Ann. Sci. Math. Québec **19** (1995), no. 2, 139–151.

STANDARD PATHS IN THE COMPOSITION POSET

FRANÇOIS BERGERON*, MIREILLE BOUSQUET-MÉLOU AND SERGE DULUCQ

RÉSUMÉ. Nous étudions différents problèmes d'énumération de chemins standard dans l'ensemble partiellement ordonné des compositions. Nous montrons comment plusieurs questions, analogues à celles que l'on étudie dans le cas du treillis des partages d'entiers, se révèlent plus simples dans ce contexte. Nous donnons des formules explicites pour les séries génératrices des chemins standard dans cet ensemble partiellement ordonné et dans certains sous-ensembles intéressants. Nous démontrons également une formule donnant le nombre de chemins standard – ou tableaux – de forme finale fixée.

ABSTRACT. We study different problems of enumeration of standard paths in the poset of compositions of integers. We show that several problems similar to those considered in the poset of partitions of integers become simpler in this context. We give explicit formulas for generating functions of standard paths in this poset and interesting subposets, and a closed formula for the number of standard paths ending at a given composition.

1. Introduction. The poset of partitions of integers, the so-called Young lattice, has been studied by many authors (see [3, 4, 10]) and it is well known that this study is closely related to the study of irreducible representations of the symmetric group and their characters, as well as other subjects in algebraic geometry and algebra. Sergey Fomin, in the footsteps of Richard Stanley, has shown that several aspects of this study can be extended to other posets [4, 10]. One of these aspects is the enumeration of up-going paths going from the minimal element of the poset to some given element. For instance, in the partition lattice these paths correspond to standard Young tableaux of a given shape. Fomin gives a general setup for the enumeration of such paths as well as for pairs of paths with same endpoint. However, the problem studied here does not fall into his framework in a straightforward manner.

We study in this paper the poset of compositions of integers. Let us recall that a *composition* P is a sequence of positive (>0) integers $(p_1, p_2, \ldots, p_k)$. The p_i's are called the *parts* of the composition and k, the number of parts, is said to be the *length*

Reçu le 3 mai 1994 et, sous forme définitive, le 11 juillet 1995.
* With support from NSERC and FCAR

$\ell(P)$ of P. The *weight* $|P|$ of a composition P is the sum of its parts

$$|P| = \sum_{i=1}^{k} p_i.$$

If $|P| = n$, we say that P is a composition of n and write $P \models n$. Similarly, a *partition* λ of n is a non-decreasing sequence of positive integers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ that sum to n, and we write $\lambda \vdash n$. The partition obtained by reordering the parts of a composition P in non-decreasing order is denoted $\lambda(P)$.

We say that a composition Q covers a composition P if Q is obtained either by adding 1 to a part of P, or by adding a part of size 1 to P. The partial order obtained by transitive closure of this covering relation is denoted $\prec$ and the poset thus obtained is denoted Γ. For partitions, the analogous order corresponds to the inclusion of Ferrers diagrams. The poset of partitions is denoted Λ and the function $\lambda : \Gamma \longrightarrow \Lambda$, defined above, is a morphism of graded posets (graded by $|P|$).

Our first objective will be the enumeration, with some parameters, of "standard" (up-going) paths starting with the composition (1) and finishing at $P \models n$. We will then consider such enumeration problems for several subposets obtained by restrictions on the compositions.

A *standard path of length* n is a sequence $\gamma = (P_1, P_2, \ldots, P_n)$ of compositions such that

$$P_1 \prec P_2 \prec P_3 \prec \cdots \prec P_n,$$

with $P_i \models i$. The path γ is said to *end* at the composition P_n. We now give a geometric representation for standard paths. First, define the *diagram* of a composition P to be the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq p_i$. It is convenient to replace the node (i, j) by the square with corners $(i-1, j-1)$, $(i-1, j)$, $(i, j-1)$ and (i, j). For a standard path ending at P, we label the squares of the diagram of P in the order of their apparition in the path. If P is obtained by adding a part of size 1 to a composition, we consider that this new part has been added at the beginning of a sequence of ones (if any), for otherwise the encoding would be ambiguous. For instance, the step

$$(2, 3, 1, 5) \prec (2, 3, 1, 1, 5)$$

is encoded by the addition of the box labeled 12 in Figure 1.

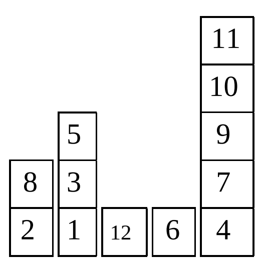


Figure 1.

The labeled diagram obtained in this manner is called the *tableau* of the path, and the underlying diagram (or composition) of the tableau is called its *shape*. This representation suggests that the length (number of parts) of the endpoint P of a standard path γ should be called the *width* of the path, and its largest part the *height* of the path.

We obtain an explicit expression for the exponential generating function of standard paths counted according to their length:

$$F(x) = \frac{\exp(-x)}{\left(\cosh\left(\frac{x}{\sqrt{2}}\right) - \sqrt{2}\,\sinh\left(\frac{x}{\sqrt{2}}\right)\right)^2}, \tag{1}$$

and show that the enumeration of paths with bounded width is very different from the enumeration of paths with bounded height. This is best illustrated by the fact that the ordinary generating function of standard paths of *width* 2 is the rational function

$$\frac{x^2 + x^3}{(1-x)\,(1-2\,x)}$$

whereas the exponential generating function of paths of *height* at most 2 is:

$$\frac{1}{1-\sin(x)}.$$

This is in sharp contrast with similar enumeration problems in the Young lattice, where the two problems coincide in view of the order preserving bijection between diagrams of height k and those of width k. In the sequel of this paper, we denote $\Gamma_{(k)}$ the subposet of compositions of width $\leq k$, and $\Gamma^{(k)}$ the subposet of compositions of height $\leq k$.

2. Standard paths in the poset Γ. Denote $\Gamma_{n,i,j}$ the set of compositions of n with i parts of size 1 and j parts of size > 1, and let $\gamma_{n,i,j}$ be the number of standard paths with endpoint in $\Gamma_{n,i,j}$. We wish to obtain an explicit expression for the following exponential generating function:

$$F(u,v,x) = \sum_{n\geq 0}\left(\sum_{i,j}\gamma_{n,i,j}u^i v^j\right)\frac{x^n}{n!}.$$

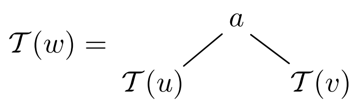
We first encode standard paths using permutations, and then encode these permutations as increasing binary trees.

As usual, a permutation σ of $[n] = \{1,2,\ldots,n\}$ is denoted by the word

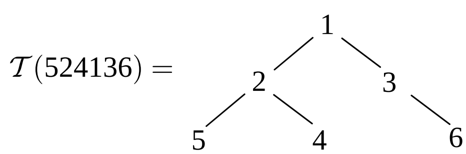
$$\sigma(1)\sigma(2)\cdots\sigma(n).$$

The *set of descents* of σ is $\mathcal{D}(\sigma) = \{i \mid \sigma(i) > \sigma(i+1)\}$. An *increasing factor* of σ of length ℓ is a word $\sigma(i+1)\sigma(i+2)\cdots\sigma(i+\ell)$ such that $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(i+\ell)$. We associate to σ the unique composition of n, denoted $\mathcal{P}(\sigma) = (p_1, p_2, \ldots, p_k)$, such that $\mathcal{D}(\sigma) = \{p_1, p_1+p_2, \ldots, p_1+p_2+\cdots+p_{k-1}\}$. Hence the number of parts of $\mathcal{P}(\sigma)$, minus one, is the cardinality of $\mathcal{D}(\sigma)$, and the greatest part of $\mathcal{P}(\sigma)$ is the length of the longest increasing factor of σ.

Let us now recall the classical *increasing binary tree* encoding of permutations. For any word $w = w_1\, w_2\, \cdots\, w_n$ with $n \geq 0$ distinct letters on an ordered alphabet, we recursively define the binary tree $\mathcal{T}(w)$ to be the empty tree if w is the empty word, and otherwise



where $a = \min(w)$ is the minimum letter in w, and u and v are the factors of w such that $w = u\, a\, v$. Thus $\mathcal{T}(u)$ is the left subtree of the vertex a, and $\mathcal{T}(v)$ is its right subtree. The *leftmost branch* of $\mathcal{T}(w)$ is $\emptyset$ if $w = \emptyset$, otherwise the subtree composed of a together with the leftmost branch of $\mathcal{T}(u)$. The *leftmost vertex* of $\mathcal{T}(w)$ is defined to be lowest vertex of its leftmost branch. Using the definition of $\mathcal{T}$, the tree corresponding to the permutation $\omega = 524136$ is



Observe that, when reading up the leftmost branch of $\mathcal{T}(w)$, starting with its leftmost vertex (in this case, 5), we obtain the sequence of left-right local minima of w (in our example: 5,2,1).

Clearly, the labels in such a tree will be in increasing order on any path going from the root to a leaf. $\mathcal{T}$ establishes a bijection between permutations of $[n]$ and increasing binary trees with labels $\{1, 2, \ldots, n\}$. A *jumping-chain* in such a tree is a sequence $(i_1, i_2, \ldots, i_\ell)$ of vertices such that i_j is the leftmost vertex of the right subtree of i_{j-1}, for $j \geq 2$. One can check recursively that $\mathcal{T}$ satisfies the following properties:

– the number of parts of $\mathcal{P}(\sigma)$, minus one, is the total number of left sons in $\mathcal{T}(\sigma)$,
– the number of parts of size 1 in $\mathcal{P}(\sigma)$ is the number of left sons in $\mathcal{T}(\sigma)$ having no brother, counting the leftmost vertex of $\mathcal{T}(\sigma)$ whenever it is a leaf,
– the greatest part of $\mathcal{P}(\sigma)$ is the length of the longest jumping-chain of $\mathcal{T}(\sigma)$.

We finally define recursively a bijection $\mathcal{S}$ between standard paths of length n and a subset of permutations of $[n]$, such that the composition associated to $\mathcal{S}(\gamma)$ is the shape of γ. Let γ be a standard path $(P_1, P_2, P_3, \ldots, P_n)$, where $P_n = (p_1, p_2, \ldots, p_k)$, and $\gamma' = (P_1, P_2, P_3, \ldots, P_{n-1})$. Then $\mathcal{S}(\gamma)$ is obtained from $\mathcal{S}(\gamma')$ by inserting n either

– in first position, if P_n is obtained by adding a new part of size 1 at the beginning of P_{n-1},
– in position $p_1 + p_2 + \cdots + p_m$, if P_n is obtained from P_{n-1} either by adding 1 to the m^{th} part of P_{n-1} or by adding a part of size 1 to P_{n-1}, just after the m^{th} part of P_{n-1} (of size > 1).

For example, the sequence of permutations associated, through this process, to the path

$$\gamma = (1) \prec (1,1) \prec (2,1) \prec (1,2,1) \prec (2,2,1) \prec (2,3,1) \prec (2,4,1) \prec (2,4,2) \prec (2,4,1,2),$$

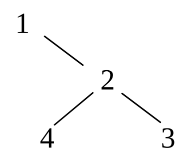
is

$$1, 21, 231, 4231, 45231, 452361, 4523671, 45236718, 452369718$$

hence $\mathcal{S}(\gamma) = 452369718$.
Note that $\{p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_{k-1}\}$ is the set of descents of $\mathcal{S}(\gamma)$. Hence, $\mathcal{S}$ is injective. However, we do not obtain all permutations of $[n]$ in this manner, since n can not be inserted in the "middle" of a maximal increasing factor, that is, in a rise that is not the last one of this increasing factor. The set of permutations actually obtained is easier to characterize in terms of increasing binary trees: the permutation σ encodes a standard path if and only if any vertex ν of $\mathcal{T}(\sigma)$ not belonging to its leftmost branch satisfies

if ν has two sons, the label of its left son is less than the label of its right son. (C)

Thus, the smallest increasing tree that does not correspond to an encoding of a standard path is:



This is the only excluded tree with four vertices, thus the coefficient of $x^4/4!$ in the expansion of (1) will be 23.
Now, since $\mathcal{P}(\mathcal{S}(\gamma)) = P_n$, we can read off the height and the width of γ on the tree $\mathcal{T}(\mathcal{S}(\gamma))$, as well as the number of parts equal to 1 in P_n.

Proposition 1. *The exponential generating function of standard paths in the composition poset is*

$$F(u, v, x) = \frac{\exp(-x)}{\left(\cos\left(\frac{\alpha}{2}\, x\right) - \frac{1+u}{\alpha}\, \sin\left(\frac{\alpha}{2}\, x\right)\right)^2}, \qquad (2)$$

where

$$\alpha = \sqrt{2\, v - (1+u)^2},$$

the variables u and v accounting respectively for the number of parts of size 1 and those of size > 1 in the endpoint.

We will give two different proofs of this proposition. The first one is short and natural, but does not explain how we got formula (2). The second one is based on the permutation encoding of standard paths, and gives equations for $F(u, v, x)$ that are easy to solve.

First proof. Let's consider a standard path $(P_1, \ldots, P_{n+1})$ such that P_{n+1} belongs to $\Gamma_{n+1,i,j}$. Then P_n either belongs to $\Gamma_{n,i,j}$, $\Gamma_{n,i+1,j-1}$, or $\Gamma_{n,i-1,j}$. Conversely, by counting the compositions of $\Gamma_{n+1,i,j}$ that cover a given composition of $\Gamma_{n,i,j}$, $\Gamma_{n,i+1,j-1}$ or $\Gamma_{n,i-1,j}$, one finds that the coefficients $\gamma_{n,i,j}$ are totally determined by the initial conditions $\gamma_{0,0,0} = 1$, $\gamma_{0,i,j} = 0$ if i or j is not zero, and the recurrence

$$\gamma_{n+1,i,j} = j\gamma_{n,i,j} + (1+i)\gamma_{n,i+1,j-1} + (1+j)\gamma_{n,i-1,j}.$$

Of course, $\gamma_{n,i,j}$ is zero if i or j is negative. This recurrence implies that $F(u,v,x)$ is the unique formal power series satisfying $F(u,v,0)=1$ and the partial differential equation:

$$\frac{\partial}{\partial x}F(u,v,x)=(1+u)\,v\,\frac{\partial}{\partial v}F(u,v,x)+uF(u,v,x)+v\frac{\partial}{\partial u}F(u,v,x). \qquad (3)$$

One can check that expression (2) satisfies equation (3) with the prescribed initial condition. In order to derive formula (2) from equation (3), one could replace the above initial condition by $F(u,0,x)=\exp(u\,x)$ and $(\frac{\partial}{\partial v}F(0,v,x))\big|_{v=0}=\exp(x)-1-x$.

Second proof. Using the increasing binary tree encoding of standard paths defined above, the problem of computing $F(u,v,x)$ becomes a classical problem of enumeration of labeled trees [8], and we obtain:

$$\begin{aligned}
\frac{\partial}{\partial x}\,F(u,v,x)&=F(u,v,x)\,(u+G(u,v,x)), && F(u,v,0)=1,\\
\frac{\partial}{\partial x}\,G(u,v,x)&=v+(1+u)\,G(u,v,x)+\frac{G(u,v,x)^2}{2}, && G(u,v,0)=0.
\end{aligned}$$

That is the actual system we solved. □

Remark. The first few terms of the series $F(u,v,x)$ are:

$$1+u\,x+(v+u^2)\,\frac{x^2}{2!}+(v+4\,v\,u+u^3)\,\frac{x^3}{3!}+(v+4\,v^2+6\,v\,u+11\,v\,u^2+u^4)\,\frac{x^4}{4!} \\ +(v+14\,v^2+34\,u\,v^2+8\,v\,u+23\,u^2v+26\,v\,u^3+u^5)\,\frac{x^5}{5!}+\ldots$$

Setting $u=v=1$, we obtain

$$1+x+2\,\frac{x^2}{2!}+6\,\frac{x^3}{3!}+23\,\frac{x^4}{4!}+107\,\frac{x^5}{5!}+586\,\frac{x^6}{6!}+3690\,\frac{x^7}{7!} \\ +26245\frac{x^8}{8!}+207997\frac{x^9}{9!}+\ldots$$

Using the Maple package `gdev` [9], with the help of Bruno Salvy, we obtained the following expression for the asymptotic expansion of the coefficient of $x^n/n!$ in $F(1,1,x)$:

$$\frac{\big((\alpha+1)\,\ln(\alpha)+\alpha\,(n+1)\big)}{\alpha^{\alpha}\ln(\alpha)^2}\left(\frac{1}{\sqrt{2}\,\ln(\alpha)}\right)^n n!, \qquad (4)$$

where $\alpha=1+\sqrt{2}$. For n from 0 to 9, formula (4) gives the following values

$$0.83,\ 0.96,\ 2.02,\ 6.02,\ 22.99,\ 106.98,\ 586.01,\ 3690.06,\ 26245.03,\ 207996.78$$

showing that this approximation is very good even for small values of n.

3. Standard paths of bounded height. The story is similar for the posets $\Gamma^{(k)}$ of compositions of height bounded by k. Once again, let $\Gamma^{(k)}_{n,i,j}$ be the set of compositions of n of height $\leq k$, having i parts of size 1 and j parts of size ≥ 2. As before, let $\gamma^{(k)}_{n,i,j}$ be the number of standard paths with endpoint in $\Gamma^{(k)}_{n,i,j}$, and

$$H_k(u,v,x) = \sum_{n\geq 0} \left(\sum_{i,j} \gamma^{(k)}_{n,i,j} u^i v^j \right) \frac{x^n}{n!}.$$

Let's begin with the simplest non-trivial case: $k = 2$. We could proceed as in the derivation of $F(u,v,x)$, writing the basic recurrence:

$$\gamma^{(2)}_{n+1,i,j} = (1+j)\gamma^{(2)}_{n,i-1,j} + (1+i)\gamma^{(2)}_{n,i+1,j-1},$$

with $n+1 = i+2j$. However, we can easily derive $H_2(u,v,1)$ directly from $F(u,v,x)$ since

$$\lim_{x\to 0} F(u/x, v/x^2, x) = H_2(u,v,1).$$

Hence we get:

$$H_2(u,v,1) = \frac{1}{\left(\cos(\frac{\beta}{2}) - \frac{u}{\beta}\,\sin(\frac{\beta}{2})\right)^2},$$

where $\beta = \sqrt{2\,v - u^2}$. Observe that for $u = x$ and $v = x^2$, this identity becomes

$$H_2(x,x^2,1) = \frac{1}{\left(\cos(x/2) - \sin(x/2)\right)^2} = \frac{1}{1-\sin(x)} = \frac{d}{dx}(\sec(x) + \tan(x)), \quad (5)$$

showing that the number of standard paths of length $n-1$ and height at most 2 coincides with the n^{th} *eulerian* number. It is interesting to observe that (5) is not D-finite (see Stanley [11, section 4 a)] and the note below). This illustrates that the problem of enumerating paths in the composition poset $\Gamma^{(k)}$ is quite different from the corresponding problem in the context of partitions since it has been shown that the generating functions for the number of tableaux of bounded height are all D-finite [6, 1].

For a general k, the study of $\Gamma^{(k)}$ becomes more intricate, but the techniques are essentially the same as those of section 2. Using the increasing binary tree encoding of standard paths, we can derive a system of differential equations with H_k as one of its solutions.

Proposition 2. *The exponential generating function $H_k(u,v,x)$ of standard paths of height bounded by k is such that*

$$\begin{aligned}
\frac{\partial}{\partial x} H_k(u,v,x) &= H_k(u,v,x)\,(u + I_{k,k-1}(u,v,x)) \\
\frac{\partial}{\partial x} I_{k,\ell}(u,v,x) &= J_{k,\ell}(u,v,x) \qquad\qquad (6)\\
\frac{\partial}{\partial x} J_{k,\ell}(u,v,x) &= J_{k,\ell}(u,v,x)\,(u + I_{k,k-1}(u,v,x)) + J_{k,\ell-1}(u,v,x),
\end{aligned}$$

for $\ell = 1, \ldots, k-1$*, with initial conditions* $I_{k,0}(u,v,x) = 0$*,* $H_k(u,v,0) = 0$*,* $I_{k,j}(u,v,0) = 0$ *and* $J_{k,j}(u,v,0) = v$*.*

Proof. We will only outline the proof, which uses a classical enumeration technique for labeled trees [8]. These are counted according to their size (variable x) and parameters accounting for the number of parts equal to 1 and those greater than 1 in the corresponding composition. We consider the following different classes of increasing binary trees:

– $\mathcal{H}_k$ is the set of increasing binary trees such that all vertices not belonging to the leftmost branch satisfy condition C, and all jumping-chain-lengths are bounded by k; the generating function for this set is denoted $H_k(u,v,x)$;
– $\mathcal{I}_{k,\ell}$ is the set of increasing binary trees such that all vertices satisfy condition C, the length of the maximal jumping-chain starting from the leftmost vertex is at most ℓ, and all jumping-chain-lengths are bounded by k; the generating function for this set is denoted $I_{k,\ell}(u,v,x)$.

System (6) is obtained by considering the effect of removing the root of such trees. This operation generates two subtrees, each belonging to one of the previous classes. □

Note. The solutions of system (6) are *constructible differentially algebraic* series as defined in [2]. Recall that a series $y = y(x)$, with coefficients in $\mathbb{K}$, is said to be constructible differentially algebraic (CDF for short) if for some $k \geq 1$, there exist k series $y_1, \ldots, y_k$ with $y_1 = y$ and polynomials $P_1, \ldots, P_k$ (with coefficients in $\mathbb{K}$) such that

$$\begin{array}{rcl} y_1' & = & P_1(y_1, \ldots, y_k) \\ & \vdots & \\ y_k' & = & P_k(y_1, \ldots, y_k). \end{array}$$

The class of CDF series contains polynomials, algebraic series, and series expansions around 0 of usual functions such as e^x, $\log(1+x)$, or trigonometric functions and their inverses. It is closed for sum, product, composition, derivation, integration, inversion ($1/y(x)$), and inversion for composition. However it is not closed under Hadamard product (term-wise product). All CDF series are analytic around 0, hence this class does not contain the class of D-finite series (see [11, 12]), which is the class of formal series satisfying some non trivial linear differential equation with polynomial coefficients

$$p_0(x)\, y + p_1(x)\, y' + p_2(x)\, y'' + \cdots + p_k(x)\, y^{(k)} = 0.$$

Conversely, the series expansion around 0 of $1/\cos(x)$ is not D-finite, but it is CDF. Thus the two classes are non-comparable.

4. Standard paths of given width. For the study of the poset $\Gamma_{(k)}$ of compositions of width k, we consider a refined weight ν on the paths in this poset, setting, for a path γ of shape $(p_1, p_2, \ldots, p_k)$

$$\nu(\gamma) = x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}.$$

We want to compute the generating function

$$f_k(x_1, x_2, \ldots, x_k) = \sum_{\gamma \text{ path of width } k} \nu(\gamma).$$

Observe that we are now using ordinary generating functions. These turn out to be more convenient in this case. We have the following simple rule for a recursive computation of the f_k's.

Proposition 3. *The generating function $f_k(x_1, x_2, \dots, x_k)$ of standard paths of width k is a rational function that can be computed recursively thanks to the following relation:*

$$f_k(x_1, x_2, \dots, x_k) = \frac{1}{1 - x_1 - \dots - x_k} \left(\Lambda_1(f_{k-1}) + \sum_{i=2}^{k} \Lambda_i \left(f_{k-1} - \Delta_{i-1}(f_{k-1}) \right) \right),$$

where, for any function $g(x_1, \dots, x_k)$,

$$\Lambda_i(g) = x_i g(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+1}),$$

and

$$\Delta_i(g) = x_i \frac{\partial g}{\partial x_i}(x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_k).$$

With $f_0 = 1$, we obtain successively

$$f_1(x_1) = \frac{x_1}{1 - x_1},$$

$$f_2(x_1, x_2) = \frac{1}{1 - x_1 - x_2} \left(\frac{x_1 x_2}{1 - x_2} - \frac{x_2 x_1^2}{1 - x_1} \right) = \frac{x_1 x_2 (1 - x_1 x_2)}{(1 - x_1)(1 - x_2)(1 - x_1 - x_2)}.$$

Denoting $L_k(x) = f(x, x, \dots, x)$, we deduce that:

$$L_1(x) = \frac{x}{1 - x},\ L_2(x) = \frac{x^2(1 + x)}{(1 - x)(1 - 2x)},$$

$$L_3(x) = \frac{x^3(1 + 4x - 3x^2)}{(1 - x)^2(1 - 2x)(1 - 3x)},\ L_4(x) = \frac{x^4(1 + x)(1 + 12x - 31x^2 + 12x^3)}{(1 - x)^2(1 - 2x)^2(1 - 3x)(1 - 4x)}.$$

Proof. Use the geometric representation of paths by tableaux described in the first section. A tableau of width k can be obtained by adding a new cell either
– at the top of a column of another tableau of width k,
– at the beginning of a tableau of width $k - 1$,
– or after the $(i - 1)^{\text{th}}$ column (of height greater than 1) of a tableau of width $k - 1$.
In terms of generating functions, these three cases correspond respectively to

$$(x_1 + x_2 + \dots + x_k) f_k, \qquad \Lambda_1(f_{k-1}), \qquad \Lambda_i \left(f_{k-1} - \Delta_{i-1}(f_{k-1}) \right). \quad \square$$

5. Standard paths of given shape. We finally derive an expression for the number of standard paths (or tableaux) of shape $(p_1, p_2, \ldots, p_k)$. This number is the coefficient of $x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}$ in the series $f_k(x_1, x_2, \ldots, x_k)$ defined in the previous section. Recall that the answer to this question, for the partition lattice, is given by the hook formula [5, 7].

To begin with, we associate to a tableau T of width k a binary tree with k vertices. This tree encodes the order in which the parts of T are created, and only depends on the labels $(a_1, a_2, \ldots, a_k)$ occurring (from left to right) in the lowest row of T. The tree $\mathcal{A}(a_1, a_2, \ldots, a_k)$ is recursively defined as follows: for $k = 0$, the tree is empty; if $k = 1$, the tree is reduced to one vertex, and, for $k > 1$, the left (resp. right) subtree of $\mathcal{A}(a_1, a_2, \ldots, a_k)$ is $\mathcal{A}(a_1, a_2, \ldots, a_\ell)$ (resp. $\mathcal{A}(a_{\ell+1}, a_{\ell+2}, \ldots, a_{k-1})$), where $\ell = \max\{j \,|\, 0 \le j < k \text{ and } a_j < a_k\}$. By convention, $a_0 = 0$. This means that the rightmost part of the tableau was created by inserting a cell labeled a_k just after the cell labeled a_ℓ, and that the parts lying between the ℓ^{th} and the k^{th} part were created later. For example, the tree associated to the tableau of Figure 1 is:

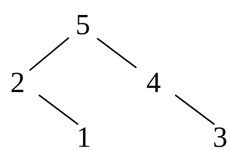


Figure 2.

Reading the tree in suffix order, we label its vertices with the integers $(k, k-1, \ldots, 1)$ (see Figure 2). From now on, a vertex will be denoted by its label.

Proposition 4. *The number of standard paths of shape $(p_1, p_2, \ldots, p_k)$ with underlying tree $\mathcal{A}$ is:*

$$N_{\mathcal{A}}(p_1, p_2, \ldots, p_k) = \frac{(p_1 + p_2 + \cdots + p_k)!}{\prod_{j \in \mathcal{A}} \left[(p_j - 2)!\, r_j\, s_j\right]}, \tag{7}$$

where

$$r_j = -1 + \sum_{i=j}^{M_j} p_i, \qquad s_j = \sum_{i=m_j}^{M_j} p_i,$$

m_j is the minimum label of the tree composed of j together with its right subtree, and M_j is the maximal label among the vertices that have j in their leftmost branch. Hence, the total number of tableaux of shape $(p_1, \ldots, p_k)$ is the sum of C_k terms $N_{\mathcal{A}}(p_1, \ldots, p_k)$, with $C_k = \binom{2k}{k}/(k+1)$ being the usual k^{th} Catalan number.

Example. If $\mathcal{A}$ is the tree of Figure 2, then $(m_1, m_2, m_3, m_4, m_5) = (1, 1, 3, 3, 3)$ and $(M_1, M_2, M_3, M_4, M_5) = (1, 5, 3, 4, 5)$, and the number of tableaux of shape $(p_1, p_2, p_3, p_4, p_5)$ associated to $\mathcal{A}$ is

$$N_{\mathcal{A}}(p_1, p_2 p_3, p_4, p_5) = \frac{(p_1 + p_2 + p_3 + p_4 + p_5 - 1)!}{p_1!(p_2 - 2)!p_3!(p_4 - 1)!(p_5 - 1)!(p_2 + p_3 + p_4 + p_5 - 1)(p_3 + p_4)(p_3 + p_4 + p_5)}.$$

Proof. We proceed by induction on k. The statement is clearly true when $k = 0$ or $k = 1$. For $k > 1$, let $\mathcal{B}$ (resp. $\mathcal{C}$) be the left (resp. right) subtree of $\mathcal{A}$. Suppose $\mathcal{B}$ has ℓ vertices. Then:

$$N_{\mathcal{A}}(p_1, \ldots, p_k) = \binom{-2 + \sum_{i=\ell}^{k} p_i}{p_\ell - 2} \binom{-1 + \sum_{i=\ell+1}^{k} p_i}{p_k - 1} N_{\mathcal{B}}(p_1, \ldots, p_{\ell-1}, p_\ell + \cdots + p_k) N_{\mathcal{C}}(p_{\ell+1}, \ldots, p_{k-1}). \quad (8)$$

This identity reflects the fact that any tableau T of shape $(p_1, \ldots, p_k)$ with underlying tree $\mathcal{A}$ can be obtained by the following procedure:

– first, build an auxiliary tableau of shape $(p_1, \ldots, p_{\ell-1}, p_\ell + \cdots + p_k)$ with underlying tree $\mathcal{B}$.

The next steps will consist in transforming the last part of this tableau. Thus,

– in the last part of the auxiliary tableau, select a set $\mathcal{L}$ of $p_{\ell+1} + \cdots + p_k$ labels not containing the two minimal ones. The set $\mathcal{L}$ will be used to label the last $k - \ell$ parts of the tableau T being constructed;
– remove from the last part of the current tableau the cells corresponding to the labels in $\mathcal{L}$.

The final result for T is obtained by adding to the current tableau $k - \ell$ columns in the following manner:

– once again, select in $\mathcal{L}$ a set $\mathcal{L}'$ of $p_{k+1} + \cdots + p_{\ell-1}$ labels not containing the minimal element of $\mathcal{L}$;
– build a tableau of shape $(p_{\ell+1}, \ldots, p_{k-1})$ with underlying tree $\mathcal{C}$, and append this tableau to the right of the current tableau;
– the final tableau T is obtained by adding a k^{th} part of size p_k labeled in increasing order by the remaining labels.

A careful verification shows that the right hand-side of (7) satisfies recurrence (8), with the same initial conditions, thus the proposition is proved. □

Résumé substantiel en français. Le treillis des partages d'entiers – ou treillis de Young – fait l'objet de nombreuses études, en liaison notamment avec la théorie des représentations du groupe symétrique. Nous étudions ici un ensemble partiellement ordonné voisin : celui des compositions d'entiers. Rappelons qu'une *composition* de l'entier n est une suite $(p_1, p_2, \ldots, p_k)$ d'entiers strictement positifs, telle que la somme des p_i soit égale à n. Les p_i sont appelés les *parts* de la composition. Le nombre de parts est la *largeur* de la composition, et la plus grande part est sa *hauteur*.

Nous définissons sur l'ensemble des compositions un ordre partiel en disant qu'une composition P *couvre* une composition Q si P s'obtient, soit en rajoutant 1 à une part de Q, soit en ajoutant à Q une nouvelle part de taille 1. Par analogie avec les tableaux de Young standard, nous appelons *chemin standard* de longueur n toute suite croissante de compositions $\gamma = (P_1, P_2, \ldots, P_n)$ telle que, pour tout i, P_i soit une composition de i. La *forme* de γ est la composition finale P_n, la *largeur* et la *hauteur* de γ sont celles de P_n. L'objet de notre étude est l'énumération de chemins standard, dont la hauteur ou la largeur vérifient éventuellement certaines contraintes.

Une bijection entre les chemins standard de longueur n et certaines permutations de n éléments nous permet tout d'abord d'obtenir la série génératrice exponentielle des

chemins standard généraux, puis celle des chemins standard de hauteur bornée par une entier k fixé (Propositions 1 et 2). Ces séries sont *constructiblement différentiellement algébriques* au sens de [2]. Rappelons que pour le treillis de Young, la série génératrice des chemins standard est celle, très simple, des involutions, tandis que les séries correspondant aux tableaux de Young de hauteur bornée sont en général assez mal connues. On sait toutefois qu'elles sont D-finies [11].

Nous considérons ensuite les chemins standard de largeur bornée. Contrairement au cas du treillis de Young, ce problème est bien différent de l'étude des chemins de hauteur bornée. Nous donnons une formule permettant de calculer récursivement les séries génératrices ordinaires correspondantes, qui sont de simples séries rationnelles (Proposition 3).

Pour finir, nous nous intéressons au nombre de chemins standard de forme donnée, c'est-à-dire que nous cherchons un analogue de la formule des équerres. Pour cela, nous associons tout d'abord à chaque chemin standard de largeur k un arbre binaire à k sommets. Puis, nous démontrons une formule donnant le nombre de chemins standard de longueur n, d'arbre sous-jacent et de forme finale fixés, qui prouve que ce nombre est encore un diviseur de $n!$ (Proposition 4).

References

1. F. Bergeron, L. Favreau and D. Krob, *Conjectures on the number of standard tableaux of bounded height*, Discrete Math. **139** (1995), 463–468.
2. F. Bergeron and C. Reutenauer, *Combinatorial resolution of systems of differential equations III: a special class of differentially algebraic series*, European J. Combin. **11** (1990), 501–512.
3. D. Gouyou-Beauchamps, *Standard Young tableaux of height 4 and 5*, European. J. Combin. **10** (1989), 69–82.
4. S. Fomin, *Schensted Algorithms for Dual Graded Graphs*, J. Algebraic Combin. **1** (1995), 5–46.
5. J. S. Frame, G. de B. Robinson et R. M. Thrall, *The Hook Graphs of the Symmetric Group*, Canad. J. Math. **6** (1954), 316–325.
6. I. M. Gessel, *Symmetric Functions and P-Recursiveness,*, J. Combin. Theory Ser. A **53** (1990), 257–285.
7. C. Greene, A. Nijenhuis and H.S. Wilf, *A Probabilistic Proof of a Formula for the Number of Young Tableaux of a Given Shape,*, Adv. Math. **31** (1979), 104–109.
8. P. Leroux and X. Viennot, *Résolution combinatoire d'équations différentielles II : calcul intégral combinatoire*, Ann. Sci. Math. Québec **12** (1988), 233–253.
9. B. Salvy, *Examples of Automatic Asymptotic Expansions*, Technical Report 114, Institut National de Recherche en Informatique et Automatique, 1989.
10. R. P. Stanley, *Differential Posets*, J. Amer. Math. Soc. **1** (1988), 919–961.
11. R. P. Stanley, *Differentiably Finite Power Series*, European J. Combin. **1** (1980), 175–188.
12. D. Zeilberger, *A Holonomic Systems Approach to Special Functions Identities,*, J. Comput. Appl. Math. **32** (1990), 321–368.

F. BERGERON
LACIM — UNIVERSITÉ DU QUÉBEC À MONTRÉAL
CASE POSTALE 8888, SUCCURSALE CENTRE-VILLE

MONTRÉAL QC H3C 3P8
CANADA

M. BOUSQUET-MÉLOU AND S. DULUCQ
LABRI – UNIVERSITÉ DE BORDEAUX I
351, COURS DE LA LIBÉRATION
33405 TALENCE CEDEX
FRANCE