SUMS OF PRODUCTS OF BERNOULLI NUMBERS OF THE SECOND KIND

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ABSTRACT

The Bernoulli numbers of the second kind b_n are defined by

$$\sum_{n=0}^{\infty} b_n t^n = \frac{t}{\log(1+t)}.$$

In this paper, we give an explicit formula for the sum

$$\sum_{\substack{j_1+j_2+\cdots+j_N=n\\ j_1,j_2,\dots,j_N\geqslant 0}} b_{j_1} b_{j_2} \cdots b_{j_N}.$$

We also establish a q-analogue for

$$\sum_{k=0}^{n} b_k b_{n-k} = -(n-1)b_n - (n-2)b_{n-1}.$$

The Bernoulli numbers B_n are defined by

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}.$$

It is well-known (cf. [5]) that for $n > 1$

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n+1)B_{2n}. \tag{1}$$

As a generalization of (1), in [2] Dilcher proved that for $n > N/2$

$$\sum_{\substack{j_1+j_2+\cdots+j_N=n\\ j_1,j_2,\dots,j_N\geqslant 0}} \binom{2n}{2j_1, 2j_2, \dots, 2j_N} B_{2j_1} B_{2j_2} \cdots B_{2j_N}$$

$$= \frac{(2n)!}{(2n-N)!} \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} c_k^{(N)} \frac{B_{2n-2k}}{2n-2k}, \tag{2}$$

where the array $\{c_k^{(N)}\}$ is given by $c_0^{(1)} = 1$ and

$$c_k^{(N+1)} = -\frac{1}{N}c_k^{(N)} + \frac{1}{4}c_{k-1}^{(N-1)}$$

with $c_k^{(N)} = 0$ for $k < 0$ and $k > \lfloor (N-1)/2 \rfloor$.

On the other hand, the Bernoulli numbers of the second kind b_n are given by

$$\sum_{n=0}^{\infty} b_n t^n = \frac{t}{\log(1+t)}.$$

And we set $b_k = 0$ whenever $k < 0$. It is easy to check that

$$\sum_{k=0}^{n} \frac{(-1)^k b_{n-k}}{k+1} = \delta_{n,0}, \tag{3}$$

where $\delta_{n,0} = 1$ or 0 according to whether $n = 0$ or not. In [3], Howard used the Bernoulli numbers of the second kind to give an explicit formula for degenerate Bernoulli numbers. And some 2-adic congruences of b_n have been investigated by Adelberg in [1].

In this short note, we shall give an analogue of (2) for the Bernoulli numbers of the second kind. Define an array of polynomials $\{a_k^{(N)}(x)\}$ by

$$a_0^{(1)}(x) = 1, \qquad a_k^{(N)}(x) = 0 \text{ for } k < 0 \text{ and } k \geqslant N,$$

and

$$a_k^{(N)}(x) = -\frac{1}{N-1}((x-N+1)a_k^{(N-1)}(x) + (x-N)a_{k-1}^{(N-1)}(x-1))$$

if $N > k \geqslant 0$.

Theorem 1: *Let $N = 1$ be an integer. Then for any non-negative integer n*

$$\sum_{\substack{j_1+j_2+\cdots+j_N=n \\ j_1,j_2,\ldots,j_N \geqslant 0}} b_{j_1}b_{j_2}\cdots b_{j_N} = \sum_{k=0}^{N-1} a_k^{(N)}(n) b_{n-k}. \tag{4}$$

Proof: Let:

$$s_N(n) = \sum_{\substack{j_1+j_2+\cdots+j_N=n \\ j_1,j_2,\ldots,j_N \geqslant 0}} b_{j_1}b_{j_2}\cdots b_{j_N}.$$

Clearly $s_1(n) = b_n$, whence (4) holds for $N = 1$. Now we make an induction on N. For arbitrary power series $f(t)$, let $[t^n]f(t)$ denot the coefficient of t^n in $f(t)$. It is easy to see that

$$\frac{t^N}{\log^N(1+t)} = \left(\sum_{j=0}^{\infty} b_j t^j\right)^N = \sum_{n=0}^{\infty} s_N(n) t^n.$$

Therefore

$$\begin{aligned} s_{N+1}(n) =&[t^n]\frac{t^{N+1}}{\log^{N+1}(1+t)} = [t^{n-N-1}]\frac{1}{\log^{N+1}(1+t)} \\ =& -[t^{n-N-1}]\left(\frac{(1+t)}{N}\frac{d}{dt}\left(\frac{1}{\log^N(1+t)}\right)\right) \\ =& -[t^{n-N-1}]\frac{1}{N}\frac{d}{dt}\left(\frac{1}{\log^N(1+t)}\right) - [t^{n-N-2}]\frac{1}{N}\frac{d}{dt}\left(\frac{1}{\log^N(1+t)}\right). \end{aligned}$$

Now

$$\frac{d}{dt}\left(\frac{1}{\log^N(1+t)}\right) = \frac{d}{dt}\left(\sum_{n=0}^{\infty} s_N(n)t^{n-N}\right) = \sum_{n=0}^{\infty}(n-N)s_N(n)t^{n-N-1}.$$

Thus by the induction hypothesis on N,

$$\begin{aligned} s_{N+1}(n) =& -\frac{1}{N}\left((n-N)s_N(n) + (n-N-1)s_N(n-1)\right) \\ =& -\frac{n-N}{N}\sum_{k=0}^{N-1} a_k^{(N)}(n)b_{n-k} - \frac{n-N-1}{N}\sum_{k=0}^{N-1} a_k^{(N)}(n-1)b_{n-1-k} \\ =& -\frac{n-N}{N}\sum_{k=0}^{N-1} a_k^{(N)}(n)b_{n-k} - \frac{n-N-1}{N}\sum_{k=1}^{N} a_{k-1}^{(N)}(n-1)b_{n-k} \\ =& -\frac{1}{N}\sum_{k=0}^{N}\Big((n-N)a_k^{(N)}(n) + (n-N-1)a_{k-1}^{(N)}(n-1)\Big)\, b_{n-k} \\ =& \sum_{k=0}^{N} a_k^{(N+1)}(n)b_{n-k}. \end{aligned}$$

We are done. □

For example, substituting $N = 2, 3$ in (4), we obtain that

$$s_2(n) = -(n-1)b_n - (n-2)b_{n-1}, \tag{5}$$

and

$$s_3(n) = \frac{1}{2}(n-1)(n-2)b_n + \frac{1}{2}(n-2)(2n-5)b_{n-1} + \frac{1}{2}(n-3)^2 b_{n-2}. \tag{6}$$

For arbitrary integer n, let

$$[n]_q = \frac{1-q^n}{1-q}.$$

We say that $[n]_q$ is a q-analogue of the integer n since $\lim_{q\to 1}[n]_q = n$. Then $[1]_q = 1$ and $[n-a]_q = [n]_q - q^{n-a}[a]_q$. Define the q-logarithm function by

$$\log_q(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{[n]_q}$$

which is convergent for $|t| < 1$. Also define a q-analogue of the Bernoulli numbers of the second kind by

$$\sum_{n=0}^{\infty} b_n(q)t^n = \frac{t}{\log_q(1+t)}.$$

Clearly we get the q-analogue of (3)

$$\sum_{k=0}^{n} \frac{(-1)^k b_{n-k}(q)}{[k+1]_q} = \delta_{n,0}. \tag{7}$$

Now we can give a q-analogue of (5).

Theorem 2: *For any integer $n \geqslant 0$, we have*

$$\sum_{k=0}^{n} q^{k-1} b_k(q) b_{n-k}(q) = -[n-1]_q b_n(q) - [n-2]_q b_{n-1}(q), \tag{8}$$

where we set $b_k(q) = 0$ if $k < 0$.

Proof: We make an induction on n. When $n = 0$, noting that $[-1]_q = -q^{-1}$ and $b_0(q) = 1$ by (7), so both sides of (8) coincide with q^{-1}. Now assume that $n > 0$ and (8) holds for smaller values of n. In view of (7), we have

$$b_{n-k}(q) = -\sum_{j=1}^{n-k} \frac{(-1)^j b_{n-k-j}(q)}{[j+1]_q}$$

when $k < n$. Then

$$\begin{aligned}
&\sum_{k=0}^{n} q^{k-1} b_k(q) b_{n-k}(q) \\
&\quad = q^{n-1} b_n(q) - \sum_{k=0}^{n-1} q^{k-1} b_k(q) \sum_{j=1}^{n-k} \frac{(-1)^j b_{n-k-j}(q)}{[j+1]_q} \\
&\quad = q^{n-1} b_n(q) - \sum_{j=1}^{n} \frac{(-1)^j}{[j+1]_q} \sum_{k=0}^{n-j} q^{k-1} b_k(q) b_{n-k-j}(q) \\
&\quad = q^{n-1} b_n(q) + \sum_{j=1}^{n} \frac{(-1)^j}{[j+1]_q} ([n-j-1]_q b_{n-j}(q) + [n-j-2]_q b_{n-j-1}(q))
\end{aligned}$$

where we apply the induction hypothesis in the last step. Now we know that

$$
\begin{aligned}
-\sum_{j=1}^{n} \frac{(-1)^j}{[j+1]_q} \sum_{k=0}^{n-j} q^{k-1} b_k(q) b_{n-k-j}(q) \\
&= \sum_{j=1}^{n} \frac{(-1)^j}{[j+1]_q} ([n-j-1]_q b_{n-j}(q) + [n-j-2]_q b_{n-j-1}(q)) \\
&= \sum_{j=1}^{n} \frac{(-1)^j}{[j+1]_q} (([n]_q - q^{n-j-1}[j+1]_q) b_{n-j}(q) + ([n-1]_q - q^{n-j-2}[j+1]_q) b_{n-j-1}(q)) \\
&= [n]_q \sum_{j=1}^{n} \frac{(-1)^j b_{n-j}(q)}{[j+1]_q} + [n-1]_q \sum_{j=1}^{n-1} \frac{(-1)^j b_{n-j-1}(q)}{[j+1]_q} - \sum_{j=1}^{n} (-1)^j q^{n-j-1} b_{n-j}(q) \\
-\sum_{j=1}^{n-1} (-1)^j q^{n-j-2} b_{n-j-1}(q) \\
&= -[n]_q b_n(q) - [n-1]_q b_{n-1}(q) - \sum_{j=1}^{n} (-1)^j q^{n-j-1} b_{n-j}(q) + \sum_{j=2}^{n} (-1)^j q^{n-j-1} b_{n-j}(q).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{k=0}^{n} q^{k-1} b_k(q) b_{n-k}(q) &= q^{n-1} b_n(q) - [n]_q b_n(q) - [n-1]_q b_{n-1}(q) + q^{n-2} b_{n-1}(q) \\
&= -[n-1]_q b_n(q) - [n-2]_q b_{n-1}(q). \qquad \square
\end{aligned}
$$

Remark: A q-analogue of (1) has been given by Satoh in [2].

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