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Sum of Prime Factors in the Prime Factorization of an Integer

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina
jakimczu@mail.unlu.edu.ar

In memory of my sister Fedra Marina Jakimczuk (1970-2010)

Abstract

In this article we obtain some results on the sequence $c(n)$, where $c(n)$ is the sum of the prime factors in the prime factorization of n.

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1 Introduction

In this note we obtain some results on the sequence $c(n)$, where $c(n)$ is the sum of the prime factors in the prime factorization of $n \geq 2$. For example if $n = 12$ then $c(n) = c(12) = c(2.2.3) = 2 + 2 + 3 = 7$ and if n is prime then $c(n) = n$.

The first few terms of the integer sequence $c(n)$ are

$$2, 3, 4, 5, 5, 7, 6, 6, 7, 11, 7, 13, 9, 8, 8, 17, 8, 19, 9, 10, 13, 23, 9, 10, 15, 9, 11, 29,$$

$$10, 31, 10, 14, 19, 12, 10, 37 \dots$$

Therefore $c(2) = 2$, $c(3) = 3$, $c(4) = 4$, $c(5) = 5$, $c(6) = 5$, $\dots$

We see that the sequence $c(n)$ is very irregular. On the other hand, we see that there are numbers repeated.

2 Main Results

First, we establish two general lemmas.

Lemma 2.1 *If a_i $(i = 1, \ldots, n)$ are positive integers such that $a_i \geq 2$ $(i = 1, \ldots, n)$ then we have the following inequality*

$$a_1 + a_2 + \cdots + a_n \leq a_1 a_2 \cdots a_n \qquad (n \geq 1). \tag{1}$$

Proof. We apply mathematical induction. If $n = 1$ we obtain $a_1 \leq a_1$. Consequently the lemma is true for $n = 1$. If $n = 2$ we can suppose that $a_1 \geq a_2$. Therefore

$$a_1.a_2 \geq a_1.2 = a_1 + a_1 \geq a_1 + a_2.$$

That is

$$a_1.a_2 \geq a_1 + a_2. \tag{2}$$

Consequently the lemma is true for $n = 2$.

Suppose that the lemma is true for $n \geq 2$, that is (inductive hypothesis)

$$a_1 + a_2 + \cdots + a_n \leq a_1 a_2 \cdots a_n. \tag{3}$$

We shall prove that the lemma is also true for $n + 1$.

Equation (2) and equation (3) give

$$a_1 \cdots a_n a_{n+1} = (a_1 \cdots a_n)\, a_{n+1} \geq (a_1 \cdots a_n) + a_{n+1} \geq a_1 + \cdots + a_n + a_{n+1}.$$

The lemma is proved.

Lemma 2.2 *If a_i $(i = 1, \ldots, n)$ are positive integers such that $a_i \geq 2$ $(i = 1, \ldots, n)$ then we have the following inequality*

$$a_1 + a_2 + \cdots + a_n \geq \frac{3}{\log 3} \log \left(a_1 a_2 \cdots a_n\right). \tag{4}$$

Proof. If we study the function $f(x) = \frac{x}{\log x}$ then we obtain the following inequality

$$\frac{k}{\log k} \geq \frac{3}{\log 3},$$

where k is a positive integer such that $k \geq 2$. Consequently we have $(n \geq 1)$

$$\frac{a_i}{\log a_i} \geq \frac{3}{\log 3} \qquad (i = 1, \ldots, n).$$

That is

$$a_i \geq \frac{3}{\log 3} \log a_i \qquad (i = 1, \ldots, n).$$

Therefore

$$a_1+a_2+\cdots+a_n \geq \frac{3}{\log 3}\left(\log a_1+\log a_2+\cdots+\log a_n\right)=\frac{3}{\log 3}\log\left(a_1a_2\cdots a_n\right)$$

The lemma is proved.

We have the following theorems.

Theorem 2.3 *We have the following inequalities*

$$c(n) \leq n \qquad (n \geq 2), \tag{5}$$

$$c(n) \geq \frac{3}{\log 3}\log n \qquad (n \geq 2). \tag{6}$$

Proof. If we consider the prime factorization of n and apply (1) and (4) then we obtain (5) and (6) respectively. The theorem is proved.

Remark 2.4 *Note that in theorem 2.3 n is the least upper bound since if n is prime then $c(n)=n$. Besides, $\frac{3}{\log 3}\log n$ is the greatest lower bound since if $n=3^k$ then $c(3^k)=\frac{3}{\log 3}\log 3^k=3k$.*

Corollary 2.5 *The following limit holds.*

$$\lim_{n\to\infty} c(n)=\infty.$$

Proof. It is an immediate consequence of equation (6).

Let n be a positive integer greater than 1. $\vartheta(n)$ denotes the number of partitions of n into positive prime numbers.

Lemma 2.6 *If $n=2,3,4$ then $\vartheta(n)=1$. If $n \geq 5$ then $\vartheta(n) \geq 2$.*

Proof. If $n \geq 6$ is even we have $n=2+\cdots+2=2+\cdots+2+3+3$. If $n \geq 9$ is odd we have $n=2+\cdots+2+3=2+\cdots+2+3+3+3$. If $n=5$ we have $5=5$ and $5=2+3$. If $n=7$ we have $7=7$, $7=2+5$ and $7=2+2+3$. The lemma is proved.

Theorem 2.7 *The equation $c(i)=n$ where $n \geq 2$ has $\vartheta(n)$ solutions. Consequently if $n=2,3,4$ then the equation has one solution. On the other hand if $n \geq 5$ the equation has at least two solutions.*

Proof. It is an immediate consequence of the definition of $c(n)$ and of lemma 2.6. The theorem is proved.

Theorem 2.8 *The sequence $c(n)$ is not decreasing. The sequence $c(n)$ is not increasing.*

Proof. Clearly the sequence $c(n)$ is not decreasing (see corollary 2.5). On the other hand, the sequence $c(n)$ is not increasing since if $p \geq 7$ is prime we have $p < 2^{\frac{p-1}{2}}$ and $c(p) = p > c\left(2^{\frac{p-1}{2}}\right) = p - 1$. The theorem is proved.

Theorem 2.9 *We have the following asymptotic formula*

$$\sum_{i=2}^{n} c(i) \sim \frac{\pi^2}{12} \frac{n^2}{\log n}. \tag{7}$$

Proof. Let $S_1(n)$ be the sum of the prime factors in the prime factorization of $n!$. In a previous article [1] we prove the asymptotic formula $S_1(n) \sim \frac{\pi^2}{12}\frac{n^2}{\log n}$ On the other hand, clearly $S_1(n) = \sum_{i=2}^{n} c(i)$. The theorem is proved.

Theorem 2.10 *Let us consider the set $\{c(2), c(3), \ldots, c(n)\}$. Let n_0 be the number of numbers in this set such that $c(i) \geq \frac{i}{\log^{1-\epsilon} i}$ where $0 < \epsilon < 1$ is fixed. We have $\lim_{n\to\infty} \frac{n_0}{n} = 0$.*

Proof. Note that there exists a positive integer $q > 2$ such that the function $f(x) = \frac{x}{\log^{1-\epsilon} x}$ is strictly increasing on the interval $[q, \infty]$.

Let us consider the set $\{c(q), c(q + 1), \ldots, c(n)\}$. Let m_0 be the number of numbers in this set such that $c(i) \geq \frac{i}{\log^{1-\epsilon} i}$. Suppose that the limit $\lim_{n\to\infty} \frac{m_0}{n} = 0$ is not fulfilled. Therefore there exists $\alpha > 0$ such that for infinite values of n we have $\frac{m_0}{n} \geq \alpha$.

Note that (L'Hospital's rule)

$$\lim_{x\to\infty} \frac{\int_q^x \frac{t}{\log^{1-\epsilon} t}\, dt}{\frac{x^2}{2\log^{1-\epsilon} x}} = 1.$$

That is

$$\int_q^x \frac{t}{\log^{1-\epsilon} t}\, dt = h_1(x) \frac{x^2}{2\log^{1-\epsilon} x},$$

where $h_1(x) \to 1$.

Therefore we have

$$\begin{aligned}
\sum_{i=q}^{n} c(i) &\geq \sum_{c(i)\geq \frac{i}{\log^{1-\epsilon} i},\ i\geq q} c(i) \geq \sum_{i=q}^{q+m_0-1} \frac{i}{\log^{1-\epsilon} i} \geq \int_q^{q+m_0-1} \frac{x}{\log^{1-\epsilon} x}\, dx \\
&\geq \int_q^{q-1+\alpha n} \frac{x}{\log^{1-\epsilon} x}\, dx = h_1\left(q - 1 + \alpha n\right) \frac{\left(q-1+\alpha n\right)^2}{2\log^{1-\epsilon}\left(q - 1 + \alpha n\right)} \\
&= h_2(n) \frac{\alpha^2}{2} \frac{n^2}{\log^{1-\epsilon} n} \geq C \frac{n^2}{\log n} \log^{\epsilon} n,
\end{aligned}$$

where $h_2(n) \to 1$ and $0 < C < \frac{\alpha^2}{2}$. That is

$$\sum_{i=q}^{n} c(i) \geq C \frac{n^2}{\log n} \log^{\epsilon} n. \tag{8}$$

Now, equation (8) and equation (7) are an evident contradiction. Therefore $\lim_{n\to\infty} \frac{m_0}{n} = 0$ and consequently $\lim_{n\to\infty} \frac{n_0}{n} = 0$. The theorem is proved.

Corollary 2.11 *Let us consider the set* $\{c(2), c(3), \ldots, c(n)\}$. *Let* n_1 *be the number of numbers in this set such that* $c(i) < \frac{i}{\log^{1-\epsilon} i}$ *where* $0 < \epsilon < 1$ *is fixed. We have* $\lim_{n\to\infty} \frac{n_1}{n} = 1$.

Theorem 2.12 *There exists* n_0 *such that if* $n \geq n_0$ *we have the following inequality*

$$\frac{1}{2} \log n < \sum_{i=2}^{n} \frac{1}{c(i)} < 2 \frac{n}{\log n}. \tag{9}$$

Proof. Theorem 2.3 gives

$$\sum_{i=2}^{n} \frac{1}{i} \leq \sum_{i=2}^{n} \frac{1}{c(i)} \leq \sum_{i=2}^{n} \frac{1}{\log i}. \tag{10}$$

Now, we have

$$\sum_{i=2}^{n} \frac{1}{i} = \int_2^n \frac{1}{x}\, dx + O(1) = h(n) \log n > \frac{1}{2} \log n, \tag{11}$$

where $h(n) \to 1$.

On the other hand, we have

$$\sum_{i=2}^{n} \frac{1}{\log i} = \int_2^n \frac{1}{\log x}\, dx + O(1) = g(n) \frac{n}{\log n} < 2 \frac{n}{\log n}, \tag{12}$$

where $g(n) \to 1$. Since (L'Hospital rule)

$$\lim_{x\to\infty} \frac{\int_2^x \frac{1}{\log t}\, dt}{\frac{x}{\log x}} = \lim_{x\to\infty} \frac{\frac{1}{\log x}}{\frac{\log x - 1}{\log^2 x}} = 1.$$

Finally, (10), (11) and (12) give (9). The theorem is proved.

Corollary 2.13 *The series* $\sum_{n=2}^{\infty} \frac{1}{c(n)}$ *is divergent and* $\sum_{i=2}^{n} \frac{1}{c(i)} = o(n)$.

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References

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