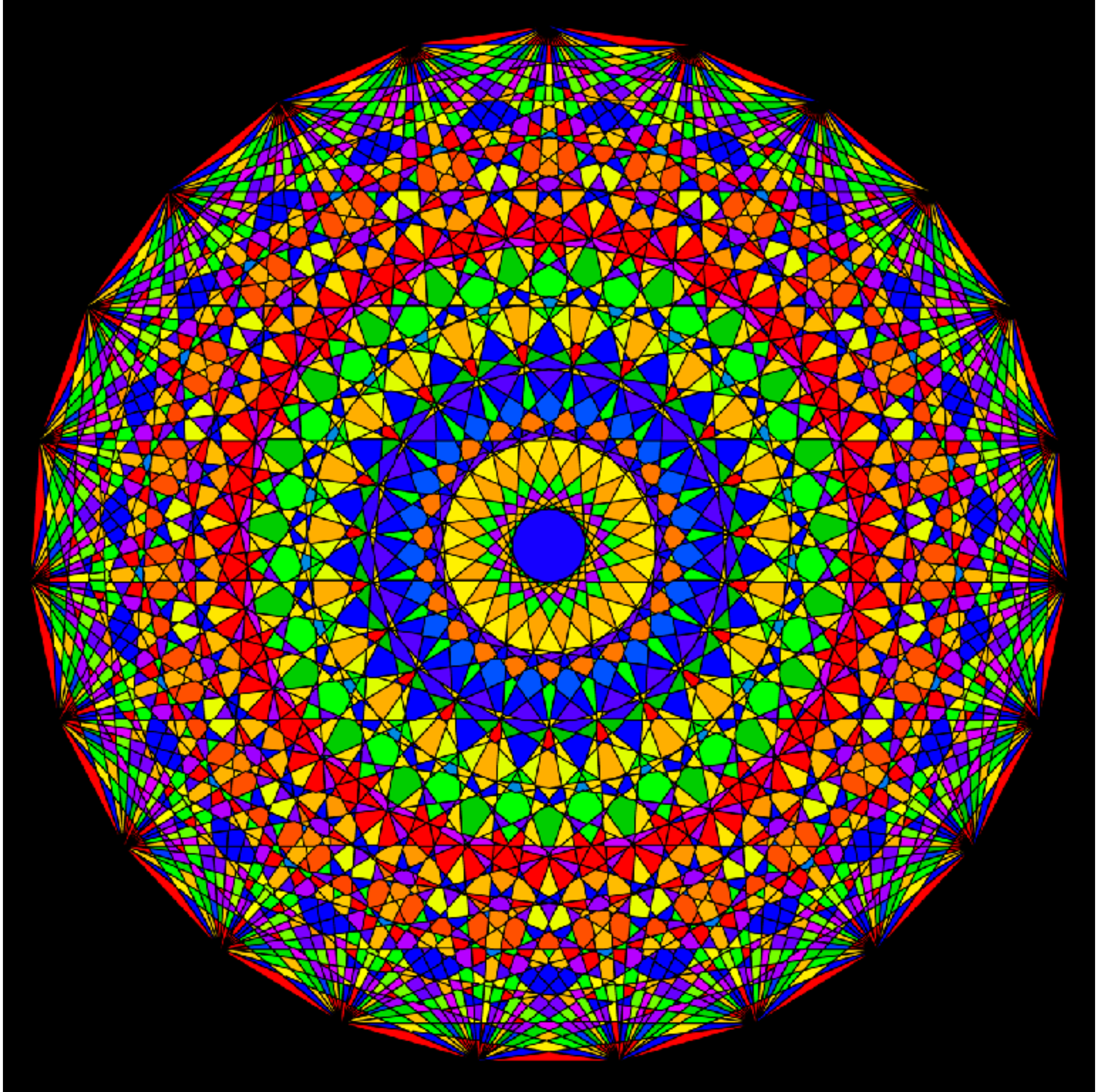
$$\alpha_1 + \cdots + \alpha_6 \ = \ 1$$

Here, $\alpha_1 \ = \ V + W - U - \frac{1}{2}$, etc.

[A trigonometric diophantine equation, solvable: Conway and Jones (1976)]

A7678
Scott
Shannon

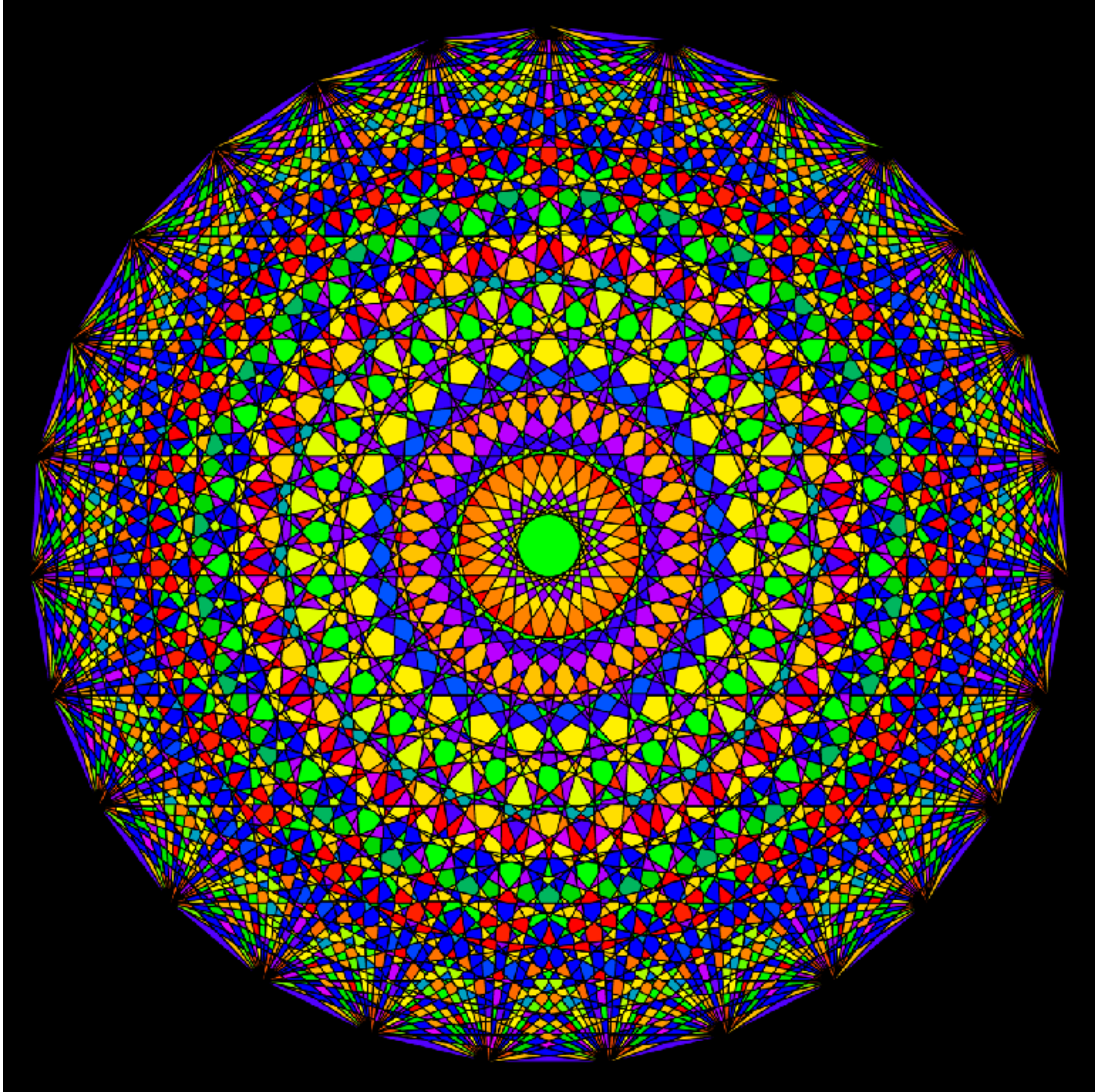
23 points



A7678

Scott Shannon

27 points

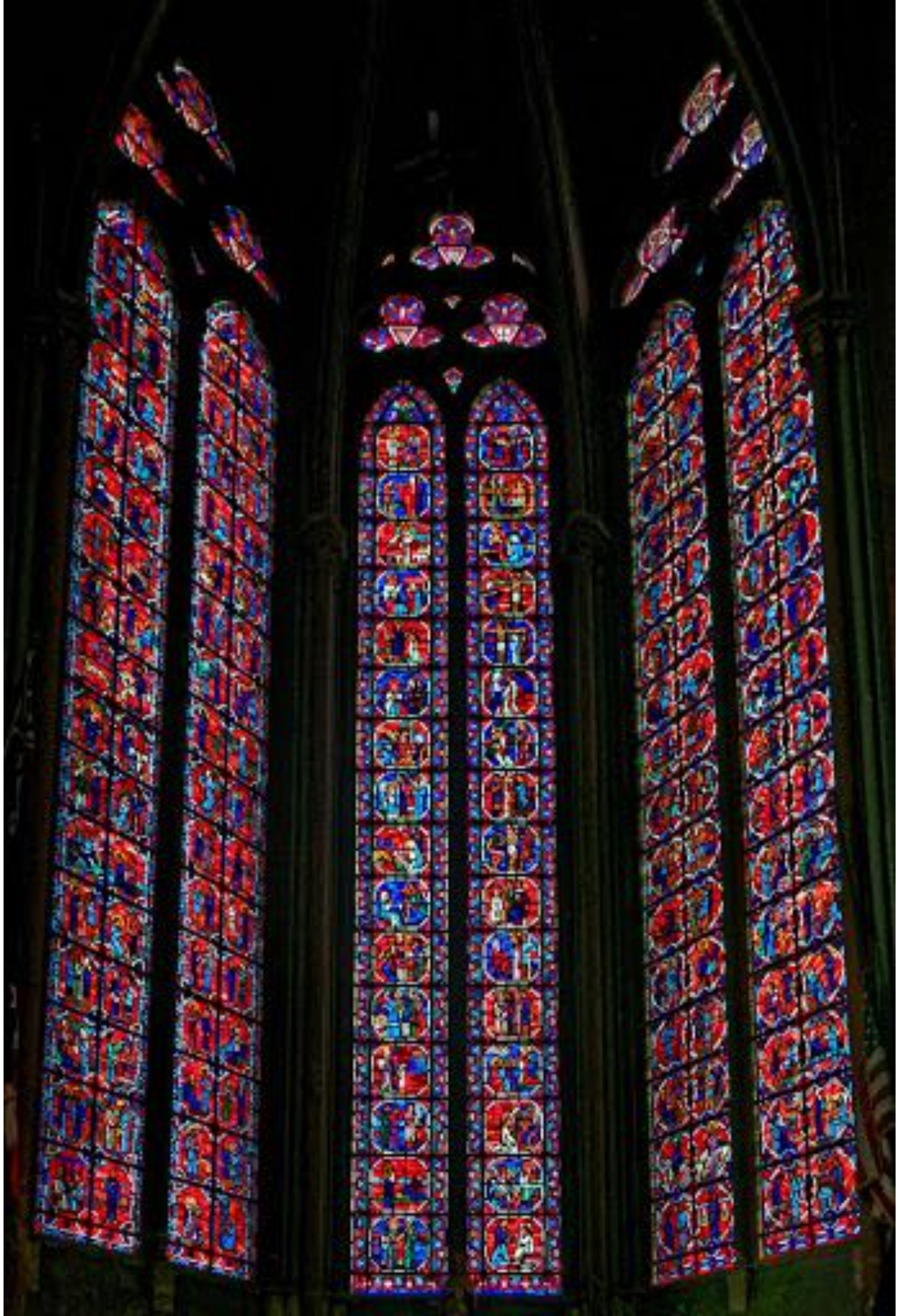


Homework 1

Program in Maple:

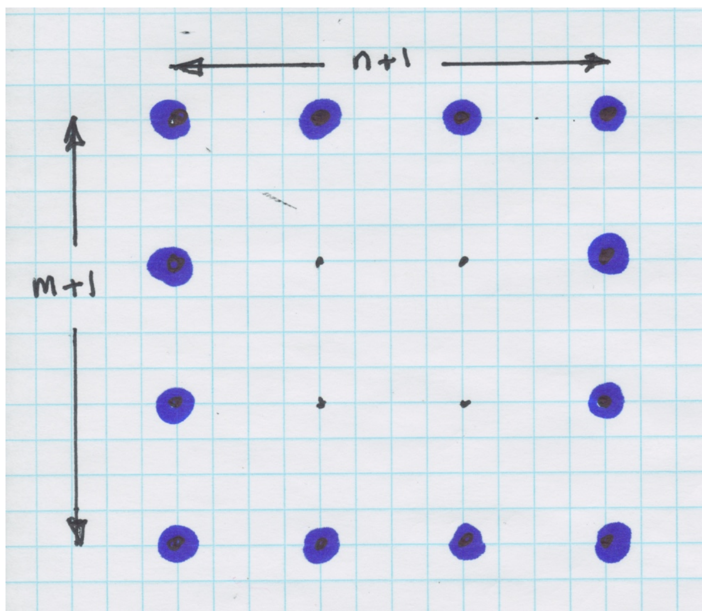
1. **Input n, construct complete graph K_n, count and output Regions, Edges, Vertices from the graph, check against the formulas in A007678. A135565, A007569.**
2. **Input n, output colored picture Rose.n.pdf (or .png)**

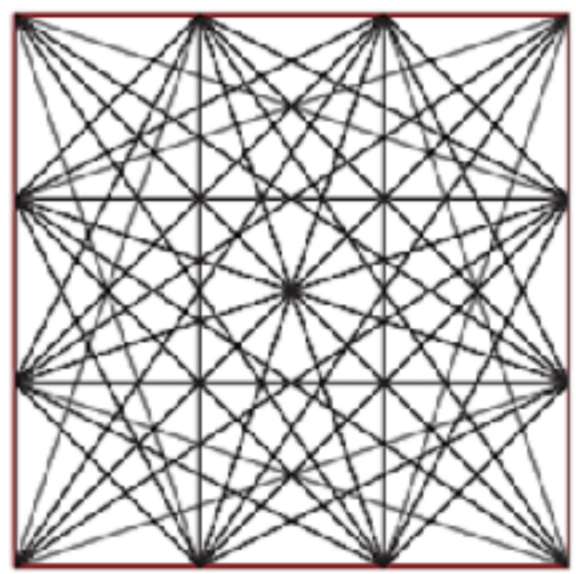
Rectangular Windows

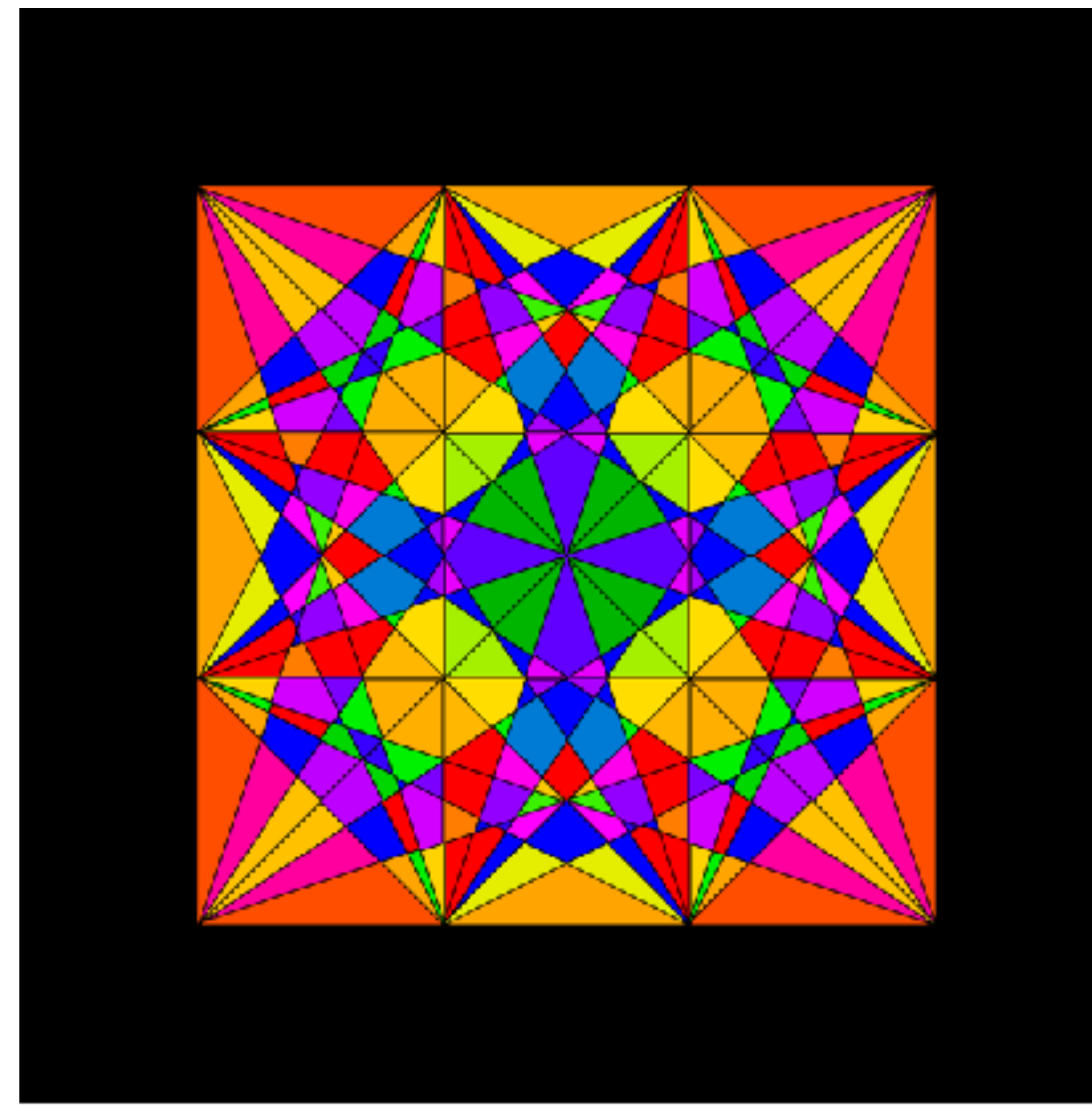


Rectangular Windows

m=3, n=3,
2(m+n) perimeter points,
join every pair by a line segment







R = 340 regions,
E = 596 edges
V = 257 vertices

Euler: R - E + V = 1

Scott Shannon

Rectangular Windows

Need more terms for this table!

TABLE OF R (REGIONS), E (EDGES), V (VERTICES)

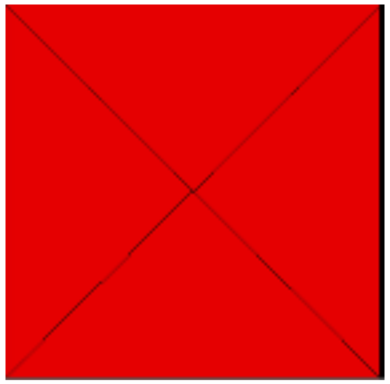
m\n	1			2			3			...
1	4	8	5							
2	16	28	13	56	92	37				
3	46	80	35	142	240	99	340	596	257	
...	...			...			...			...

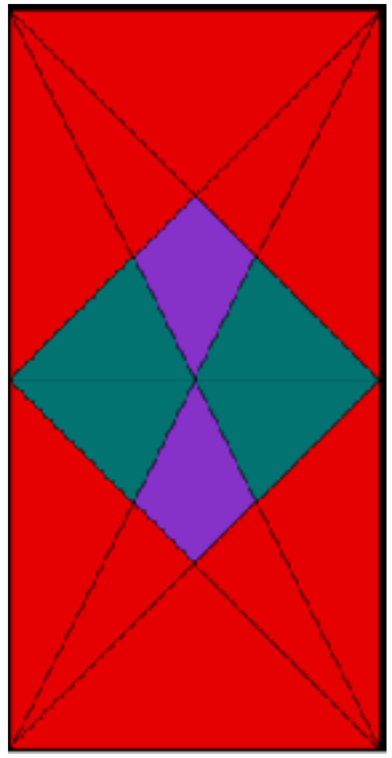
OEIS ENTRIES: REGIONS A331452

EDGES A331454

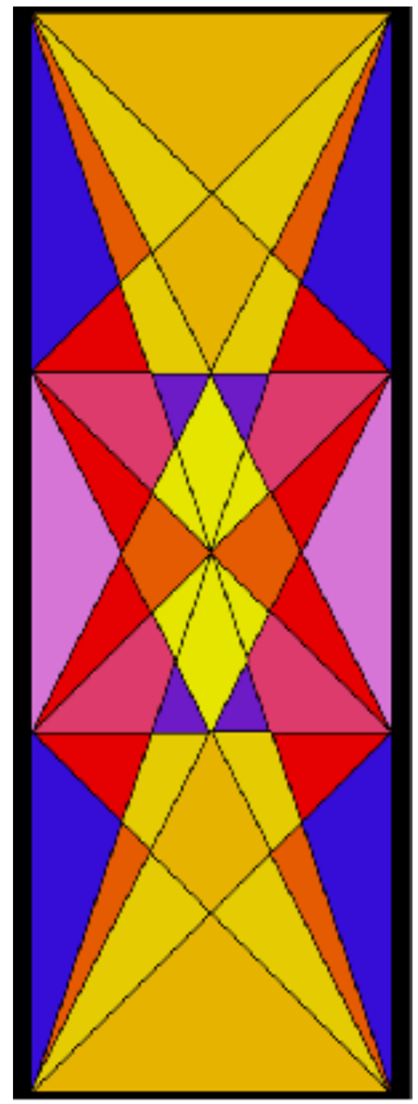
VERTICES A331453

Only the first column (the m X 1 windows) is solved

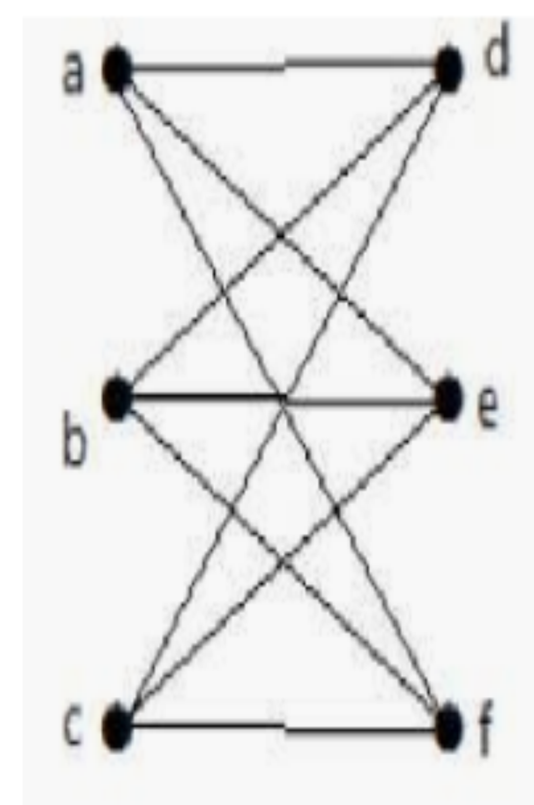




m=2



etc



m=2

Studied via the

complete bipartite graph

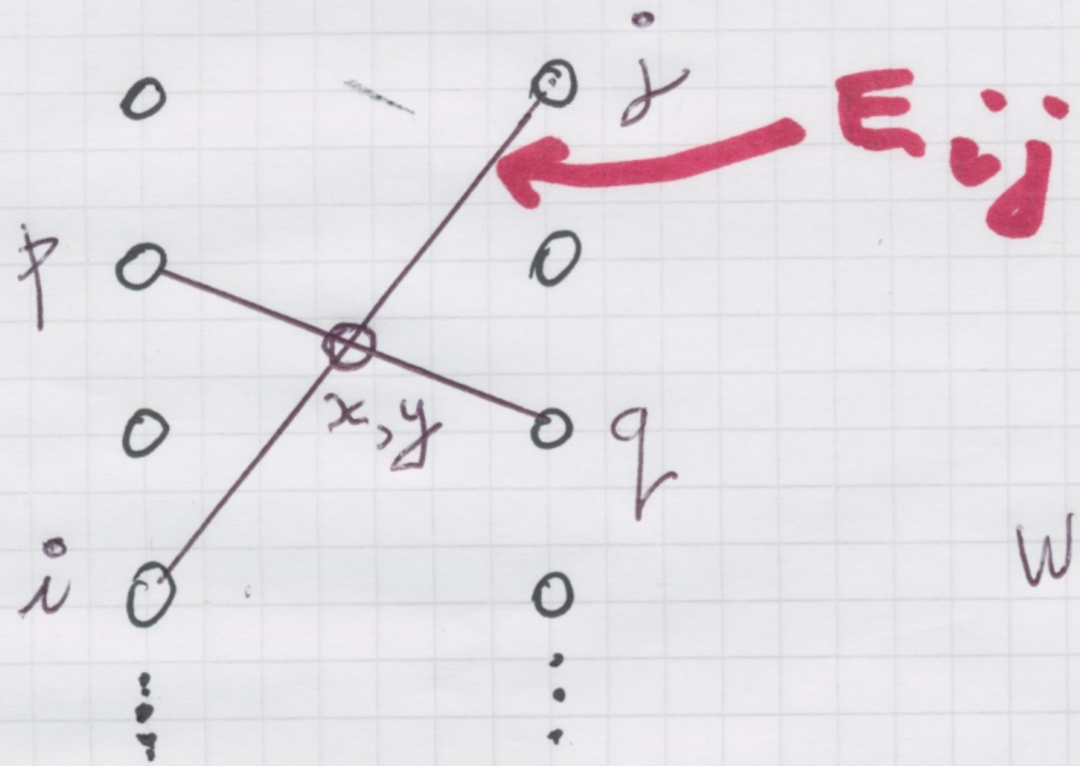
K_{n+1,n+1}

Solved by:

S. Legendre, J. Integer Seqs. 12 (2009);
M. Griffiths, J. Integer Seqs. 13 (2010);
Max Alekseyev, SIAM J. Discr. Math. 24 (2010);
Alekseyev, Basova, Zolotykh, SIAM J. Disc. Math. 29 (2015

The $K_{n,n}$ graph

From the Griffiths article



IF E_{ij} MEETS E_{pq}

EITHER $1 \le p < i,\ j \le q$

OR $i < p,\ 1 \le q < j$

WRITE

$$p = i + ka$$
$$q = j - kb$$
$$\gcd(a, b) = 1$$

THEN

$$x = \frac{a}{a+b}, \quad y = \frac{aj + bi}{a+b}$$

REGIONS IN $K_{n,n}$ GRAPH IS

$$R(n) = (n-1)^2 + \sum_{\substack{1 \le a, b, < n \\ \gcd(a,b) = 1}} (n-a)(n-b).$$

The m X 1 windows, continued

Number of regions R(m) = V(m,m) + m^2 +2m where

$$V(m,m) \; = \; \sum_{i=1..m} \quad \sum_{j=1..m;gcd(i,j)=1} (m+1-i)(m+1-j).$$

Similar formulas for E = no of edges, V = no of vertices

The m X 1 windows, continued

R = no. of regions = A306302 = 4, 16, 46, 104, …

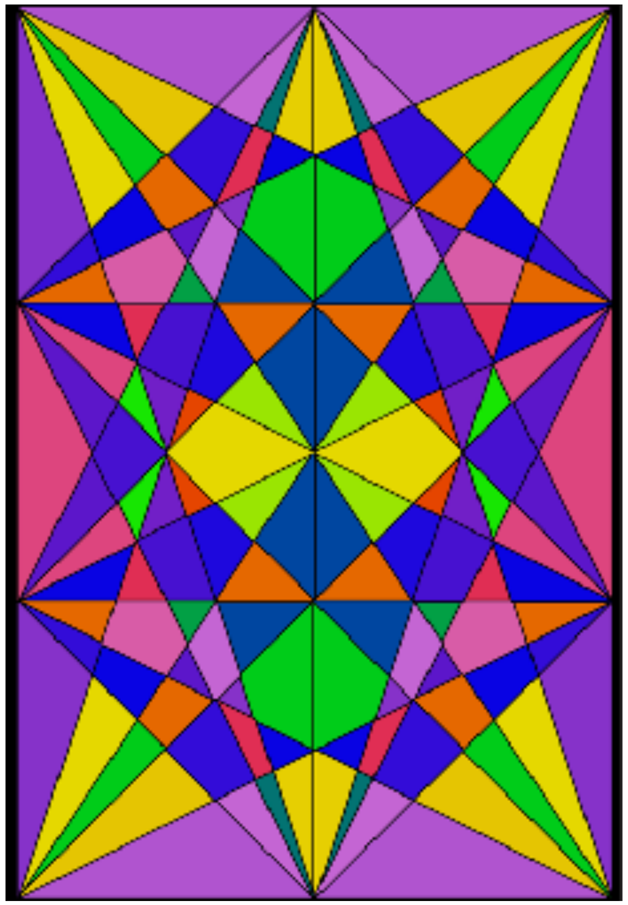
E = no. of edges = A331757 = 8, 28, 80, 178, …

V = no. of vertices = A331755 = 5, 13, 35, 75, …

Remarkable: There are 8 sequences in OEIS that are equivalent to the R sequence:
A306302, A290131, A114043, A115004*, A115005, A141255, A088658, A114146

[No of ways to divide mXm grid into two parts by a straight line; threshold functions; no. of triangles of area 1/2 in grid; etc.]

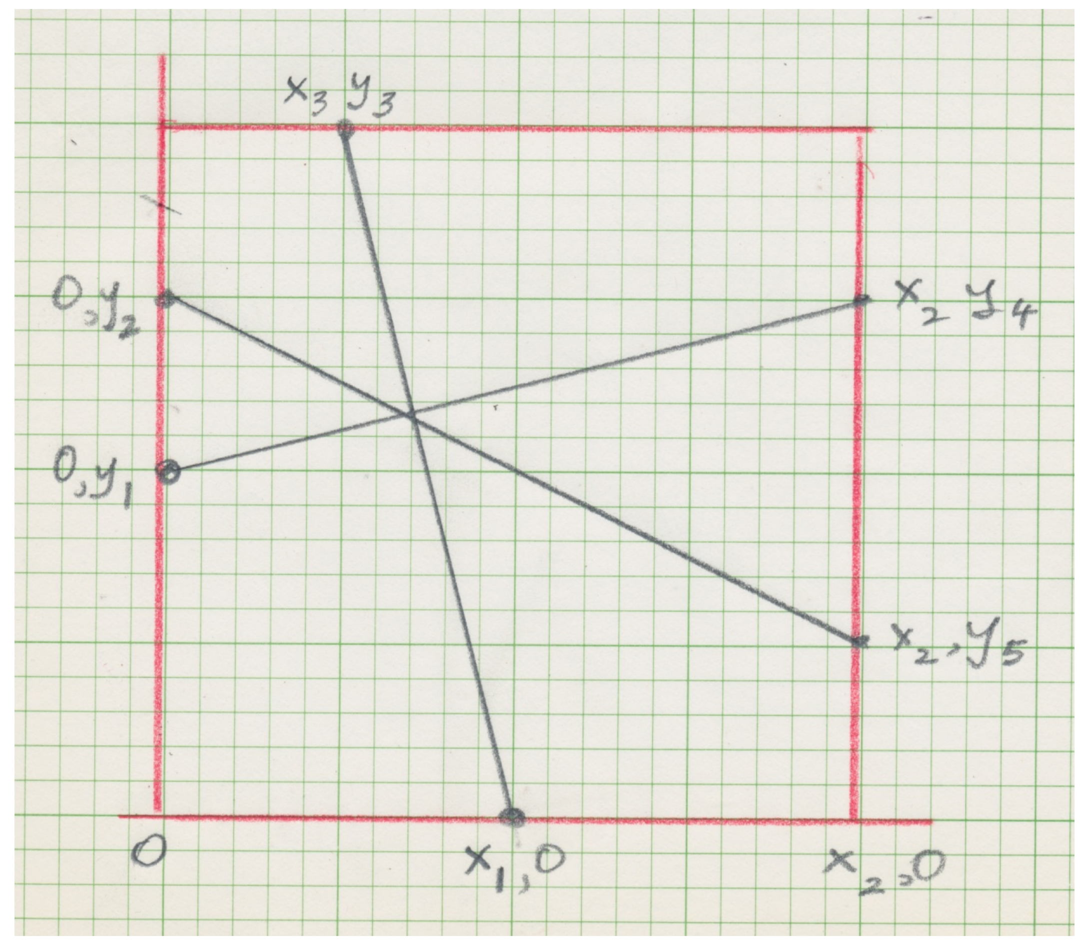
The general m X n rectangular window is unsolved



3 X 2

(Scott Shannon)

Need versions of the triple point lemma for integer points on boundary of square grid



The m X 2 case might be very interesting!

See A331763, A331765, A331766

Homework 2

Program in Maple:

Given m and n, build the m X n rectangular window, output the number of Regions, Edges, Vertices

Draw graph underlying the window

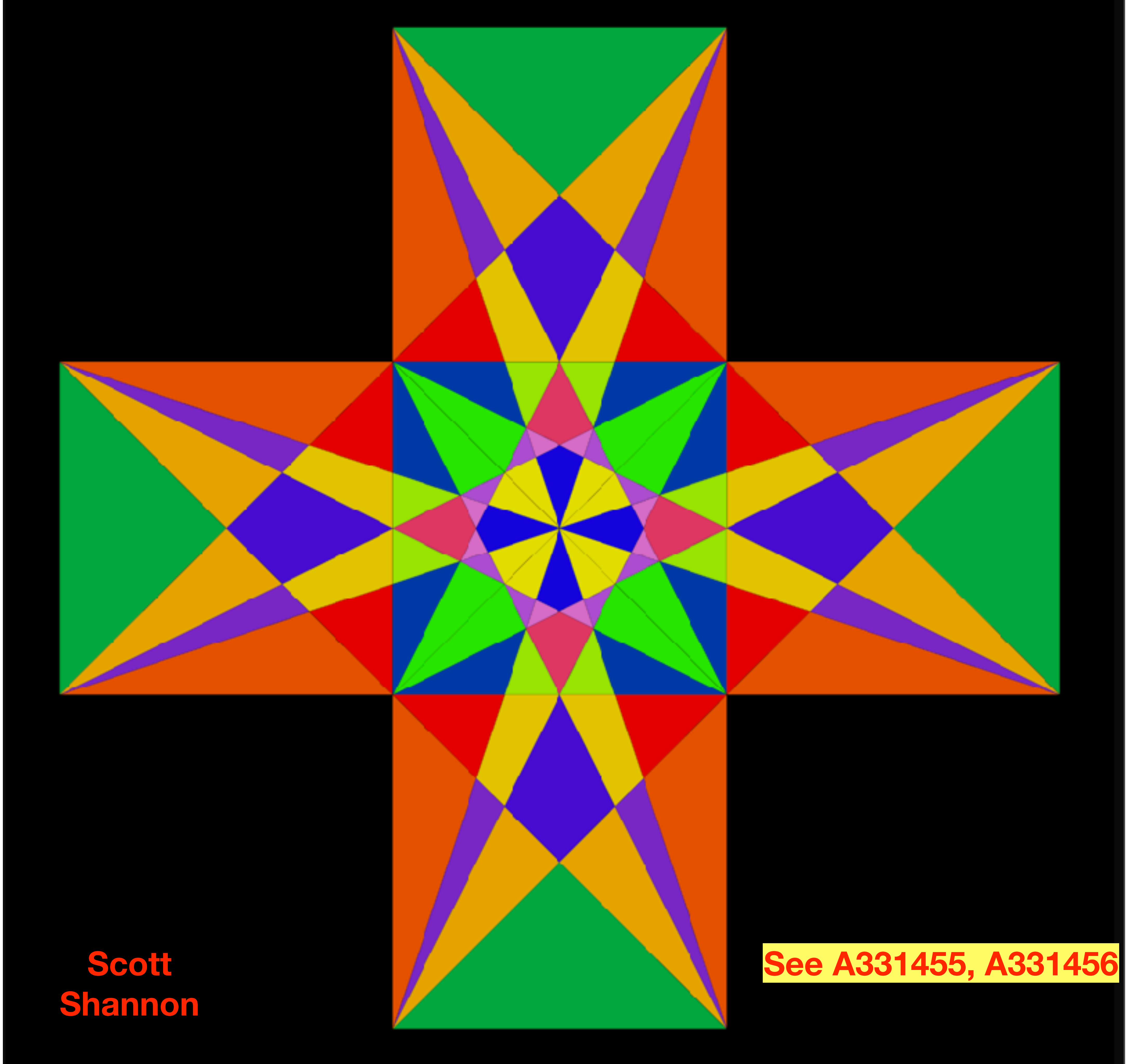
Produce colored graph

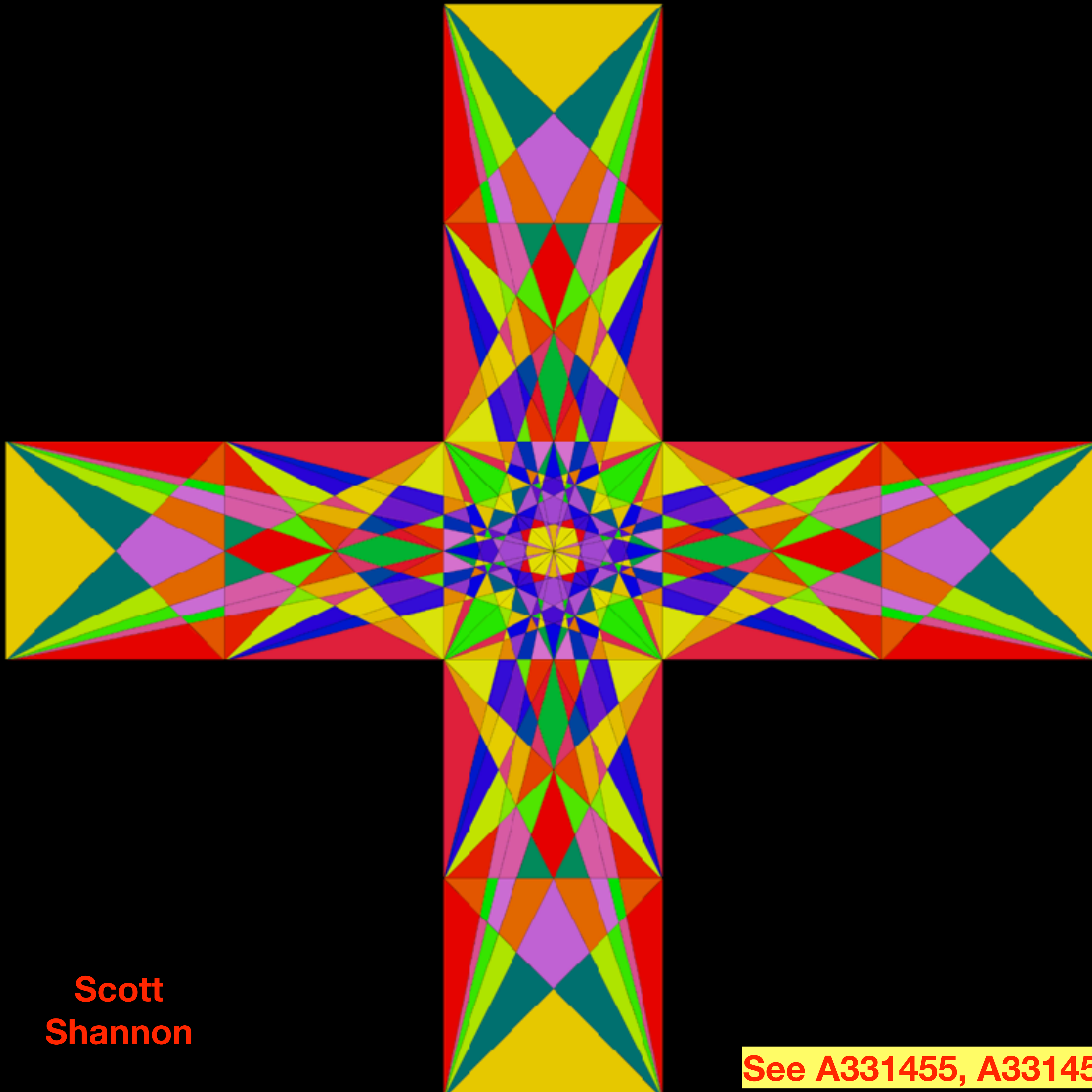
Find formulas for the numbers!

Especially the first unsolved case, n=2

Other Shapes (Crosses, Stars, etc)

(No formulas known!)


Scott
Shannon
See A331455, A331456



Scott
Shannon

See A331455, A331456

Points on a line

Take n equally-spaced points on a line and join by semi-circles: how many intersection points?

The math problems web site http://www.zahlenjagd.at

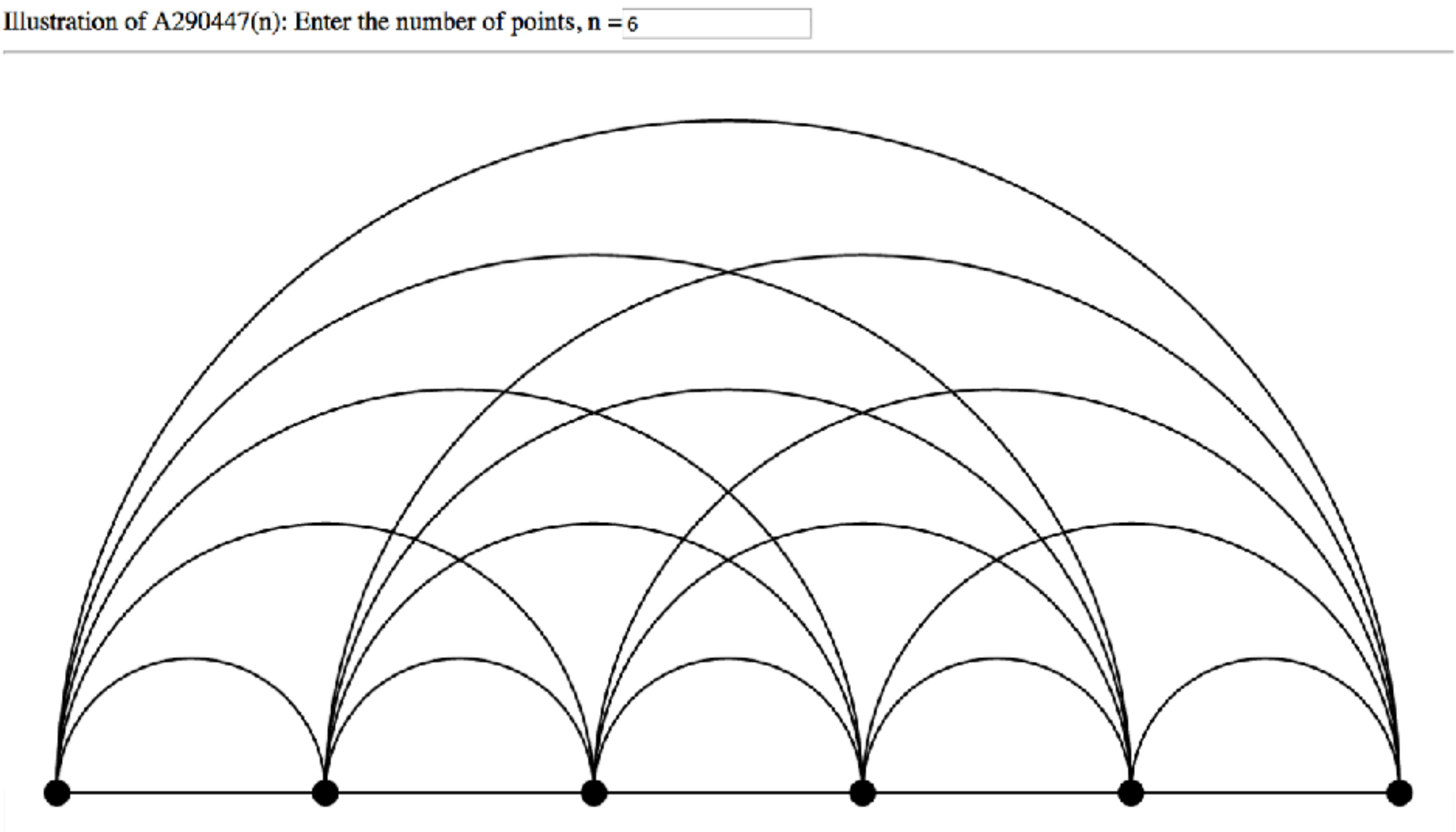
Problem for Winter 2010 says:

Gegeben sind 10 Punkte in gleichem Abstand auf einer Geraden. Darüber sind alle möglichen Halbkreise errichtet, deren Durchmesser jeweils 2 der 10 Punkte verbindet.

Wieviele Schnittpunkte haben diese Halbkreise?

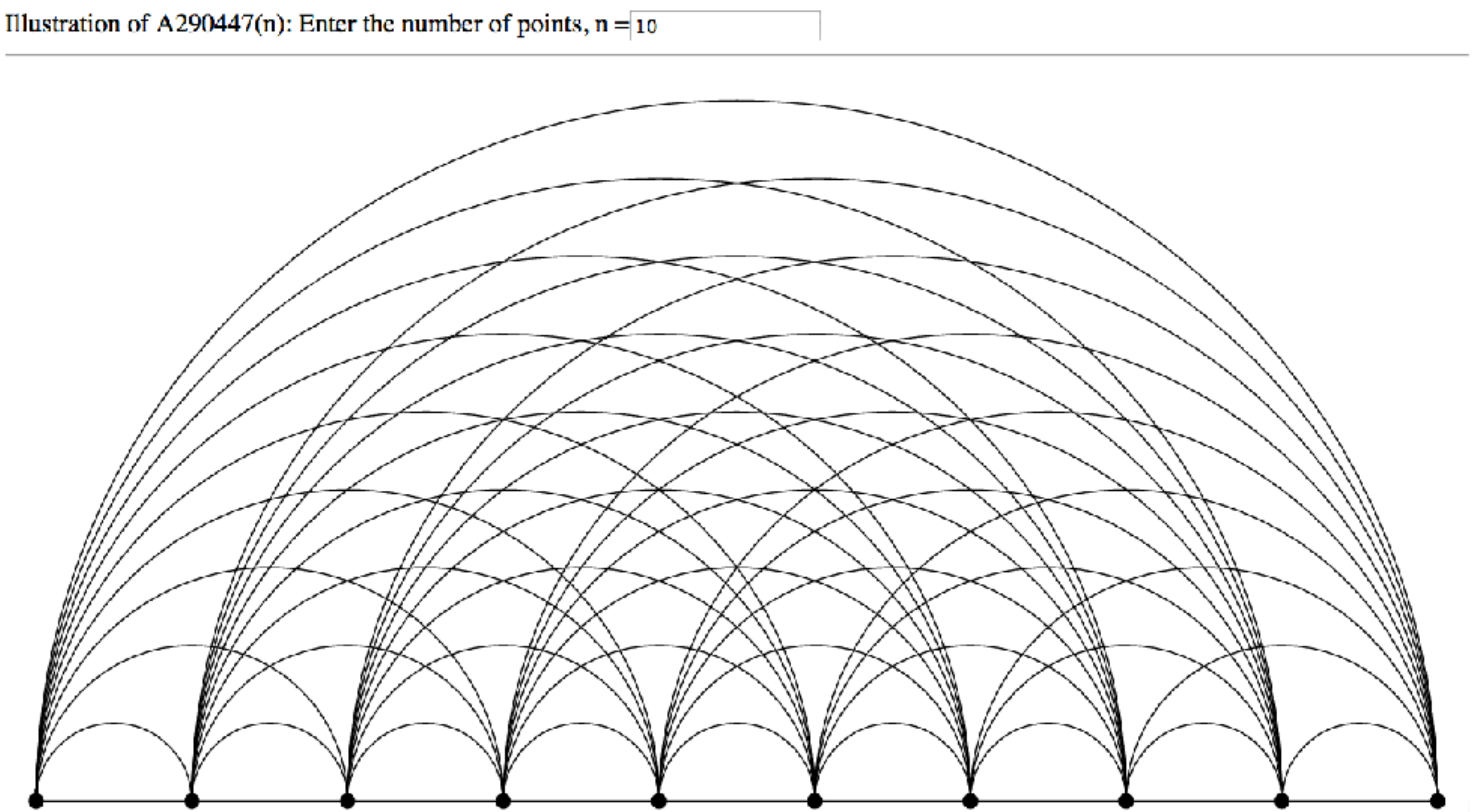
A290447

6 points on line, A290447(6) = 15 intersection points



[Torsten Sillke, Maximilian Hasler]

10 points on line, A290447(10) = 200 intersection points

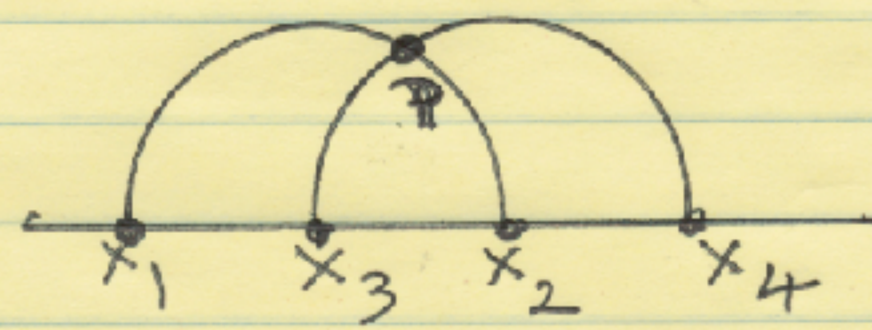


David Applegate found first 500 terms:

0, 0, 0, 1, 5, 15, 35, 70, 124, 200, 300, 445, 627, 875, 1189, 1564, 2006, 2568, 3225, ...

A290447

Lemma (David Applegate)



$P = (x, y)$ with

$$x = \frac{x_3 x_4 - x_1 x_2}{x_3 + x_4 - x_1 - x_2}$$

$$y^2 = \frac{(x_3 - x_1)(x_4 - x_1)(x_2 - x_3)(x_4 - x_2)}{(x_3 + x_4 - x_1 - x_2)^2}$$

A290447 continued

No formula or recurrence is known

$$a(n) \leq \binom{n}{4} \quad \text{with} = \text{iff } n \leq 8$$

Comparison	rose window	semicircles
# points	A6561	A290447
# regions	A6533	A290865
# k-fold inter. points	A292105	A290867

Reference: Scott R. Shannon and N. J. A. Sloane, Graphical Enumeration Problems and Stained Glass Windows, In preparation, 2020