A GEOMETRIC INTRODUCTION TO K-THEORY

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Preface

I first learned Serre's definition of intersection multiplicity from Mel Hochster, back when I was an undergraduate. I was immediately intrigued by this surprising connection between homological algebra and geometry. As it has always been for me when learning mathematics, I wanted to know how I could have guessed this definition for myself—what are the underlying principles that *tell us* to go looking in homological algebra for a definition of multiplicity. This question has been in the back of my mind for most of my mathematical life. It took me a long time to accept that the answers to such questions are not often readily available; one has to instead make do with vague hints and partial explanations. I still believe, though, that the answers exist *somewhere*—and that it is the ultimate job of mathematicians to uncover them. So perhaps it is better said this way: those questions often don't have simple answers *yet*.

During my first year of graduate school I tried to puzzle out for myself the secrets behind Serre's definition. Thanks to the Gillet-Soulé paper [GS] I was led to K-theory, and similar hints of topology seemed to be operating in work of Roberts [R]. Coincidentally, MIT had a very active community of graduate students in topology, and I soon joined their ranks. Although there were other factors, it is not far from the truth to say that I became a topologist in order to understand Serre's definition.

In Winter quarter of 2012 I taught a course on this material at the University of Oregon. The graduate students taking the course converted my lectures into LaTeX, and then afterwards I both heavily revised and added to the resulting document. The present notes are the end result of this process. I am very grateful to the attending graduate students for the work they put into typesetting the lectures. These students were: Jeremiah Bartz, Christin Bibby, Safia Chettih, Emilio Gardella, Christopher Hardy, Liz Henning, Justin Hilburn, Zhanwen Huang, Tyler Kloefkorn, Joseph Loubert, Sylvia Naples, Min Ro, Patrick Schultz, Michael Sun, and Deb Vicinsky.

Introduction

1. Algebraic intersection multiplicities

Let Z be the parabola $y = x^2$ in $\mathbb{R}^2$, and let W be the tangent line at the vertex: the line $y = 0$. Then Z and W have an isolated point of intersection at $(0,0)$. Since high school you have known how to associate a *multiplicity* with this intersection: it is multiplicity 2, essentially because the polynomial x^2 has a double root at $x = 0$. This multiplicity also has a geometric interpretation, coming from intersection theory. If you perturb the intersection a bit, say by moving either Z or W by some small amount, then you get two points of intersection that are near $(0,0)$—and these points both converge to $(0,0)$ as the perturbation gets smaller and smaller.

You might object, rightly so, that I am lying to you. If we perturb $y = 0$ to $y = \epsilon$, with $\epsilon > 0$, then indeed we get two points of intersection: $(\sqrt{\epsilon}, \epsilon)$ and $(-\sqrt{\epsilon}, \epsilon)$. And these do indeed converge to $(0,0)$ as $\epsilon \to 0$. But if we perturb the line in the other direction, by taking ϵ to be negative, then we get no points of intersection at all! To fix this, it is important to work over the complex numbers rather than the reals: the connection between geometry and algebra works out best (and simplest) in this case. If we work over $\mathbb{C}$, then it is indeed true that almost all small perturbations of our equations yield two solutions close to $(0,0)$.

Our goal will be to vastly generalize the above phenomenom. Let $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$, and let Z be the algebraic variety defined by the vanishing of the f_j's. We write

$$Z = V(f_1, \ldots, f_k) = \{x \in \mathbb{C}^n \mid f_1(x) = f_2(x) = \cdots = f_k(x) = 0\}.$$

Likewise, let $g_1, \ldots, g_l \in \mathbb{C}[x_1, \ldots, x_n]$ and let $W = V(g_1, \ldots, g_l)$. Assume that P is an isolated point of the intersection $Z \cap W$. Our goal is to determine an algebraic formula, in terms of the f_i's and g_j's, for an intersection multiplicity $i(Z, W; P)$. This multiplicity should have the basic topological property that it coincides with the number of actual intersection points under almost all small deformations of Z and W.

Here are some basic properties, by no means comprehensive, that we would want such a formula to satisfy:

(1) $i(Z, W; P)$ should depend only on local information about Z and W near P.
(2) $i(Z, W; P) \geq 0$ always.
(3) If $\dim Z + \dim W < n$ then $i(Z, W; P) = 0$ (because in this case there is enough room in the ambient space to perturb Z and W so that they don't intersect at all).
(4) If $\dim Z + \dim W = n$ then $i(Z, W; P) > 0$.
(5) If $\dim Z + \dim W = n$ and Z and W meet transversely at P (meaning that $T_P Z \oplus T_P W = \mathbb{C}^n$), then $i(Z, W; P) = 1$.

Note that because of property (1) we can extend the notion of intersection multiplicity to varieties in $\mathbb{C}P^n$, simply by looking locally inside an affine chart for projective space that contains the point P. From now on we will do this without comment. The two statements below are not exactly 'basic properties' along the lines of (1)–(5) above, but they are basic results that any theory of intersection multiplicities should yield as consequences.

(6) Suppose that $X \hookrightarrow \mathbb{C}P^n$ is the vanishing set of a homogeneous polynomial, that is $X = V(f)$. Let L be a projective line in $\mathbb{C}P^n$ that meets X in finitely-many points. Then
$$\sum_{P \in X \cap L} i(X, L; P) = \deg(f).$$

(7) (Bezout's Theorem) Suppose that $X, Y \hookrightarrow \mathbb{C}P^2$ are the vanishing sets of homogeneous polynomials f and g, and that $X \cap Y$ consists of finitely-many points. Then
$$\sum_{P \in X \cap Y} i(X, Y; P) = (\deg f)(\deg g).$$

Note that (6), for the particular case $n = 2$, is a special case of (7).

If you play around with some simple examples, an idea for defining intersection multiplicities comes up naturally. It is

$$i(Z, W; P) = \dim_{\mathbb{C}} \Big[\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k, g_1, \ldots, g_l)\Big]_P. \tag{1.1}$$

Here the subscript P indicates localization of the given ring at the maximal ideal $(x_1 - p_1, \ldots, x_n - p_n)$ where $P = (p_1, \ldots, p_n)$. The localization is necessary because $Z \cap W$ might have points other than P in it, and our definition needs to only depend on what is happening near P.

The best way to get a feeling for the above definition is via some easy examples:

Example 1.2. Let $f = y - x^2$ and $g = y$. This is our example of the parabola and the tangent line at its vertex. The point $P = (0, 0)$ is the only intersection point, and our definition tells us to look at the ring
$$\mathbb{C}[x, y]/(y - x^2, y) \cong \mathbb{C}[x]/(x^2).$$
As a vector space over $\mathbb{C}$ this is two-dimensional, with basis 1 and x. So our definition gives $i(Z, W; P) = 2$ as desired. [Note that technically we should localize at the ideal (x, y), which corresponds to localization at (x) in $\mathbb{C}[x]/(x^2)$; however, this ring is already local and so the localization has no effect].

Example 1.3. Let $f = y^2 - x^3 - 3x$ and $g = y - \frac{3}{2}x - \frac{1}{2}$. Then $Z = V(f)$ is an elliptic curve, and one can check that $W = V(g)$ is the tangent line at the point $P = (1, 2)$. Let us recall how this works: the gradient vector to the curve is
$$\nabla f = [-3x^2 - 3, 2y]$$
and this is normal to the curve at (x, y). A tangent vector is then $[2y, 3x^3 + 3]$ (since this is orthogonal to ∇f), which means the slope of the curve at (x, y) is $(3x^3 + 3)/2y$. At the point $(1, 2)$ we then get slope $\frac{3}{2}$, and $V(g)$ is the line passing through $(1, 2)$ with this slope.

The line $V(g)$ intersects the curve at one other point, which we find by simultaneously solving $y^2 = x^3 + 3x$ and $y = \frac{3}{2}x + \frac{1}{2}$. This yields the cubic
$$0 = x^3 + 3x - (\tfrac{3}{2}x + \tfrac{1}{2})^2.$$
Since we know that $x = 1$ is a root, we can factor this out and then solve the resulting quadratic. One finds that the cubic factors as
$$0 = (x - 1)^2 \cdot (x - \tfrac{1}{4}).$$
The second point of intersection is found to be $Q = (\frac{1}{4}, \frac{7}{8})$.

Note the appearance of $(x-1)$ with multiplicity two in the above factorization. The fact that we had a tangent line at $x=1$ guaranteed that the multiplicity would be strictly larger than one. Likewise, the fact that $(x-\frac{1}{4})$ has multiplicity one tells us that $V(g)$ intersects the curve transversally. These facts suggest that $i(Z,W;P)=2$ and $i(Z,W;Q)=1$. Let us consider these in terms of point-counting under small deformations. We can perturb either Z or W, but it is perhaps easiest to perturb the line W: we can write $g'=y-Ax-B$ and then consider what happens for all (A,B) near $(\frac{3}{2},\frac{1}{2})$. We will need to find the intersection of Z and $W'=V(g')$, which as before leads to a cubic. To save us from the unpleasantness of having to solve the cubic, let us again arrange for there to be a known solution which we can factor out. It is possible to have this solution be either $(1,2)$ or $(\frac{1}{4},\frac{7}{8})$. The calculations turn out to be a little easier for the latter, despite the annoying fractions. So we assume $\frac{7}{8}=\frac{A}{4}+B$, or $g'=y-A(x-\frac{1}{4})-\frac{7}{8}$. Since we want to look at A near $\frac{3}{2}$, it is convenient to write $A=\frac{3}{2}+\epsilon$ where ϵ is near zero.

Finding common solutions of $f=0$ and $g'=0$ yields a cubic with $(x-\frac{1}{4})$ as a factor, and dividing this out we obtain the quadratic

$$0=x^2-x(2+3\epsilon+\epsilon^2)+(1-\epsilon+\tfrac{\epsilon^2}{4}).$$

The discriminant of this quadratic is $D=\epsilon(\epsilon^3+6\epsilon^2+4\epsilon+16)$, so the quadratic has a double root when $\epsilon=0$ (as expected) but simple roots for values of ϵ near but not equal to zero. So for these values of ϵ we get two points of intersection of $V(f)$ and $V(g')$ near P, and it is easy to see that they converge to P as ϵ approaches zero.

Let us now see what our provisional definition from (1.1) gives. The quotient ring in our definition is

$$\begin{aligned}\mathbb{C}[x,y]/\big(y^2-x^3-3x,\, y-\tfrac{3}{2}x-\tfrac{1}{2}\big) &\cong \mathbb{C}[x]/\big((\tfrac{3}{2}x+\tfrac{1}{2})^2-x^3-3x\big)\\ &\cong \mathbb{C}[x]/\big((x-1)^2(x-\tfrac{1}{4})\big).\end{aligned}$$

Here we are killing a cubic in $\mathbb{C}[x]$, and so we get a three-dimensional vector space with basis $1,x,x^2$. Note that this is, in some sense, seeing all of the information at P and Q together—this demonstrates the importance of localization. Localization at P corresponds to localizing at $(x-1)$, which turns $(x-\frac{1}{4})$ into a unit. So our localized ring is

$$\mathbb{C}[x]_{(x-1)}/((x-1)^2)\cong\mathbb{C}[t]_{(t)}/(t^2)$$

(where we set $t=x-1$), and this has dimension 2 over $\mathbb{C}$. So $i(Z,W;P)=2$, as desired.

If we localize at $(x-\frac{1}{4})$ then the $(x-1)^2$ factor becomes a unit, and our localized ring becomes $\mathbb{C}[x]_{(x-\frac{1}{4})}/(x-\frac{1}{4})\cong\mathbb{C}[t]_{(t)}/(t)$, which is just a copy of $\mathbb{C}$. So $i(Z,W;Q)=1$.

Note that both Example 1.2 and Example 1.3 involve a key step where the variable y is eliminated, thus bringing the problem down to the multiplicity of a root in a one-variable polynomial. One cannot always do such an elimination—in fact it happens only rarely. So these examples are very special, although they still serve to give some sense of how things are working.

It turns out that our provisional definition from (1.1) is enough to prove Bezout's Theorem for curves in $\mathbb{C}P^2$. But in some sense one is getting lucky here, and it works only because the dimensions of the varieties are so small. When one starts to look at higher-dimensional varieties it doesn't take long to find examples where the definition clearly gives the wrong answers:

Example 1.4. Let $\mathbb{C}^4$ have coordinates u, v, w, y, and let $X, Y \subseteq \mathbb{C}^4$ be given by

$$X = V(u^3 - v^2, u^2y - vw, uw - vy, w^2 - uy^2), \qquad Y = V(u, y).$$

Note that X is somewhat complicated, but Y is just a plane. If a point (u, v, w, y) is on $X \cap Y$ then $u = y = 0$ and therefore the equations for X say that

$$v^2 = 0, \quad vw = 0, \quad \text{and} \quad w^2 = 0$$

as well. So $X \cap Y$ consists of the unique point $(0, 0, 0, 0)$. Our provisional definition of intersection multiplicities would have us look at the ring

$$\mathbb{C}[u, v, w, y]/(u, y, u^3 - v^2, u^2y - vw, uw - vy, w^2 - uy^2) \cong \mathbb{C}[v, w]/(v^2, vw, w^2)$$

which is three-dimensional over $\mathbb{C}$. If this were the correct answer, then perturbing the plane Y should generically give three points of intersection. However, this is not the case. If we perturb Y to $V(x - \epsilon, y - \delta)$ then the intersection with X is given by the equations

$$u = \epsilon, \quad y = \delta, \quad \epsilon^3 = v^2, \quad \epsilon^2\delta = vw, \quad \epsilon w = v\delta, \quad w^2 = \epsilon\delta^2.$$

As long as $\epsilon \neq 0$ we have two solutions for v, and then the fourth equation determines w completely. So we only have two points on the intersection, after small perturbations. This is, in fact, the correct answer: $i(Z, W; P) = 2$, and our provisional definition is a failure.

Serre discovered the correct formula for the interesection multiplicity [S]. His formula is as follows. If we set $R = \mathbb{C}[x_1, \ldots, x_n]$ then

$$(1.5) \qquad i(Z, W; P) = \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{C}} \Big[\mathrm{Tor}_j^R\Big(R/(f_1, \ldots, f_k), R/(g_1, \ldots, g_l)\Big)\Big]_P.$$

There are several things to say here. First, although the sum is written to infinity it turns out that the Tor modules vanish for all $j > n$ (we will prove this later). So it is, in fact, a finite sum. Secondly, the condition that P be an isolated point of intersection forces the $\mathbb{C}$-dimension of all the Tor's to be finite. So the formula does make sense. As to why this gives the "correct" numbers, it will take us a while to explain this. But note that the $j = 0$ term is the dimension of

$$\begin{aligned}\mathrm{Tor}_0(R/(f_1, \ldots, f_k), R/(g_1, \ldots, g_l)) &\cong R/(f_1, \ldots, f_k) \otimes_R R/(g_1, \ldots, g_l) \\ &\cong R/(f_1, \ldots, f_k, g_1, \ldots, g_l).\end{aligned}$$

So our provisional definition from (1.1) is just the $j = 0$ term. One should think of the higher terms as "corrections" to this initial term; in a certain sense these corrections get smaller as j increases (this is not obvious).

An algebraist who looks at (1.5) will immediately notice some possible generalizations. The $R/(\underline{f})$ and $R/(\underline{g})$ terms can be replaced by any finitely-generated module M and N, as long as the $\mathrm{Tor}_j(M, N)$ modules are finite-dimensional over $\mathbb{C}$. For this it turns out to be enough that $M \otimes_R N$ be finite-dimensional over $\mathbb{C}$. Also, we can replace $\mathbb{C}[x_1, \ldots, x_n]$ with any ring having the property that all finitely-generated modules have finite projective dimension—necessary so that the alternating sum of (1.5) is finite. Such rings are called **regular**. Also, instead of localizing the Tor-modules we can just localize the ring R at the very beginning.

And finally, in this generality we need to replace $\dim_{\mathbb{C}}$ with a similar invariant: the notion of *length* (meaning the length of a composition series for our module). This leads to the following setup.

Let R be a regular, local ring (all rings are assumed to be commutative and Noetherian unless otherwise noted). Let M and N be finitely-generated modules over R such that $M \otimes_R N$ has finite length. This implies that all the $\mathrm{Tor}_j(M, N)$ modules also have finite length. Define

$$e(M,N) = \sum_{j=0}^{\infty} (-1)^j \ell\big(\mathrm{Tor}_j(M,N)\big) \tag{1.6}$$

and call this the **intersection multiplicty** of the modules M and N.

Based on geometric intuition, Serre made the following conjectures about the above situation:

(1) $\dim M + \dim N \leq \dim R$ always
(2) $e(M, N) \geq 0$ always
(3) If $\dim M + \dim N < \dim R$ then $e(M, N) = 0$.
(4) If $\dim M + \dim N = \dim R$ then $e(M, N) > 0$.

Serre proved all of these in the case that R contains a field, the so-called "geometric case" (some non-geometric examples for R include power series rings over the p-adic integers $\mathbb{Z}_p$). Serre also proved (1) in general. Conjecture (3) was proven in the mid 80s by Roberts and Gillet-Soule (independently), using some sophisticated topological ideas that were imported into algebra. Conjecture (2) was proven by Gabber in the mid 90s, using some high-tech algebraic geometry. Conjecture (4) is still open.

1.7. **Where we are headed.** Our main goal in these notes is to describe a particular subset of the mathematics surrounding Serre's definition of multiplicity. It is possible to explore this subject purely in algebraic terms, and that is basically what Serre did in his book [S]. In contrast, our main focus will be topological. Although both commutative algebra and algebraic geometry play a large role in our story, we will always adopt a perspective that concentrates on their relations to topology—and in particular, to K-theory.

Here is a brief summary of some of the main points that we will encounter:

(1) There are certain generalized cohomology theories—called *complex-oriented*—which have a close connection to geometry and intersection theory. Any such cohomology can be used to detect intersection multiplicities.
(2) Topological K-theory is a complex-oriented cohomology theory. Elements of the groups $K^*(X)$ are specified by vector bundles on X, or more generally by bounded chain complexes of vector bundles on X. Fundamental classes for complex submanifolds of X are given by *resolutions*.
(3) When X is an algebraic variety there is another version of K-theory called *algebraic K-theory*, which we might denote $K^*_{alg}(X)$. The analogs of vector bundles are locally free coherent sheaves, or just finitely-generated projective modules when X is affine. Thus, in the affine case elements of $K^*_{alg}(X)$ can be specified by bounded chain complexes of finitely-generated projective modules. This is the main connection between homological algebra and K-theory.

(4) Serre's definition of intersection multiplicities essentially comes from the intersection product in K-homology, which is the cup product in K-cohomology translated to homology via Poincaré Duality.

We will spend a large chunk of these notes filling in the details behind (1)–(4). But whereas we take our motivation from Serre's definition of multiplicity, that is not the only subject we cover in these notes. Once we have the K-theory apparatus up and running, there are lots of neat things to do with it. We have attempted, for the most part, to chose topics that accentuate the relationship between K-theory and geometry in the same way that Serre's definition of multiplicity does.

Part 1. K-theory in algebra

In this first part of the notes we investigate the K-theory of modules over a ring R. There are two main varieties: one can study the K-theory of all finitely-generated modules, leading to the group $G(R)$, or one can study the K-theory of finitely-generated *projective* modules—leading to the group $K(R)$. In the following sections we get a taste for these groups and the relations between them.

For the duration of these notes, all rings are commutative with identity unless otherwise stated. Some of the theory we develop works in greater generality, but we will stay focused on the commutative case.

2. A first look at K-theory

Understanding Serre's alternating-sum-of-Tor's formula for intersection multiplicities will be a gradual process. In particular, there is quite a bit of nontrivial commutative algebra that is needed for the story; we will need to develop this as we go along. We will continue to sweep some of these details under the rug for the moment, but let us at least get a couple of things out in the open. To begin with, we will need the following important result:

Theorem 2.1 (Hilbert Syzygy Theorem)**.** *Let k be a field and let R be $k[x_1,\dots,x_n]$ (or any localization of this ring). Then every finitely-generated R-module has a free resolution of length at most n.*

We will prove this theorem in Section 17 below. We mention it here because it implies that $\mathrm{Tor}_j(M,N)=0$ for $j>n$. Therefore the sum in Serre's formula is actually finite. More generally, a ring is called **regular** if every finitely-generated module has a finite, projective resolution. It is a theorem that localizations of regular rings are again regular. Hilbert's Syzygy Theorem simply says that polynomial rings over a field are regular. We will find that regular rings are the 'right' context in which to explore Serre's formula.

We will also need the following simple observation. If P is a prime ideal in any ring R, then

$$[\mathrm{Tor}_R(M,N)]_P=\mathrm{Tor}^{R_p}(M_P,N_P).$$

To see this, let $Q_\bullet\to M\to 0$ be an R-free resolution of M. Since localization is exact, $(Q_\bullet)_P$ is an R_P-free resolution of M_P. Hence

$$\begin{aligned}\mathrm{Tor}_j^{R_P}(M_P,N_P)=H_j\big((Q_\bullet)_P\otimes_{R_P}N_P\big)&=H_j\big(Q_\bullet\otimes_R R_P\otimes_{R_P}N\otimes_R R_P\big)\\&=H_j\big(Q_\bullet\otimes_R N\otimes R_P\big)\\&=H_j\big(Q_\bullet\otimes_R N\big)\otimes R_P\\&=\mathrm{Tor}_j^R(M,N)\otimes_R R_P.\end{aligned}$$

The importance of this observation is that it tells us that the Tor's in Serre's formula may all be taken over the ring R_P. So we might as well work over this ring from beginning to end. Moreover, without loss of generality we might as well assume that our point of intersection is the origin, which makes the corresponding maximal ideal $(x_1,\dots,x_n)$.

Let $R=\mathbb{C}[x_1,\dots,x_n]_{(x_1,\dots,x_n)}$, and let M and N be finitely-generated modules over R. Assume that $\dim_{\mathbb{C}}(M\otimes_R N)<\infty$. It turns out that this implies that

$\dim_{\mathbb{C}} \mathrm{Tor}_j(M, N) < \infty$ for every j, so that we can define

$$e(M, N) = \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{C}} \mathrm{Tor}_j(M, N).$$

The above definition generalizes the notion of intersection multiplicity from pairs $(R/I, R/J)$ to pairs of modules (M, N). The reason for making this generalization might not be clear at first, but the following nice property provides some justification:

Lemma 2.2. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. Then $e(M, N) = e(M', N) + e(M'', N)$, assuming all three multiplicities are defined (that is, under the assumption that $\dim_{\mathbb{C}}(M \otimes N) < \infty$ and similarly with M replaced by M' and M'').*

Proof. Consider the long exact sequence

$$\cdots \to \mathrm{Tor}_j(M', N) \to \mathrm{Tor}_j(M, N) \to \mathrm{Tor}_j(M'', N) \to \cdots$$

This sequence terminates after a finite number of steps, by Hilbert's Syzygy Theorem. By exactness, the alternating sum of the dimensions is zero. This is precisely the desired formula. □

Lemma 2.2 is referred to as the *additivity of intersection multiplicities*. Of course the additivity holds equally well in the second variable, by the same argument.

While exploring ideas in this general area, Grothendieck hit upon the idea of inventing a group that captures **all** the additive invariants of modules. Any invariant such as $e(-, N)$ would then factor through this group. Here is the definition:

Definition 2.3. *Let R be any ring. Let $\mathcal{F}(R)$ be the free abelian group with one generator $[M]$ for every isomorphism class of finitely-generated R-module M. Let $G(R)$ be the quotient of $\mathcal{F}(R)$ by the subgroup generated by all elements $[M] - [M'] - [M'']$ for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely-generated R-modules. The group $G(R)$ is called the* ***Grothendieck group*** *of finitely-generated R-modules.*

Remark 2.4. It is important in the definition of $G(R)$ that one use only *finitely-generated* R-modules, otherwise the group would be trivial. To see this, if M is any module then let $M^\infty = M \oplus M \oplus M \cdots$. Note that there is a short exact sequence

$$0 \to M \hookrightarrow M^\infty \to M^\infty \to 0$$

where M is included as the first summand. If we had defined $G(R)$ without the finite-generation condition, we would have $[M^\infty] = [M] + [M^\infty]$ and therefore $[M] = 0$. Since this holds for every module M, the group $G(R)$ would be zero. This is called the "Eilenberg Swindle".

The following lemma records some useful ways of obtaining relations in $G(R)$:

Lemma 2.5. *Let R be any ring.*

(a) If $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ is an exact sequence of finitely-generated R-modules, then $\sum (-1)^i [C_i] = 0$ in $G(R)$.

(b) If $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} = 0$ is a filtration of M by finitely-generated modules, then $[M] = \sum_i [M_i/M_{i+1}]$ in $G(R)$.

(c) *Assume that R is Noetherian, and let $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ be any chain complex of R-modules. Then $\sum_i (-1)^i [C_i] = \sum_i (-1)^i [H_i(C)]$ in $G(R)$.*

Proof. We prove (a) and (c) at the same time. If $C_\bullet$ is a chain complex, note that one has the short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ where Z_i and B_i are the cycles and boundaries in each dimension. One also has $0 \to B_i \hookrightarrow Z_i \to H_i(C) \to 0$. Assuming everything in sight is finitely-generated, one gets a series of relations in $G(R)$ that immediately yield $\sum(-1)^i[C_i] = \sum(-1)^i[H_i(C)]$. So if R is Noetherian we are done, because everything indeed is finitely-generated; this proves (c). In the general case where R is not necessarily Noetherian, we know that each B_i is finitely-generated because it is the image of C_{i+1}. But if $C_\bullet$ is exact then $B_i = Z_i$ and so the Z_i's are also finitely-generated. We have the relations $[C_i] = [Z_i] + [B_{i-1}] = [Z_i] + [Z_{i-1}]$, and from this it is evident that $\sum(-1)^i[C_i] = 0$. This proves (a).

The proof of (b) is similarly easy; one considers the evident exact sequences $0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0$ and the resulting relations in $G(R)$. $\square$

Here are a series of examples:

(1) Suppose $R = F$, a field. Clearly $G(F)$ is generated by $[F]$, since every finitely-generated F-module has the form F^n. If we observe the existence of the group homomorphism $\dim\colon G(F) \to \mathbb{Z}$, which is clearly surjective because it sends $[F]$ to 1, then it follows that $G(F) \cong \mathbb{Z}$.

(2) More generally, suppose that R is a domain. The **rank** of an R-module M is defined to be the dimension of $M \otimes_R QF(R)$ over $QF(R)$, where $QF(R)$ is the quotient field. The rank clearly gives a homomorphism $G(R) \to \mathbb{Z}$, which is surjective because $[R] \mapsto 1$. So $G(R)$ has $\mathbb{Z}$ as a direct summand.

(3) Next consider $R = \mathbb{Z}$. Then $G(\mathbb{Z})$ is generated by the classes $[\mathbb{Z}]$ and $[\mathbb{Z}/n]$ for $n > 1$, by the classification of finitely-generated abelian groups. The short exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \to 0$ shows that $[\mathbb{Z}/n] = 0$ for all n, hence $G(\mathbb{Z})$ is cyclic. Using (b), it follows that $G(\mathbb{Z}) = \mathbb{Z}$. This computation works just as well for any PID.

(4) So far we have only seen cases where $G(R) \cong \mathbb{Z}$. For a case where this is not true, try $R = F \times F$ where F is a field. You should find that $G(R) \cong \mathbb{Z}^2$ here.

(5) Let G be a finite group, and let $R = \mathbb{C}[G]$ be the group algebra. So R-modules are just representations of G on complex vector spaces. The basic theory of such finite-dimensional representations says that each is a direct sum of irreducibles, in an essentially unique way. Moreover, each short exact sequence is split. A little thought shows that this is saying that $G(R)$ is a free abelian group with basis consisting of the isomorphism classes of irreducible representations.

(6) So far all the examples we have computed have $G(R)$ equal to a free abelian group. This is not always the case, although I don't know an example where it is really easy to see this. For a not-so-simple example, let R be the ring of integers in a number field. It turns out that $G(R) \cong \mathbb{Z} \oplus \mathrm{Cl}(R)$, where $\mathrm{Cl}(R)$ is the ideal class group of R. This class group contains some sophisticated number-theoretic information about R. It is known to always be torsion, and it is usually nontrivial. We will work out a simple example when we have more tools under out belt: see Example 4.2.

(7) As another simple example, we look at $R = F[t]/(t^2)$ where F is a field. For any module M over R we have the filtration $M \supseteq tM$, and so $[M] = [M/tM] + [tM]$. But both M/tM and tM are killed by t, hence are direct sums of copies of F (where t acts as zero). This shows that $G(R)$ is generated by $[F]$. We also have the function $\dim_F(-)\colon G(R) \to \mathbb{Z}$. Since this function sends $[F]$ to 1, it must be an isomorphism.

(8) The final example we consider here is a variation of the previous one. Let us look at $R = \mathbb{Z}/p^2$. The R-modules are simply abelian groups killed by p^2. Given any such module A one can consider the sequence $0 \to pA \hookrightarrow A \to A/pA \to 0$, and observe that the first and third terms are $\mathbb{Z}/p$-vector spaces. So $[\mathbb{Z}/p]$ generates $G(R)$. We claim that $G(R) \cong \mathbb{Z}$, and as in the previous example the easiest way to see this is to write down an additive invariant of R-modules taking its values in $\mathbb{Z}$. All finitely-generated R-modules have a finite composition series, and so we can take the Jordan-Hölder length; this is the same as $\ell(A) = \dim_{\mathbb{Z}/p} A/pA + \dim_{\mathbb{Z}/p} pA$. With some trouble one can check that this is indeed an additive invariant (or refer to the Jordan-Hölder theorem), and of course $\ell(\mathbb{Z}/p) = 1$. This completes the calculation.

Exercise 2.6. Prove that $G(R) \cong \mathbb{Z}$ for $R = F[t]/(t^n)$ or $R = \mathbb{Z}/p^n$.

The above examples help establish some basic intuition. In general, though, it is very hard to compute $G(R)$.

We can adapt our definition of intersection multiplicity of two modules to define a product on $G(R)$, at least when R is regular. For finitely-generated modules M and N, define

$$[M] \odot [N] = \sum_j (-1)^j [\mathrm{Tor}_j(M, N)].$$

The long exact sequence for Tor shows that this definition is additive in the two variables, and hence passes to a pairing $G(R) \otimes G(R) \to G(R)$. It is not *at all* clear that this is associative, although we will prove this shortly.

The above product on $G(R)$ is certainly not the first thing one would think of. It is more natural to try to define a product by having $[M] \cdot [N] = [M \otimes_R N]$, but of course this is not additive in the two variables because of the failure of the tensor product to be exact. The higher Tor's are correcting for this. However, we *can* make this naive definition work if we restrict to a certain class of modules. To that end, let us introduce the following definition:

Definition 2.7. *Let R be any ring. Let $\mathcal{F}_K(R)$ be the free abelian group with one generator $[P]$ for every isomorphism class of finitely-generated, projective R-module M. Let $K(R)$ be the quotient of $\mathcal{F}_K(R)$ by the subgroup generated by all elements $[P] - [P'] - [P'']$ for every short exact sequence $0 \to P' \to P \to P'' \to 0$ of finitely-generated projectives. The group $K(R)$ is called the **Grothendieck group of finitely-generated projective modules**.*

Every short exact sequence of projectives is actually split, so we could also have defined $K(R)$ by imposing the relations $[P \oplus Q] = [P] + [Q]$ for every two finitely-generated projectives P and Q. This makes it a little easier to understand when two modules represent the same class in $K(R)$:

Proposition 2.8. *Let P and Q be finitely-generated projective R-modules. Then $[P] = [Q]$ in $K(R)$ if and only if there exists a finitely-generated projective module W such that $P \oplus W \cong Q \oplus W$. In fact, the same remains true if we require W to be free instead of projective.*

Proof. The 'if' part of the proposition is trivial; we concentrate on the 'only if' part. Let $\mathcal{R}el \subseteq \mathcal{F}(R)$ be the subgroup generated by all elements $[J] - [J'] - [J'']$ for short exact sequences $0 \to J' \to J \to J'' \to 0$. If $[P] - [Q] \in \mathcal{R}el$ then there exists two collections of such sequences $0 \to P'_i \to P_i \to P''_i \to 0$, $1 \le i \le k_1$, and $0 \to Q'_i \to Q_i \to Q''_i \to 0$, $1 \le i \le k_2$, such that

$$[P] - [Q] = \sum_{i=1}^{k_1}\big([P_i] - [P'_i] - [P''_i]\big) + \sum_{i=1}^{k_2}\big([Q'_i] + [Q''_i] - [Q_i]\big)$$

in $\mathcal{F}(R)$. Rearranging, this gives

$$[P] + \sum_{i=1}^{k_1}\big([P'_i] + [P''_i]\big) + \sum_{i=1}^{k_2}[Q_i] = [Q] + \sum_{i=1}^{k_1}[P_i] + \sum_{i=1}^{k_2}\big([Q'_i] + [Q''_i]\big).$$

The only way such sums of basis elements can give the same element of $\mathcal{F}(R)$ is if the collection of summands on the two sides are the same up to permutation. But in that case one can write

$$P \oplus \bigoplus_{i=1}^{k_1}(P'_i \oplus P''_i) \oplus \bigoplus_{i=1}^{k_2} Q_i \cong Q \oplus \bigoplus_{i=1}^{k_1} P_i \oplus \bigoplus_{i=1}^{k_2}(Q'_i \oplus Q''_i).$$

But note that $P_i \cong P'_i \oplus P''_i$ and similarly for Q_i. So if we let W be the module $\bigoplus_{i=1}^{k_1}(P'_i \oplus P''_i) \oplus \bigoplus_{i=1}^{k_2} Q_i$ then we have $P \oplus W \cong Q \oplus W$.

For the last statement in the proposition, just observe that since W is projective it is a direct summand of a free module. That is, there exists a module W' such that $W \oplus W'$ is finitely-genereated and free. Certainly $P \oplus (W \oplus W') \cong Q \oplus (W \oplus W')$. □

Since projective modules are flat, the product $[P] \cdot [Q] = [P \otimes_R Q]$ is additive and so extends to a product $K(R) \otimes K(R) \to K(R)$. Note that this product is obviously associative, and so makes $K(R)$ into a ring. This is true without any assumptions on R whatsoever (except our standing assumption that R be commutative).

Remark 2.9. Given the motivation of having the tensor product give a ring structure, one might wonder why we used projective modules to define $K(R)$ rather than flat modules. We could have done so, but for finitely-generated modules over commutative, Noetherian rings, being flat and projective are equivalent notions—see [E, Corollary 6.6]. For various reasons it is more common to make the definition using the projective hypothesis.

There is an evident map $\alpha\colon K(R) \to G(R)$ which sends $[P]$ to $[P]$ (note that these two symbols, while they look the same, denote elements of different groups). This brings us to our first important theorem:

Theorem 2.10. *If R is regular, then $\alpha\colon K(R) \to G(R)$ is an isomorphism.*

Proof. Surjectivity is easy to see: if M is a finitely-generated module, choose a finite, projective resolution $P_\bullet \to M \to 0$. Then $\sum_j (-1)^j [P_j] = [M]$ in $G(R)$, and this clearly proves that $[M]$ is in the image of α.

Proving injectivity is slightly harder, and it will be most convenient just to define an inverse for α. The above paragraph gives us the definition: for a finitely-generated R-module M, define

$$\beta([M]) = \sum_j (-1)^j [P_j]$$

where $P_\bullet \to M \to 0$ is some finite, projective resolution. We need to show that this is independent of the choice of P, and that it is additive: these facts will show that β defines a map $G(R) \to K(R)$. It is then obvious that this is a two-sided inverse to α.

Suppose $Q_\bullet \to M \to 0$ is another finite, projective resolution of M. Use the Comparison Theorem of homological algebra to produce a map of chain complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Let $T_\bullet$ be the mapping cone of $f\colon P_\bullet \to Q_\bullet$. Recall this means that $T_j = Q_j \oplus P_{j-1}$, with the differential defined by

$$d_T(a,b) = \Big(d_Q(a) + (-1)^{|b|} f(b),\ d_P(b)\Big).$$

There is a short exact sequence of chain complexes

$$0 \to Q \hookrightarrow T \to \Sigma P \to 0$$

where ΣP denotes a copy of P in which everything has been shifted up a dimension (so that $(\Sigma P)_n = P_{n-1}$). The long exact sequence on homology groups shows readily that T is exact, hence we have $\sum_j (-1)^j [T_j] = 0$ in $K(R)$. Since $[T_j] = [Q_j] + [P_{j-1}]$ in $K(R)$ this gives that $\sum_j (-1)^j [P_j] = \sum_j (-1)^j [Q_j]$. Hence our definition of β does not depend on the choice of resolution.

A similar argument can be used to show additivity. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, and let $P_\bullet \to M'$ and $Q_\bullet \to M$ be finite, projective resolutions. Lift the map $M' \to M$ to a map of complexes $f\colon P_\bullet \to Q_\bullet$, and let $T_\bullet$ be the mapping cone of f. The long exact sequence for homology readily shows that T is a projective resolution of M''. So

$$\beta(M'') = \sum (-1)^j [T_j] = \sum (-1)^j [Q_j] - \sum (-1)^j [P_j] = \beta(M) - \beta(M')$$

and this proves additivity. This completes our proof. □

Using the isomorphism $K(R) \to G(R)$ (when R is regular), we can transplant the ring structure on $K(R)$ to the group $G(R)$. We claim that this gives the product $\odot$ defined via Tor's. In the following result, $\beta\colon G(R) \to K(R)$ is the inverse to α defined in the proof of Theorem 2.10.

Proposition 2.11. *Assume that R is regular. Then for any two finitely-generated modules M and N, we have*

$$\alpha\Big[\beta([M]) \otimes \beta([N])\Big] = \sum (-1)^j [\mathrm{Tor}_j(M,N)] = [M] \odot [N].$$

Proof. Let $P_\bullet \to M$ and $Q_\bullet \to N$ be finite, projective resolutions. Fix j, and consider the complex $P_\bullet \otimes Q_j$. This is a resolution of $M \otimes Q_j$, since Q_j is flat. So $\sum_i (-1)^i [P_i \otimes Q_j] = [M \otimes Q_j]$ in $G(R)$. Using this for each j, we have that

$$\begin{aligned}
\alpha\Big[\beta([M]) \otimes \beta([N])\Big] &= \sum_{i,j} (-1)^{i+j} [P_i \otimes Q_j] \\
&= \sum_j (-1)^j [M \otimes Q_j] \\
&= \sum_j (-1)^j [H_j(M \otimes Q)] \quad \text{using Lemma 2.5(c)} \\
&= \sum_j (-1)^j [\operatorname{Tor}_j(M, N)].
\end{aligned}$$

□

Corollary 2.12. *When R is regular, the product $\odot$ on $G(R)$ is associative.*

Proof. This follows immediately from the fact that the tensor product gives an associative multiplication on $K(R)$. □

Let us review the above situation. For any ring R, we have the group $K(R)$ which also comes to us with an easily-defined ring structure $\otimes$. We also have the group $G(R)$—but this does not have any evident ring structure. When R is regular, there is an isomorphism $K(R) \to G(R)$ which allows one to transplant the ring structure from $K(R)$ onto $G(R)$: and this leads us directly to our alternating-sum-of-Tors.

This situation is very reminiscent of something you have seen in a basic algebraic topology course. When X is a (compact, oriented) manifold, there were early attempts to put a ring structure on $H_*(X)$ coming from the intersection product. This is technically very difficult. In modern times one avoids these technicalities by instead introducing the cohomology groups $H^*(X)$, and here it is easy to define a ring structure: the cup product. When X is a compact, oriented manifold one has the Poincaré Duality isomorphism $H^*(X) \to H_*(X)$ given by capping with the fundamental class, and this lets one transplant the cup product onto $H_*(X)$. This is the modern approach to intersection theory.

The parallels here are intriguing: $K(R)$ is somehow like $H^*(X)$, and $G(R)$ is somehow like $H_*(X)$. The regularity condition is like being a manifold. We will spend the rest of this course exploring these parallels. [The reader might wonder what happened to the assumptions of compactness and orientability. Neither of these is really needed for Poincaré Duality, as long as one does things correctly. For the version of Poincaré Duality for noncompact manifolds one needs to replace ordinary homology with Borel-Moore homology—this is similar to singular homology, but chains are permitted to have infinitely many terms if they stretch out to infinity. For non-orientable manifolds one needs to use twisted coefficients.]

2.13. **Some very basic algebraic geometry.** To further develop the analogies between $(K(R), G(R))$ and $(H^*(X), H_*(X))$ we need more of a geometric understanding of the former groups. This starts to require some familiarity with the language of algebraic geometry.

At its most basic level, algebraic geometry attempts to study the geometry of affine n-space $\mathbb{C}^n$ by seeing how it is reflected in the algebra of the ring of polynomial functions $R = \mathbb{C}[x_1, \dots, x_n]$. Hilbert's Nullstellensatz says that points of $\mathbb{C}^n$ are in bijective correspondence with maximal ideals in R: the bijection sends $q = (q_1, \dots, q_n)$ to $m_q = (x_1 - q_1, \dots, x_n - q_n)$. With a little work one can generalize this bijection. If $S \subseteq \mathbb{C}^n$ is any subset, define $\mathcal{I}(S) = \{f \in R \mid f(x) = 0 \text{ for all } x \in S\}$. This is an ideal in R, in fact a radical ideal (meaning that if $f^n \in \mathcal{I}(S)$ then $f \in \mathcal{I}(S)$). In the other direction, if $I \subseteq R$ is any ideal then define $V(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in I\}$. Notice that $V(m_q) = \{q\}$ and $\mathcal{I}(\{q\}) = m_q$.

An **algebraic set** in $\mathbb{C}^n$ is any subset of the form $V(I)$ for some ideal $I \subseteq R$. The algebraic sets form the closed sets for a topology on $\mathbb{C}^n$, called the **Zariski topology**. One form of the Nullstellensatz says that V and $\mathcal{I}$ give a bijection between algebraic sets and radical ideals in R. Under this bijection the prime ideals correspond to **irreducible** algebraic sets—ones that cannot be written as $X \cup Y$ where both X and Y are proper closed subsets. Algebraic sets are also called algebraic **subvarieties**.

The above discussion is summarized in the following table:

Geometry	**Algebra**
$\mathbb{C}^n$ or $\mathbb{A}^n_{\mathbb{C}}$	$\mathbb{C}[x_1, \dots, x_n] = R$
Points $(q_1, \dots, q_n)$	Maximal ideals $(x_1 - q_1, \dots, x_n - q_n)$
Algebraic sets	Radical ideals
Irreducible algebraic sets	Prime ideals

The ring R is best thought of as the set of maps of varieties $\mathbb{A}^n \to \mathbb{A}^1$, with pointwise addition and multiplication. If we restrict to some irreducible subvariety $X = V(P) \subseteq \mathbb{A}^n$ instead, then the ring of functions $X \to \mathbb{A}^1$ is R/P. This ring of functions is commonly called the **coordinate ring** of X. Much of the dictionary between $\mathbb{A}^n$ and R discussed above adapts verbatim to give a dictionary between X and its coordinate ring:

Geometry	**Algebra**
$X = V(P)$	$\mathbb{C}[x_1, \dots, x_n]/P = R/P$
Points in X	Maximal ideals in R/P
Algebraic subsets $V(I) \subseteq X$	Radical ideals in R/P
Irreducible algebraic sets $V(Q) \subseteq X$	Prime ideals in R/P.

Note that ideals in R/P correspond bijectively to ideals in R containing P, and likewise for prime (respectively, radical) ideals.

We need one last observation. Passing from $\mathbb{A}^n$ to $\mathbb{A}^{n+1}$ corresponds algebraically to passing from R to $R[t]$. If $X = V(P) \subseteq \mathbb{A}^n$ is an irreducible algebraic set, then $X \times \mathbb{A}^1 \subseteq \mathbb{A}^{n+1}$ is $V(P[t])$ where $P[t] \subseteq R[t]$. That is, the coordinate ring of X is R/P and the coordinate ring of $X \times \mathbb{A}^1$ is $R[t]/P[t] = (R/P)[t]$. We supplement our earlier tables with the following line:

Geometry	**Algebra**
$X \rightsquigarrow X \times \mathbb{A}^1$	$S \rightsquigarrow S[t]$

We have defined $G(-)$ and $K(-)$ as functors taking rings as their inputs, but we could also think of them as taking varieties (or schemes) as their inputs. We will write $G(R)$ and $G(\operatorname{Spec} R)$ interchangeably, and similarly for the K-groups. It turns out that the geometric perspective and notation is very useful—many properties of these functors take on a familiar "homological" form when written geometrically. But for the moment we will mostly keep with the algebraic notation, writing $G(R)$ more often than $G(\operatorname{Spec} R)$.

2.14. **Further properties of $G(R)$.** We return to the study of the groups $G(R)$ and $K(R)$, for the moment concentrating on the former.

Theorem 2.15. *If R is Noetherian, the Grothendieck group $G(R)$ is generated by the set of elements $[R/P]$ where $P \subseteq R$ is prime.*

Before proving this result let us comment on the significance. When X is a topological space, the groups $H_*(X)$ have a geometric presentation in terms of "cycles" and "homologies". The cycles are, of course, generators for the group. The definition of $G(R)$ doesn't look anything like this, but Theorem 2.15 says that the group is indeed generated by classes that have the feeling of "algebraic cycles" on the variety $\operatorname{Spec} R$. One thinks of $G(R)$ as having a generator $[R/P]$ for every irreducible subvariety of R, and then there are some relations amongst these that we don't yet understand. It is worth pointing out that in $H_*(X)$ the cycles are strictly separated by dimension—the dimensions i cycles are confined to the single group $H_i(X)$—whereas in $G(R)$ the cycles of different dimensions are all inhabiting the same group. This is one of the main differences between K-theory and singular homology/cohomology.

To prove Theorem 2.15 we first need a lemma from commutative algebra:

Lemma 2.16. *Let R be a Noetherian ring. For any finitely-generated R-module M, there exists a prime ideal $P \subseteq R$ and an embedding $R/P \hookrightarrow M$. Equivalently, there is some $z \in M$ whose annihilator is prime.*

Proof. Pick any nonzero $x \in M$ and consider the family of ideals

$$S_x = \{\operatorname{Ann}(rx) \,|\, r \in R \ \text{ and } \ rx \neq 0\}.$$

Since R is Noetherian, S_x has a maximal element $\operatorname{Ann}(rx)$. We claim that $\operatorname{Ann}(rx)$ is prime, in which case taking $z = rx$ completes the proof. To justify the claim, suppose that $ab \in \operatorname{Ann}(rx)$ and $b \notin \operatorname{Ann}(rx)$. Then $abrx = 0$ but $brx \neq 0$. So $a \in \operatorname{Ann}(brx)$. But $\operatorname{Ann}(brx) \supseteq \operatorname{Ann}(rx)$, so the maximality of $\operatorname{Ann}(rx)$ in S_x implies that $\operatorname{Ann}(brx) = \operatorname{Ann}(rx)$. Hence $a \in \operatorname{Ann}(rx)$, and this completes the proof that $\operatorname{Ann}(rx)$ is prime. □

Proof of Theorem 2.15. Let M be a finitely-generated R-module. We will use repeated applications of the lemma to construct a so-called **prime filtration** of M. Pick an embedding $R/P_0 \hookrightarrow M$, and let $M_0 = R/P_0$. Next consider M/M_0. If $M/M_0 = 0$, our filtration is complete. If $M/M_0 \neq 0$, then there exists a prime P_1 and an embedding $R/P_1 \hookrightarrow M/M_0$. Let $\pi\colon M \to M/M_0$ denote the projection and define $M_1 = \pi^{-1}(R/P_1)$. Then $\pi\colon M_1 \to R/P_1$ also has kernel M_0; that is, $M_0 \subseteq M_1$ and $M_1/M_0 \cong R/P_1$. Next consider M/M_1 and repeat. This process yields a filtration of M

$$0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

such that $M_{i+1}/M_i \cong R/P_i$. The filtration must be finite since R is Noetherian. Therefore $[M] = \sum [M_{i+1}/M_i] = \sum [R/P_i]$, and we have proven that the set $\{[R/P] \,|\, P \text{ is prime in } R\}$ generates $G(R)$. □

Remark 2.17. The prime filtrations constructed in the above proof are very useful, and will appear again in our proofs. For future use we note that if an ideal $I \subseteq R$ is such that $IM = 0$, then I also kills any subquotient of M. Consequently, I will be contained in any P_i for which R/P_i appears as a subquotient in a prime filtration of M.

If M is an R-module, write $M[t]$ for the $R[t]$-module $M \otimes_R R[t]$. The functor $M \mapsto M[t]$ is exact, because $R[t]$ is flat over R (in fact, it is even free). So we have an induced map $\alpha\colon G(R) \to G(R[t])$ given by $[M] \mapsto [M[t]]$.

Theorem 2.18 (Homotopy invariance)**.** *If R is Noetherian, $\alpha\colon G(R) \to G(R[t])$ is an isomorphism.*

We comment on the name "homotopy invariance" for the above result. If $X = \operatorname{Spec} R$ then $\operatorname{Spec} R[t] = X \times \mathbb{A}^1$, so the result says that $G(-)$ gives the same values on X and $X \times \mathbb{A}^1$. This is reminiscent of a functor on topological spaces giving the same values on X and $X \times I$.

Proof. We will first construct a left inverse $\beta\colon G(R[t]) \to G(R)$. A naive possibility for the map β is $J \mapsto J/tJ = J \otimes_{R[t]} R[t]/(t)$, but this doesn't preserve short exact sequences in general. So we correct this using Tor, and instead define

$$\beta([J]) = [\operatorname{Tor}_0^{R[t]}(J, R[t]/(t))] - [\operatorname{Tor}_1^{R[t]}(J, R[t]/(t))].$$

Before checking that this is well-defined, let us analyze the two Tor-groups. Recall that we can calculate Tor by taking an $R[t]$-resolution of either variable. In this case, it is easier to resolve $R[t]/(t)$:

$$0 \to R[t] \xrightarrow{t} R[t] \to R[t]/(t) \to 0.$$

Tensoring with J yields $0 \to J \xrightarrow{t} J \to 0$, so that $\operatorname{Tor}_0^{R[t]}(J, R[t]/(t)) = J/tJ$ and $\operatorname{Tor}_1^{R[t]}(J, R[t]/(t)) = \operatorname{Ann}_J(t)$. Notice also that $\operatorname{Tor}_i^{R[t]}(J, R[t]/(t)) = 0$ for $i > 1$. We have

$$\beta([J]) = [J/tJ] - [\operatorname{Ann}_J(t)] = \sum_{i=0}^{\infty} (-1)^i [\operatorname{Tor}_i^{R[t]}(J, R[t]/(t))].$$

The fact that β is a well-defined group homomorphism now follows by the usual argument: a short exact sequence of modules induces a long exact sequence of Tor groups, and the alternating sum of these is zero in $G(R)$. It is immediate that $\beta\alpha = Id$: this follows from the fact that for any R-module M one has $M[t]/tM[t] \cong M$ and $\operatorname{Ann}_{M[t]}(t) = 0$. Consequently, α is injective.

The difficult part of the proof is showing that α is surjective. We will use the fact, from Theorem 2.15, that $G(R[t])$ is generated by elements of the form $[R[t]/Q]$ for primes $Q \subseteq R[t]$. It suffices to show that each $[R[t]/Q]$ is in the image of α. Let us write S for $R[t]$, and define

$$T = \{Q \cap R \mid Q \subseteq S \text{ is prime and } [S/Q] \notin \operatorname{im}(\alpha)\}.$$

Our goal is to show that T must be empty.

If $T \neq \emptyset$ then since S is Noetherian it has a maximal element $P = Q \cap R$ for some prime $Q \subseteq S$. Using this P and this Q, we will construct an S-module W which forces $[S/Q]$ to lie in $\operatorname{im}(\alpha)$, thus obtaining a contradiction.

First, some observations:

(1) If $I \subseteq R$ is any ideal then the expansion IS equals $I[t]$, the set of polynomials with coefficients in I. One has $S/IS \cong (R/I)[t]$.
(2) Any S-module M which is killed by $P+u$ for some $u \in R-P$ must lie in $\mathrm{im}(\alpha)$. This is because for each prime Q_i appearing in a prime filtration of M, we have $Q_i \supseteq \mathrm{Ann}_R(M) \supseteq P+u$. In particular, none of these Q_i can be in T since P was chosen to be maximal. So $[S/Q_i] \in \mathrm{im}(\alpha)$ for all these Q_i, and hence $[M] \in \mathrm{im}(\alpha)$ as well.
(3) For any prime $J \subseteq R$ we have $[S/JS] \in \mathrm{im}(\alpha)$, since $S/JS = (R/J)[t] = \alpha([R/J])$.
(4) If $f \in S - JS$ where $J \subseteq R$ is prime, then $[S/(JS+f)] = 0$ in $G(S)$ since $S/(JS+f)$ fits into the short exact sequence
$$0 \longrightarrow S/JS \xrightarrow{f} S/JS \longrightarrow S/(JS+f) \longrightarrow 0.$$
Note that $S/JS \cong (R/J)[t]$, which is a domain—and this is why multiplication by f is injective.

Consider the maps
$$S \twoheadrightarrow S/PS \hookrightarrow (R-P)^{-1}(S/PS).$$
Observe that $(R-P)^{-1}(S/PS) = (R_P/PR_P)[t]$. But R_P/PR_P is a field, so the ring $(R-P)^{-1}(S/PS)$ is a PID. Therefore the image of Q in $(R-P)^{-1}(S/P[t])$ is generated by a single element. Let $f \in Q$ be some lifting of this generator to S.

Consider the S-module $W = Q/(PS+f)$. Since Q and f have the same image in the ring $(R-P)^{-1}(S/PS)$, we have $(R-P)^{-1}W = 0$. Now, W is finitely generated (as an S-module), so there exists some $u \in R-P$ such that $uW = 0$. Since $PW = 0$ by the definition of W, we have that W is killed by $P+u$. By observation (2) above, $[W] \in \mathrm{im}(\alpha)$.

At the same time, W fits into the exact sequence $0 \to W \to S/(PS+f) \to S/Q \to 0$, and we know $[S/(PS+f)] = 0$ in $G(S)$ by observation (4). But this implies that $[W]$ and $[S/Q]$ are additive inverses, and hence $[S/Q]$ lies in $\mathrm{im}(\alpha)$, contradicting our choice of Q. □

Here is an interesting consequence of homotopy invariance:

Corollary 2.19. *Let F be a field. Then $K(F[x_1,\dots,x_n]) \cong \mathbb{Z}$.*

Proof. We have $K(F[x_1,\dots,x_n]) \cong G(F[x_1 \dots, x_n])$ by Theorem 2.10, since the ring $F[x_1,\dots,x_n]$ is regular by Hilbert's Syzyzy Theorem. We also have $G(F[x_1,\dots,x_n]) \cong G(F)$ by homotopy invariance, and $G(F) \cong \mathbb{Z}$ via the dimension map. □

In the next section we will see what Corollary 2.19 says about projectives over $F[x_1,\dots,x_n]$. See Proposition 3.1.

3. A closer look at projectives

Recall that a module is projective if and only if it is a direct summand of a free module. So free modules are projective, and for almost all applications in homological algebra one can get by with using *only* free modules. Consequently, it is common not to know many examples of non-free projectives. We begin this section by remedying this.

Before considering our examples we need one small tool. Let R be a commutative ring, P a projective over R, and $m \subseteq R$ a maximal ideal. Define $\mathrm{rank}_m(P) = \dim_{R/mR}(P/mP)$. Note that $\mathrm{rank}_m(-)$ is additive.

(1) Let $R = \mathbb{Z}/6$. Since $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$, both $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are projective R-modules—and they are clearly not free.

(2) Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (2, 1+\sqrt{-5})$. For convenience let us write $\mu = \sqrt{-5}$. Let K be the kernel of the map $R^2 \to I$ sending e_1 to 2 and e_2 to $1+\mu$. A little work shows that K is spanned by $(1+\mu, -2)$ and $(-3, 1-\mu)$. If one defines $\chi\colon R^2 \to K$ by
$$\chi(e_1) = (3, -1+\mu), \quad \chi(e_2) = (1+\mu, -2),$$
it is readily verified that χ is a splitting for the sequence $0 \to K \to R^2 \to I \to 0$. So $K \oplus I \cong R^2$, and hence both K and I are projective.

Note that $\pi_2\colon R^2 \to R$ restricts to a map $K \to I$, which is clearly a surjection. It is easy to check that this is actually an isomorphism. So for every maximal ideal $m \subseteq R$ we have $\mathrm{rank}_m(K) = \mathrm{rank}_m(I)$, and of course $\mathrm{rank}_m(K) + \mathrm{rank}_m(I) = 2$. So $\mathrm{rank}_m(K) = \mathrm{rank}_m(I) = 1$.

If I were free, the above rank calculation would show that $I \cong R$. However, the ideal I is not principal so this would be a contradiction. So I is a non-free projective.

This example generalizes: if D is a Dedekind domain (such as the ring of integers in an algebraic number field) then every ideal $I \subseteq D$ is projective. Non-principal ideals are never free.

(3) Let $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$. If $C(S^2)$ denotes the ring of continuous functions $S^2 \to \mathbb{R}$, note that we may regard R as sitting inside of $C(S^2)$: it is the subring of polynomial functions on the 2-sphere. The connections with the topology of the 2-sphere will be important below.

Let $\pi\colon R^3 \to R$ be the map $\pi(f,g,h) = xf + yg + zh$. That is, π is left-multiplication by the matrix $\begin{bmatrix} x & y & z \end{bmatrix}$. Let T be the kernel of π:
$$0 \to T \hookrightarrow R^3 \xrightarrow{\pi} R \to 0.$$
The map π is split via $\chi\colon R \to R^3$ sending $1 \mapsto (x,y,z)$. We conclude that $T \oplus R \cong R^3$, so T is projective.

We claim that T is not free. Suppose, towards a contradiction, that T is free. For any maximal ideal $m \subseteq R$ we have
$$T/mT \oplus R/m \cong (R/m)^3$$
and therefore $T/mT \cong (R/m)^2$ by linear algebra. So T must be isomorphic to the free module R^2. Choose an isomorphism $R^2 \to T$, let e_1 and e_2 be the standard basis for R^2, and let the image of e_1 under our isomorphism be (f,g,h). So f, g, and h are polynomial functions on S^2 and
$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} f(p) \\ g(p) \\ h(p) \end{bmatrix} = 0$$
for all $p = (p_1, p_2, p_3) \in S^2$. So $p \mapsto (f(p), g(p), h(p))$ is a tangent vector field on S^2. By the Hairy Ball Theorem we can find a point $q = (q_1, q_2, q_3) \in S^2$ such that $f(q) = g(q) = h(q) = 0$. Let $m = (x - q_1, y - q_2, z - q_3) \subseteq R$ and

consider the commutative diagram

$$\begin{array}{ccccc} R^2 & \xrightarrow{\cong} & T & \rightarrowtail & R^3 \\ \downarrow & & \downarrow & & \downarrow \\ (R/mR)^2 & \xrightarrow[\cong]{} & T/mT & \rightarrowtail & (R/mR)^3. \end{array}$$

Note that $R/mR \cong \mathbb{R}$ via $F \mapsto F(q)$. Start with e_1 in the upper left corner and compute its image in $(R/mR)^3 \cong \mathbb{R}^3$ under the two outer ways of tracking around the diagram. Along the top route e_1 maps to $(f(q), g(q), h(q))$ which is just $(0, 0, 0)$. On the other hand, along the bottom route e_1 first maps to $(1, 0) \in \mathbb{R}^2$ and then the bottom composite is an injection—so the image in $\mathbb{R}^3$ is nonzero. This is a contradiction, so we conclude that T is not free. (In fact, we have proven more: we have proven that T does not contain R as a direct summand).

Note that T is, in some sense, an algebraic analog of the tangent bundle of S^2. These parallels between projective modules and vector bundles are very important, and we will see much more about them in Section 10.

(4) Let us do one more example where we use topology to produce an example of a non-free projective. This example is based on the Möbius bundle over S^1. Let

$$S = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$$

and let $R \subseteq S$ be the span of the even degree monomials. One should regard S as the ring of polynomial functions on the circle, and R is the ring of polynomial functions $f(x, y)$ satisfying $f(x, y) = f(-x, -y)$. So R is trying to be the ring of polynomial functions on $\mathbb{R}P^1$ (which happens to be homeomorphic to S^1).

Let $P \subseteq S$ be the $\mathbb{R}$-linear span of the homogeneous polynomials with odd total degree. Observe that P is a finitely generated R-module and we have $\pi\colon R^2 \twoheadrightarrow P$ via $\pi(e_1) = x$ and $\pi(e_2) = y$. Define $\chi\colon P \to R^2$ via

$$h \mapsto \chi(h) = \begin{bmatrix} xh \\ yh \end{bmatrix}.$$

One checks that $\pi \circ \chi = id$, so P is projective. We leave it as an exercise for the reader to show that P is not free.

The topological examples (3) and (4), as well as many similar ones, can be found in the lovely paper [Sw]. See also Section 10 below.

A projective module P is called **stably free** if there exists a free module F such that $P \oplus F$ is free. The example in (3) gives a projective that is stably-free but not free. It turns out that $K(R)$ can be used to tell us whether such modules exist or not. To see this, recall that if $m \subseteq R$ is a maximal ideal then $\text{rank}_m(-)$ is an additive function on finitely-generated, projective modules. So it induces a map $\text{rank}_m(-)\colon K(R) \to \mathbb{Z}$, which is evidently surjective because $\text{rank}_m(R) = 1$. This shows that $K(R)$ always contains $\mathbb{Z}$ as a direct summand.

Define the **reduced Grothendieck group of** R to be

$$\widetilde{K}(R) = K(R)/\langle [R] \rangle.$$

Here is another way to define this group. Take the set of isomorphism classes of finitely-generated projectives and impose the equivalence relation $P \sim P \oplus R$

for every P. Such equivalences classes are called **stable projectives**. Define a monoid structure on this set by $[P] + [Q] = [P \oplus Q]$, and note that $[0] = [R]$ is the unit. If P is any projective then there exists a Q such that $P \oplus Q$ is free, and therefore $[P] + [Q] = 0$ in this monoid; hence, we have a group. This is called the **Grothendieck group of stable projectives**. One readily checks that this group is isomorphic to $\widetilde{K}(R)$, with the equivalence class $[P]$ corresponding to the element $[P] \in \widetilde{K}(R)$ (we apologize for the multiple uses of the notation $[P]$ here).

Proposition 3.1. *Let R be a commutative ring. The following are equivalent:*
(1) $K(R) \cong \mathbb{Z}$
(2) $\widetilde{K}(R) = 0$
(3) Every finitely-generated, projective R-module is stably-free.

Proof. Immediate. □

Example 3.2. Recall from Corollary 2.19 that if F is a field then $K(F[x_1, \ldots, x_n]) = \mathbb{Z}$. Thus, every finitely-generated, projective $F[x_1, \ldots, x_n]$-module is stably-free.

In the 1950s, Serre conjectured that every finitely-generated projective over $F[x_1, \ldots, x_n]$ is actually free. As we will see later (Remark 11.5 below), the motivation for this conjecture is inspired by topology and the connection between vector bundles and projective modules. Quillen and Suslin independently proved Serre's conjecture in the 1970s.

Example 3.3. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let I be the ideal $(2, 1 + \sqrt{-5})$. This ideal is known not to be principal. We saw in example (2) from the beginning of this section that I is a rank one projective that is not free. Could I be stably free? If it were, then we would have $I \oplus R^k \cong R^{k+1}$, for some k. Apply the exterior product $\Lambda^{k+1}(-)$ to deduce that

$$R \cong \Lambda^{k+1}(R^{k+1}) \cong \Lambda^{k+1}(I \oplus R^k) \cong \Lambda^1(I) \otimes \Lambda^k(R^k) \cong I \otimes R \cong I$$

(in the third isomorphism we have used the formula for the exterior product of a direct sum, together with the general fact that $\Lambda^j(P) = 0$ for $j > \mathrm{rank}(P)$). However, this is a contradiction; we would have $I \cong R$ only if I were principal. Hence, I is not stably free and so $[I]$ determines a nonzero class in $\tilde{K}(R)$.

Again, this example generalizes to any Dedekind domain D. If $I \subseteq D$ is a non-principal ideal then I is a rank one projective that is not stably free. So a Dedekind domain has $K(D) \cong \mathbb{Z}$ if and only if D is a PID. As another consequence, we observe that over any ring a rank one projective P cannot be stably free unless it is actually free.

4. A brief tour of localization and dévissage

It would be nice if we could compute the K-groups of more rings. For example, we haven't even computed $K(R)$ for a simple ring like $R = \mathbb{Z}[\sqrt{-5}]$. But so far we don't have many techniques to tackle such a computation. An obvious thing to try to do is to relate the K-groups of R to those of simpler rings made from R, for example quotient rings R/I and localizations $S^{-1}R$. We will start to explore these ideas in the present section. For the moment it will be easier to do this for G-theory, though, rather than K-theory. Note that $R = \mathbb{Z}[\sqrt{-5}]$ is a regular ring, and so $K(R) \cong G(R)$; hence, the focus on G-groups still gets us what we want in this case.

Let R be a commutative ring and let $f \in R$. Consider the maps

$$G(R/f) \xrightarrow{d_1} G(R) \xrightarrow{d_0} G(f^{-1}R)$$

where $d_1([M]) = [M]$ and $d_0([W]) = f^{-1}W$. Clearly $d_0 \circ d_1 = 0$. We claim that d_0 is also surjective. To see this, let Z be an $f^{-1}R$-module with generators $z_1, \ldots, z_n$. Let $W = R\langle z_1, \ldots, z_n\rangle \subseteq Z$ be the R-submodule generated by the z_i's. One checks that $f^{-1}W \cong Z$, and so d_0 is surjective.

Theorem 4.1. *The sequence*

$$G(R/f) \xrightarrow{d_1} G(R) \xrightarrow{d_0} G(f^{-1}R) \longrightarrow 0$$

is exact.

We will delay the proof of this theorem for the moment, as it is somewhat involved. Let us first look at an example.

Example 4.2. Let $R = \mathbb{Z}[\sqrt{-5}]$ and $f = (2)$. Note that R is not a PID but $f^{-1}R$ is. Thus $G(f^{-1}R) \cong \mathbb{Z}$. Now we compute

$$R/f = \mathbb{Z}/2[x]/(x^2+5) = \mathbb{Z}/2[x]/(x^2+1) = \mathbb{Z}/2[x]/((x+1)^2) \cong \mathbb{Z}/2[t]/(t^2).$$

We calculated in example (7) from Section 2 that $G(\mathbb{Z}/2[t]/(t^2)) \cong \mathbb{Z}$ and is generated by the module $\mathbb{Z}/2$ with t acting as zero. Translated into the present situation, we are saying $G(R/f) \cong \mathbb{Z}$ with the group being generated by $R/(2, x+1) = R/(2, 1-\sqrt{-5})$. Note that $(2, 1-\sqrt{-5}) = (2, 1+\sqrt{-5})$.

We have computed that the exact sequence from Theorem 4.1 has the form

$$\mathbb{Z} \xrightarrow{d_1} G(R) \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

where $d_1(1) = [R/(2, 1+\sqrt{-5})]$ and $d_0([R]) = 1$. Let $I = (2, 1+\sqrt{-5})$ and notice that $G(R)$ is generated by $[R]$ and $[R/I]$.

Now look at the short exact sequence $0 \to K \longrightarrow R^2 \xrightarrow{\phi} I \to 0$ where $\phi(e_1) = 2$, $\phi(1+\sqrt{-5})$, and $K = ker(\phi) = \{(x,y) \,|\, 2x + (1+\sqrt{-5})y = 0\}$. In example (2) from Section 3 we indicated that $K \cong I$. So we have $[I] + [I] = [R^2]$ in $G(R)$, or $2([R] - [I]) = 0$. But $[R] - [I] = [R/I]$, hence $2[R/I] = 0$. It follows that $G(R)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$, depending on whether the class $[R/I] = [R] - [I]$ is zero or not.

Now use that R is regular, so that $G(R) \cong K(R)$. Recall that we saw in Example 3.3 that $\tilde{K}(R) \neq 0$, or equivalently $K(R) \neq \mathbb{Z}$. In fact we saw precisely that $[R] - [I]$ is not zero in $K(R)$. We conclude that $G(R) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, with generators $[R]$ and $[R/I]$ for each of the two summands.

Remark 4.3. Theorem 4.1 gives another parallel between $G(-)$ and singular homology. If $X = \operatorname{Spec} R$ then $A = \operatorname{Spec} R/f$ is a closed subscheme, and $\operatorname{Spec} f^{-1}R = X - A$ is the open complement. So the sequence in Theorem 4.1 can be written as

$$G(A) \to G(X) \to G(X - A) \to 0.$$

This is somewhat reminiscent of the long exact sequence in singular homology $\cdots \to H_*(A) \to H_*(X) \to H_*(X, A) \to \cdots$ but with some important differences.

One obvious difference is that our sequence does not yet extend to the left to give a long exact sequence, but that turns out to be just a lack of knowledge on our part: we will eventually see that there are 'higher G-groups' completing the picture. The other evident difference is the presence of $G(X-A)$ as the 'third term' in the long exact sequence, rather than a relative group $G(X,A)$. There are lots of things to say about this that are not worth going into at the moment, but perhaps the most relevant is that $H_*(-)$ is really the wrong analogy to be looking at. If we instead consider Borel-Moore homology, then there are indeed long exact sequences that look like $\cdots \to H_*^{BM}(A) \to H_*^{BM}(X) \to H_*^{BM}(X-A) \to \cdots$

Remark 4.4. It is important in Theorem 4.1 that we are using G-theory rather than K-theory. In K-theory we have maps $K(R) \to K(R/f)$ and $K(R) \to K(f^{-1}R)$, both given by tensoring, but in neither case do we have an evident 'third group' that might form an exact sequence. In essence this is because we need relative K-groups; we will start to encounter these in the next section.

We will now work towards proving Theorem 4.1. The proof is somewhat involved, and the result is actually not going to be used much in the rest of the notes. But the proof is very interesting, as it demonstrates many general issues that arise in the subject of K-theory. So it is worth spending time on this.

The proof comes in two parts. For the first part, let us introduce the multiplicative system $S = \{1, f, f^2, f^3, \dots\}$. Write

$$G(M \,|\, S^{-1}M = 0)$$

for the Grothendieck group of all finitely-generated R-modules M such that $S^{-1}M = 0$. The notation is slightly slack, but it is very convenient. There are evident maps

$$G(M \,|\, S^{-1}M = 0) \to G(R) \to G(S^{-1}R) \to 0,$$

and we will prove that this is exact for *any* multiplicative system S. This is called the **localization sequence** for G-theoery.

The second step is to notice that if M is an R/f-module then as an R-module it has the property that $S^{-1}M = 0$. So we have a map

$$G(R/f) \to G(M|S^{-1}M = 0). \tag{4.5}$$

If M is an arbitrary finitely-generated R-module, the condition $S^{-1}M = 0$ just says that M is killed by a power of f. So we would have a filtration

$$M \supseteq fM \supseteq f^2M \supseteq \cdots \supseteq f^NM = 0$$

where the factors are all R/f-modules. This shows that the map in (4.5) is surjective, and in fact these ideas allow one to define an inverse. The fact that

$$G(R/f) \cong G(M|S^{-1}M = 0)$$

is an example of a general principle known as **dévissage**. When we come to prove this in a bit we will develop the generalization and get a better understanding of what is going on here.

So those are the two pieces for the proof of Theorem 4.1: a general localization sequence where the third term is something we had not considered before—in essence, a relative G-group—and a dévissage theorem identifying that third term with something more familiar.

4.6. **The localization sequence.** To begin with we will need a lemma giving several facts about the localization functor $\gamma\colon \langle\langle R - \text{Mod}\rangle\rangle \to \langle\langle S^{-1}R - \text{Mod}\rangle\rangle$. These facts are easy to prove, and it seems like they should be encapsulated in some kind of general statement about the functor γ—but I don't know what this might be.

Lemma 4.7. *Let $S \subseteq R$ be a multiplicative system. In all parts of this lemma, the modules are always assumed to be finitely-generated.*

(a) For any $S^{-1}R$-module W, there exists an R-module A and an isomorphism $S^{-1}A \cong W$.

(b) For any R-modules A_1 and A_2 and map of $S^{-1}R$-modules $f\colon S^{-1}A_1 \to S^{-1}A_2$, there exists a map of R-modules $g\colon A_1 \to A_2$ and a diagram of $S^{-1}R$-modules

$$\begin{array}{ccc} S^{-1}A_1 & \xrightarrow{S^{-1}g} & S^{-1}A_2 \\ \| & & \downarrow{\cong} \\ S^{-1}A_1 & \xrightarrow{f} & S^{-1}A_2. \end{array}$$

(c) For any short exact sequence of $S^{-1}R$-modules

$$0 \to W_1 \to W_2 \to W_3 \to 0,$$

there exists a short exact sequence of R-modules

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

and isomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{-1}A_1 & \longrightarrow & S^{-1}A_2 & \longrightarrow & S^{-1}A_3 & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & 0 \end{array}$$

Proof. Part (a) was proven near the beginning of Section 4. The proofs of the other parts are reasonably easy exercises in algebra that we leave to the reader. □

Corollary 4.8. *The following subgroups of $G(R)$ are all equal:*

(1) $\big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle$

(2) $\big\langle [A] - [B] \,\big|\,$ *there exists a map $f\colon A \to B$ such that $S^{-1}f$ is an isomorphism*$\big\rangle$

(3) $\big\langle [J] \,\big|\, S^{-1}J = 0 \big\rangle$.

Proof. Let S_1, S_2, and S_3 be the subgroups listed in (1)–(3). Clearly $S_1 \supseteq S_2 \supseteq S_3$. The opposite subset $S_1 \subseteq S_2$ follows directly from Lemma 4.7(b). To prove $S_2 \subseteq S_3$, let $f\colon A \to B$ be a map of R-modules such that $S^{-1}f$ is an isomorphism. Consider the short exact sequence

$$0 \to \ker f \to A \to B \to \operatorname{coker} f \to 0,$$

and note that our hypothesis implies that $S^{-1}(\ker f) = 0 = S^{-1}(\operatorname{coker} f)$. But $[A] - [B] = [\ker f] - [\operatorname{coker} f]$ in $G(R)$, so we have that $[A] - [B] \in S_3$. □

Proposition 4.9. *Let $S \subseteq R$ be a multiplicative system. The sequence*

$$G(M|S^{-1}M = 0) \xrightarrow{a} G(R) \xrightarrow{b} G(S^{-1}R) \to 0$$

is exact, where a and b are the evident maps.

Proof. Part (a) of Lemma 4.7 gives surjectivity. The somewhat tricky thing is to get the exactness in the middle. Let $\mathcal{F}(R)$ denote the free abelian group on isomorphism classes of finitely-generated R-modules, and let $\mathcal{R}el(R) \subseteq \mathcal{F}(R)$ denote the subgroup generated by elements $[M_i'] + [M_i''] - [M_i]$ for short exact sequences $0 \to M_i' \to M_i \to M_i'' \to 0$. Note that $[0] \neq 0$ in $\mathcal{F}(R)$; we could have imposed this as an extra condition, but it is slightly more convenient to not do so. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R}el(R) & \longrightarrow & \mathcal{F}(R) & \longrightarrow & G(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi|_{\mathcal{R}el}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & \mathcal{R}el(S^{-1}R) & \longrightarrow & \mathcal{F}(S^{-1}R) & \longrightarrow & G(S^{-1}R) & \longrightarrow & 0, \end{array}$$

which we wish to regard as a short exact sequence of chain complexes (the columns become chain complexes by adding zeros above and below). Lemma 4.7(a) gives surjectivity of π, and Lemma 4.7(b) gives surjectitivity of $\pi|_{\mathcal{R}el}$. The long exact sequence in homology then becomes

$$0 \to \ker(\pi|_{\mathcal{R}el}) \to \ker(\pi) \to \ker b \to 0. \tag{4.10}$$

We next analyze the kernel of π.

Assume that $x \in \ker(\pi)$. One can write x in the form

$$x = \Big([M_1] + [M_2] + \cdots + [M_k]\Big) - \Big([J_1] + \cdots + [J_l]\Big)$$

for some modules $M_1, \dots, M_k, J_1, \dots, J_l$. We then have

$$0 = \pi(x) = \Big([S^{-1}M_1] + [S^{-1}M_2] + \cdots + [S^{-1}M_k]\Big) - \Big([S^{-1}J_1] + \cdots + [S^{-1}J_l]\Big)$$

in $\mathcal{F}(S^{-1}R)$. How can this happen? It can only be that $k = l$ and that for each module $S^{-1}M_j$ there is some i for which $S^{-1}M_j \cong S^{-1}J_i$. By pairing the terms up two by two we find that

$$x \in \big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle \subseteq \mathcal{F}(R).$$

So $\ker \pi = \big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle$. It then follows from (4.10) and Corollary 4.8 that

$$\ker b = \langle [J] \,|\, S^{-1}J = 0 \rangle \subseteq G(R).$$

This is what we wanted to prove. ◻

Remark 4.11. The above proof represents the first time we have really had to get our hands dirty with the relations defining $G(R)$.

4.12. **Dévissage.** Now we move to the second stage of the proof of Theorem 4.1. We can rephrase what needs to be shown as saying that the map

$$G(M|M \text{ is killed by } f) \to G(M|M \text{ is killed by a power of } f)$$

is an isomorphism. We have seen a baby version of this argument before, namely back in Section 2 when we showed that

$$G(\mathbb{Z}/p) \to G(\mathbb{Z}/p^2) \quad \text{and} \quad G(F) \to G(F[t]/(t^2))$$

are both isomorphisms. These are both maps of the form

$$G(M|M \text{ is killed by } f) \to G(M|M \text{ is killed by } f^2),$$

for the rings $R = \mathbb{Z}$ and $R = F[t]$, respectively. Iterating the same idea we used to prove these—filter by powers of f—allows one to prove the required generalization. But while we're at it, let us generalize even further.

Let $\mathcal{B}$ be an exact category. I will not say exactly what the definition of such a thing is, except that $\mathcal{B}$ is an additive category with a collection of sequences $M' \to M \to M''$ called "exact", and the collection must satisfy a reasonable list of axioms. Any abelian category with its intrinsic notion of short exact sequence is an example. The complete definition is in [Q]. We are not giving it here in part because the reader can manufacture a suitable definition for himself: just figure out what axioms one needs to make the following proof work.

Theorem 4.13 (Dévissage)**.** *Let $\mathcal{B}$ be an exact category, and let $\mathcal{A} \hookrightarrow \mathcal{B}$ be an exact subcategory such that any object in $\mathcal{B}$ has a finite filtration whose factors are in $\mathcal{A}$. Then $G(\mathcal{A}) \to G(\mathcal{B})$ is an isomorphism.*

Proof. The inclusion $i\colon \mathcal{A} \to \mathcal{B}$ induces a map $\alpha\colon G(\mathcal{A}) \to G(\mathcal{B})$, and we want to define an inverse $\beta\colon G(\mathcal{B}) \to G(\mathcal{A})$. To do so, for $M \in \mathcal{B}$ choose a filtration

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = 0,$$

whose quotients M_i/M_{i+1} are in $\mathcal{A}$, and define

$$\beta([M]) = \sum [M_i/M_{i+1}].$$

We must check that β well defined, because it seems to depend on the choice of filtration. There are two things that need to be said. The first and easier thing is to check that our formula gives the same class in $G(\mathcal{A})$ if we refine the filtration, meaning that we replace one of the links $M_i \supseteq M_{i+1}$ with a longer chain $M_i \supseteq M_i^1 \supseteq \cdots \supseteq M_i^r = M_{i+1}$. This is trivial.

The second thing is to recall something you probably learned in a basic algebra class, namely the Jordan-Hölder Theorem. This says that given any two filtrations of M we can refine each one so that the two refinements have the same quotients up to reindexing. If you accept this, it shows that $\beta([M])$ does not depend on the choice of filtration. It is a simple exercise to prove that β is additive, which we leave to the reader.

At this point we have the map β. It is immediate that $\beta\alpha = \mathrm{id}$ and $\alpha\beta = \mathrm{id}$. $\square$

Remark 4.14. We will not prove the Jordan-Hölder Theorem, as this is something that can be found in basic algebra textbooks, but let us at least recall the main idea for why it is true. Suppose $M \supseteq A \supseteq 0$ and $M \supseteq B \supseteq 0$ are two filtrations for M. Consider the refinement of the first given by

$$M \supseteq A + B \supseteq A \supseteq A \cap B \supseteq 0,$$

having quotients $M/(A+B)$, $(A+B)/A \cong B/(A\cap B)$, $A/(A\cap B) \cong (A+B)/A$, and $A \cap B$, Interchanging the roles of A and B gives a similar filtration refining $M \supseteq B \supseteq 0$, having the same set of filtration quotients.

Once one has the above basic idea, it is not hard to extend to longer filtrations.

Note that it is often true in mathematics that the hard work goes into showing that something is well-defined, and afterwards the rest is easy. We saw this in the case of the Dévissage Theorem, where all the hard work went into constructing the map β.

5. K-theory of complexes and relative K-theory

Recall that there is always a map $K(R) \to G(R)$ sending the K-class of a projective to the G-class of the same projective. We proved in Theorem 2.10 that when R is regular this map is an isomorphism, and we did this by constructing the inverse: it sends a class $[M]$ to $\sum(-1)^i[P_i]$, where $P_\bullet \to M$ is any bounded resolution of M by finitely-generated projectives. If you go back and examine the proof of that result, you might notice that the alternating sums are largely an *annoyance* in the proof—all the key ideas are best expressed without them, and they are only forced into the proof so that we get actual elements of $K(R)$. If you think about this enough, it might eventually occur to you to try to make a definition of $K(R)$ that uses chain complexes instead of modules, thus eliminating the need for these alternating sums. We will show how to do this in the present section.

The importance of using chain complexes extends much further that simply changing language to simplify a proof. We will see that defining K-theory in terms of complexes allows us to write down natural definitions for relative K-groups as well.

Throughout this section let R be a fixed commutative ring. We begin by making the following definition:

Definition 5.1.

$$K^{cplx}(R) = \frac{\mathbb{Z}\langle\, [P_\bullet] \,\big|\, P_\bullet \textit{ is a bounded chain complex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[P_\bullet] = [P'_\bullet]$ *if* $P_\bullet$ *and* $P'_\bullet$ *are chain homotopy equivalent,*

(2) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ *if there is a short exact sequence* $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

The second relation is the one that by now we would expect in a K-group, but the first relation is new to us. If one goes back and thinks about the proof of Theorem 2.10, the need for this first relation quickly becomes clear: it guarantees, for instance, that two projective resolutions of a module will represent the same class in the K-group.

Regarding relation (1), let us introduce some common terminology:

Definition 5.2. *A map of chain complexes* $C_\bullet \to D_\bullet$ *is a* ***quasi-isomorphism*** *if the induced maps* $H_i(C_\bullet) \to H_i(D_\bullet)$ *are isomorphisms for all* $i \in \mathbb{Z}$. *Two chain complexes* $C_\bullet$ *and* $D_\bullet$ *are* ***quasi-isomorphic****, written* $C_\bullet \simeq D_\bullet$*, if there is a zig-zag of quasi-isomorphisms*

$$C_\bullet \xrightarrow{\sim} J^1_\bullet \xleftarrow{\sim} J^2_\bullet \xrightarrow{\sim} \cdots \xrightarrow{\sim} J^n_\bullet \xleftarrow{\sim} D_\bullet.$$

The following lemma is basic homological algebra. We omit the proof, but it is very similar to the proof that two projective resolutions of the same module are chain homotopy equivalent.

Lemma 5.3. *If P and Q are bounded below complexes of projectives, then every quasi-isomorphism is a chain homotopy equivalence.*

This lemma lets us replace the words "chain homotopy equivalence" with "quasi-isomorphism" in any statement about bounded, projective complexes. In particular, we do this in relation (1) from the definition of $K^{cplx}(R)$. The advantage of doing this is simply that quasi-isomorphisms are somewhat easier to identify than chain homotopy equivalences.

Here is our main result concerning the K-theory of complexes:

Proposition 5.4. *$K(R) \cong K^{cplx}(R)$.*

Before giving the proof we record two useful results:

Lemma 5.5. *Let P and Q be bounded complexes of finitely-generated projectives. Let ΣP denote the complex obtained by shifting every module up one degree: $(\Sigma P)_n = P_{n-1}$.*
(a) $[\Sigma P] = -[P]$ in $K^{cplx}(R)$.
(b) Let $f\colon P \to Q$ and let Cf denote the mapping cone of f. Then $[Cf] = [Q] - [P]$ in $K^{cplx}(R)$.

Proof. Recall that there is a short exact sequence of complexes

$$0 \to Q \hookrightarrow Cf \to \Sigma P \to 0,$$

which shows immediately that $[Cf] = [Q]+[\Sigma P]$ in $K^{cplx}(R)$. Let T be the mapping cone of the identity $P \to P$. Note that T is exact, hence quasi-isomorphic to the zero complex. So $0 = [T] = [P] + [\Sigma P]$, from which we get $[\Sigma P] = -[P]$. It then follows that $[Cf] = [Q] + [\Sigma P] = [Q] - [P]$. □

Exercise 5.6. Prove that if relation (1) in the definition of $K^{cplx}(R)$ is replaced with
(1') $[P_\bullet] = 0$ for every exact complex $P_\bullet$,
then the resulting quotient group is also equal to $K^{cplx}(R)$.

We now have enough tools to prove the main result of this section:

Proof of Proposition 5.4. If P is a projective R-module, let $P[n]$ denote the chain complex that has P in degree n and in all other degrees is equal to 0. There is an obvious map $\alpha\colon K(R) \to K^{cplx}(R)$ defined by

$$[P] \mapsto \big[P[0]\big].$$

It is somewhat less obvious, but one can define a map back $\beta\colon K^{cplx}(R) \to K(R)$ by

$$\beta\Big([P_\bullet]\Big) = \sum (-1)^i [P_i].$$

To see that this is well-defined we need to check that it respects the two defining relations for $K^{cplx}(R)$. Relation (2) is obvious, but for the other relation it is convenient to use Exercise 5.6 to replace (1) with (1'). The fact that β respects (1') is immediate, being a consequence of Lemma 2.5(a) (or really, the analog of this result for $K(R)$).

It is clear that $\beta \circ \alpha = \mathrm{id}$, so α is injective and β is surjective. To finish the proof, it is easiest to prove that α is surjective; we will do this in several steps. If P is a finitely-generated projective then $P[0]$ is obviously in the image of α, and we know that $P[n] = (-1)^n[P[0]]$ by iterated application of Lemma 5.5(a). So $P[n] \in \operatorname{im} \alpha$ for all $n \in \mathbb{Z}$; said differently, any complex of projectives of length 0 belongs to the image of α. We next extend this to all bounded complexes by an induction on the length.

Let $P_\bullet$ be a bounded complex of finitely-generated projectives, bounded between degrees k and $n+k$, say. Then $P_k[k]$ is a subcomplex of $P_\bullet$, and the quotient $Q_\bullet$ has length at most $n-1$. We have $[P_\bullet] = [P_k[k]] + [Q_\bullet]$, and both $[P_k[k]]$ and $[Q_\bullet]$ belong to $\operatorname{im}\alpha$ by induction. So $[P_\bullet] \in \operatorname{im}\alpha$, and we are done. □

We will use our identification of $K^{cplx}(R)$ and $K(R)$ implicitly from now on. For example, if P is a bounded complex of projectives we will often write $[P]$ to denote an element of $K(R)$—although of course we mean $\beta([P])$.

Exercise 5.7. Assume that R is Noetherian, and let $G_{fpd}(R)$ denote the Grothendieck group of finitely-generated modules having finite projective dimension. Prove that $K^{cplx}(R) \cong G_{fpd}(R)$.

5.8. ***G*-theory and chain complexes.** One can, of course, prove an analog of Proposition 5.4 in which the 'projective' hypothesis is left out everywhere. This would show that $G(R)$ is isomorphic to a Grothendieck group made from bounded chain complexes of arbitrary finitely-generated modules. What is more interesting, however, is a variant that again uses chain complexes of projectives. Precisely, consider chain complexes $P_\bullet$ such that

(1) Each P_i is a finitely-generated projective,
(2) $P_\bullet$ is bounded-below, in the sense that $P_i = 0$ for all $i \ll 0$.
(3) $P_\bullet$ has bounded homology, in the sense that $H_i(P) \neq 0$ only for finitely many values of i.

Start with the free abelian group on isomorphism classes of such complexes, and define $G^{cplx}(R)$ to be the quotient by the analogs of relations (1) and (2) in the definition of $K^{cplx}(R)$.

Note that one readily obtains maps $\alpha\colon G^{cplx}(R) \to G(R)$ and $\beta\colon G(R) \to G^{cplx}(R)$ by

$$\alpha([P_\bullet]) = \sum_i (-1)^i [H_i(P)] \quad \text{and} \quad \beta([M]) = [Q_\bullet]$$

where $Q_\bullet \to M$ any resolution by finitely-generated projectives.

Proposition 5.9. *The maps α and β give inverse isomorphisms $G^{cplx}(R) \cong G(R)$.*

Proof. It is immediate that $\alpha\beta = \mathrm{id}$, so that α is surjective and β is injective. The proof will be completed by showing that β is surjective. Let $P_\bullet$ be a bounded-below, homologically bounded chain complex of finitely-generated projectives. We will prove by induction on the number of nonzero homology groups of $P_\bullet$ that $[P_\bullet] \in \operatorname{im}\beta$. The base is trivial, for if all the homology groups are zero then $P_\bullet \simeq 0$ and so $[P_\bullet] = 0$.

Without loss of generality assume that $P_i = 0$ for $i < 0$. Let n be the smallest integer for which $H_n(P) \neq 0$. If $n > 0$ then $P_1 \to P_0$ is surjective, so there exists a splitting. Using this splitting one sees that P is quasi-isomorphic to a chain complex concentrated in degrees strictly larger than zero. Repeating this argument if necessary, one concludes that P is actually quasi-isomorphic to a chain complex (of f.g. projectives) concentrated in degrees n and higher. So we may assume that P has this property, and then by shifting indices we may assume $n = 0$.

Let $Q_\bullet \to H_0(P)$ be a resolution by finitely-generated projectives. Standard homological algebra gives us a map $f\colon P_\bullet \to Q_\bullet$ inducing an isomorphism on H_0.

Let T be the mapping cone of f. The long exact homology sequence shows that T has one fewer non-vanishing homology group than P, and hence we may assume by induction that $[T] \in \operatorname{im}\beta$. But we know from Lemma 5.5 (really, its analog for $G^{cplx}(R)$) that $[T] = [Q] - [P]$. Since $[Q] \in \operatorname{im}\beta$ by the definition of β, it follows that $[P] \in \operatorname{im}\beta$ as well. □

When we first learned the definitions of $K(R)$ and $G(R)$, the difference seemed to be about projective versus arbitrary modules. When we look at these groups as $K^{cplx}(R)$ and $G^{cplx}(R)$, however, the difference is about bounded versus bounded-below chain complexes.

5.10. **Relative K-theory.** It may seem like we have introduced an unnecessary level of complexity (no pun intended) by introducing $K^{cplx}(R)$. After all, the proof of Proposition 5.4 shows that for any bounded complex P the class $[P]$ is just the alternating sum $\sum(-1)^i[P_i[0]]$. That is, in $K^{cplx}(R)$ we may decompose any complex into its constituent modules; one really only needs modules, not chain complexes. But we will get some mileage out of these ideas by defining similar K-groups but restricting to complexes *subject to certain conditions*. In these cases we might not be able to 'unravel' the complexes anymore. We give a few examples:

[1]. Let S be a multiplicative system in R. Start with the free abelian group on isomorphism classes of bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that $S^{-1}P_\bullet$ is exact. Define $K(R, S)$ to be the quotient of this free abelian group by the analogs of relations (1) and (2) defining $K^{cplx}(R)$.

[2]. Let $I \subseteq R$ be an ideal. Start with the free abelian group on isomorphism classes of bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that each $H_k(P)$ is annihilated by I. Define $K(R, I)$ to be the quotient of this free abelian group by the analogs of relations (1) and (2) defining $K^{cplx}(R)$.

[3]. Fix an $n \geq 0$. Start with the free abelian group on isomorphism classes of bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that each $H_k(P)$ has Krull dimension at most n. Define $K(R, \leq n)$ to be the quotient of this free abelian group by the usual relations (1) and (2).

Exercise 5.11. In analogy to [3], define a group $K(R, \geq n)$. Prove that if $n \leq \dim R$ then $K(R, \geq n) \cong K(R)$, and that if $n > \dim R$ then $K(R, \geq n) = 0$.

Here is a lemma that will be very useful later on:

Lemma 5.12. *Let $\alpha\colon P \to Q$ and $\beta\colon Q \to W$ be maps between finitely-generated projectives, and assume both become isomorphisms after localization at S. Then*

$$[P \xrightarrow{\beta\alpha} W] = [P \xrightarrow{\alpha} Q] + [Q \xrightarrow{\beta} W]$$

in $K(R, S)$.

Proof. Use the following short exact sequence of maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{f} & Q \oplus P & \xrightarrow{g} & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}_Q \oplus \beta\alpha} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & Q & \xrightarrow{f'} & Q \oplus W & \xrightarrow{g'} & W & \longrightarrow & 0,
\end{array}$$

where $f(x) = (\alpha(x), x)$, $f'(y) = (y, \beta(y))$, $g(a, b) = a - \alpha(b)$, and $g'(c, d) = \beta(c) - d$. This gives that

$$[Q \xrightarrow{\text{id}} Q] + [P \xrightarrow{\beta\alpha} W] = [P \xrightarrow{\alpha} Q] + [Q \xrightarrow{\beta} W],$$

but of course the first term on the left is zero in $K(R, S)$. □

Note that there is an evident map $K(R, S) \to K(R)$ that sends a class $[P]$ in $K(R, S)$ to the similarly-named (but different) class $[P]$ in $K(R)$; in colloquial terms, the map simply 'forgets' that a complex P is S-exact. The composite $K(R, S) \to K(R) \to K(S^{-1}R)$ is clearly zero.

Proposition 5.13. *For any multiplicative system in a commutative ring R the sequence $K(R, S) \to K(R) \to K(S^{-1}R)$ is exact in the middle.*

Proof. Suppose $x \in K(R)$ is in the kernel of the map to $K(S^{-1}R)$. Every element of $K(R)$ may be written as $x = [P] - [Q]$ for some finitely-generated projectives P and Q. Then $[S^{-1}P] = [S^{-1}Q]$ in $K(S^{-1}R)$, so by Proposition 2.8 there exists an n such that $S^{-1}P \oplus (S^{-1}R)^n \cong S^{-1}Q \oplus (S^{-1}R)^n$. Alternatively, write this as $S^{-1}(P \oplus R^n) \cong S^{-1}(Q \oplus R^n)$. By Lemma 4.7(b) there exists a map of R-modules $Q \oplus R^n \to P \oplus R^n$ that becomes an isomorphism after S-localization. Regarding this map as a chain complex concentrated in degrees 0 and 1, it gives an element in $K(R, S)$; the image of this element under $K(R, S) \to K(R)$ is clearly x. □

The reader might have noticed that in the above proof we didn't encounter any kind of complicated chain complex when trying to construct our preimage in $K(R, S)$; in fact, we accomplished everything with chain complexes of length 1. This is a general phenomenon, similar to the fact that elements of $K^{cplx}(R)$ can all be decomposed into modules. For the relative K-groups one can't quite decompose *that* far, but one can always get down to complexes of length 1. To state a theorem along these lines, consider maps $f \colon P \to Q$ where P and Q are finitely-generated R-projectives and $S^{-1}f$ is an isomorphism (it is convenient to regard such maps as chain complexes concentrated in degrees 0 and 1). Let $K(R, S)_{\le 1}$ be the quotient of the free abelian group on such maps by the following relations:

(1) $[f] = 0$ if f is an isomorphism;
(2) $[f] = [f'] + [f'']$ if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0 \end{array}$$

where the rows are exact.

Notice that there is an evident map $K(R, S)_{\le 1} \to K(R, S)$.

Theorem 5.14. *For any multiplicative system S in a commutative ring R, the map $K(R, S)_{\le 1} \to K(R, S)$ is an isomorphism.*

The proof of this theorem is a bit difficult, and the techniques are too distant from the topics at hand to merit spending time on them. We give the proof in an appendix (???), for the interested reader.

Theorem 5.14 naturally suggests the following question: why use chain complexes at all, for relative K-theory? That is to say, if one can access the same groups using only chain complexes of length one, why complicate things by making the defintion using complexes of arbitrary length? The answer comes from algebraic geometry. Let X be a scheme and let U be an open subset of X. Then the 'correct' way to define a relative K-theory group $K(X,U)$ is to use bounded chain complexes of locally free sheaves on X that are exact on U. When $X = \operatorname{Spec} R$ and $U = \operatorname{Spec} S^{-1}R$ then it happens that one can get the same groups using only complexes of length one—as we saw above. But even for $X = \operatorname{Spec} R$ not every open subset is of this form. A general subset will have the form $U = (\operatorname{Spec} S_1^{-1}R) \cup (\operatorname{Spec} S_2^{-1}R) \cup \cdots \cup (\operatorname{Spec} S_d^{-1}R)$, but to get the same relative K-group here one must use complexes of length at most d. See [FH] and [D3] for the proof in this case.

When R is a regular ring all localizations $S^{-1}R$ are also regular. So the groups $K(R)$ and $K(S^{-1}R)$ can be identified with $G(R)$ and $G(S^{-1}R)$, by Theorem 2.10. Comparing the localization sequence in K-theory to the one in G-theory from Proposition 4.9 suggests an identification of the relative terms. Indeed, observe that the usual Euler characteristic map $\chi(P_\bullet) = \sum (-1)^i [H_i(P)]$ gives a well-defined map $K(R,S) \to G(M \,|\, S^{-1}M = 0)$. We have the following:

Theorem 5.15. *If R is regular then $\chi\colon K(R,S) \to G(M \,|\, S^{-1}M = 0)$ is an isomorphism.*

Proof. Define $\beta\colon G(M \,|\, S^{-1}M = 0) \to K(R,S)$ by sending $[M]$ to $[P_\bullet]$ for some finite resolution of M by finitely-generated projectives (which exists because R is regular). The exact same steps as in the proof of Theorem 2.10 shows that this is well-defined and a two-sided inverse to χ. □

5.16. **Relative K-theory and intersection multiplicities.** We now wish to tie several themes together, and use everything we have learned so far to give a complete, K-theoretic perspective on Serre's definition of intersection multiplicity. This perspective is from the paper [GS].

Let R be a Noetherian ring, and let $Z \subseteq \operatorname{Spec} R$ be a Zariski closed set. Recall that an R-module M is said to be **supported** on Z if $M_P = 0$ for all primes $P \notin Z$. One usually defines $\operatorname{Supp} M$, the **support** of M, to be $\{P \in \operatorname{Spec} R \,|\, M_P \neq 0\}$. This is known to be a closed subset of $\operatorname{Spec} R$, and to say that M is supported on Z is just the requirement that $\operatorname{Supp} M \subseteq Z$. Let $G(R)_Z$ be the Grothendieck group of all finitely-generated R-modules that are supported on Z.

Similarly, if $C_\bullet$ is a chain complex of R-modules then $\operatorname{Supp} C$ is defined to be $\{P \in \operatorname{Spec} R \,|\, H_*(C_P) \neq 0\}$. We say that $C_\bullet$ is supported on Z if $\operatorname{Supp} C \subseteq Z$, or if C_Q is exact for every $Q \notin Z$. Note that $C_\bullet$ is supported on Z if and only if all the homology modules $H_*(C)$ are supported on Z.

Similar to our definitions of $K^{cplx}(R)$ and $K(R,S)$, define $K(R)_Z$ to be the Grothendieck-style group of bounded complexes $P_\bullet$ of finitely-generated projective R-modules having the property that $\operatorname{Supp} P_\bullet \subseteq Z$. Note that if $Z = \operatorname{Spec} R - \operatorname{Spec} S^{-1}R$ then $K(R)_Z$ is precisely the group $K(R,S)$ previously defined.

The following statements should be easy exercises for the reader:

(1) The Euler characteristic $\chi(P_\bullet) = \sum_i (-1)^i [H_i(P)]$ defines a group homomorphism $K(R)_Z \to G(R)_Z$.
(2) If R is regular then the map $\chi\colon K(R)_Z \to G(R)_Z$ is an isomorphism.

(3) Tensor product of chain complexes gives pairings
$$\otimes\colon K(R)_Z \otimes K(R)_W \to K(R)_{Z\cap W}$$
for all pairs of closed subsets $Z, W \subseteq \operatorname{Spec} R$.

(4) If $Z = V(I)$ then a module M is supported on Z if and only if M is killed by a power of I.

(5) If M and N are R-modules then $\operatorname{Supp}(M \otimes N) = \operatorname{Supp} M \cap \operatorname{Supp} N$.

(6) Assume that R is regular, and transplant the tensor product of chain complexes to a pairing
$$G(R)_Z \otimes G(R)_W \to G(R)_{Z\cap W}.$$
This sends $[M] \otimes [N]$ to $\sum(-1)^i[\operatorname{Tor}_i(M, N)]$. (Note that this makes sense on the level of supports: if $Z = V(I)$ and $W = V(J)$, M is killed by a power of I and N is killed by a power of J, then $M \otimes N$ is killed by a power of $I + J$).

(7) Let $Z = \{m\}$ where m is a maximal ideal of R. Then the assignment $M \mapsto \ell(M_m)$ gives an isomorphism $G(R)_Z \xrightarrow{\cong} \mathbb{Z}$.

(8) Let M and N be R-modules such that $\operatorname{Supp}(M \otimes N) = \{m\}$ where m is a maximal ideal of R (geometrically, $\operatorname{Supp} M$ and $\operatorname{Supp} N$ have an isolated point of intersection). Then Serre's intersection multiplicity $e(M, N)$ is the image of $[M] \otimes [N]$ under the composite
$$G(R)_Z \otimes G(R)_W \longrightarrow G(R)_{Z\cap W} \xrightarrow{\ \ell\ } \mathbb{Z},$$
where we have written $Z = \operatorname{Supp} M$ and $W = \operatorname{Supp} N$ (and the map labelled ℓ is in fact an isomorphism).

Remark 5.17. We will understand this better after seeing how intersection multiplicities fit into algebraic topology, but it is worth noting that the group $K(R)_Z$ would be better written as $K(X, X - Z)$, where $X = \operatorname{Spec} R$. For comparison, relative products in a cohomology theory would give pairings
$$K(X, X-Z) \otimes K(X, X-W) \to K(X, (X-Z)\cup(X-W)) = K(X, X-(Z\cap W)),$$
which is what we saw above in the form $K(R)_Z \otimes K(R)_W \to K(R)_{Z\cap W}$.

6. K-theory of exact complexes

We have seen the isomorphism of groups $K(R) \cong K^{cplx}(R)$. If $P_\bullet$ is a bounded, exact complex of projectives then it gives rise to a relation in $K(R)$, and (equivalently) represents the zero object in $K^{cplx}(R)$. Given this, it might seem surprising to learn that there is yet another model for $K(R)$ in which exact complexes can represent nonzero elements—and even more, *all nonzero elements* can be represented this way. The goal of the present section is to explain this model, as well as some variations. This material is adapted from [Gr].

Note: The contents of this section are only needed once in the remainder of the notes, for a certain perspective on Adams operations in Section 30. While the material is intriguing, it can certainly be skipped if desired.

As in the last section, let R be a fixed commutative ring.

Definition 6.1.
$$K^{exact}(R) = \frac{\mathbb{Z}\langle\, [P_\bullet] \,\big|\, P_\bullet \textit{ is a bounded, exact chain complex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ *if there is a short exact sequence* $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$,

(2) $[\Sigma P_\bullet] = -[P_\bullet]$.

If P is a projective module let CP denote the mapping cone of the identity map $P \to P$. Specifically, CP is a chain complex concentrated in dimensions 0 and 1 where the only nonzero differential is the identity map on P. Observe that there is a group homomorphism $K(R) \to K^{exct}(R)$ that sends $[P]$ to $[CP]$.

Proposition 6.2. *The map* $K(R) \to K^{exct}(R)$ *is an isomorphism. The inverse is denoted* $\chi' \colon K^{exct}(R) \to K(R)$ *and called the* ***derived*** *(or* ***secondary****)* ***Euler characteristic****. If* $P_\bullet$ *is a bounded complex of finitely-generated projectives then*

$$\chi'(P_\bullet) = \sum_j (-1)^{j+1} j[P_j] = \sum_j (-1)^{j-1}[\operatorname{im} d_j]$$

where $d_j \colon P_j \to P_{j-1}$.

Technically speaking the second formula given for χ' doesn't make sense unless we know that each $\operatorname{im} d_j$ is a finitely-generated projective module. This is a simple exercise, but let us record it in a lemma.

Lemma 6.3. *Let* $P_\bullet$ *be a bounded, exact complex of projectives. Then each* $\operatorname{im} d_j$ *is projective, and finitely-generated if* P_j *is.*

Proof. Without loss of generality we can assume that $P_\bullet$ has the form $0 \to P_n \to \cdots \to P_0 \to 0$. So $d_1 \colon P_1 \to P_0$ is surjective, hence $\operatorname{im} d_1 = P_0$ and there is nothing to prove here. Exactness gives us short exact sequences $0 \to \operatorname{im} d_{j+1} \to P_j \to \operatorname{im} d_j \to 0$, for each j. We can assume by induction that $\operatorname{im} d_j$ is projective, hence the sequence is split-exact and therefore $\operatorname{im} d_{j+1}$ is also projective.

Since $\operatorname{im} d_j$ is a quotient of P_j, it is finitely-generated if P_j is. □

The above proof of course gives more than was explicity stated: by choosing splittings one level at a time one can see that $P_\bullet$ decomposes as a direct sum of exact complexes of length 1. This decomposition is non-canonical, however, depending on the choices of splitting. For variety we will see a weaker, but more canonical, version of this decomposition in the next proof.

Proof of Proposition 6.2. Let α denote the map $K(R) \to K^{exct}(R)$. It is easy to see that α is surjective, because if $0 \to P_n \to \cdots \to P_0 \to 0$ is an exact complex then there is an evident short exact sequence

$$0 \to \Sigma^{n-1}(CP_n) \to P_\bullet \to Q_\bullet \to 0$$

where $Q_\bullet$ is exact and has length at most $n-1$. Since $[P_\bullet] = [\Sigma^{n-1}(CP_n)] + [Q_\bullet] = (-1)^{n-1}[CP_n] + [Q_\bullet]$, an immediate induction shows that $K^{exct}(R)$ is generated by the classes $[CP]$ as P ranges over all finitely-generated projectives.

Note that $P_n \cong \operatorname{im} d_n$, and $Q_{n-1} = \operatorname{coker}(P_n \to P_{n-1}) \cong \operatorname{im} d_{n-1}$. The induction mentioned in the preceding paragraph shows that $[P_\bullet] = \sum_j (-1)^{j-1}[C(\operatorname{im} d_j)]$. From this it is clear that if an inverse to α exists it must send $[P_\bullet]$ to $\sum_j (-1)^{j-1}[\operatorname{im} d_j]$. It is only left to check that this formula does indeed define a map $K^{exct}(R) \to K(R)$.

Let $P_\bullet$ be any bounded, exact complex of finitely-generated projectives, and assume that the smallest degree containing a nonzero module is degree n. Write

$I_j = \operatorname{im} d_j$. Since $\cdots \to P_{j+1} \to P_j \to I_j \to 0$ is exact, we have that $[I_j] = \sum_{k\geq j}(-1)^{k-j}[P_k]$ in $K(R)$. So in $K(R)$ we have

$$\begin{aligned}\sum_{j\geq n}(-1)^{j-1}[I_j] = \sum_{j\geq n}\sum_{k\geq j}(-1)^{j-1}(-1)^{k-j}[P_k] &= \sum_k (-1)^{k-1}\sum_{n\leq j\leq k}[P_k] \\ &= \sum_k (-1)^{k-1}(k-n+1)[P_k] \\ &= \sum_k (-1)^{k-1}k[P_k] + (n-1)\chi(P_\bullet) \\ &= \sum_k (-1)^{k-1}k[P_k].\end{aligned}$$

In the last equality we have used that $\chi(P_\bullet)=0$ since $P_\bullet$ is exact.

Define $\chi'(P_\bullet) = \sum_k(-1)^{k-1}k[P_k]$. One easily checks that this satisfies relations (1) and (2) in the definition of $K^{exct}(R)$, and hence defines a map $\beta\colon K^{exct}(R) \to K(R)$. It is trivial to check that $\beta\circ\alpha = \mathrm{id}$. The above calculation (and preceding remarks) verifies that $\alpha\circ\beta = \mathrm{id}$. □

6.4. **Derived Euler characteristics.** Now that we have encountered the derived Euler characteristic it seems worthwhile to take a moment and place it into a broader context. Consider the definition

$$\chi_t(P_\bullet) = \sum t^j[P_j] \in K(R)[t,t^{-1}].$$

This function is additive, and in fact it is clearly the universal additive invariant for bounded complexes of finitely-generated projectives. The usual Euler characteristic is $\chi(P_\bullet) = \chi_t(P_\bullet)|_{t=-1}$. Of course we do not have $\chi_t(\Sigma P_\bullet) = -\chi_t(P_\bullet)$, this only becomes true after the substitution $t=-1$; what we have instead is the identity

$$\chi_t(\Sigma P_\bullet) = t\cdot\chi_t(P_\bullet). \tag{6.5}$$

If we differentiate χ_t with respect to t then we obtain $\chi_t'(P_\bullet) = \sum jt^{j-1}[P_j]$. Clearly this is also an additive invariant of complexes. The invariant we called χ' is just $\chi_t'(P_\bullet)|_{t=-1}$. Differentiating (6.5) yields the formula

$$\chi_t'(\Sigma P_\bullet) = \chi_t(P_\bullet) + t\cdot\chi_t'(P_\bullet), \tag{6.6}$$

and consequently $\chi'(\Sigma P_\bullet) = \chi(P_\bullet) - \chi'(P_\bullet)$. This is not the kind of behavior we are used to, but notice that if we restrict to complexes $P_\bullet$ with $\chi(P_\bullet)=0$ then we get the nicer behavior $\chi'(\Sigma P_\bullet) = -\chi(P_\bullet)$.

One can, of course, iterate this procedure. Let $\chi_t^{(n)}(P_\bullet)$ denote the nth derivative of $\chi_t(P_\bullet)$, and white $\chi^{(n)}(P_\bullet) = \chi_t^{(n)}(P_\bullet)|_{t=-1}$. Call this the ***n*th derived Euler characteristic**. It is an additive function, and if one restricts to complexes such that $0=\chi(P_\bullet)=\chi'(P_\bullet)=\cdots=\chi^{(n-1)}(P_\bullet)$ then it satifies $\chi^{(n)}(\Sigma P_\bullet) = -\chi^{(n)}(P_\bullet)$.

6.7. **Doubly-exact complexes.** A bicomplex $C_{\bullet,\bullet}$ will be called **bounded** if the modules $C_{i,j}$ are nonzero for only finitely many values of (i,j). The bicomplex will be called **doubly-exact** if every row and every column is exact. By abuse of terminology an ordinary chain complex $D_\bullet$ will be called doubly-exact if it is isomorphic to the total complex of a bounded, doubly-exact bicomplex. Doubly exact complexes all represent zero in $K^{exct}(R)$:

Proposition 6.8. *If $P_\bullet$ is a bounded, doubly-exact complex of finitely-generated projectives then $[P_\bullet] = 0$ in $K^{exct}(R)$.*

Proof. Let $M_{\bullet,\bullet}$ be a doubly-exact bicomplex of finitely-generated projectives. Without loss of generality let us assume that $M_{i,j}$ is nonzero only for $0 \le i \le n$ and $0 \le j \le k$. Write $M_{i,*}$ for the ordinary complex whose jth term is $M_{i,j}$, and write $M_{\ge i,*}$ for the sub-bicomplex of $M_{\bullet,\bullet}$ consisting of all $M_{a,j}$ for $a \ge i$. Observe that there are short exact sequences

$$0 \to \Sigma^i M_{i,*} \hookrightarrow \mathrm{Tot}(M_{\ge i,*}) \twoheadrightarrow \mathrm{Tot}(M_{\ge (i+1),*}) \to 0,$$

for all i. Induction shows that each $\mathrm{Tot}(M_{\ge i,*})$ is exact, and therefore in $K^{exct}(R)$ we have

$$[\mathrm{Tot}\, M_{\bullet,\bullet}] = \sum_i [\Sigma^i M_{i,*}] = \sum_i (-1)^i [M_{i,*}].$$

But $M_{\bullet,\bullet}$ may be regarded as an exact sequence of chain complexes

$$0 \to M_{n,*} \to M_{n-1,*} \to \cdots \to M_{1,*} \to M_{0,*} \to 0.$$

The image of each map in this sequence is a chain complex of finitely-generated projectives (using Lemma 6.3), and each of these image complexes is exact by a straightforward induction. So the above exact sequence breaks up into a collection of short exact sequences of exact complexes of finitely-generated projectives, and hence shows that $\sum_i (-1)^i [M_{i,*}]$ is zero in $K^{exct}(R)$. We have therefore shown that $[\mathrm{Tot}\, M_{\bullet,\bullet}] = 0$ in $K^{exct}(R)$. $\square$

The reader will notice the beginnings of a pattern here. Exact complexes $P_\bullet$ represent zero in $K^{cplx}(R)$, but then we produced a new model for this same group where the exact complexes were our generators. In this new group $K^{exct}(R)$ the doubly-exact complexes represent zero. It is natural, then, to wonder if there is yet another model for this group where the doubly-exact complexes are the generators. Indeed, this works out in what is now a completely straightforward manner, and can be repeated ad infinitum.

Let us use the term *multicomplex* for the evident generalization of bicomplexes to n dimensions. We will denote a multicomplex by $M_\star$, where the symbol $\star$ stands for an n-tuple of integers. Say that the multicomplex is ***n*-exact** if every linear 'row' (obtained by fixing $n-1$ of the indices) is exact.

Definition 6.9.

$$K^{n-exct}(R) = \frac{\mathbb{Z}\langle\, [M_\star] \,\big|\, M_\star \textit{ is a bounded, } n\textit{-exact multicomplex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[M_\star] = [M'_\star] + [M''_\star]$ *if there is an exact sequence* $0 \to M'_\star \to M_\star \to M''_\star \to 0$,

(2) $[\Sigma M_\star] = -[M_\star]$, *where* Σ *stands for any of the* n *suspension operators on* n*-multicomplexes.*

Given an $(n+1)$-multicomplex $M_\star$ there are $\binom{n+1}{2}$ ways to totalize it to get an n-multicomplex—one needs to choose two of the $n+1$ directions to combine. One can follow the proof of Proposition 6.8 to show that if $M_\star$ is $(n+1)$-exact then each of these totalizations represents zero in $K^{n-exct}(R)$.

If $M_\star$ is an n-multicomplex then let $CM_\star$ denote the cone on the identity map $M_\star \to M_\star$. This is an $(n+1)$-multicomplex, defined in the evident manner. This cone construction induces a group homomorphism $K^{n-exct}(R) \to K^{(n+1)-exct}(R)$.

Proposition 6.10. *The map $K^{n-exct}(R) \to K^{(n+1)-exct}(R)$ is an isomorphism, with inverse given by*

$$\chi'(M_\star) = \sum (-1)^{j+1} j[M_{j,\star}]$$

where the symbols $M_{j,\star}$ represent the various slices of $M_\star$ in any fixed direction.

Proof. Follow the proof of Proposition 6.2 almost verbatim, but where each P_i represents an n-exact multicomplex rather than an R-module. □

We have the sequence of isomorphisms

$$K(R) \to K^{exct}(R) \to K^{2-exct}(R) \to \cdots$$

The composite map $K(R) \to K^{n-exct}(R)$ sends $[P]$ to the n-dimensional cube consisting of P's and identity maps. The composite $K^{n-exct}(R) \to K(R)$ sends an n-multicomplex $M_\star$ to

$$M_\star \mapsto \sum_{j_1,\ldots,j_n} (-1)^{j_1+\cdots+j_n+n} j_1 \cdots j_n [M_{j_1,\ldots,j_n}].$$

If one considers the formal Laurent polynomial

$$\chi_{t_1,\ldots,t_n}(M) = \sum_{j_1,\ldots,j_n} t_1^{j_1} \cdots t_n^{j_n} [M_{j_1,\ldots,j_n}]$$

then this is the nth order partial derivative $\partial_{t_1} \cdots \partial_{t_n} \chi_{t_1,\ldots,t_n}(M)$ evaluated at $t_1 = t_2 = \cdots = t_n = -1$.

Remark 6.11. Grayson [Gr] suggests a perspective where exact complexes are analogous to the formal infinitesimals from nonstandard analysis. Doubly-exact complexes are analogues of products of infinitesimals, and so forth. ????

7. A taste of K_1

Note: The material in this section will not be needed for most of what follows in these notes. We include it for general interest, and because the material fits naturally here. But this section can safely be skipped.

Given a ring R and a multiplicative system $S \subseteq R$, we have seen the exact sequences

$$G(M \mid S^{-1}M = 0) \to G(R) \to G(S^{-1}R) \to 0$$

and

$$K(R, S) \to K(R) \to K(S^{-1}R).$$

It is natural to wonder if these extend to long exact sequences, and the answer is that they do: in the first case there is an extension to the left, and in the latter two cases there is an extension in both directions. These extensions are not easy to produce, however—they are the subject of 'higher algebraic K-theory', an area that involves some very deep and difficult mathematics. Our aim here is not to start a long journey into that subject, but rather to just give some indications of the very beginnings.

Remark 7.1. From now on the groups $K(R)$ and $G(R)$ will be written $K_0(R)$ and $G_0(R)$.

7.2. **The basic theory of $K_1(R)$.** Let us adopt the perspective that $K_0(R)$ is, in essence, constructed with the goal of generalizing the familiar notion of dimension for vector spaces. The key property of dimension is additivity for short exact sequences, so consequently one forms the universal group with that property. The group $K_1(R)$ is obtained similarly but with the goal of generalizing the *determinant*.

Determinants are invariants of self-maps—maps with the same domain and target—and we need some language for dealing with such things. Given two self-maps $f\colon A \to A$ and $g\colon B \to B$, we define a map from f to g to be a map $u\colon A \to B$ giving a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{u} & B. \end{array}$$

Likewise, an exact sequence of self-maps is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{u_0} & A & \xrightarrow{u_1} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & A' & \xrightarrow{u_0} & A & \xrightarrow{u_1} & A'' & \longrightarrow & 0 \end{array} \tag{7.3}$$

in which the rows are short exact sequences of modules.

Definition 7.4. *Form the free abelian group generated by isomorphism classes of maps $[P \xrightarrow{\alpha} P]$ where P is a finitely-generated projective and α is an isomorphism. Let $K_1(R)$ be the quotient of this group by the following relations:*

(a) $[P \xrightarrow{\alpha} P] = [P' \xrightarrow{\alpha'} P'] + [P'' \xrightarrow{\alpha''} P'']$ whenever there is a short exact sequence as in (7.3);

(b) $[P \xrightarrow{\alpha\beta} P] = [P \xrightarrow{\alpha} P] + [P \xrightarrow{\beta} P]$ for all self-maps $\alpha, \beta\colon P \to P$.

Note that as a consequence of relation (b) one has that $[P \xrightarrow{\mathrm{id}} P] = 0$ for any finitely-generated projective module P. Note also that if $\alpha\colon P \to P$ and $\beta\colon Q \to Q$ are automorphsms then

$$[P \oplus Q \xrightarrow{\alpha\oplus\beta} P \oplus Q] = [P \xrightarrow{\alpha} P] + [Q \xrightarrow{\beta} Q], \tag{7.5}$$

as a consequence of relation (a).

The use of projective modules in the definition of $K_1(R)$ turns out to be unnecessarily complicated—one can get the same group by only using automorphisms of free modules. Even more, the use of short exact sequences in relation (a) is unnecessarily complicated; one can get the same group by only imposing the weaker relation from (7.5). We will prove both of these claims in just a moment.

Observe that there is a map of groups $GL_n(R) \to K_1(R)$ that sends a matrix A to the class $[R^n \xrightarrow{A} R^n]$ (left-multiplication-by-A). Relation (b) guarantees that this is indeed a group homomorphism. If we let $j\colon GL_n(R) \hookrightarrow GL_{n+1}(R)$ be the usual inclusion, obtained by adding an additional row and column and a 1 along the diagonal, then it is clear that $[R^{n+1} \xrightarrow{j(A)} R^{n+1}] = [R^n \xrightarrow{A} R^n]$. This follows from (7.5) and the fact that $[R \xrightarrow{\mathrm{id}} R] = 0$. Let $GL(R)$ denote the colimit

$$GL(R) = \operatorname{colim}[GL_1(R) \to GL_2(R) \to GL_3(R) \to \cdots],$$

and call this the **infinite general linear group** of R. We have obtained a map $GL(R) \to K_1(R)$, and of course this will factor through the abelianization to give

$$GL(R)_{ab} = GL(R)/[GL(R), GL(R)] \to K_1(R).$$

Theorem 7.6. *The map $GL(R)_{ab} \to K_1(R)$ is an isomorphism.*

It will be convenient to prove this at the same time that we give other descriptions for $K_1(R)$. In particular, we make the following definitions:

(1) $K_1^{fr}(R)$ is the group defined similarly to $K_1(R)$ but changing all occurrences of 'projective' to 'free'.
(2) $K_1^{sp}(R)$ is the group defined similarly to $K_1(R)$ but replacing relation (a) by the direct sum relation of (7.5). The "sp" stands for "split".
(3) $K_1^{sp,fr}(R)$ is the group defined by making both the changes indicated in (1) and (2).

One obtains a large diagram as follows:

(7.7)
$$\begin{array}{ccccccc}
 & & \operatorname{colim}_P \operatorname{Aut}(P)_{ab} & \twoheadrightarrow & K_1^{sp}(R) & \twoheadrightarrow & K_1(R) \\
 & & \uparrow & & \uparrow & & \uparrow \\
GL(R)_{ab} & = & \operatorname{colim}_n GL_n(R)_{ab} & \twoheadrightarrow & K_1^{sp,fr}(R) & \twoheadrightarrow & K_1^{fr}(R).
\end{array}$$

The maps labelled as surjections are obviously so. Let us explain the colimit over projectives P. Let $\mathcal{M}$ denote the monoid of isomorphism classes of finitely-generated projectives, with the operation of $\oplus$. The *translation category* $T(\mathcal{M})$ of this monoid has object set equal to $\mathcal{M}$, and the maps from A to B are the elements $C \in \mathcal{M}$ such that $A + C = B$. This is the indexing category for our colimit. Given an isomorphism $f\colon P \to Q$, there is an induced map of groups $\operatorname{Aut}(P) \to \operatorname{Aut}(Q)$ sending α to $f\alpha f^{-1}$. Changing f gives a different induced map, but it gives the same induced map on $\operatorname{Aut}(P)_{ab} \to \operatorname{Aut}(Q)_{ab}$; this is an easy exercise. It follows that there is a functor $T(\mathcal{M}) \to \mathcal{A}b$ sending $[P]$ to $\operatorname{Aut}(P)_{ab}$ and having the property that the map $[Q]$ from $[P]$ to $[P \oplus Q]$ yields the map $\operatorname{Aut}(P)_{ab} \to \operatorname{Aut}(P \oplus Q)_{ab}$ induced by direct sum with id_Q. The upper left term in our diagram is the colimit of this functor. The map from this colimit to $K_1^{sp}(R)$ is induced by the one sending an element $\alpha \in \operatorname{Aut}(P)$ to the class $[P \xrightarrow{\alpha} P]$.

Theorem 7.6 will follow as an immediate consequence of the following stronger result:

Theorem 7.8. *All of the maps in (7.7) are isomorphisms.*

We are almost ready to prove this theorem, but we do need one key lemma. Let $E(R) \subseteq GL(R)$ be the subgroup generated by the elementary matrices—matrices that have ones along the diagonal and a single nonzero, off-diagonal entry. Note that right multiplication by such a matrix amounts to performing a column operation where a multiple of one column is added to another; similarly, left multiplication amounts to performing the analogous row operation. One very useful way to recognize a matrix as belonging to $E(R)$ is to observe that it can be obtained from the identity matrix by using these types of row and column operations. It is useful to say that a column or row operation is **allowable** if the corresponding elementary matrix belongs to $E(R)$.

Lemma 7.9.
(a) For any $X \in M_n(R)$ the matrix $\left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ and its transpose belong to $E(R)$.
(b) If $A \in GL_n(R)$ then $\left[\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right] \in E(R)$.
(c) Let A be a matrix obtained from the identity by switching two colums and multiplying one of the switched columns by -1. Then $A \in E(R)$, and similarly for the transpose of A.

Proof. For part (a) just note that $\left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ can be obtained from the identity matrix by a sequence of allowable column operations of the type discussed above. For the transpose, use row operations.

For (b) consider the following chain of matrices:

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} I & A \\ A^{-2} - A^{-1} & A^{-1} \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ A^{-2} - A^{-1} & A^{-1} \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}.$$

Passage from each matrix to the next can be done by allowable row and column operations; alternatively, each matrix can be obtained from its predecessor by left or right multiplication by a matrix of the type considered in (a).

Finally, for (c) we argue directly in terms of column operations. If v and w are two columns consider the following chain

$$v, w \quad \mapsto \quad v, w - v \quad \mapsto \quad w, w - v \quad \mapsto \quad w, -v.$$

Each link involves adding a multiple of one column to another, and is therefore allowable; therefore the composite operation is allowable. □

The following is the key lemma that we will need in our proof of Theorem 7.8:

Lemma 7.10 (Whitehead Lemma)**.** $E(R) = [GL(R), GL(R)]$

Proof. For the $\subseteq$ direction we consider three columns u, v, w, and the following chain of operations (where $r, s \in R$):

$$\begin{aligned} u, v, w \quad \mapsto \quad u, v + ru, w \quad \mapsto \quad u, v + ru, w + sv + sru \quad &\mapsto \quad u, v, w + sv + sru \\ &\mapsto \quad u, v, w + sru. \end{aligned}$$

It should be clear what column operation is being used in each step. Note that the third and fourth operations are the inverses of the first and second, so the composite it a commutator. This shows that any column operation of the type "add a multiple of one column to another" is a commutator, and therefore $E(R) \subseteq [GL(R), GL(R)]$. (We have actually shown $E_n(R) \subseteq [GL_n(R), GL_n(R)]$ for $n \geq 3$).

For the other subset, let $A, B \in GL_n(R)$. Consider the following identity:

$$\begin{bmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{bmatrix}.$$

The first matrix is identified with the commutator of A and B inside $GL(R)$, and all of the other matrices are in $E(R)$ by Lemma 7.9(b). So $[A, B] \in E(R)$ as well. □

Corollary 7.11. *For any $A \in GL_n(R)$, $B \in GL_k(R)$, and $X \in M_{n\times k}(R)$, $\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in $GL(R)_{ab}$. If $n = k$ then this matrix also equals $\begin{bmatrix} AB & 0 \\ 0 & I \end{bmatrix}$ in $GL(R)_{ab}$.*

Proof. For the first claim simply observe that

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} I & A^{-1}X \\ 0 & I \end{bmatrix}.$$

The second matrix in the product is in $E(R)$ by Lemma 7.9(a), and hence in $[GL(R), GL(R)]$ by the Whitehead Lemma.

For the second claim notice that $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & I \end{bmatrix}$ and use Lemma 7.9(b) together with the Whitehead Lemma. □

We are now ready to prove that all the descriptions of $K_1(R)$ give the same group:

Proof of Theorem 7.8. Let $\alpha\colon P \to P$ be an automorphism of a finitely-generated projective, and let Q be a free complement to P: that is, $P \oplus Q \cong R^n$ for some n. Then

$$[P \xrightarrow{\alpha} P] = [P \oplus Q \xrightarrow{\alpha\oplus \mathrm{id}_Q} P \oplus Q]$$

in $K_1^{sp}(R)$, which shows that $K_1^{sp,fr}(R) \to K_1^{sp}(R)$ is surjective. The same proof works for all of the vertical maps in diagram (7.7).

The fact that $\mathrm{colim}_n\, GL_n(R)_{ab} \to \mathrm{colim}_P\, \mathrm{Aut}(P)_{ab}$ is an isomorphism is very easy: it is just because the subcategory of $T(\mathcal{M})$ consisting of the free modules is cofinal in $T(\mathcal{M})$.

Define a map $K_1^{fr}(R) \to GL(R)_{ab}$ by sending $[R^n \xrightarrow{A} R^n]$ to the matrix A. To see that this is well-defined we need to verify that it respects relations (a) and (b) from Definition 7.4. Relation (b) is self-evident. For (a), suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \end{array}$$

is a short exact sequence of automorphisms between free modules. Then there is a basis for F with respect to which the matrix for α has the form $\begin{bmatrix} \alpha' & * \\ 0 & \alpha'' \end{bmatrix}$. Corollary 7.11 verifies that this matrix equals $\begin{bmatrix} \alpha' & 0 \\ 0 & \alpha'' \end{bmatrix}$ in $GL(R)_{ab}$.

Now that we have the map $K_1^{fr}(R) \to GL(R)_{ab}$, it is trivial to check that this is a two-sided inverse for the map from (7.7). It follows that all the maps in the bottom row of that diagram are isomorphisms.

The proof for the maps along the top row proceeds in a similar manner. Define a map $K_1(R) \to \mathrm{colim}_P\, \mathrm{Aut}(P)_{ab}$ by sending $[P \xrightarrow{\alpha} P]$ to the element $\alpha \in \mathrm{Aut}(P)_{ab}$. One has to check that this respects relations (a) and (b) in the definition of $K_1(R)$, and relation (b) is again trivial. Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \end{array}$$

is a short exact sequence of automorphisms between finitely-generated projectives. Choose free complements Q' for P', and Q'' for P''. Consider the new short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'\oplus Q' & \longrightarrow & P\oplus Q'\oplus Q'' & \longrightarrow & P''\oplus Q'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'\oplus \mathrm{id}_{Q'}} & & \downarrow{\scriptstyle \alpha\oplus \mathrm{id}_{Q'}\oplus \mathrm{id}_{Q''}} & & \downarrow{\scriptstyle \alpha''\oplus \mathrm{id}_{Q''}} & & \\ 0 & \longrightarrow & P'\oplus Q' & \longrightarrow & P\oplus Q'\oplus Q'' & \longrightarrow & P''\oplus Q'' & \longrightarrow & 0. \end{array}$$

All of the modules in this diagram are free (recall that $P \cong P' \oplus P''$), and so this diagram gives a relation in $K_1^{fr}(R)$. Using the map $K_1^{fr}(R) \to GL(R)_{ab}$ already constructed, we find that

$$\alpha \oplus \mathrm{id}_{Q'} \oplus \mathrm{id}_{Q''} = \big(\alpha' \oplus \mathrm{id}_{Q'}\big) + \big(\alpha'' \oplus \mathrm{id}_{Q''}\big)$$

in $GL(R)_{ab}$ and hence also in $\operatorname{colim}_P \operatorname{Aut}(P)_{ab}$. But this says precisely that $\alpha = \alpha' + \alpha''$ as elements in $\operatorname{colim}_P \operatorname{Aut}(P)_{ab}$, and this is what we needed to check. We have now constructed our map $K_1(R) \to \operatorname{colim}_P \operatorname{Aut}(P)_{ab}$, and it readily follows that it is an inverse for the map in the other direction from (7.7). So all the maps in the top horizontal row of (7.7) are isomorphisms.

We have shown that all horizontal maps in (7.7) are isomorphisms, and that the left vertical map is an isomorphism. So all the maps are isomorphisms. □

Observe that $\det\colon GL(R) \to R^*$ factors through the abelianization and therefore yields an induced map $\det\colon K_1(R) \to R^*$. This map is split, since we can send any $r \in R^*$ to the class of the automorphism $R \xrightarrow{r} R$ (this is a group homomorphism using relation (2) of Definition 7.4). So we always have $K_1(R) \cong R^* \oplus (???)$. The mystery factor is often called $SK_1(R)$.

We will not calculate K_1 for many rings, but let us at least do the easiest examples.

Proposition 7.12. *If F is a field then $K_1(F) = F^*$.*

Proof. One must show that if $A \in GL(F)$ satisfies $\det(A) = 1$ then $A \in [GL(F), GL(F)] = E(F)$.

We first observe that if A is a diagonal matrix of determinant 1 then A lies in $E(F)$. This can be proven by matrix manipulation, but the following argument is a bit easier to write. We use that $GL(F)/E(F) \cong K_1(F)$. Let $d_1, \dots, d_n$ be the diagonal entries of A. Working in $K_1(F)$ we write

$$[F^n \xrightarrow{A} F^n] = [F \xrightarrow{d_1} F] + \cdots + [F \xrightarrow{d_n} F] = [F \xrightarrow{d_1 \cdots d_n} F] = [F \xrightarrow{1} F] = 0$$

where the first equality is by relation (a) in Definition 7.4 and the second equality is by relation (b).

Now let A be an arbitrary $n \times n$ matrix of determinant 1. We will use two types of column (and row) operations: adding a multiple of one column/row to another, and switching two columns together with a sign change of one of them. Both types of operation are allowable, the latter by Lemma 7.9(c). Using these allowable column operations we can transform A into a lower diagonal matrix; then by using allowable row operations one further transforms A into a diagonal matrix. That is, there exist matrices $E_1, E_2 \in E(F)$ such that E_1AE_2 is diagonal. But by the preceding paragraph we then have $E_1AE_2 \in E(F)$, and so $A \in E(F)$. □

Essentially the same proof as above also shows the following:

Proposition 7.13. *Let R be a Euclidean domain. Then* $\det\colon K_1(R) \to R^*$ *is an isomorphism. In particular,* $K_1(\mathbb{Z}) = \{1, -1\} \cong \mathbb{Z}/2$ *and when F is a field* $K_1(F[t]) \cong F^*$*.*

Proof. We must again show that if $A \in GL_n(R)$ has $\det(A) = 1$ then $A \in E(R)$. For any fixed row of A, the ideal generated by the elements in that row contains $\det(A)$ and is therefore the unit ideal. Pick an element x of smallest degree in this row and then use column operations (and the Euclidean division property) to arrange all other elements in this row to be either zero or have degree smaller than x. By repeating this process, eventually the row will contain a unit. Do a signed transposition to switch this unit into position $(1, 1)$, and then do row operations to clear out all other terms in the first column. Repeat this process for the submatrix obtained by deleting the first row and column, and so forth. Eventually the matrix will be reduced to a diagonal matrix, necessarily of determinant 1. Such a diagonal matrix lies in $E(R)$, so this proves A also lies in $E(R)$. □

7.14. **Longer localization sequences.**

Proposition 7.15. *There is a unique map* $\partial\colon K_1(S^{-1}R) \to K_0(R, S)$ *having the property that if* $\alpha\colon R^n \to R^n$ *is such that* $S^{-1}\alpha$ *is an isomorphism, then* ∂ *sends* $[S^{-1}R^n \xrightarrow{S^{-1}\alpha} S^{-1}R^n]$ *to the class of the chain complex* $0 \to R^n \xrightarrow{\alpha} R^n \to 0$ *(concentrated in degrees* 0 *and* 1*).*

Proof. First, assume that $\beta\colon (S^{-1}R)^n \to (S^{-1}R)^n$. Let A be the matrix for β with respect to the standard basis, and let $u \in S$ be an element such that uA has entries in R (e.g., take u to be the product of all the denominators of the entries in A). Then uA represents a map $\beta'\colon R^n \to R^n$, and we have the commutative diagram

$$\begin{array}{ccccc} (S^{-1}R)^n & \xrightarrow{\beta} & (S^{-1}R)^n & \xrightarrow{uI_n} & (S^{-1}R)^n \\ \uparrow & & & & \uparrow \\ R^n & & \xrightarrow{\qquad\beta'\qquad} & & R^n \end{array}$$

where the vertical maps are localization. This diagram gives $uI_n \circ \beta \cong S^{-1}\beta'$, and so $[uI_n] + [\beta] = [S^{-1}\beta']$ in $K_1(S^{-1}R)$. Note that $[uI_n] = n[uI_1]$, and uI_1 is itself the localization of the multiplication-by-u map on R; so we can write

$$[\beta] = [S^{-1}\beta'] - n[S^{-1}u]. \tag{7.16}$$

This shows that $K_1(R)$ is generated by classes $[S^{-1}\alpha]$ for $\alpha\colon R^n \to R^n$, and we have thereby proven the uniqueness part of the proposition.

For existence, we will define a map $\partial\colon K_1^{sp,fr}(R) \to K_0(R, S)$ and then appeal to Theorem 7.8. Given a map $\beta\colon (S^{-1}R)^n \to (S^{-1}R)^n$, choose a $u \in S$ such that the standard matrix representing $u\beta$ has entries in R. Consider the assignment

$$\beta \mapsto F(\beta, u) = [R^n \xrightarrow{u\beta} R^n] - n[R \xrightarrow{u} R] \in K_0(R, S).$$

Note that this expression doesn't come out of thin air: the expected homomorphism ∂, if it exists, must have this form by (7.16). It remains to show that the above formula does indeed define a homomorphism.

We first show that $F(\beta, u)$ does not depend on the choice of u. It suffices to prove that $F(\beta, tu) = F(\beta, u)$ for any $t \in S$; for if u' is another choice for u then we would have $F(\beta,u)=F(\beta,u'u)=F(\beta,u')$. But now we just compute that

$$\begin{aligned} F(\beta,tu) &= [R^n \xrightarrow{tu\beta} R^n] - n[R \xrightarrow{tu} R] \\ &= [R^n \xrightarrow{t} R^n] + [R^n \xrightarrow{u\beta} R^n] - n\Big[[R \xrightarrow{t} R] + [R \xrightarrow{u} R]\Big] \\ &= [R^n \xrightarrow{u\beta} R^n] - n[R \xrightarrow{u} R] \end{aligned}$$

(the second equality is by Lemma 5.12, applied twice).

Let us now write $F(\beta)$ instead of $F(\beta,u)$. The last thing that must be checked is that $F(\beta\oplus\beta')=F(\beta)+F(\beta')$, but this is obvious. So we have established the existence of $\partial\colon K_1(R)\to K_0(R,S)$ having the desired properties. □

Theorem 7.17 (Localization sequence for K-theory)**.** *Let R be a commutative ring and $S\subseteq R$ a multiplicative system. The following sequence is exact:*

$$K_1(R) \longrightarrow K_1(S^{-1}R) \xrightarrow{\partial} K_0(R,S) \longrightarrow K_0(R) \longrightarrow K_0(S^{-1}R).$$

Proof. We will not prove exactness at $K_1(S^{-1}R)$, as this is a bit difficult and would take us too far afield. Exactness at $K_0(R)$ was already proven in Proposition 5.13, so it only remains to verify exactness at $K_0(R,S)$.

Let $x\in K_0(R,S)$. We know by Theorem 5.14 that x can be written in the form $x=[P_1\to P_0]-[Q_1\to Q_0]$ for finitely-generated projectives P_0,P_1,Q_0,Q_1 over R and maps $\alpha\colon P_1\to P_0$, $\beta\colon Q_1\to Q_0$ that become isomorphisms after S-localization. Consider the isomorphism $S^{-1}Q_0\to S^{-1}Q_1$ that is the inverse to $S^{-1}\beta$. By Lemma 4.7(b) there is a map $\gamma\colon Q_0\to Q_1$ whose localization is isomorphic to this map. Notice that

$$\begin{aligned} x &= [P_1\to P_0]+[Q_0\to Q_1]-\big([Q_0\to Q_1]+[Q_1\to Q_0]\big) \\ &= [P_1\oplus Q_0\to P_0\oplus Q_1]-[Q_0\oplus Q_1\to Q_1\oplus Q_0]. \end{aligned}$$

So by replacing our original P's and Q's we can assume that $Q_0=Q_1$.

Let G be a projective such that $Q_0\oplus G$ is free, and observe that

$$x = x-[G\xrightarrow{\mathrm{id}} G] = [P_1\to P_0]-[Q_0\oplus G\to Q_0\oplus G].$$

So again, by replacing our chosen $Q_0=Q_1$ we can actually assume that $Q_0=Q_1$ is free. That is, $x=[P_1\xrightarrow{\alpha} P_0]-[R^n\xrightarrow{\beta} R^n]$.

Now assume that x maps to zero in $K_0(R)$. This just says that $[P_0]=[P_1]$ in $K_0(R)$, and so there exists a free module G such that $P_0\oplus G\cong P_1\oplus G$. Since $x=x+[G\xrightarrow{id} G]$ we see that we can write x as

$$x=[R^k\xrightarrow{\alpha} R^k]-[R^n\xrightarrow{\beta} R^n]$$

where α and β become isomorphisms after S-localization. It is now immediate that x is in the image of ∂; to be completely specific,

$$x=\partial\Big([S^{-1}R^k\xrightarrow{S^{-1}\alpha} S^{-1}R^k]-[S^{-1}R^n\xrightarrow{S^{-1}\beta} S^{-1}R^n]\Big).$$

□

Example 7.18. This example will be a "reality check". We won't learn anything new, but we will see that the localization sequence is doing something sensible. Let R be a discrete valuation ring (a regular local ring whose maximal ideal is principal), and let F be the quotient field. Let π be a generator for the maximal ideal, and let $S = \{1, x, x^2, \ldots\}$. Note that $S^{-1}R = F$. The localization sequence takes on the form

$$K_1(R) \to F^* \xrightarrow{\partial} K_0(R,S) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$$

where we are using $K_1(F) \cong F^*$, $K_0(R) \cong \mathbb{Z}$ (because R is a PID), and the map $K_0(R) \to K_0(F) \cong \mathbb{Z}$ sends $[R]$ to $[F]$ and is therefore an isomorphism.

Although we have not calculated $K_1(R)$, the commutative diagram

$$\begin{array}{ccc} K_1(R) & \longrightarrow & K_1(F) \\ {\scriptstyle \det}\downarrow & & {\scriptstyle \det}\downarrow{\scriptstyle \cong} \\ R^* & \rightarrowtail & F^* \end{array}$$

shows that the image of $K_1(R)$ in F^* is just R^*. Our localization sequence distills into a single isomorphism

$$F^*/R^* \xrightarrow{\partial} K_0(R,S).$$

The group F^*/R^* is readily checked to be $\mathbb{Z}$, where the isomorphism $\mathbb{Z} \cong F^*/R^*$ sends n to $[\pi^n]$. On the other hand, we also know by Theorem 5.15 that $K_0(R,S) \cong G(M \mid S^{-1}M = 0)$. A finitely-generated module M satisfies $S^{-1}M = 0$ if and only if M is killed by a power of x, or equivalently if M has finite length over R. The map $\ell\colon G(M \mid S^{-1}M = 0) \to \mathbb{Z}$ is easily checked to be an isomorphism.

Finally, let us analyze the map ∂. Given an element $a \in F^*$, we write $a = r/\pi^n$ for some $r \in R^*$ and $n \geq 0$. The description of ∂ given in the proof of Proposition 7.15 shows that

$$\partial(a) = [R \xrightarrow{r} R] - [R \xrightarrow{\pi^n} R] = [R \xrightarrow{r} R] - n[R \xrightarrow{\pi} R].$$

The isomorphism $K_0(R,S) \to G(M \mid S^{-1}M = 0)$ sends a complex $P_\bullet$ to the alternating sum of its homology modules, so under this isomorphism we would write

$$\partial(a) = [R/rR] - n[R/\pi R].$$

Note that $\ell(R/\pi R) = 1$. We can write $r = u\pi^k$ for some unit $u \in R$ and $n \geq 0$, in which case $R/rR \cong R/\pi^k R$ and so $\ell(R/rR) = k$. It follows that the composite

$$F^* \xrightarrow{\partial} K_0(R,S) \xrightarrow{\cong} G(M \mid S^{-1}M = 0) \xrightarrow{\cong} \mathbb{Z}$$

is just the usual π-adic valuation on F^*.

The following example generalizes the previous one, but is a bit more interesting.

Example 7.19. Let D be a Dedekind domain—a regular ring of dimension one. In such a ring all nonzero primes are maximal ideals. Let $S = D - \{0\}$ and let $F = S^{-1}D$ be the quotient field. Our localization sequence looks like

$$K_1(D) \to F^* \to K_0(D,S) \to K_0(D) \to \mathbb{Z}.$$

Just as in the previous example, the image of $K_1(D)$ in F^* is just D^*. The map $K_0(D) \to \mathbb{Z}$ is just the usual rank map, so its kernel is $\widetilde{K}_0(D)$. So we get a short exact sequence

$$0 \to F^*/D^* \xrightarrow{\partial} K_0(D,S) \to \widetilde{K}_0(D) \to 0.$$

We know $K_0(D,S) \cong G(M \,|\, S^{-1}M = 0)$. The condition $S^{-1}M = 0$ just says that M is a torsion module. Consider the evident map

$$j\colon \bigoplus_{P\neq 0} G(D/P) \to G(M \,|\, S^{-1}M = 0)$$

where the direct sum is over all nonzero prime ideals and where the map just forgets that a module is defined over D/P and instead regards it as a D-module. This map is clearly surjective: a torsion D-module M will have a prime filtration in which the primes appearing are all maximal, and $[M]$ will be the sum of the corresponding $[D/P]$'s by the usual argument (see Theorem 2.15 and its proof).

Note that each D/P is a field, and so $G(D/P) \cong \mathbb{Z}$. If M is a torsion D-module then M_P is a torsion D_P-module. Since D_P is a discrete valuation ring, this means that M_P has finite length. Define

$$\chi\colon G(M \,|\, S^{-1}M = 0) \to \bigoplus_{P\neq 0} G_0(R/P)$$

by sending $[M]$ to the tuple of integers $\ell_{D_P}(M_P)$, as P runs over all maximal ideals of D (the only ones that give nonzero lengths are the ones containing $\operatorname{Ann} M$, and there are only finitely-many of these since they are precisely the minimal primes of $\operatorname{Ann} M$). It is easy to check that $\chi \circ j = \mathrm{id}$; since j was already known to be surjective this means they are inverse isomorphisms. So we can rewrite our short exact sequence as

$$0 \to F^*/D^* \xrightarrow{\partial} \bigoplus_{P\neq 0} \mathbb{Z} \to \widetilde{K}_0(D) \to 0.$$

It will be convenient to write e_P for the basis element of the free abelian group in the middle corresponding to the maximal ideal P. Note that these basis elements correspond to the closed points of $\operatorname{Spec} D$, and so we are looking at a group of 0-cycles.

It remains to analyze the map ∂. By Proposition 7.15, if $r \in D - \{0\}$ then $\partial(r) = [D \xrightarrow{r} D] \in K_0(D,S)$. Under the isomorphisms described above this corresponds to the tuple of integers $\ell_{D_P}(D_P/rD_P)$. This is usually called the **divisor class** of r, and written

$$\operatorname{div}(r) = \sum \ell_{D_P}(D_P/rD_P)e_P.$$

It should be thought of as listing all the zeros of the "function" r, together with their orders of vanishing (see below for an example). For a general element $x \in F^*$ we would just write $x = r/s$ for $r, s \in D - \{0\}$, and then $\partial(x) = \operatorname{div}(r) - \operatorname{div}(s)$; this gives the zeros and poles of x, with multiplicities.

The quotient of $\bigoplus_{P\neq 0} \mathbb{Z}$ by the classes $\operatorname{div}(x)$ is called the **divisor class group** of D; it is isomorphic to the ideal class group from algebraic number theory. Our short exact sequence shows that $\widetilde{K}_0(D)$ is also isomorphic to this group.

To demonstrate the geometric intuition behind $\operatorname{div}(r)$, consider the case $D = F[t]$ where F is algebraically closed. If $r = p(t)$ then the maximal ideals containing r are the ones $(t - a_i)$ where a_i is a root of $p(t)$. If we write $r = \prod (t - a_j)^{m_j}$ and we localize at $P = (t - a_i)$, then r becomes a unit multiple of $(t - a_i)^{m_i}$ and the number $\ell_{D_P}(D_P/rD_P)$ is precisely m_i. So

$$\operatorname{div}(r) = \sum_i m_i \cdot e_{(t-a_i)},$$

as expected. Note that the divisor class group is not very interesting in this case: clearly div is surjective, and so the group is zero. We already knew this for another reason, because $\widetilde{K}_0(D) = 0$ whenever D is a PID.

Remark 7.20. The localization sequence of Theorem 7.17 can be extended further to the left, by definining K-groups $K_n(R)$ and $K_n(R, S)$ for all $n \geq 1$. This is the subject of higher algebraic K-theory.

Part 2. K-theory in topology

Finite-dimensional linear algebra is a subject that mathematicians understand very well. There aren't that many isomorphism types of objects (one for each dimension), and we have a pretty good understanding of the maps between them. Our next goal in these notes is to explore the idea of doing linear algebra *locally* over a fixed base space X. To be slightly more precise, our objects of interest will be maps of spaces $E \to X$ where the fibers carry the structure of vector spaces; a map from $E \to X$ to $E' \to X$ is a continuous map $F\colon E \to E'$, commuting with the maps down to X, such that F is a linear transformation on each fiber. It turns out that much of linear algebra carries over easily to this enhanced setting. But there are more isomorphism types of objects here, because the topology of X allows for some twisting in the vector space structure of the fibers. The surprise is that studying these 'twisted vector spaces' over a base space X quickly leads to interesting homotopy invariants of X! Topological K-theory is a cohomology theory for topological spaces that arises out of this study of fibrewise linear algebra.

8. Vector bundles

A (real) vector space is a set V together with operations $+\colon V \times V \to V$ and $\cdot\colon \mathbb{R} \times V \to V$ satisfying a familiar (but long) list of properties. If X is a topological space, a **family of vector spaces** over X will be a continuous map $p\colon E \to X$ together with extra data making each fiber $p^{-1}(x)$ into a vector space, with the operations varying in a continuous manner. The easiest way to say this is as follows:

Definition 8.1. *A **family of (real) vector spaces** is a map $p\colon E \to X$ together with operations $+\colon E \times_X E \to E$ and $\cdot\colon \mathbb{R} \times E \to E$ making the two diagrams*

commute, and such that the operations make each fiber $p^{-1}(x)$ into a real vector space over X.

One could write down the above definition completely category-theoretically, in terms of maps and commutative diagrams. Essentially one is defining a "vector space object" in the category of spaces over X.

The space X is called the **base** of the family. If $x \in X$ we will usually write E_x for the fiber $p^{-1}(x)$ regarded with its vector space structure. The dimension of E_x is called the **rank** of the family at x, and denoted $\text{rank}_x(E)$. The rank of E is defined to be

$$\text{rank}(E) = \sup\{\text{rank}_x(E) \,|\, x \in X\},$$

where we include the possibility $\text{rank}(E) = \infty$ (although we will never need this).

We leave it to the reader to define a map between families of vector spaces, an isomorphism between families of vector spaces, a "subspace" of a family of vector spaces, and so on. All of the definitions from linear algebra can easily be adapted.

Example 8.2.

(a) The simplest example is $E = X \times \mathbb{R}^n$, with the projection map $X \times \mathbb{R}^n \to X$. This is called the **trivial family of rank** n, and it is often denoted simply by $\underline{n}$ (with the space X understood).

(b) Let $E = \{(x, v) \mid x \in \mathbb{R}^2, v \in \mathbb{R}.\langle x\rangle\}$, and let $p\colon E \to \mathbb{R}^2$ be projection onto the first coordinate. Define $(x, v) + (x, v') = (x, v + v')$ and $r.(x, v) = (x, rv)$. This makes $E \to \mathbb{R}^2$ into a family of vector spaces. Note that the fiber $p^{-1}(x)$ is one-dimensional for $x \neq 0$, but 0-dimensional when $x = 0$.

(c) Let $X = \mathbb{R}$. Let e_1, e_2 be the standard basis for $\mathbb{R}^2$. Let $E \subseteq X \times \mathbb{R}^2$ be the union of $\{(x, re_1) \mid x \in \mathbb{Q}, r \in \mathbb{R}\}$ and $\{(x, re_2) \mid x \in X \backslash \mathbb{Q}, r \in \mathbb{R}\}$. Recall from (a) that $X \times \mathbb{R}^2 \to X$ is a family of vector spaces, and note that E becomes a sub-family of vector spaces under the same operations.

The family of vector spaces from Example 8.2(c) perhaps makes it clear that this notion is too wild to be of much use: there are too many 'crazy' families of vector spaces like this one. One fixes this by adding a condition that forces the fibers to vary continuously, in a certain sense. This is done as follows:

Definition 8.3. *A* ***vector bundle*** *is a family of vector spaces $p\colon E \to X$ such that for each $x \in X$ there is a neighborhood $x \in U \subseteq X$, an $n \in \mathbb{Z}_{\geq 0}$, and an isomorphism of families of vector spaces*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \cong\ } & U \times \mathbb{R}^n \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Usually one simply says that a vector bundle is a family of vector spaces that is locally trivial. The isomorphism in the above diagram is called a "local trivialization".

Remark 8.4. Note that the n appearing in Definition 8.3 depends on the point x. It is called the **rank** of the vector bundle at x, and denoted $\mathrm{rank}_x(E)$. It is easy to prove that the rank is constant on the connected components of X. Vector bundles of rank 1 are often called **line bundles**, and bundles of rank 2 are called **plane bundles**.

Notation 8.5. If $p\colon E \to X$ is a family of vector spaces and $A \hookrightarrow X$ is a subspace, then $p^{-1}(A) \to A$ is also a family of vector spaces. We will usually write this restriction as $E|_A$. Note that if E is a vector bundle then so is $E|_A$, by a simple argument. The construction $E|_A$ is a special case of a *pullback bundle*, which we will discuss in Section 8.9.

Of the families of vector spaces we considered in Example 8.2, only the trivial family from (a) is a vector bundle. Before discussing more interesting examples, it will be useful to have a mechanism for deciding when a family of vector spaces is trivial. If $p\colon E \to X$ is a family of vector spaces, a **section** of p is a map $s\colon X \to E$ such that $ps = \mathrm{id}$. The set of sections is denoted $\Gamma(E)$, and this becomes a vector space using pointwise addition and multiplication in the fibers of E. A collection of sections $s_1, \ldots, s_r$ is **linearly independent** if the vectors $s_1(x), s_2(x), \ldots, s_r(x)$ are linearly independent in E_x for every $x \in X$.

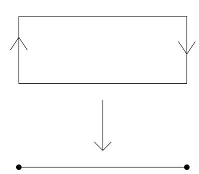
Proposition 8.6. *Let $E \to X$ be a family of vector spaces of constant rank n. Then the family is trivial if and only if there is a linearly independent collection of sections $s_1, s_2, \dots, s_n$.*

Proof. We mostly leave this to the reader. The map $X \times \mathbb{R}^n \to E$ given by $(x, t_1, \dots, t_n) \mapsto t_1 s_1(x) + \cdots + t_n s_n(x)$ gives the desired trivialization. □

Example 8.7.

(a) Let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector space isomorphism. Let $E' = [0,1] \times \mathbb{R}^n$ and let E be the quotient of E' by the relation $(0, v) \sim (1, \phi(v))$. Identifying S^1 with the quotient of $[0,1]$ by $0 \sim 1$, we obtain a map $E \to S^1$ that is clearly a family of vector spaces. We claim this is a vector bundle. If $x \in (0,1)$ then it is evident that E is locally trivial at x, so the only point of concern is $x = 0 = 1 \in S^1$. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$, and let $s_i\colon [0, \frac{1}{4}] \to E'$ be the constant section whose value is e_i. Likewise, let $s_i'\colon (\frac{3}{4}, 1] \to E'$ be the constant section whose value is $\phi(e_i)$. Projecting into E we obtain $s_i(0) = s_i'(1)$, and so the sections s_i and s_i' patch together to give a section $S_i\colon U \to E$, where $U = [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$. The sections $S_1, \dots, S_n$ are independent and therefore give a local trivialization of E over U.

When $n = 1$ and $\phi(x) = -x$ the resulting bundle is the *Möbius bundle* M, depicted below:



We further discuss the case of general n and ϕ in Example 11.3 below.

(b) Let $X = \mathbb{R}P^n$, and let $L \subseteq X \times \mathbb{R}^{n+1}$ be the set

$$L = \{(l, v) \,|\, l \in \mathbb{R}P^n, x \in l\}.$$

Then L is a subfamily of the trivial family, and we claim that it is a line bundle over X. To see this, for any $l \in X$ we must produce a local trivialization. By symmetry it suffices to do this when $l = \langle e_1 \rangle$. Let $U \subseteq \mathbb{R}P^n$ be the set of lines whose orthogonal projection to $\langle e_1 \rangle$ is nonzero. Such a line contains a unique vector of the form $e_1 + u$ where $e_1 \cdot u = 0$. Define $s\colon U \to L$ by sending l to $(l, e_1 + u)$ where $e_1 + u$ is the unique point on l described above. This section is clearly nonzero everywhere, so it gives a trivialization of $L|_U$. Thus, we have proven that L is locally trivial and hence a vector bundle.

The bundle L is called the **tautological line bundle** over $\mathbb{R}P^n$. Do not confuse this with the *canonical* line bundle over $\mathbb{R}P^n$ that we will define shortly (they are duals of each other). Note that when $n = 1$ the bundle L is isomorphic to the Möbius bundle on S^1.

(c) One may generalize the previous example as follows. Let V be a vector space and fix an integer $k > 0$. Consider the Grassmannian $\mathrm{Gr}_k(V)$ of k-planes in V. Let

$$\eta = \{(W, x) \,|\, W \in \mathrm{Gr}_k(V), x \in W\}.$$

Projection to the first coordinate $\pi\colon \eta \to \mathrm{Gr}_k(V)$ makes η into a rank k vector bundle, called the **tautological bundle** over $\mathrm{Gr}_k(V)$. To see that it is indeed a bundle, let $W \in \mathrm{Gr}_k(V)$ be an arbitrary k-plane. By choosing an appropriate basis for V we can just assume $W = \langle e_1, \ldots, e_k\rangle$. Equip V with the standard dot product with respect to the e-basis, and let $U \subseteq \mathrm{Gr}_k(V)$ be the collection of all k-planes whose orthogonal projection onto W is surjective (equivalently, an isomorphism). One readily checks that this is an open set of W. For each $J \in U$ let $s_1(J), \ldots, s_k(J)$ be the unique vectors in J that orthogonally project onto $e_1, \ldots, e_k$. One checks that these are continuous sections of $\eta|_U$, and of course they are clearly independent and hence give a local trivialization.

(d) Let M be a smooth manifold, and let $TM \to M$ be its tangent bundle. So the fiber over each $x \in M$ is the tangent space at x. Let $x \in M$ and let U be a local coordinate patch about x. Let $x_1, \ldots, x_n$ be local coordinate in U, and let $\partial_1, \ldots, \partial_n$ be the associated vector fields (giving the tangent vectors to the coordinate curves in this system). Then $\partial_1, \ldots, \partial_n$ are independent sections of TM, and hence give a local trivialization.

Definition 8.8. *A* ***map of vector bundles*** *is just a map of the underlying families of vector spaces.*

Note that if $f\colon E \to F$ is a map of vector bundles over X then neither $\ker f$ nor $\operatorname{coker} f$ will necessarily be a vector bundle. For an example, let $X = [-1, 1]$ and let $E = \underline{1}$. Define $f\colon E \to E$ by letting it be multiplication-by-t on the fiber over $t \in X$.

8.9. **Pullback of vector bundles.** Suppose that $p\colon E \to X$ is a family of vector spaces and $f\colon Y \to X$ is a map. One may form the pullback $Y \times_X E$, more commonly denoted f^*E in bundle theory:

$$\begin{array}{ccccc} f^*E & = & Y \times_X E & \longrightarrow & E \\ & & \downarrow & & \downarrow \\ & & Y & \longrightarrow & X. \end{array}$$

A point in f^*E is a pair (y, e) such that $f(y) = p(e)$, and one defines addition and scalar multiplication on f^*E by $(y, e) + (y, e') = (y, e + e')$ and $r \cdot (y, e) = (y, re)$. This makes $f^* \to Y$ into a family of vector spaces, called the **pullback family**. If $y \in Y$ then there is an evident map of vector spaces $(f^*E)_y \to E_{f(y)}$ which one readily checks is an isomorphism.

It is easy to see that if E is a vector *bundle* then so is f^*E; in this case f^*E is called the **pullback bundle**.

Example 8.10.

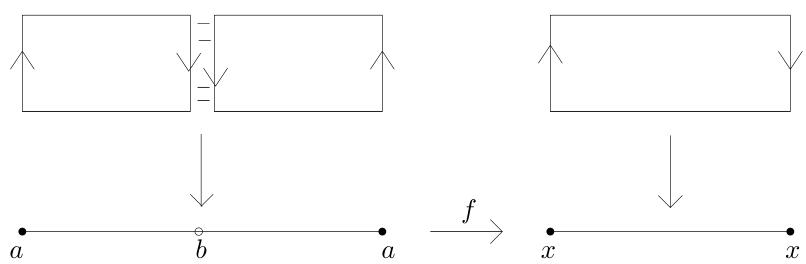
(a) If f is an inclusion $Y \hookrightarrow X$ then f^*E is just the restriction $E|_Y$ that we have discussed before.

(b) Pullback bundles can be slightly non-intuitive. Let $M \to S^1$ be the Möbius bundle, and let $f\colon S^1 \to S^1$ be the map $z \mapsto z^2$. We claim that $f^*M \cong \underline{1}$. This is easiest to see if one uses the following model for M:

$$M = \Big\{ \big(e^{i\theta}, re^{i\frac{\theta}{2}}\big) \,\Big|\, \theta \in [0, 2\pi], r \in \mathbb{R} \Big\}.$$

The bundle map is projection onto the first coordinate $\pi\colon M \to S^1$. Then $f^*M = \{(e^{i\theta}, re^{i\theta}) \mid \theta \in [0, 2\pi], r \in \mathbb{R}\}$. This is clearly isomorphic to $S^1 \times \mathbb{R}$, via the map $(e^{i\theta}, r) \mapsto (e^{i\theta}, re^{i\theta})$.

We can also demonstrate the isomorphism $f^*M \cong \underline{1}$ by the following picture:



Here f is the evident map that wraps the circle around itself twice, so that $f^{-1}(x) = \{a, b\}$. We see that f^*M can be thought of as two copies of M that are cut open and then sewn together as shown, thereby producing a cylinder.

Remark 8.11. Given composable maps $Z \xrightarrow{g} Y \xrightarrow{f} X$, there is an evident natural isomorphism $(fg)^*E \cong g^*(f^*E)$. For each topological space X and each integer $k \geq 0$, let $\text{Vect}_k(X)$ denote the set of isomorphism classes of vector bundles of rank k on X. The pullback construction then makes $\text{Vect}_k(-)$ into a contravariant functor from $\mathcal{T}op$ into $\mathcal{S}et$.

8.12. **Constructing vector bundles out of old ones.** Let $p\colon E \to X$ and $p'\colon E' \to X$ be two vector bundles, say of constant ranks k and l, respectively. We may form a new bundle $E \oplus E'$, whose underlying topological space is just the pullback $E \times_X E'$. So a point in $E \oplus F$ is a pair (e, e') where $p(e) = p'(e')$. The rules for vector addition and scalar multiplication are the evident ones. Note that the fiber of $E \oplus E'$ over a point x is simply $E_x \oplus E'_x$.

More generally, any canonical construction one can apply to vector spaces may be extended to apply to vector bundles. So one can talk about the bundles $E \otimes E'$, the dual bundle E^*, the hom-bundle $\underline{\text{Hom}}(E, E')$, the exterior product bundle $\bigwedge^i E$, and so on. We will only carefully define $E \otimes E'$, and leave the other definitions to the reader.

Set-theoretically define

$$E \otimes E' = \{(x, v) \mid x \in X, v \in E_x \otimes E'_x\}.$$

This is clear enough, and it is clear how to define addition and scalar multiplication in the fibers. The only thing that takes thought is how to define the topology on $E \otimes E'$, and to check that the operations are continuous. But it is enough to define the topology *locally*, and to check continuity locally. If $x \in X$, let U be a neighborhood of x over with both E and E' are trivializable. Choose isomorphisms $\phi\colon U \times \mathbb{R}^k \to E|_U$ and $\phi'\colon U \times \mathbb{R}^l \to E'|_U$. Then one gets a bijection of sets $U \times (\mathbb{R}^k \otimes \mathbb{R}^l) \to (E \otimes E')|_U$ which is a linear isomorphism on each fiber: one sends $(u, v \otimes w)$ to $(u, \phi(u, v) \otimes \phi'(u, w))$ and then extends linearly. Finally, one uses this bijection to transplant the topology from $U \times (\mathbb{R}^k \otimes \mathbb{R}^l)$ to $(E \otimes E')|_U$. We leave the reader to fill in all the details here.

Remark 8.13 (External sums and products)**.** Let $E \to X$ and $F \to Y$ be two vector bundles, but this time over possibly different base spaces. One may construct an external direct sum $E\hat{\oplus}F \to X \times Y$ whose fiber over (x,y) is $E_x \oplus F_y$. The underlying topological space of $E\hat{\oplus}F$ is just $E\times F$, and it has the evident operations. Note that $E\hat{\oplus}F$ can also be constructed as $\pi_1^*(E)\oplus\pi_2^*(F)$, where π_1 and π_2 are the projections from $X \times Y$ onto the two factors.

In the case $X = Y$ we can construct the (internal) direct sum from the external one: namely $E \oplus F = \Delta^*(E\hat{\oplus}F)$ where $\Delta\colon X \to X \times X$ is the diagonal map. Thus, the internal and external direct sums determine each other.

One can tell a similar story about external tensor products, or external hom-bundles.

8.14. **Constructing vector bundles by patching.** Let X be a space and let A and B be subspaces such that $A \cup B = X$. Recall that if $f_A\colon A \to Y$ and $f_B\colon B \to Y$ are continuous maps that agree on $A \cap B$ then we may patch these together to get a continuous map $f\colon X \to Y$ provided that either (i) A and B are both closed, or (ii) A and B are both open. This is a basic fact about topological spaces. The analogous facts for vector bundles are very similar in the case of an open cover, but more subtle for closed covers.

Proposition 8.15. *Let $E \to X$ be a family of vector spaces.*

(a) If $\{U_\alpha\}$ is an open cover of X and each $E|_{U_\alpha}$ is a vector bundle, then E is a vector bundle.

(b) Suppose $\{A, B\}$ is a cover of X by closed subspaces, and that for every $x \in A\cap B$ and every open neighborhood $x \in U \subseteq X$ there exists a neighborhood $x \in V \subseteq U$ such that $V \cap A \cap B \hookrightarrow V \cap B$ has a retraction. Then if $E|_A$ and $E|_B$ are both vector bundles, so is E.

Proof. Part (a) is trivial, so we focus on (b). Let $x \in X$, with the goal of producing a local trivialization around x. There are three cases: $x \in X - A$, $x \in X - B$, and $x \in A \cap B$. If $x \in X - B$ then $x \in A$, and since $E|_A$ is a vector bundle there exists an open subset U of X such that $E|_A$ is trivializable over $U\cap A$. If $V = U\cap(X-B)$ then V is an open neighborhood of x in X and $V \subseteq U \cap A$, so E is trivializable on V. A similar argument works if $x \in X - A$.

Finally, we analyze the case $x \in A \cap B$. The fact that $E|_A$ is a vector bundle implies that there exists an open set $x \in U \subseteq X$ such that E is trivializable over $U \cap A$. So there exist independent sections $s_1, \ldots, s_n$ defined on $U \cap A$. By our assumption there exists a neighborhood $x \in U' \subseteq U$ such that $U' \cap A \cap B \hookrightarrow U' \cap B$ has a retraction. Pre-composing the maps $s_i|_{U'\cap A\cap B}$ with this retraction, we obtain maps $s_1', \ldots, s_n'\colon U' \cap B \to E$ that are everywhere linearly independent.

Now patch the section $s_i|_{U'\cap A}$ together with $s_i'|_{U'\cap B}$ (these agree on the intersection by construction) to form a section $t_i\colon U' \to E$. These sections are everywhere independent, and so give a local trivialization of E on U'. □

Remark 8.16. A good example of Proposition 8.15(b) is the covering of a sphere S^n by its upper and lower hemispheres.

Corollary 8.17 (Patching vector bundles)**.** *Let $\{A, B\}$ be a cover of the space X. Suppose given vector bundles $E_A \to A$ and $E_B \to B$, together with a vector bundle isomorphism $\phi\colon E_A|_{A\cap B} \to E_B|_{A\cap B}$. Then there exists a vector bundle $E \to X$*

such that $E|_A$ is isomorphic to E_A and $E|_B$ is isomorphic to E_B provided that one of the following conditions holds:

(i) A and B are both open, or

(ii) A and B is a closed cover satisfying the hypotheses in part (b) of Proposition 8.15.

Proof. Define E to be the pushout of the following diagram:

$$\begin{array}{ccccc} E_A|_{A\cap B} & \xrightarrow{\phi} & E_B|_{A\cap B} & \rightarrowtail & E_B \\ \downarrow & & & & \vdots \\ E_A & \cdots & \cdots & \cdots\!\!> & E. \end{array}$$

The maps $E_A \to X$ and $E_B \to X$ yield a map $E \to X$, and one readily checks that this inherits the structure of a family of vector spaces. It is also evident that $E|_A \cong E_A$ and $E|_B \cong E_B$. It only remains to verify that E is a vector bundle, and this is a direct application of Proposition 8.15. □

Corollary 8.17 admits a generalization to arbitrary open coverings. Suppose $\{U_\alpha\}$ is an open cover of X, and assume given a collection of vector bundles $E_\alpha \to U_\alpha$. For each α and β further assume given an isomorphism

$$\phi_{\beta,\alpha} \colon E_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\cong} E_\beta|_{U_\alpha\cap U_\beta}.$$

Let E be the quotient of $\amalg_\alpha E_\alpha$ by the equivalence relation generated by saying $(\alpha, v_\alpha) \sim (\beta, \phi_{\beta,\alpha}(v_\alpha))$ for every α, β, and $v_\alpha \in E_\alpha|_{U_\alpha\cap U_\beta}$. Here we are writing (α, v_α) for the element v_α in $\amalg_\gamma E_\gamma$ that lies in the summand indexed by α.

It is easy to see that in this generality E is a family of vector spaces. It is not necessarily the case, however, that $E|_{U_\alpha} \cong E_\alpha$. If this were true for all α then of course E would be a vector bundle and we would be done. Here is the trouble, though. Suppose $\alpha_0, \alpha_1, \ldots, \alpha_n$ are a sequence of indices such that $\alpha_0 = \alpha_n = \alpha$. If $v \in E_\alpha$ then we identify v with $\phi_{\alpha_1,\alpha_0}(v)$, which is in turn identified with $\phi_{\alpha_2,\alpha_1}(\phi_{\alpha_1,\alpha_0}(v))$, and so forth—so that v ends up being identified with

$$\Big(\phi_{\alpha_n,\alpha_{n-1}} \circ \phi_{\alpha_{n-1},\alpha_{n-2}} \circ \cdots \circ \phi_{\alpha_1,\alpha_0}\Big)(v). \tag{8.18}$$

Note that, like v, this expression is an element of E_α. So identifications are possibly being made within individual summands of $\amalg_\alpha E_\alpha$, rather than just between different summands. The fibers of $E|_{U_\alpha}$ are quotients of those in E_α, but they might not be identical. To prohibit this from happening we impose some extra conditions: for any indices α, β, γ we require that

(i) $\phi_{\alpha,\alpha} = \mathrm{id}$,

(ii) The two isomorphisms $\phi_{\gamma,\alpha}$ and $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}$ agree on their common domain of definition, which is $E_\alpha|_{U_\alpha\cap U_\beta\cap U_\gamma}$.

We leave it to the reader to check that these conditions force any expression as in (8.18), with $\alpha_0 = \alpha_n$, to just be equal to v (in particular, note that they force $\phi_{\alpha,\beta} = \phi_{\beta,\alpha}^{-1}$). So the fibers of E coincide with the fibers of the E_α's, we get isomorphisms $E|_{U_\alpha} \cong E_\alpha$, and hence E is a vector bundle.

Condition (ii) above is usually called the **cocycle condition**. To see why, consider the case where all of the E_α's are trivial bundles of rank n. Then the data in the $\phi_{\alpha,\beta}$ maps is really just the data of a map $g_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. These

$g_{\alpha,\beta}$ maps are called **transition functions**. Condition (ii) is the requirement that the transition functions assemble to give a Čech 1-cocycle with values in the group $GL_n(\mathbb{R})$. Condition (i) is just a normalization condition, so that we are dealing with 'normalized' Čech 1-cocycles. Elements of the Čech cohomology group $\check{H}^1(U_\bullet; GL_n(\mathbb{R}))$ can be seen to be in bijective correspondence with isomorphism classes of vector bundles on X that are trivializable over the U_α's; if we take the direct limit over all open coverings then we obtain a bijection between isomorphism classes of vector bundles on X and elements of the Čech cohomology group $\check{H}^1(X; GL_n(\mathbb{R}))$. But we are getting ahead of ourselves here; see Section 11.6 for more discussion.

8.19. **Dual bundles.** Let $E \to X$ be a vector bundle of rank n. Using the method of Section 8.12 we can define the dual bundle E^*, which set-theoretically is $\{(x, v) \mid x \in X, v \in E_x^*\}$. One can examine this construction in terms of patching trivial bundles. Choose an open cover $\{U_\alpha\}$ of X with respect to which E is trivializable; a choice of trivialization over each U_α then yields a collection of gluing maps $\phi_{\alpha,\beta}$. We think of E as being built from the trivial bundles $E_\alpha = U_\alpha \times \mathbb{R}^n$ via these gluing maps. Then the dual bundle E^* is built from the trivial bundles $U_\alpha \times (\mathbb{R}^n)^*$ via the duals of the gluing maps: that is, $(\phi^{E^*})_{\beta,\alpha} = \big(\phi^E_{\alpha,\beta}\big)^*$.

We will see in a moment (Corollary 8.23) that for real vector bundles over paracompact Hausdorff spaces one always has $E \cong E^*$, although the isomorphism is not canonical. This is not true for complex or quaternionic bundles, however.

Let $L \to \mathbb{C}P^n$ be the tautological complex line bundle over $\mathbb{C}P^n$. Its (complex) dual L^* is called the **canonical line bundle** over $\mathbb{C}P^n$. Whereas from a topological standpoint neither L nor L^* holds a preferential position over the other, in algebraic geometry there is an important difference between the two. The difference comes from the fact that L^* has certain "naturally defined" sections, whereas L does not. For a point $z = [z_0 : \cdots : z_n] \in \mathbb{C}P^n$, L_z is the complex line in $\mathbb{C}^{n+1}$ spanned by $(z_0, \ldots, z_n)$. Given only $z \in \mathbb{C}P^n$ there is no evident way of writing down a point on L_z, without making some kind of arbitrary choice; said differently, the bundle L does not have any easily-described sections. In contrast, it is much easier to write down a *functional* on L_z. For example, let ϕ_i be the unique functional on L_z that sends the point $(z_0, \ldots, z_n)$ to z_i. Notice that this description depends only on $z \in \mathbb{C}P^n$, not the point $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ that represents it; that is, the functional sending $(\lambda z_0, \ldots, \lambda z_n)$ to λz_i is the same as ϕ_i. In this way we obtain an entire $\mathbb{C}^{n+1}$'s worth of sections for L^*, by taking linear combinations of the ϕ_i's.

To be clear, it is important to realize that L has plenty of sections—it is just that one cannot describe them by simple formulas. The slogan to remember is that the bundle L^* has *algebraic* sections, whereas L does not. In algebraic geometry the bundle L^* is usually denoted $\mathcal{O}(1)$, whereas L is denoted $\mathcal{O}(-1)$. More generally, $\mathcal{O}(n)$ denotes $(L^*)^{\otimes n}$ when $n \geq 0$ (so that $\mathcal{O}(0)$ is the trivial line bundle), and denotes $L^{\otimes(-n)}$ when $n < 0$.

8.20. **Inner products on bundles.** It is nearly possible to develop everything we need from bundle theory without using inner products, and in the rest of the text we do try to minimize our use of them. But for some results the use of inner products does provide significant simplifications of proofs, and so it is good to know about them.

Definition 8.21. *Let $E \to X$ be a real vector bundle. An **inner product** on E is a map of vector bundles $E \otimes E \to \underline{1}$ that induces a positive-definite, symmetric, bilinear form on each fiber E_x. A vector bundle with an inner product is usually called an **orthogonal** vector bundle.*

There is a similar notion for Hermitian inner products on complex vector bundles, but here we cannot phrase things in terms of the tensor product because of conjugate-linearity in one variable. So perhaps the simplest thing is just to say that if $E \to X$ is a complex bundle then a Hermitian inner product is a map $E \times_X E \to \underline{1}$ (over X) which induces a Hermitian inner product on each fiber.

Proposition 8.22. *Assume that X is paracompact and Hausdorff. Then any real bundle on X admits an inner product, and any complex bundle on X admits a Hermitian inner product.*

Proof. The idea is to produce the necessary inner products locally, and then use a partition of unity to average the results into a global inner product.

Let $E \to X$ be a real vector bundle, and let $\{U_\alpha\}$ be an open cover over which the bundle is trivial. Choose bundle isomorphisms $f_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$, for each α. Equip $\mathbb{R}^n$ with the standard Euclidean inner product, and let $\langle -, - \rangle_\alpha$ be the inner product on $E|_{U\alpha}$ obtained by transplanting the Euclidean product across the isomorphisms f_α.

Let $\{\phi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. For $x \in X$ and $v, w \in E_x$ define

$$\langle v, w \rangle = \sum_\alpha \phi_\alpha(x) \cdot \langle v, w \rangle_\alpha.$$

It is clear that this is continuous in v and w, bilinear, symmetric, and positive-definite—these follow from the corresponding properties of the forms $\langle -, - \rangle_\alpha$. So this completes the construction.

The proof for Hermitian inner products on a complex bundle is basically identical. □

Corollary 8.23. *Let $E \to X$ be a real vector bundle on a paracompact Hausdorff space X. Then E is isomorphic to its dual E^*.*

Proof. Start by equipping E with an inner product $E \otimes E \to \underline{1}$, and note that the fiberwise forms are nondegenerate (since they are positive-definite). The adjoint of the above bundle map is a map $E \to E^*$, and nondegeneracy of the fiberwise forms shows that this is a fiberwise isomorphism. □

Remark 8.24. Here is an illuminating exercise. Every complex vector space may be equipped with a nondegenerate, symmetric bilinear form. Check that the proof of Proposition 8.22 does *not* generalize to show that every complex vector bundle may be equipped with a symmetric bilinear form that is nondegenerate on the fibers—in particular, find the point where the proof breaks down. Note that if the proof did generalize, one could show just as in Corollary 8.23 that every complex bundle was isomorphic to its own dual. This is false, as we will see in Example 8.26 below. The complex version of Corollary 8.23 says that if $E \to X$ is a complex bundle over a paracompact space then E is isomorphic to the conjugate of E^* (the bundle obtained from E^* by changing the complex structure so that $z \in \mathbb{C}$ acts as $\bar{z}$).

Consider a trivial bundle $X \times \mathbb{R}^n \to X$ and equip it with the standard inner product. This bundle may be considered as trivial in two different ways: the vector bundle structure is trivial, and the inner product structure is also trivial. It is not clear *a priori* that the former property implies the latter, but in fact it does:

Proposition 8.25. *Let X be a space. Every inner product on $X \times \mathbb{R}^n$ is isomorphic to the 'constant' inner product provided by the standard Eucliden metric.*

Proof. Consider $\mathbb{R}^n$ with its standard basis $e_1, \dots, e_n$. Inner products on $\mathbb{R}^n$ are in bijective correspondence with symmetric, positive-definite matrices $A \in M_{n\times n}(\mathbb{R})$, by sending an inner product $\langle -, - \rangle$ to the matrix $a_{ij} = \langle e_i, e_j \rangle$. Let $M^{sym,+}$ denote the space of such matrices. To give an inner product on the trivial bundle $X \times \mathbb{R}^n$ is therefore equivalent to giving a map $X \to M^{sym,+}$.

Given an isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ we may transplant an inner product from the target onto the domain; this gives rise to an action of $GL_n(\mathbb{R})$ on the space of inner products. If $P \in GL_n(\mathbb{R})$ and $A \in M^{sym,+}$ then the action is $P.A = PAP^T$. The fact that every inner product on $\mathbb{R}^n$ has an orthonormal basis shows that $M^{sym,+}$ equals the orbit of the identity matrix I_n under this action. The stabilizer of the identity is of course the orthogonal group O_n, and so we obtain the homeomorphism $GL_n(\mathbb{R})/O_n \cong M^{sym,+}$.

Now consider the fibration sequence $O_n \hookrightarrow GL_n(\mathbb{R}) \to GL_n(\mathbb{R})/O_n$. The projection map sends a matrix P to $PI_nP^T = PP^T$. The inclusion $O_n \hookrightarrow GL_n(\mathbb{R})$ is a homotopy equivalence by Gram-Schmidt, and so $GL_n(\mathbb{R})/O_n$ is weakly contractible. Standard techniques show that this homogeneous space may be given the structure of a CW-complex. The lifting diagram

$$\begin{array}{ccc} & & GL_n(\mathbb{R}) \\ & \overset{r}{\nearrow} & \downarrow \\ GL_n(\mathbb{R})/O_n & \xrightarrow{\text{id}} & GL_n(\mathbb{R})/O_n \end{array}$$

therefore has a lift as indicated.

As we have discussed, our given inner product on $X \times \mathbb{R}^n$ is represented by a map $X \to GL_n(\mathbb{R})/O_n$. Compose with r to obtain $X \to GL_n(\mathbb{R})$. This map specifies a bundle isomorphism $X \times \mathbb{R}^n \to X \times \mathbb{R}^n$; if we equip the domain with our given inner product and the codomain with the standard inner product, this map preserves the inner products and therefore proves the proposition. □

Suppose that $E \to X$ is a rank n real vector bundle with an inner product. Choose a trivializing open cover $\{U_\alpha\}$, and for each α fix an inner-product-preserving trivialization $f_\alpha\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ where the codomain has the standard inner product (this is possible by Proposition 8.25). The transition functions $g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ therefore factor through O_n, as they must preserve the inner product. This process is usually referred to as *reduction of the structure group*.

We may use these ideas to give another proof of Corollary 8.23, one that is perhaps more down-to-earth. Let $E \to X$ be a real vector bundle on a compact space, and choose a trivializing cover $\{U_\alpha\}$ with respect to which there exist local trivializations where the transition functions are maps $g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to O_n$. So we obtain E by gluing together the spaces $E_\alpha = U_\alpha \times \mathbb{R}^n$ via the maps $g_{\alpha,\beta}$. But then we obtain E^* by gluing together the spaces $U_\alpha \times (\mathbb{R}^n)^*$ via the maps

$$h_{\beta,\alpha} = g^*_{\alpha,\beta}.$$

Recall that in terms of matrices the dual is represented by the transpose. Since each $g_{\alpha,\beta}(x)$ is in O_n we can write $h_{\beta,\alpha}(x) = g_{\alpha,\beta}(x)^{-1}$, or

$$h_{\alpha,\beta}(x) = h_{\beta,\alpha}(x)^{-1} = g_{\alpha,\beta}(x).$$

In other words, the transition functions for E and E^* are exactly the same, and that is why the bundles are isomorphic.

We close this section with the promised example of a complex bundle that is not isomorphic to its dual:

Example 8.26. Let D_+ and D_- denote the upper and lower hemisphere of S^2; let S^1 be the equator, which we identify with the unit complex numbers. Given a map $f\colon S^1 \to GL_n(\mathbb{C})$ we may construct a complex bundle on S^2 by taking two trivial bundles $\underline{n}_{D_+}$ and $\underline{n}_{D_-}$ and gluing them together using the map f: precisely, for $z \in S^1$ an element $v \in \left(\underline{n}_{D_+}\right)_z$ is glued to $f(z) \cdot v \in \left(\underline{n}_{D_-}\right)_z$. Here we are using Corollary 8.17(b). Let $E(f)$ denote the resulting bundle.

Observe that giving an isomorphism $E(f) \to E(g)$ is equivalent to giving two maps $A\colon D_+ \to GL_n(\mathbb{R})$ and $B\colon D_- \to GL_n(\mathbb{C})$ such that $g(z) \cdot A(z) = B(z) \cdot f(z)$ for all $z \in S^1$. Let us rewrite this as $A(z) = g(z)^{-1}B(z)f(z)$. Now, the map $B|_{S^1}\colon S^1 \to GL_n(\mathbb{C})$ is null-homotopic because it extends over D_-; so it is (unbased) homotopic to the constant map at 1. Therefore the map $z \mapsto g(z)^{-1}B(z)f(z)$ is homotopic to $g(z)^{-1}f(z)$. But $A|_{S^1}$ is also (unbased) homotopic to the constant map at 1, because it extends over D_+. So we have proven that if $E(f) \cong E(g)$ then $z \mapsto g(z)^{-1}f(z)$ is unbased homotopic to a constant.

Next, observe that the dual of $E(f)$ is $E(f')$, where $f'(z) = \left[f(z)^T\right]^{-1}$. So if $E(f)$ is isomorphic to its dual then the map $z \mapsto f(z)^T \cdot f(z)$ is null-homotopic.

Consider the case $f(z) = z$. Since $z \mapsto z^2$ is not null-homotopic, we see that $E(f)$ is not isomorphic to its dual. The reader may wish to check that $E(f)$ is the tautological line bundle $L \to \mathbb{C}P^1$.

9. Some results from fiberwise linear algebra

Recall that our basic goal is to learn to do linear algebra "over a base space". The fundamental objects in this setting are the vector bundles, and the maps are the bundle maps. This section contains a miscellany of results that are frequently useful. This material can be safely skipped the first time through and referred back to as needed.

Lemma 9.1. *Let X be any space, and let $f\colon \underline{n} \twoheadrightarrow \underline{k}$ be a surjective map of bundles. Then f has a splitting.*

Note that the result is not immediately obvious. Of course one can choose a splitting in each fiber, but what guarantees that these can be chosen in a continuous manner?

Proof. Let $W = \{A \in M_{k\times n} \mid \operatorname{rank} A = k\}$, which is the space of surjective maps $\mathbb{R}^n \to \mathbb{R}^k$ (our matrices act on the left). Let Z be the space

$$Z = \{(A, B) \mid A \in M_{k\times n}, B \in M_{n\times k}, AB = I\},$$

which is the space of surjective maps with a chosen splitting. We claim that the projection map $p_1 \colon Z \to W$ is a fiber bundle with fiber $\mathbb{R}^{k(n-k)}$, but defer the proof for just a moment. The fact that the fiber is contractible then shows that p_1 is weak homotopy equivalence.

Consider the diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow {\scriptstyle p_1}\, \sim \\ W & = & W. \end{array}$$

The Lie group $GL_n(\mathbb{R}) \times GL_k(\mathbb{R})$ acts on W via change-of-basis in the domain and range, and this action is transitive. So W is homeomorphic to a homogeneous space $(GL_n(\mathbb{R}) \times GL_k(\mathbb{R}))/H$ for some subgroup H. From this it is not hard to see that W can be given the structure of a CW-complex. The standard lifting theorems now show that there is a lifting $r \colon W \to Z$ in the above diagram.

Our surjective bundle map $f \colon \underline{n} \to \underline{k}$ is determined by a map $X \to W$. Composing with $W \to Z$, and then projecting to the second coordinate of Z, gives the desired splitting for f.

It remains to prove the claim about p_1 being a fiber bundle. Let $A \in W$. Since $\operatorname{rank}(A) = k$ there is a $k \times k$ minor of A that is nonzero; without loss of generality let us assume that it is minor made up of the first k columns of the matrix. Let $U \subseteq W$ be the subspace consisting of all matrices where this same minor is nonzero, which is an open neighborhood of A in W. Writing matrices in block form, U consists of matrices $[X|Y]$ where where $det(X) \neq 0$. Then $p_1^{-1}(U)$ consists of pairs

$$\alpha_{X,Y,J,K} = \left(\begin{bmatrix} X & Y \end{bmatrix}, \begin{bmatrix} J \\ K \end{bmatrix} \right)$$

having the property that $\det(X) \neq 0$ and $XJ + YK = I_k$. We obtain an isomorphism $U \times M_{n-k,k}(\mathbb{R}) \cong \pi^{-1}(U)$ by sending $([X|Y], K)$ to $\alpha_{X,Y,J,K}$ with $J = X^{-1}(I_k - YK)$. □

Note the significance of the map $W \to Z$ that is produced in the above proof. This assigns to every surjection $\mathbb{R}^n \to \mathbb{R}^k$ a splitting, and it does so in a continuous manner. Of course there is no claim that there is a nice formula for how to do this, and in fact there almost certainly is not—but the proof shows that there does exist *some* way of doing so.

The following proposition is the generalization to arbitrary bundles:

Proposition 9.2. *Let X be a paracompact space. Then any surjection of bundles $E \twoheadrightarrow F$ has a splitting.*

Proof. Briefly, we choose local splittings and then use a partition of unity to patch them together.

Choose an open cover $\{U_\alpha\}$ such that both E and F are trivializable over each U_α. Lemma 9.1 shows that there are splittings $\chi_\alpha \colon F_{U_\alpha} \to E_{U_\alpha}$. Now choose a partition of unity $\{\phi_\alpha\}$ subordinate to our open cover. Set $\chi = \sum_\alpha \phi_\alpha \chi_\alpha$. This sum makes sense because the partition of unity is locally finite, and one readily checks that it is a splitting for f. □

The next result gives a useful tool for recognizing vector bundles. The proof follows the same pattern of the previous two results.

Proposition 9.3. *Let X be any space, and let $f\colon E \to F$ be a map of vector bundles over X. If f has constant rank then* $\ker f$, $\operatorname{coker} f$, *and* $\operatorname{im} f$ *are vector bundles.*

Proof. We first prove the result for $\ker f$ and $\operatorname{coker} f$. Let $x \in X$, let $n = \operatorname{rank}_x(E)$, let $k = \operatorname{rank}_x(F)$, and let $r = \operatorname{rank}(f)$. It will suffice to produce a neighborhood U of x together with $n-r$ independent sections of $\ker f$ over U and $k-r$ independent sections of $\operatorname{coker} f$ over U. In particular, this makes it clear that we might as well assume that E and F are both trivial bundles; in this case f is specified by a map $X \to W_r$ where $W_r = \{A \in M_{k\times n} \mid \operatorname{rank}(A) = r\}$.

Let Z_r be the space

$$Z_r = \{(A, v_1, \ldots, v_{n-r}) \mid A \in M_{k\times n}, \operatorname{rank}(A) = r, \text{ and } v_1, \ldots, v_{n-r} \text{ span the kernel of } A\}.$$

One can check that the projection $Z_r \to W_r$ is a fiber bundle with fiber $GL_{n-r}(\mathbb{R})$, but this is stronger than what we actually need. We only need that the map is locally split: any point in W_r has a neighborhood over which there exists a section. Given a map $X \to W_r$, it will then follow that every point in x has a neighborhood over which there exists a lifting into Z_r, and this will give the $n-r$ independent local sections of $\ker f$.

So let A be a point in W_r. Since $\operatorname{rank}(A) = r$, some $r \times r$ minor of A is nonzero. Without loss of generality we might as well assume it is the upper left $r \times r$ minor. Since $\operatorname{rank}(A) = r$, then for $j > r$ the jth column of A is a linear combination of the first r columns in a unique way; said differently, there is a unique vector of the form

$$v_j = e_j - s_1e_1 - s_2e_2 - \cdots - s_re_r$$

that is in the kernel of A. Here the s_i's are certain rational expressions in the matrix entries of A that can be determined using Cramer's Rule. These formulas define sections on the neighborhood U of A consisting of all $k \times n$ matrices of rank r whose upper left $r \times r$ minor is nonzero. This finishes the proof of our claim.

We have established that $\ker f$ is a vector bundle. The proof for $\operatorname{coker} f$ is entirely similar. Finally, consider the projection $F \to \operatorname{coker} f$. This map has constant rank, and so by what has already been established its kernel is a vector bundle. Notice that this kernel is precisely $\operatorname{im} \alpha$. □

The result below is an easy variation on Proposition 9.2; it will be used often, and so it is useful to have it stated explicitly.

Corollary 9.4. *Let X be a paracompact space. Then any injection of bundles $E \hookrightarrow F$ has a splitting.*

Proof. Let Q be the quotient, which is a vector bundle by Proposition 9.3. By Proposition 9.2 the map $F \twoheadrightarrow Q$ has a splitting, which then induces an isomorphism $F \cong E \oplus Q$. The composition $F \xrightarrow{\cong} E \oplus Q \xrightarrow{\pi_1} E$ gives the required splitting of $E \hookrightarrow F$. □

The next result is of a somewhat different nature:

Proposition 9.5. *Suppose that X is compact and Hausdorff. Then every bundle is a subbundle of some trivial bundle.*

Proof. Let $\pi\colon E \to X$ be a vector bundle on X. Choose a finite cover $U_1, \dots, U_s$ over which E is trivializable, which exists because of compactness. For each i choose a trivialization $f_i\colon E|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{R}^n$. Write $F_i = \pi_2 f_i$.

Let $\{\phi_i\}$ be a partition of unity subordinate to the open cover. Define a map

$$\beta\colon E \longrightarrow X \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$$

(where there are s copies of $\mathbb{R}^n$) by the formula

$$\beta(v) = \Big(\pi v,\ \phi_1(\pi v)F_1(v),\ \dots,\ \phi_s(\pi v)F_s(v)\Big).$$

We have written πv instead of $\pi(v)$ here, to avoid being overwhelmed by parentheses. Note that if v is not in $E|_{U_i}$ then $F_i(v)$ is undefined, but in this case $\phi_i(\pi v)$ equals 0 and so the formula still makes sense. It is routine to check that this formula gives an embedding of bundles. □

Finally, we close this section with a few useful results related to ranks and exactness:

Lemma 9.6. *Let $\alpha\colon E \to F$ be a map of vector bundles over X. Then for any $n \in \mathbb{Z}_{\geq 0}$, the set $\mathcal{R}_n = \{x \in X \mid \operatorname{rank}(\alpha_x) \geq n\}$ is an open subset of X.*

Proof. Let $k = \operatorname{rank}(E)$ and $l = \operatorname{rank}(F)$. Let $x \in \mathcal{R}_n$. We can choose a neighborhood V of x over which both E and F are trivial. The map α is then specified by a continuous function $\alpha\colon V \to \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^l) = M_{l\times k}(\mathbb{R})$. Since $\operatorname{rank}(\alpha_x) \geq n$, some $n \times n$ minor of $\alpha(x)$ is nonzero. If $U \subseteq M_{l\times k}(\mathbb{R})$ is the subspace of all matrices for which the corresponding minor is nonzero, this is an open subset of $M_{l\times k}(\mathbb{R})$. Then $\phi^{-1}(U)$ is a neighborhood of x that is completely contained in $\mathcal{R}_n$. □

Lemma 9.7. *Let $E \xrightarrow{\alpha} F \xrightarrow{\beta} G$ be an exact sequence of vector bundles. Then $\operatorname{im}\alpha$ (which equals $\ker\beta$) is a vector bundle.*

Proof. Using Proposition 9.3 it suffices to prove that α has constant rank on each connected component of X. Without loss of generality we can assume that X is connected. Since the question is local on X, we can assume that E, F, and G are all trivial bundles. Let $n = \operatorname{rank}(F)$.

Pick an $x \in X$ and let $p = \operatorname{rank}(\alpha_x)$. Then $\operatorname{rank}(\beta_x) = n - p$ by exactness. Let $U = \{z \in X \mid \operatorname{rank}(\alpha_z) \geq p\}$ and $V = \{z \in X \mid \operatorname{rank}(\beta_z) \geq n - p + 1\}$. Note that exactness implies that $U = X - V$. But both U and V are open by Lemma 9.6, which means they are also both closed. By connectedness, U is either empty or the whole of X. Since $x \in U$, we must have $U = X$.

A similar argument proves that $\{z \in X \mid \operatorname{rank}(\alpha_z) \leq p\} = X$. So for every $z \in X$ we have $p \leq \operatorname{rank}(\alpha_z) \leq p$; that is, the rank of α is constant on X. □

If $E_\bullet$ is a chain complex of vector bundles on X and $x \in X$, write $(E_x)_\bullet$ for the chain complex of vector spaces formed by the fibers over x. Define the **support** of $E_\bullet$, denoted $\operatorname{Supp} E_\bullet$, to be the subspace $\{x \in X \mid (E_x)_\bullet \text{ is not exact}\} \subseteq X$. We will occasionally write $\operatorname{Supp}_i E_\bullet$ for $\{x \in X \mid H_i((E_x)_\bullet) \neq 0\}$. Note that $\operatorname{Supp} E_\bullet = \bigcup_i \operatorname{Supp}_i E_\bullet$.

Proposition 9.8. *Let $E_\bullet$ be a chain complex of vector bundles on X. Then for any $i \in \mathbb{Z}$, the subspace $\operatorname{Supp}_i E_\bullet$ is closed in X. If $E_\bullet$ is a bounded chain complex then $\operatorname{Supp} E_\bullet$ is closed.*

Proof. We will prove that $X - \operatorname{Supp}_i E_\bullet$ is open, so assume x belongs to this set. Write the maps in the chain complex as

$$E_{i+1} \xrightarrow{\alpha} E_i \xrightarrow{\beta} E_{i-1}.$$

Let $n = \operatorname{rank}_x(E_i)$, $a = \operatorname{rank}_x(\alpha)$, and $b = \operatorname{rank}_x(\beta)$. Since the complex is exact at x in the ith spot we have $a + b = n$. By Lemma 9.6 applied twice, there is an open neighborhood U of x such that $\operatorname{rank}_y(\alpha) \geq a$ and $\operatorname{rank}_y(\beta) \geq b$ for all $y \in U$. Then we can write

$$a \leq \operatorname{rank}_y(\alpha) \leq n - \operatorname{rank}_y(\beta) \leq n - b$$

where the middle inequality follows from the fact that $(E_y)_\bullet$ is a chain complex. Since $a = n - b$ all the inequalities are in fact equalities, and so we have exactness at y for all $y \in U$. That is, $U \subseteq X - \operatorname{Supp}_i E_\bullet$.

The final statement follows from the fact that $\operatorname{Supp} E_\bullet$ is a finite union of the $\operatorname{Supp}_i E_\bullet$ spaces. □

10. Swan's Theorem

In this section we explore our first connection between topology and algebra. We will see that vector bundles are closely related to projective modules.

When X is a space let $C(X)$ denote the ring of continuous functions from X to $\mathbb{R}$, where the addition and multiplication are pointwise. Recall that if $E \to X$ is a family of vector spaces, then $\Gamma(E)$ denotes the vector space of sections. In addition to being a vector space, it is easy to see that this is actually a module over $C(X)$: if $f \in C(X)$ and $s \in \Gamma(E)$ then fs is the section $x \mapsto f(x)s(x)$. The assignment $E \mapsto \Gamma(E)$ gives a functor from vector bundles to $C(X)$-modules.

It is easy to check that Γ is a left-exact functor: if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of families of vector spaces then $0 \to \Gamma(E') \to \Gamma(E) \to \Gamma(E'')$ is exact.

If $E \to X$ is a vector bundle then of course the modules of the form $\Gamma(E)$ are not just arbitrary $C(X)$-modules; there is something special about them. It is easiest to say what this is under some assumptions on X:

Proposition 10.1. *If X is compact and Hausdorff, and E is a vector bundle over X, then $\Gamma(E)$ if a finitely-generated, projective module over $C(X)$.*

Proof. By Proposition 9.5 we can embed E into a trivial bundle $\underline{N}$. This embedding has constant rank, so by Proposition 9.3 the quotient Q is also a vector bundle. So we have the exact sequence $0 \to E \rightarrowtail \underline{N} \twoheadrightarrow Q \to 0$ of vector bundles on X. Now apply $\Gamma(-)$, which yields the exact sequence

$$0 \to \Gamma(E) \to \Gamma(\underline{N}) \to \Gamma(Q)$$

of $C(X)$-modules. This much is for free. But by Proposition 9.2 the map $\underline{N} \to Q$ has a splitting, and this splitting shows that $\Gamma(\underline{N}) \to \Gamma(Q)$ is split-surjective. So

$$\Gamma(E) \oplus \Gamma(Q) \cong \Gamma(\underline{N}) = C(X)^n.$$

That is, $\Gamma(E)$ is a direct summand of a free module; hence it is projective. □

For the rest of this section we will assume that our base spaces are compact and Hausdorff. Let $\langle\!\langle \operatorname{Vect}(X) \rangle\!\rangle$ denote the category of vector bundles over X, and let $\langle\!\langle \operatorname{Mod} - C(X) \rangle\!\rangle$ denote the category of modules over the ring $C(X)$. Let

$\langle\langle \mathrm{Proj}-C(X) \rangle\rangle$ denote the full subcategory of finitely-generated, projective modules. Then Γ is a functor $\langle\langle \mathrm{Vect}(X) \rangle\rangle \to \langle\langle \mathrm{Proj}-C(X) \rangle\rangle$. It is proven in [Sw] that this is actually an equivalence:

Theorem 10.2 (Swan's Theorem)**.** *Let X be a compact, Hausdorff space. Then*

$$\Gamma\colon \langle\langle \mathrm{Vect}(X) \rangle\rangle \to \langle\langle \mathrm{Proj}-C(X) \rangle\rangle$$

is an equivalence of categories.

To prove this result we need to verify two things:

- Every finitely-generated projective over $C(X)$ is isomorphic to $\Gamma(E)$ for some vector bundle E.
- For every two vector bundles E and F, the induced map
$$\Gamma\colon \mathrm{Hom}_{\mathrm{Vect}(X)}(E,F) \to \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$$
is a bijection.

That is to say, we need to prove that Γ is surjective on isomorphism classes, and is fully faithful. Here is the first part:

Proposition 10.3. *If X is paracompact Hausdorff and P is a finitely-generated projective module over $C(X)$, then $P \cong \Gamma(E)$ for some vector bundle $E \to X$.*

Proof. Choose a surjection $p\colon C(X)^n \twoheadrightarrow P$. Since P is projective, there is a splitting χ. Then $e = \chi p$ satisfies $e^2 = e$, and P is isomorphic to $\mathrm{im}(e)$.

Since $e\colon C(X)^n \to C(X)^n$ we can represent e by an $n \times n$ matrix whose elememts are in $C(X)$. Denote the entries of this matrix as e_{ij}. Note that for any $x \in X$ we can evaluate all these functions at x to get an element $e(x) \in M_{n\times n}(\mathbb{R})$.

Define a map of vector bundles $\alpha\colon \mathbb{R}^n \times X \to \mathbb{R}^n \times X$ by the formula $\alpha(v,x) = e(x)\cdot v$. Then the sequence

$$\underline{n} \xrightarrow{\alpha} \underline{n} \xrightarrow{1-\alpha} \underline{n}$$

is an exact sequence of vector bundles. Let $E = \mathrm{im}(\alpha)$, which by Lemma 9.7 is a vector bundle on X; the proof of that lemma also shows that α has constant rank. We claim that $\Gamma(E) \cong P$. To see this, consider the following diagram of vector bundles:

$$\begin{array}{ccccc} \ker\alpha & \rightarrowtail & \underline{n} & \xrightarrow{\alpha} & \underline{n} \\ & & \searrow & & \nearrow \\ & & & \mathrm{im}\,\alpha. & \end{array}$$

The map $\underline{n} \to \mathrm{im}\,\alpha$ is split by Proposition 9.2, because X is paracompact. Applying Γ to the above diagram gives

$$\begin{array}{ccccc} \Gamma(\ker\alpha) & \rightarrowtail & C(X)^n & \xrightarrow{e} & C(X)^n \\ & & \searrow & & \nearrow \\ & & & \Gamma(\mathrm{im}\,\alpha). & \end{array}$$

The sequence $0 \to \Gamma(\ker\alpha) \to C(X)^n \to \Gamma(\mathrm{im}\,\alpha) \to 0$ is exact because it was split-exact before applying Γ, and the identification $\Gamma(\ker\alpha) = \ker(\Gamma\alpha)$ shows that $\Gamma(\ker\alpha)$ is the kernel of e. It now follows that $\Gamma(\mathrm{im}\,\alpha)$ is isomorphic to the image of e, which is P. $\square$

Our final goal is to prove that Γ is fully faithful. To do this, it is useful to relate the fibers E_x of our bundle to an algebraic construction based on the module $\Gamma(E)$. For each $x \in X$ consider the evaluation map $\mathrm{ev}_x\colon C(X) \to \mathbb{R}$, and let m_x be the kernel. The ideal $m_x \subseteq C(X)$ is maximal, since the quotient is a field.

Note that we have the evaluation map $\mathrm{ev}_x\colon \Gamma(E) \to E_x$. This map clearly sends the submodule $m_x\Gamma(E)$ to zero.

Lemma 10.4. *Assume that X is paracompact Hausdorff. Then for any vector bundle $E \to X$ and any $x \in X$, the map $\mathrm{ev}_x\colon \Gamma(E)/m_x\Gamma(E) \xrightarrow{\cong} E_x$ is an isomorphism.*

Proof. We first record the following important fact, which we label (*): if s is a section of E defined on some open neighborhood U of x, then there exists a section s' defined on all of X such that s and s' agree on some (potentially smaller) neighborhood of x. To see this, first choose an open neighborhood V of x such that $\overline{V} \subseteq U$ (this exists because X is normal). By Urysohn's Lemma there is a continuous function $f\colon X \to \mathbb{R}$ such that $f|\overline{V} = 1$ and $f|_{X-U} = 0$. The assignment $z \mapsto f(z)\cdot s(z)$ is readily checked to be a continuous section of E that agrees with s on V.

To prove surjectivity of ev_x, let $v \in E_x$. Since E is locally trivial, one can find a section s defined locally about x such that $s(x) = v$. By principle (*) there is a section s' defined on all of X that agrees with s near x; in particular, $s'(x) = v$.

For injectivity we must work a little harder. Suppose that $s \in \Gamma(E)$ and $s(x) = 0$. We must prove that $s \in m_x\Gamma(E)$. Choose independent sections $e_1, \ldots, e_n$ defined on U. Fact (*) says that by replacing U by a smaller neighborhood of x we can assume that the sections are defined on all of X (but only independent on U).

Using that $e_1(y), \ldots, e_n(y)$ is a basis for E_y when $y \in U$, we can write $s(y) = a_1(y)e_1(y) + \cdots + a_n(y)e_n(y)$ for uniquely defined numbers $a_1(y), \ldots, a_n(y) \in \mathbb{R}$. The functions a_i are continuous, since they may be expressed by determinantal formulas via Cramer's Rule. Regarding the a_i's as local sections of the trivial bundle $X \times \mathbb{R}$, (*) shows we may assume the a_i's are defined on all of X (again, we need to replace U with a smaller neighborhood here). Since $s(x) = 0$ note that $0 = a_1(x) = a_2(x) = \cdots = a_n(x)$.

Let $t = s - a_1e_1 - \cdots - a_ne_n \in \Gamma E$. Note that t vanishes throughout the neighborhood U of x. Again using the Urysohn Lemma, choose a continuous function $b\colon X \to \mathbb{R}$ such that $b(x) = 0$ and $b|_{X-U} = 1$. Observe that

$$s(y) = b(y)t(y) + a_1(y)e_1(y) + \ldots + a_n(y)e_n(y),$$

for every $y \in X$: for if $y \in U$ then $t(y)$ vanishes, and if $y \notin U$ then $b(y) = 1$. So $s = bt + a_1e_1 + \cdots + a_ne_n$, and the expression on the right is manifestly in $m_x\Gamma(E)$. $\square$

Proposition 10.5. *Assume that X is paracompact Hausdorff. Then for any vector bundles E and F over X, the map $\Gamma\colon \mathrm{Hom}_{\mathrm{Vect}(X)}(E, F) \to \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$ is a bijection.*

Proof. First of all, it is easy to check this when E and F are both trivial. A map of vector bundles $\mathbb{R}^k \times X \to \mathbb{R}^l \times X$ is uniquely specified by a map $X \to M_{l\times k}(\mathbb{R})$, and likewise a map of $C(X)$-modules $C(X)^k \to C(X)^l$ is specified by an $l \times k$ matrix with entries in $C(X)$. One observes that continuous maps $X \to M_{l\times k}(\mathbb{R})$ bijectively correspond with $l \times k$ matrices with entries in $C(X)$.

For the general case, consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Vect}(X)}(E,F) & \xrightarrow{\Gamma} & \mathrm{Hom}_{C(X)}(\Gamma E,\Gamma F) \\ \downarrow & & \downarrow \\ \prod_{x\in X}\mathrm{Hom}(E_x,F_x) & \xrightarrow{\cong} & \prod_{x\in X}\mathrm{Hom}(\Gamma E/m_x\Gamma E,\Gamma F/m_x\Gamma_F). \end{array}$$

The bottom horizontal map is an isomorphism by Lemma 10.4. The left vertical arrow sends a bundle map $\alpha\colon E\to F$ to the collection of its restrictions to each fiber; surely this map is an injection. It follows at once that Γ is also an injection.

It remains to show that the top horizontal map is a surjection, so let $\beta\in \mathrm{Hom}_{C(X)}(\Gamma E,\Gamma F)$. We can apply the right vertical arrow to β, and then find a unique preimage in $\prod_x \mathrm{Hom}(E_x,F_x)$ using that the bottom map is an isomorphism. This gives us a map of sets $\alpha\colon E\to F$, by defining it on each of the fibers. We need to prove that α is continuous. However, this is a local question: so it suffices to do so in the case that E and F are trivial, and this case has already been verified. So we have produced a bundle map $\alpha\colon E\to F$ whose restriction to each fiber agrees with the map β. It follows that α is sent to β by Γ. □

Note that we have now completed the proof of Swan's Theorem, via Propositions 10.3 and 10.5.

10.6. **Variants of Swan's Theorem.** ????

11. Homotopy invariance of vector bundles

For a fixed n, let $\mathrm{Vect}_n(X)$ denote the set of isomorphism classes of vector bundles on X. It turns out that when X is a finite complex this set is always countable, and often finite. It actually gives a homotopy invariant of the space X.

Write i_0 and i_1 for the two inclusions $X\hookrightarrow X\times I$ coming from the boundary points of the interval. The key to homotopy invariance is the following result.

Proposition 11.1. *Let X be paracompact, and let $E\to X\times I$ be a vector bundle. Then there is an isomorphism $i_0^*(E)\cong i_1^*(E)$.*

Before proving this let us give the evident corollaries:

Corollary 11.2. *Fix $n\geq 0$.*

(a) If $f,g\colon X\to Y$ are homotopic then f^* and g^* give the same map $\mathrm{Vect}_n(Y)\to \mathrm{Vect}_n(X)$.
(b) If $f\colon X\to Y$ is a homotopy equivalence then $f^*\colon \mathrm{Vect}_n(Y)\to\mathrm{Vect}_n(X)$ is a bijection, for all $n\geq 0$.
(c) If X is contractible then all vector bundles on X are trivializable.

Proof. For (a), let $H\colon X\times I\to Y$ be a homotopy and consider the diagram

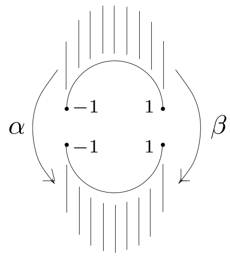
$$\mathrm{Vect}_n(X) \underset{i_1^*}{\overset{i_0^*}{\leftleftarrows}} \mathrm{Vect}_n(X\times I) \xleftarrow{H^*} \mathrm{Vect}_n(Y).$$

One of the compositions is f^*, the other is g^*, and Proposition 11.1 says that the two compositions are the same.

Parts (b) and (c) are simple consequences of (a). □

Example 11.3. To give an idea how we will apply these results, let us think about vector bundles on S^1. Divide S^1 into an upper hemisphere D_+ and a lower hemisphere D_-, intersecting in two points. Each of D_+ and D_- are contractible, so any vector bundle will be trivializable when restricted to these subspaces.

Given two elements $\alpha, \beta \in GL_n(\mathbb{R})$, let $E_n(\alpha, \beta)$ be the vector bundle on S^1 obtained by taking $\underline{n}_{D_+}$ and $\underline{n}_{D_-}$ and gluing them together via α and β at the two points on the equator. The considerations of the previous paragraph tell us that every vector bundle on S^1 is of this form. The following picture depicts the construction of $E_n(\alpha, \beta)$:

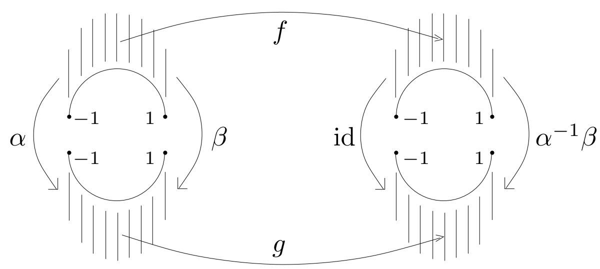


Note that $E_n(\mathrm{id}, \mathrm{id}) = \underline{n}$, and $E_1(\mathrm{id}, -1) = M$ (the Möbius bundle). It is easy to check the following:

(1) $E_n(\alpha, \beta) \cong E_n(\mathrm{id}, \alpha^{-1}\beta)$
(2) $E_n(\mathrm{id}, \beta) \cong E_n(\mathrm{id}, \beta')$ if and only if β and β' are in the same path component of $GL_n(\mathbb{R})$ (or equivalently, if $\det(\beta)$ and $\det(\beta')$ have the same sign).

In (2) we have used the fact that $\pi_0(GL_n(\mathbb{R})) = \mathbb{Z}/2$, with the isomorphism being given by the sign of the determinant.

Let us explain the above facts. The isomorphism in (1) can be depicted as



Here f and g are maps $D_+ \to GL_n(\mathbb{R})$ and $D_- \to GL_n(\mathbb{R})$ giving the isomorphisms on each fiber; compatibility with the gluing requires that we have $g(-1)\alpha = f(-1)$ and $\alpha^{-1}\beta f(1) = g(1)\beta$. This can be achieved by letting $f(t) = I_n$ and $g(t) = \alpha^{-1}$, for all t.

The proof of (2) is a little more subtle. To give an isomorphism $E(\mathrm{id}, \beta) \cong E(\mathrm{id}, \beta')$ we must again specify maps f and g as above, but this time satisfying $g(-1) = f(-1)$ and $\beta' f(1) = g(1)\beta$. If we paste D_+ and D_- together at -1 and identify the resulting interval with $[0, 1]$, then we are just asking for a map $h\colon [0, 1] \to GL_n(\mathbb{R})$ such that $\beta' h(1) = h(0)\beta$.

If β and β' are in the same path component then choose a path $h\colon I \to GL_n(\mathbb{R})$ such that $h(0) = \beta'$ and $h(1) = \beta$. Since we then have $\beta' h(1) = h(0)\beta$, this yields the desired isomorphism. Conversely, if we have a map h satisfying $\beta' h(1) = h(0)\beta$

then we can rearrange this as $\beta' = h(0)\beta h(1)^{-1}$. The term on the right is path-connected to $h(0)\beta h(0)^{-1}$, using the homotopy $t \mapsto h(0)\beta h(t)^{-1}$. But $h(0)\beta h(0)^{-1}$ has the same determinant as β, so these are also in the same path-component. Hence, β' and β are themselves path-connected and this proves (2).

To summarize, from (1) and (2) it follows that isomorphism types for rank n bundles over S^1 are in bijective correspondence with the path components of $GL_n(\mathbb{R})$. We know that for $n > 0$ there are two such path components, which can be represented by the identity matrix and the diagonal matrix J whose diagonal entries are $-1, 1, 1, \ldots, 1$. The corresponding bundles $E_n(\mathrm{id}, \beta)$ are $\underline{n}$ and $M \oplus \underline{(n-1)}$.

Most of the basics of this discussion generalize readily from S^1 to S^k. We discuss this in Proposition 12.2.

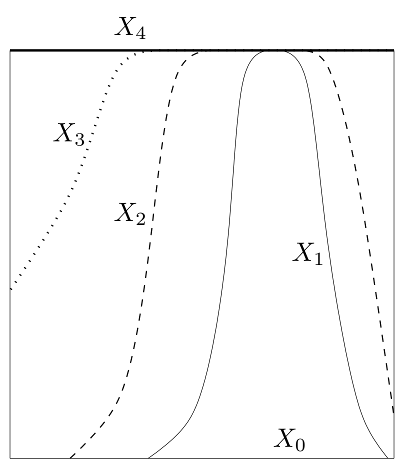
The methods of the above example apply in much greater generality, and with little change allow one to get control over vector bundles on any suspension. We will return to this topic in the next section.

At this point let us now give the proof of Proposition 11.1. This proof is from [Ha2].

Proof of Proposition 11.1. Pick an $x \in X$. Using the compactness of I and the definition of vector bundle, we may find an open neighborhood $U \subseteq X$ of x and values $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = 1$ such that E is trivial over each $U \times [a_i, a_{i+1}]$. Patching these together gives a trivialization of the vector bundle over $U \times I$.

Now assume for a moment that X is compact. Then we can cover X by open sets $U_1, \ldots, U_n$ such that E is trivial over each $U_i \times I$. Fix trivializations of each $E|_{U_i}$. Choose a partition of unity $\phi_1, \ldots, \phi_n$ subordinate to this cover, and set $\beta_0 = 0$, $\beta_i = \phi_1 + \ldots + \phi_i$. Define X_i to be the graph of β_i in $X \times I$. Observe that $\beta_n = 1$ and thus $X_n = X \times \{1\}$, $X_0 = X \times \{0\}$.

There are homeomorphisms $X \to X_i$ given by $x \mapsto (x, \beta_i(x))$. Via these we can think of each X_i as a copy of X.



There are maps $f_i \colon X_i \to X_{i-1}$ defined by "pushing down until you hit the next graph", and each of these is a homeomorphism. We may restrict E over each X_i, and a little effort yields diagrams

$$\begin{array}{ccc} E|_{X_i} & \longrightarrow & E|_{X_{i-1}} \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{f_i} & X_{i-1}. \end{array}$$

Here, the top map uses the fact X_i and X_{i-1} coincide except over the open set U_i, and that $E|_{U_i}$ is trivial. Each map $E|_{X_i} \to E|_{X_{i-1}}$ is an isomorphism on the fibers.

Via the identifications $X \cong X_i$, each $E|_{X_i}$ is a vector bundle on X and we have isomorphisms

$$i_1^*(E) = E|_{X_n} \xrightarrow{\cong} E|_{X_{n-1}} \xrightarrow{\cong} E|_{X_{n-2}} \xrightarrow{\cong} \cdots \xrightarrow{\cong} E|_{X_1} \xrightarrow{\cong} E|_{X_0} = i_0^*(E).$$

This gives us what we wanted.

The paracompact case is similar, except now we only have a countable covering rather than a finite one. One can still make sense of the countable composition of the resulting isomorphisms, and essentially the same proof goes through. The reader is referred to [Ha2] for complete details. □

Remark 11.4. The isomorphism $i_0^*(E) \cong i_1^*(E)$ is not canonical, as is clear from the proof of the theorem.

Remark 11.5. We have seen that all bundles on contractible spaces are trivial, and that there is a close connection between vector bundles and projective modules. Recall that when k is a field then $k[x_1, \ldots, x_n]$ is the algebraic analog of affine space $\mathbb{A}^n$, and that projectives over this ring correspond to algebraic vector bundles. The analogy with topology is what led Serre to conjecture that all finitely-generated projectives over $k[x_1, \ldots, x_n]$ are free, as we discussed in Example 3.2.

We have proven that if E is a vector bundle on $X \times I$ then $i_0^*(E) \cong i_1^*(E)$. It is natural to wonder if this result has a converse, but stating such a thing is somewhat tricky. Here is one possibility: if F and F' are isomorphic vector bundles on X, is there a vector bundle E on $X \times I$ such that $i_0^*(E) \cong F$ and $i_1^*(E) \cong F'$? Unfortunately, this has a trivial answer: yes, just take $E = \pi^*(F)$ where $\pi \colon X \times I \to X$ is the projection. So this phrasing of the question was not very informative.

Here is another possibility: if F and F' are isomorphic vector bundles on X, is there a vector bundle E on $X \times I$ such that $i_0^*(E) = F$ and $i_1^*(E) = F'$? Note the presence of equalities here, as opposed to isomorphisms. This question does not have an obvious answer, but it is also the kind of question that one really doesn't want to be asking: saying that two abstract gadgets are *equal*, rather than just isomorphic, is going to force us down a path that requires us to keep track of too much data.

So we find ourselves in somewhat of a muddle. Perhaps there is an interesting question here, but we don't quite know how to ask it. One approach is to restrict to a class of bundles where "equality" is something we can better control. For example, one can restrict to bundles on X that sit inside of $X \times \mathbb{R}^\infty$. Here, finally, we have an interesting question: if F and F' are two such bundles, which are abstractly isomorphic, is there a bundle E inside of $(X \times I) \times \mathbb{R}^\infty$ that restricts to F and F' at times 0 and 1? The answer is yes, and we will next explain why.

11.6. **Classifying spaces.** We may view a vector bundle as a family of vector spaces indexed by the base space. In general, we may view a map $X \to Y$ as a family of *blah* if each fiber is a *blah*. We naively hope that families of some mathematical object over X are in bijection with maps from X to some space, called the moduli space corresponding to that mathematical object. With this naive idea, we would hope that families over $*$ are in bijective correspondence with points of our moduli space. However, this does not work since the moduli space of $\mathbb{R}^n$'s is $*$.

If $V \subseteq W$ then we get an induced inclusion of Grassmannians $\mathrm{Gr}_k(V) \hookrightarrow \mathrm{Gr}_k(W)$. Consider the standard chain of inclusions of Euclidean spaces $\mathbb{R}^k$, and define the infinite Grassmannian $\mathrm{Gr}_n(\mathbb{R}^\infty)$ to be the colimit of the induced sequence of finite Grassmannians:

$$\mathrm{Gr}_n(\mathbb{R}^\infty) = \operatorname*{colim}_{k\to\infty}[\mathrm{Gr}_n(\mathbb{R}^k)].$$

Define $\gamma_n \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ by $\gamma_n = \{(V,x) \mid V \subset \mathbb{R}^\infty, \dim(V) = n, x \in V\}$. This is the tautological vector bundle on the infinite grassmanian.

To any map $f \colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ we associate the pullback bundle

$$\begin{array}{ccc} f^*\gamma_n & \longrightarrow & \gamma_n \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathrm{Gr}_n(\mathbb{R}^\infty). \end{array}$$

The assignment $f \mapsto f^*\gamma_n$ gives a map $\mathrm{Hom}(X, \mathrm{Gr}_n(\mathbb{R}^\infty)) \to \mathrm{Vect}_n(X)$. Observe that if $f, g \colon X \to Gr_n(\mathbb{R}^\infty)$ are homotopic maps, then $f^*\gamma_n \cong g^*\gamma_n$ by Corollary 11.2(a). In this way we have constructed a map $\phi \colon [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to \mathrm{Vect}_n(X)$. We will show that this is an isomorphism when X is compact and Hausdorff.

Lemma 11.7. *Let $j^{ev}, j^{odd} \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be given by $j^{ev}(x_1, x_2, ...) = (0, x_1, 0, x_2, ...)$ and $j^{odd}(x_1, x_2, ...) = (x_1, 0, x_2, 0, ...)$. Then $j^{ev} \simeq j^{odd} \simeq \mathrm{id}$, via homotopies H having the property that each H_t is a linear embedding.*

Proof. We prove the claim for j^{ev}; the proof for j^{odd} is analogous. Define a homotopy $H \colon \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$ by $H(x,t) = tj^{ev}(x) + (1-t)x$. This is clearly a homotopy between j^{ev} and id. It remains to be shown that this is a homotopy through linear embeddings. Let $t \in (0,1)$ and suppose that $H(x,t) = 0$. We need to show that $x = 0$. Our assumption yields $0 = ((1-t)x_1, tx_1 + (1-t)x_2, (1-t)x_3, tx_2 + (1-t)x_4, ...)$. Therefore $(1-t)x_i = 0$ for all odd i; but since $t \neq 1$, this means that $x_i = 0$ for all odd i. Likewise, observe that $tx_n + (1-t)x_{2n} = 0$ for all $n \in \mathbb{N}$. So $x_n = 0$ implies $x_{2n} = 0$. Since we have $x_i = 0$ for all odd i and every natural number n can be written in the form $n = 2^e i$, it follows that $x = 0$. □

Theorem 11.8. *The map $\phi \colon [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to \mathrm{Vect}_n(X)$ is always injective, and is bijective when X is compact and Hausdorff.*

Proof. For injectivity, assume $f, g \colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ are such that $f^*(\gamma_n) \cong g^*(\gamma_n)$ as vector bundles over X. We will show that f is homotopic to g. Let $\alpha \colon f^*(\gamma_n) \to g^*(\gamma_n)$ be an isomorphism. By Lemma 11.7, we may replace f by $j^{ev} \circ f$ and g by $j^{odd} \circ g$. In doing so, we are effectively assuming that $f(x) \subseteq \mathbb{R}^\infty_{ev}$ and $g(x) \subseteq \mathbb{R}^\infty_{odd}$, for each $x \in X$. Now simply define $H \colon X \times I \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ by setting $H(x,t) = \{tv + (1-t)\alpha(v) \mid v \in f(x)\}$. It is easy to see that $H(x,t)$ is a subspace of $\mathbb{R}^\infty$. Since

$f(x)$ and $g(x)$ are disjoint one readily argues that $H(x,t)$ is n-dimensional, for all t. So we really do have a homotopy, and clearly $H_0 = g$ and $H_1 = f$. CONTINUITY OF H? PICTURE HERE

For surjectivity, assume X is compact and Hausdorff and let $E \to X$ be a vector bundle of rank n. Then by Proposition 9.5 there exists an embedding $j\colon E \to \underline{\mathrm{N}}$, for large enough N (this is where we use the assumption on X). Note that $\underline{\mathrm{N}} = X \times \mathbb{R}^N \subseteq X \times \mathbb{R}^\infty$. Now define $f\colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ by $f(x) = j(E_x) \subseteq \mathbb{R}^N \subseteq \mathbb{R}^\infty$. It is left to the reader to check that f is continuous and that $f^*(\gamma_n) \cong E$. □

11.9. **Stabilization of vector bundles.** Here is a simple application of classifying spaces that we will occasionally find useful. Fix a space X. If $E \to X$ is a vector bundle of rank n, then of course $E \oplus \underline{1}$ is a vector bundle of rank $n+1$. We get a sequence of maps

$$\mathrm{Vect}_0(X) \xrightarrow{\oplus 1} \mathrm{Vect}_1(X) \xrightarrow{\oplus 1} \mathrm{Vect}_2(X) \xrightarrow{\oplus 1} \cdots$$

Are these maps injective? Surjective? Are there more and more isomorphism classes of vector bundles as one goes up in rank, or is it the case that all "large" rank vector bundles actually come from smaller ones via addition of a trivial bundle? A homotopical analysis of classifying spaces allow us give some partial answers here. We handle both the case of real and complex bundles:

Proposition 11.10. *Let X be a finite-dimensional CW-complex. For real vector bundles,* $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ *is a bijection for* $n \geq \dim X + 1$ *and a surjection for* $n = \dim X$*. For complex bundles,* $\mathrm{Vect}_n^{\mathbb{C}}(X) \to \mathrm{Vect}_{n+1}^{\mathbb{C}}(X)$ *is a bijection for* $n \geq \frac{1}{2}\dim X$ *and a surjection for* $n \geq \frac{1}{2}(\dim X - 1)$*.*

Proof. The map $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ is represented by a map of spaces $f\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$. A little thought shows this to be the map that sends a subspace $V \subseteq \mathbb{R}^\infty$ to $\mathbb{R} \oplus V \subseteq \mathbb{R} \oplus \mathbb{R}^\infty$ and then uses a fixed isomorphism $\mathbb{R} \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$ to obtain a point in $\mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$. To establish the proposition we must analyze when $[X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \xrightarrow{f_*} [X, \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)]$ is injective/surjective.

Now, the inclusion $\mathrm{Gr}_n(\mathbb{R}^\infty) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$ is n-connected. This can be argued in different ways, but one way is to examine the Schubert cell decompositions of each space and observe that they are identical until one reaches dimension $n+1$. This connectivity result implies that $[B, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to [B, \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)]$ is bijective for CW-complexes with $\dim B \leq n-1$, and surjective for CW-complexes with $\dim B = n$. We simply apply this to $B = X$.

For the complex case, $\mathrm{Gr}_n(\mathbb{C}^\infty) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{C}^\infty)$ is now $(2n+1)$-connected. So we get the analogous bijection for CW-complexes B of dimension at most $2n$, and the surjection when $\dim B = 2n+1$. □

12. Vector bundles on spheres

In this section we explore the set of isomorphism classes $\mathrm{Vect}_n(S^k)$ for various values of k and n. There are two important points. First, for a fixed k these sets stablize for $n \gg 0$. Secondly, Bott was able to compute these stable values completely and found an 8-fold periodicity (with respect to k) in the case of real vector bundles, and a 2-fold periodicity in the case of complex bundles. Bott's periodicity theorems are of paramount importance in modern algebraic topology.

12.1. **The clutching construction.** Let X be a pointed space, and let C_+ and C_- be the positive and negative cones in ΣX. Fix $n \geq 0$. For a map $f\colon X \to GL_n(\mathbb{R})$, let $E_n(f)$ be the vector bundle obtained by gluing $\underline{n}|_{C_+}$ and $\underline{n}|_{C_-}$ via the map f (we use Corollary 8.17(b) here). Precisely, if $x \in X$ and v belongs to the fiber of $\underline{n}_{C_+}$ over x then we glue v to $f(x) \cdot v$ in the fiber of $\underline{n}_{C_-}$ over x. This procedure for constructing vector bundles on ΣX is called *clutching*, and every bundle on ΣX arises in this way. By changing basis in one of the trivial bundles one sees we can always require $f(*) = I_n$; that is, we can require f to be a based map.

Proposition 12.2.

(a) If $f, f'\colon X \to GL_n(\mathbb{R})$ are homotopic relative to the basepoint, then $E_n(f) \cong E_n(f')$.

(b) The induced map $E_n\colon [X, GL_n(\mathbb{R})]_* \to \mathrm{Vect}_n(\Sigma X)$ is a bijection.

Proof. Note that (b) follows immediately from (a), since our discussion above showed that E_n is a surjection. To prove (a), let $f, g\colon X \to GL_n(\mathbb{R})$ be based maps with $E_f \cong E_g$. A choice of isomorphism α amounts to giving maps $\alpha_+\colon U_+ \to GL_n(\mathbb{R})$ and $\alpha_-\colon U_- \to GL_n(\mathbb{R})$ such that

$$f \cdot (\alpha_+|_X) = (\alpha_-|_X) \cdot g.$$

Since $\alpha_+|_X$ can be extended to C_+ there is a basepoint preserving homotopy between $\alpha_+|X$ and the map sending X to the basepoint of $GL_n(\mathbb{R})$. The basepoint of $GL_n(\mathbb{R})$ is just the identity map so we have $f \cdot (\alpha_+|_X) \simeq f$. The same argument shows that $\alpha_-|_X \simeq g$ and hence $f \simeq g$. ∎

Let us apply the above result when X is a sphere S^{k-1}. We obtain a bijection $\mathrm{Vect}_n(S^k) \simeq \pi_{k-1} GL_n(\mathbb{R})$. For $k > 2$ note that the right-hand-side is a group, although there is no evident group structure on the left-hand-side. It will be convenient to replace $GL_n(\mathbb{R})$ with its subgroup O_n. Recall that $O_n \hookrightarrow GL_n(\mathbb{R})$ is a deformation retraction, as a consequence of the Gram-Schmidt process. When $k > 2$ any based map $S^{k-1} \to O_n$ must actually factor through the connected component of the identity, which is SO_n. So we have

$$\mathrm{Vect}_n(S^k) \cong \pi_{k-1} GL_n(\mathbb{R}) \cong \pi_{k-1} O_n \cong \pi_{k-1} SO_n$$

(where the last isomorphism needs $k > 2$).

12.3. **Vector bundles on S^1.** For $k = 1$ and $n > 0$ we have that $\mathrm{Vect}_n(S^1) \cong \pi_0 GL_n(\mathbb{R}) = \mathbb{Z}/2$, and we have previously seen in Example 11.3 that the two isomorphism classes are represented by $\underline{n}$ and $M \oplus \underline{(n-1)}$ where M is the Möbius bundle.

12.4. **Vector bundles on S^2.** Here we have $\mathrm{Vect}_n(S^2) \cong \pi_1 SO_n$. Recall that $SO_2 \cong S^1$, and so we get $\mathrm{Vect}_2(S^2) \cong \mathbb{Z}$. We claim that for $n > 2$ one has $\pi_1 SO_n \cong \mathbb{Z}/2$, so that we have the following:

Proposition 12.5. $\mathrm{Vect}_n(S^2) \cong \pi_1(SO_n) \cong \begin{cases} 1 & \textit{if } n = 1, \\ \mathbb{Z} & \textit{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } n \geq 3. \end{cases}$

Proof. First of all $SO_1 = \{1\}$ and $SO_2 \cong S^1$, so this takes care of $n \leq 2$. For $n = 3$ recall that $SO_3 \cong \mathbb{R}P^3$, so that $\pi_1(SO_3) \cong \mathbb{Z}/2$. To see the homeomorphism use the model $\mathbb{R}P^3 \cong D^3/\!\sim$ where the equivalence relation has $x \sim -x$ for $x \in \partial D^3$.

Map $D^3 \to SO_3$ by sending a vector v to the rotation of $\mathbb{R}^3$ with axis $\langle v\rangle$, through $|v|\cdot\pi$ radians, in the direction given by a right-hand-rule with the thumb pointed along v. Note that this makes sense even for $v=0$, since the corresponding rotation is through 0 radians. For $x \in \partial D^3$ this map sends x and $-x$ to the same rotation, and so induces a map $\mathbb{R}P^3 \to SO_3$. This is clearly a continuous bijection, and therefore a homeomorphism since the spaces are compact and Hausdorff.

For $n \geq 4$ one can use the long exact sequence associated to the fibration $SO_{n-1} \hookrightarrow SO_n \twoheadrightarrow S^{n-1}$ to deduce that $\pi_1(SO_n) \cong \pi_1(SO_{n-1})$. □

Definition 12.6. *Let $\mathcal{O}(n) \in \mathrm{Vect}_2(S^2)$ be the vector bundle E_{f_n} where $f_n\colon S^1 \to SO_2$ is a map of degree n. Note that $\mathcal{O}(0) \cong \underline{2}$.*

The bundles $\mathcal{O}(n)$, $n \in \mathbb{Z}$, give a complete list of the rank 2 bundles on S^2. To get to rank 3 we consider the operation of adding on a trivial line bundle, and note that we have commutative diagrams

$$\begin{array}{ccc} \mathrm{Vect}_{n-1}(S^2) & \xrightarrow{\oplus 1} & \mathrm{Vect}_n(S^2) \\ \cong\downarrow & & \downarrow\cong \\ \pi_1(SO_{n-1}) & \xrightarrow{i_*} & \pi_1(SO_n) \end{array}$$

where the bottom map is induced by the inclusion $i\colon SO_{n-1} \hookrightarrow SO_n$. We saw in the proof of Proposition 12.5 that the bottom horizontal map (and therefore the top one as well) is an isomorphism for $n \geq 4$. For $n=3$ we need to analyze $\pi_1(SO_2) \to \pi_1(SO_3)$, but this is readily seen to be the projection $\mathbb{Z} \to \mathbb{Z}/2$ (use the fibration sequence $SO_2 \hookrightarrow SO_3 \to S^2$). This shows that $\mathcal{O}(j) \oplus 1$ is trivial when j is even, and is isomorphic to the nontrivial bundle $\mathcal{O}(1) \oplus 1$ when j is odd.

Putting all of this information together, the following table shows all the vector bundles on S^2:

n	1	2	3	4	5	6
$\mathrm{Vect}_n(S^2)$	$\underline{1}$	$\mathcal{O}(n)$, $n \in \mathbb{Z}$	$\underline{3}$, $\mathcal{O}(1) \oplus \underline{1}$	$\underline{4}$, $\mathcal{O}(1) \oplus \underline{2}$	$\underline{5}$, $\mathcal{O}(1) \oplus \underline{3}$	$\cdots$

The operation $(-) \oplus \underline{1}$ moves us from one column of the table to the next, and is completely clear except from column 2 to column 3; as we saw above, there it is given by $\mathcal{O}(j) \oplus \underline{1} \cong \underline{3}$ if j is even, and $\mathcal{O}(j) \oplus \underline{1} \cong \mathcal{O}(1) \oplus \underline{1}$ if j is odd.

To complete our study of these bundles there is one final question that we should answer, namely what happens when one adds two rank 2 bundles (all other sums can be figured out once one knows how to do these):

Theorem 12.7. $\mathcal{O}(j) \oplus \mathcal{O}(k) \cong \begin{cases} \underline{4} & \text{if } j+k \text{ is even,} \\ \mathcal{O}(1) \oplus \underline{2} & \text{if } j+k \text{ is odd.} \end{cases}$

Proof. Let $f_j\colon S^1 \to SO_2$ and $f_k\colon S^1 \to SO_2$ be the clutching functions for $\mathcal{O}(j)$ and $\mathcal{O}(k)$, respectively. The clutching function for the bundle $\mathcal{O}(j) \oplus \mathcal{O}(k)$ is the map $f_j \oplus f_k\colon S^1 \to SO_4$, where $\oplus$ is the (pointwise) block diagonal sum $SO_2 \times SO_2 \to SO_4$, given by

$$(A,B) \mapsto \begin{bmatrix} A & O \\ 0 & B \end{bmatrix}.$$

We can factor $f_j \oplus f_k = (f_j \oplus f_0)\cdot(f_0 \oplus f_k)$ where $\cdot$ is pointwise multiplication. It is a standard fact in topology that the group structure on $[S^1, SO_4]_*$ given by pointwise multiplication agrees with the group structure given by concatenation of loops (this is true with SO_4 replaced by any topological group). Note that the homotopy classes of $f_0 \oplus f_k$ and $f_k \oplus f_0$ are the same, since these clutching functions give rise to isomorphic bundles. So we have

$$[f_j \oplus f_k] = [f_j \oplus f_0] + [f_k \oplus f_0]$$

where this is a statement abouts sums of homotopy classes in $\pi_1(SO_4)$.

But $\pi_1(SO_4) = \mathbb{Z}/2$. The function $f_j \oplus f_0$ is the nontrivial element of $\pi_1 SO_4$ precisely when j is odd, and similarly for $f_k \oplus f_0$. It follows that the sum of these elements is trivial/non-trivial when $j + k$ is even/odd. □

12.8. **Vector bundles on S^3.** Now we have to calculate $\pi_2 SO_n$. This is trivial for $n \leq 2$ (easy), and for $n = 3$ it also trivial: use $SO_3 \cong \mathbb{R}P^3$ and the fibration sequence $\mathbb{Z}/2 \hookrightarrow S^3 \twoheadrightarrow \mathbb{R}P^3$. Finally, the fibration sequences $SO_{n-1} \hookrightarrow SO_n \twoheadrightarrow S^{n-1}$ now show that $\pi_2 SO_n = 0$ for all n. We have proven

Proposition 12.9. $\operatorname{Vect}_n(S^3) \cong \pi_2(SO_n) \cong 0$. *That is, every vector bundle on S^3 is trivializable.*

12.10. **Vector bundles on S^4.** Once again, we are reduced to calculating $\pi_3 SO_n$. Eventually one expects to get stuck here, but so far we have been getting lucky so let's keep trying. The group is trivial for $n \leq 2$, and for $n = 3$ it is $\mathbb{Z}$ using $SO_3 \cong \mathbb{R}P^3$ and $\mathbb{Z}/2 \to S^3 \to \mathbb{R}P^3$. Next look at the long exact homotopy sequence for the fibration $SO_3 \hookrightarrow SO_4 \to S^3$:

$$\cdots \to \mathbb{Z}/2 = \pi_4(S^3) \to \mathbb{Z} \to \pi_3 SO_r \to \mathbb{Z} \to \pi_2(SO_3) = 0.$$

It follows that $\pi_3 SO_4 \cong \mathbb{Z}^2$. Next do the same thing for $SO_4 \hookrightarrow SO_5 \to S^4$:

$$\mathbb{Z} = \pi_4 SO_4 \to \pi_3 SO_4 \to \pi_3 SO_5 \to 0.$$

Unfortunately we cannot go further without calculating the map $\pi_4 S^4 \to \pi_3 SO_4$, which is $\mathbb{Z} \to \mathbb{Z}^2$. So now we are indeed stuck, unless we can resolve this issue. Note, however, that the fibrations $SO_{n-1} \hookrightarrow SO_n \to S^{n-1}$ show that $\pi_3 SO_5 = \pi_3 SO_n$ for $n \geq 5$, so once we've figured this one out we know everything. We will not justify it here, but it turns out that the map $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$ is an inclusion. So we get that

Proposition 12.11. $\operatorname{Vect}_n(S^4) \simeq \pi_3(SO_n) \cong \begin{cases} 1 & n \leq 2 \\ \mathbb{Z} & n = 3 \\ \mathbb{Z}^2 & n = 4 \\ \mathbb{Z} & n > 5. \end{cases}$

With some additional work one can write down a table of all bundles on S^4, much as we did for S^2, and figure out how all the direct sums behave. We won't bother with this.

12.12. **Vector bundles on S^k.** Although we can not readily do the calculations for $k > 4$, at this point one sees the general pattern. One must calculate $\pi_{k-1} SO_n$ for each n, and these groups vary for a while but eventually stabilize. In fact, $\pi_i SO_n \cong \pi_i SO_{n+1}$ for $i + 1 < n$. The calculation of these stable groups was an important problem back in the 1950s, that was eventually solved by Bott.

Let us phrase things as follows. Consider the inclusions

$$O_1 \hookrightarrow O_2 \hookrightarrow O_3 \hookrightarrow \cdots$$

that send a matrix A to $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. The colimit of this sequence is denoted O and called the **stable orthogonal group**. The homotopy groups of O are the stable values that we encountered above. We computed the first few: $\pi_0 O = \mathbb{Z}/2$, $\pi_1 O = \mathbb{Z}/2$, $\pi_2 O = 0$. And we stated, without proof, that $\pi_3 O = \mathbb{Z}$. Bott's calculation showed the following:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi_i O$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$

The pattern is 8-fold periodic: $\pi_{i+8} O \cong \pi_i O$ for all $i \geq 0$. One is supposed to remember the pattern of groups to the tune of "Twinkle, Twinkle, Little Star":

zee - two - zee - two - ze - ro - zee ze - ro - ze - ro - ze - ro - zee.

We will eventually have to understand Bott's computations at a deeper level; in particular, we will need to get our hands on explicit generators. But for now we will just accept that the values are as given above.

12.13. **Complex vector bundles on spheres.** One can repeat the above analysis for complex vector bundles on a sphere. One finds that

$$\mathrm{Vect}_n^{\mathbb{C}}(S^k) \cong \pi_{k-1}(GL_n(\mathbb{C})) \cong \pi_{k-1}(U_n),$$

where $U_n \hookrightarrow GL_n(\mathbb{C})$ is the unitary group. Analogously to the real case, one has fiber bundles $U_{n-1} \hookrightarrow U_n \twoheadrightarrow S^{2n-1}$. Using that $U_1 \cong S^1$ one can again compute $\mathrm{Vect}_n^{\mathbb{C}}(S^k)$ for small values of k. Here is what you get:

n	1	2	3	4	5	6	$\cdots$
$\mathrm{Vect}_n(S^1)$	0	0	0	0	0	0	$\cdots$
$\mathrm{Vect}_n(S^2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
$\mathrm{Vect}_n(S^3)$	0	0	0	0	0	0	$\cdots$
$\mathrm{Vect}_n(S^4)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
$\mathrm{Vect}_n(S^5)$	0	$\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	$\cdots$

The stable value in the last row turns out to be 0, although one cannot figure this out without computing a connecting homomorphism in the long exact homotopy sequence.

The fiber bundles $U_{n-1} \hookrightarrow U_n \twoheadrightarrow S^{2n-1}$ again imply that $\pi_i U_n$ stabilizes as n grows. In fact, $\pi_i U_n \cong \pi_i U_{n+1}$ for $n > \frac{i}{2}$. We can write the stable value as $\pi_i U$ where U is the **infinite unitary group** defined as the colimit of

$$U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \cdots$$

Bott computed the homotopy groups of U to be 2-fold periodic, with

$$\pi_i U = \begin{cases} \mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

Again, for now we will just accept this result; but eventually we will have to understand the computation in more detail, and in particular we will need to get our hands on specific generators.

13. Topological K-theory

For a compact and Hausdorff space X, let $KO(X)$ denote the Grothendieck group of real vector bundles over X. Swan's Theorem gives that $KO(X) \cong K_{alg}(C(X))$, where the latter denotes the Grothendieck group of finitely-generated projectives. We can repeat this definition for both complex and quaternionic bundles, to define groups $KU(X)$ and $KSp(X)$, respectively. The group $KU(X)$ is most commonly just written $K(X)$ for brevity. In this section we start to develop the general theory of these groups, mostly concentrating on $KO(X)$ because the story is very analogous in the three cases.

Until we explicitly mention otherwise, all spaces in this section are assumed to be compact and Hausdorff.

13.1. **Initial observations on KO.** Observe that $KO(-)$ is a contravariant functor: if $f\colon X \to Y$ then $f^*\colon KO(Y) \to KO(X)$ sends $[E]$ to $[f^*E]$. In particular, the squash map $p\colon X \to *$ yields a split-inclusion $p^*\colon KO(*) \to KO(X)$, where the splitting is induced by any choice of basepoint in X. One has $KO(*) \cong \mathbb{Z}$, so $\mathbb{Z}$ is a direct summand of $KO(X)$. To analyze the complement we can take two different approaches:

Definition 13.2. *For $x \in X$ let $\widetilde{KO}(X,x) = \ker[KO(X) \xrightarrow{i^*} KO(x)]$ where $i\colon \{x\} \hookrightarrow X$. Further, define $KO^{st}(X) = KO(X)/p^*KO(*)$.*

The group $\widetilde{KO}(X,x)$ is called the **reduced KO-group** of the pointed space X. We call $KO^{st}(X)$ the **Grothendieck group of stable vector bundles on X**. The reason for the latter terminology will be clear momentarily. These two groups are isomorphic; algebraically, this is coming from the split-exact sequence

$$0 \to KO(*) \to KO(X) \to KO^{st} \to 0.$$

If $i\colon \{x\} \hookrightarrow X$ is the inclusion then i^* is a splitting for the first map in the sequence. One gets an isomorphism between $KO^{st}(X)$ and $\ker i^*$ in the evident way, by sending a class $[E]$ to $[E] - p^*i^*[E]$. This isomorphism is used so frequently that it is worth recording more visibly:

$$KO^{st}(X) \cong \widetilde{KO}(X,x) \text{ via } [E] \mapsto [E] - [\underline{\mathrm{rank}_x(E)}]. \tag{13.3}$$

Remark 13.4. Both $KO^{st}(X)$ and $\widetilde{KO}(X,x)$ appear often in algebraic topology, and topologists are somewhat cavalier about mixing them up. We give here one example where this can cause confusion.

Tensor product of bundles makes $KO(X)$ into a ring, via the formula $[E]\cdot[F] = [E \otimes F]$ and extending linearly. Then $\widetilde{KO}(X,x)$ is an ideal of this ring. Therefore $KO^{st}(X)$ may be given a product via the above isomorphism, but this product is *not* $[E]\cdot[F] = [E \otimes F]$. Indeed, it is clear that this definition would not be invariant under $E \mapsto E \oplus 1$. The product on $KO^{st}(X)$ is instead $[E]\cdot[F] = [E \otimes F] - (\mathrm{rank}\, E)[F] - (\mathrm{rank}\, F)[E] + (\mathrm{rank}\, E)(\mathrm{rank}\, F)$.

We offer the following alternative description of $KO^{st}(X)$. Let $\mathrm{Vect}(X)$ be the set of isomorphism classes of vector bundles on X, and impose the equivalence relation $E \simeq E \oplus \underline{1}$ for every vector bundle E. The set of equivalence classes is obviously a monoid under direct sum (this would be true even without taking equivalence classes), but it is actually more than a monoid: it is a group. To see this, recall that if E is any vector bundle over X then there exists an embedding $E \hookrightarrow \underline{N}$ for sufficiently large N (Proposition 9.5). If Q is the quotient then we have the exact sequence $0 \to E \to \underline{N} \to Q \to 0$, which is split by Proposition 9.2. So $E \oplus Q \cong \underline{N}$. Yet $\underline{N} = 0$ under our equivalence relation, and so E has an additive inverse. It is easy to see that $KO^{st}(X)$ is precisely this set of equivalence classes.

Finally, here is a third description of $KO^{st}(X)$. Consider the chain of maps

$$\mathrm{Vect}_0(X) \xrightarrow{\oplus 1} \mathrm{Vect}_1(X) \xrightarrow{\oplus 1} \mathrm{Vect}_2(X) \xrightarrow{\oplus 1} \cdots$$

The colimit is clearly the set of equivalence classes described in the preceding paragraph, and therefore coincides with $KO^{st}(X)$.

Recall that $\mathrm{Vect}_n(X) = [X, \mathrm{Gr}_n(\mathbb{R}^\infty)]$, and one easily sees that the $\oplus 1$ map is represented by the map of spaces

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_{n+1}(\mathbb{R} \oplus \mathbb{R}^\infty) = \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$$

that sends a subspace $U \subseteq \mathbb{R}^\infty$ to $\mathbb{R} \oplus U \subseteq \mathbb{R} \oplus \mathbb{R}^\infty$. Let $\mathrm{Gr}_\infty(\mathbb{R}^\infty)$ denote the colimit of these maps

$$\mathrm{Gr}_1(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \mathrm{Gr}_2(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \mathrm{Gr}_3(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \cdots$$

(we really want the homotopy colimit, if you know what that is, but in this case the colimit has the same homotopy type and is good enough). You might recall that $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is also called BO_n, and likewise $\mathrm{Gr}_\infty(\mathbb{R}^\infty)$ is also called BO.

Then (for compact Hausdorff spaces X) we have a bijection

$$\operatorname*{colim}_n\, [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \longrightarrow [X, \mathrm{Gr}_\infty(\mathbb{R}^\infty)].$$

So we have learned that $KO^{st}(X) \simeq [X, BO]$.

If X has a basepoint then we can consider $[X, BO]_*$ instead of $[X, BO]$. There is the evident map $[X, BO]_* \to [X, BO]$. Typically there would be no reason for this to be a bijection, but BO is a path-connected H-space: and in this setting the map *is* a bijection. So in fact we can write

$$KO^{st}(X) \simeq [X, BO]_*.$$

NOTE: PROBLEM WITH $X = S^0$.

Applying this in particular to $X = S^k$ we have that for $k \geq 1$

$$\widetilde{KO}(S^k) \cong KO^{st}(S^k) \cong [S^k, BO] \cong [S^k, BO]_* = \pi_k(BO) = \pi_{k-1}(O).$$

The calculations of Bott therefore give us the values of $\widetilde{KO}(S^k)$. For $k = 0$ observe that $KO(S^0) = KO(* \sqcup *) \cong \mathbb{Z} \oplus \mathbb{Z}$, so we have $\widetilde{KO}(S^0) \cong \mathbb{Z}$. This lets us fill out the table:

TABLE 13.4. Reduced KO-theory of spheres

k	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\widetilde{KO}(S^k)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\cdots$

Now let X be an arbitrary CW-complex, not necessarily compact. We define

$$KO(X) = [X_+, \mathbb{Z} \times BO]_* = [X, \mathbb{Z} \times BO],$$

where X_+ denotes X with a basepoint added. For a pointed space X we define $\widetilde{KO}(X) = [X, \mathbb{Z} \times BO]_*$.

As we have seen it before, Bott Periodicity shows that the homotopy groups of $\mathbb{Z} \times BO$ are 8-fold periodic. This is a consequence of the following stronger statement:

Theorem 13.5 (Bott Periodicity, Strong version)**.** *There is a weak equivalence of spaces $\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO)$.*

Using Bott Periodicity we can then calculate that for every pointed space X one has

$$\widetilde{KO}(\Sigma^8 X) = [\Sigma^8 X, \mathbb{Z} \times BO]_* = [X, \Omega^8(\mathbb{Z} \times BO)]_* = [X, \mathbb{Z} \times BO]_* = \widetilde{KO}(X).$$

Remark 13.6. In the complex case, Bott Periodicity gives the weak equivalence $\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU)$. Consequently one obtains $\widetilde{K}(\Sigma^2 X) \cong \widetilde{K}(X)$ for all pointed spaces X.

13.7. ***K*-theory as a cohomology theory.** When X is compact and Hausdorff we have seen that $KO(X) \cong [X_+, \mathbb{Z} \times BO]_*$, where X_+ is X with a disjoint basepoint added. The point of this isomorphism is that it immediately gives us several tools for computing $KO(X)$ that we didn't have before. These are tools that work for homotopy classes of maps in reasonable generality, so let us discuss them in that broader context.

Let X and Z be pointed spaces. Then $[X, Z]_*$ is just a pointed set, but if we suspend the space in the domain then we get a bit more structure: $[\Sigma X, Z]_*$ is a group, where ΣX is the reduced suspension of X. One way to see this is to collapse the equatorial copy of X in ΣX, to get $\Sigma X \vee \Sigma X$; write this collapse map as

$$\nabla \colon \Sigma X \to \Sigma X \vee \Sigma X.$$

The operation on $[\Sigma X, Z]_*$ is defined by precomposing the wedge of two homotopy classes with ∇. With some trouble one checks that ΣX is a cogroup object in the homotopy category of pointed spaces, which yields that $[\Sigma X, Z]_*$ is a group.

Here is another way to think about this, which relates it to something we already know. Let $F(X, Z)$ be the set of functions from X to Z, equipped with the compact open topology. We can write

$$[\Sigma X, Z]_* = [S^1, F(X, Z)]_* = \pi_1(F(X, Z))$$

where the basepoint of $F(X, Z)$ is the map sending all of X to the basepoint of Z. Now just use that $\pi_1(F(X, Z))$ is a group.

When $k \geq 2$ then we have $[\Sigma^k X, Z]_* = \pi_k(F(X, Z))$ by a similar argument, and so $[\Sigma^k X, Z]_*$ is an abelian group. Alternatively, one proves that now $\Sigma^k X$ is a *cocommutative* cogroup object in the homotopy category.

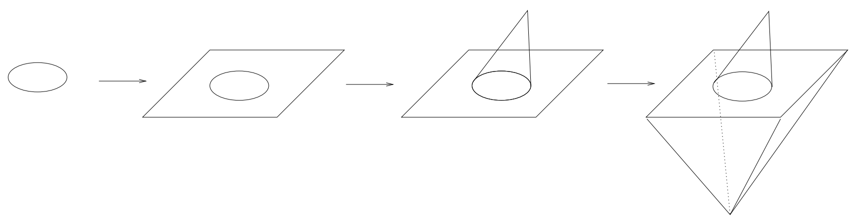
Similar results are obtained by putting conditions on Z rather than X. If Z is a loop space, say $Z \simeq \Omega Z_1$, then $[X, Z]_* \cong [X, \Omega Z_1]_* \cong [\Sigma X, Z_1]_*$, and this is a group by the above arguments. Similarly, if Z is a k-fold loop space for $k \geq 2$, say $Z \simeq \Omega^k Z_1$, then $[X, Z]_* \cong [\Sigma^k, Z_1]_*$ and this is an abelian group.

Homotopy classes of maps into a fixed space Z always give rise to exact sequences:

Proposition 13.8. *Let X, Y be pointed spaces, and let $f \colon X \to Y$ be a pointed map. Consider the mapping cone Cf and the natural map $p \colon Y \to Cf$. For any pointed space Z, the sequence of pointed sets $[X, Z]_* \leftarrow [Y, Z]_* \leftarrow [Cf, Z]_*$ is exact in the middle.*

Proof. Let $h\colon Y \to Z$ and suppose $h \circ f$ is homotopic to the constant map. Choose a pointed homotopy $H\colon X \times I \to Z$ so that $H(X \times \{1\}) = *$. Then H induces a map from $CX \to Z$. Let $g\colon Cf \to Z$ be given by H on CX and h on Y. Then clearly $g \circ p = h$. □

Given $f\colon X \to Y$ we form the mapping cone Cf, which comes to us with an inclusion $j_0\colon Y \hookrightarrow Cf$. Next form the mapping cone on i, which comes with an inclusion $j_1\colon Cf \hookrightarrow Cj_0$. Keep doing this forever to get the sequence of spaces $X \to Y \to Cf \to Cj_0 \to Cj_1 \to \cdots$ depicted below:



Note that $Cj_0 \simeq \Sigma X$ and $Cj_1 \simeq \Sigma Y$ (this is clear from the pictures). Up to sign the map $Cj_0 \to Cj_1$ is just Σf, so that the sequence of spaces becomes periodic:

$$X \to Y \to Cf \to \Sigma X \to \Sigma Y \to \Sigma(Cf) \to \Sigma^2 X \to \dots$$

This is called the **Puppe sequence**. Note that the composition of two subsequent maps is null-homotopic, and that every three successive terms form a cofiber sequence.

Now let Z be a fixed space and apply $[-, Z]_*$ to the Puppe sequence. We obtain the sequence of pointed sets

$$[X, Z]_* \leftarrow [Y, Z]_* \leftarrow [Cf, Z]_* \leftarrow [\Sigma X, Z]_* \leftarrow [\Sigma Y, Z]_* \leftarrow [\Sigma(Cf), Z]_* \leftarrow \dots$$

By Proposition 13.8 this sequence is exact at every spot where this makes sense (everywhere except at $[X, Z]_*$). At the left end this is just an exact sequence of pointed sets, but as one moves to the right at some point it becomes an exact sequence of groups (namely, at $[\Sigma Y, Z]_*$). As one moves further to the right, it becomes an exact sequence of *abelian* groups by the time one gets to $[\Sigma^2 Y, Z]_*$.

If $Z \simeq \Omega Z_1$ then we can extend the above sequence a little further to the left, by noticing that the sequences for $[-, Z_1]_*$ and $[-, Z]_*$ mesh together:

$$\begin{array}{ccccccccccc}
[X, Z_1]_* & \leftarrow & [Y, Z_1]_* & \leftarrow & [Cf, Z_1]_* & \leftarrow & [\Sigma X, Z_1]_* & \leftarrow & [\Sigma Y, Z_1]_* & \leftarrow & [\Sigma(Cf), Z_1] \leftarrow \\
 & & & & & & \| & & \| & & \| \\
 & & & & & & [X, \Omega Z_1]_* & \leftarrow & [Y, \Omega Z_1]_* & \leftarrow & [Cf, \Omega Z_1] \leftarrow \\
 & & & & & & \| & & \| & & \| \\
 & & & & & & [X, Z]_* & \leftarrow & [Y, Z]_* & \leftarrow & [Cf, Z] \leftarrow
\end{array}$$

Note that the leftmost cycle of the original sequence, which we had thought consisted just of pointed sets, in fact consisted of groups! If in turn we have $Z_1 \simeq \Omega Z_2$ then we can play this game again and extend the sequence one more cycle to the left, and so forth. If we are *really* lucky then we can do this forever:

Definition 13.9. *An **infinite loop space** is a space Z_0 together with spaces $Z_1, Z_2, Z_3, \dots$ and weak homotopy equivalences $Z_n \simeq \Omega Z_{n+1}$ for all $n \geq 0$.*

Note that if Z is an infinite loop space then we really do get a long exact sequence—infinite in both directions—consisting entirely of abelian groups, having the form

$$\cdots \leftarrow [Cf, Z_{i+1}] \leftarrow [X, Z_i]_* \leftarrow [Y, Z_i]_* \leftarrow [Cf, Z_i]_* \leftarrow [X, Z_{i-1}] \leftarrow \cdots$$

where it is convenient to use the indexing convention $Z_{-n} = \Omega^n Z$ for $n > 0$.

This situation is very reminiscent of a long exact sequence in cohomology, so let us adopt the following notation: write

$$E^i_Z(X) = [X_+, Z_i]_* = \begin{cases} [X_+, Z_i]_* & i \geq 0, \\ [\Sigma^{-i}(X_+), Z_0]_* & i < 0. \end{cases}$$

For an inclusion of subspaces $j\colon A \hookrightarrow X$ write

$$E^i_Z(X, A) = [Cj, Z_i]_+ = \begin{cases} [Cj, Z_i]_* & i \geq 0, \\ [\Sigma^i(Cj), Z_0]_* & i < 0. \end{cases}$$

It is not hard to check that this *is* a generalized cohomology theory. So we get a generalized cohomology theory whenever we have an infinite loop space. (You may know that it works the other way around, too: every generalized cohomology comes from an infinite loop space. But we won't need that fact here.)

For us the importance of all of this is that by Bott's theorem we have

$$\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO) \simeq \Omega^{16}(\mathbb{Z} \times BO) \simeq \dots.$$

Thus, $\mathbb{Z} \times BO$ is an infinite loop space and the above machinery applies. We obtain a cohomology theory KO^*. Moreover, periodicity gives us that $KO^{i+8}(X, A) \cong KO^i(X, A)$, for any i.

This all works in the complex case as well. There we have $\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU)$, so $\mathbb{Z} \times BU$ is again an infinite loop space. We get a cohomology theory K^* that is 2-fold periodic.

13.10. **Afterward.** The point of this section was to construct the cohomology theories KO and K, having the properties that when X is compact and Hausdorff the groups $KO^0(X)$ and $K^0(X)$ coincide with the Grothendieck groups of real and complex vector bundles over X. We have now accomplished this! We will spend the rest of these notes exploring what one can do with such cohomology theories, i.e., what they are good for. We have already said that one thing they are good for is calculation; we close this section with an example demonstrating the benefits and limitations here.

Let us try to compute $KO(\mathbb{R}P^2)$. Recall the ubiquitous decomposition $KO(\mathbb{R}P^2) = \mathbb{Z} \oplus \widetilde{KO}(\mathbb{R}P^2) = \mathbb{Z} \oplus KO^{st}(\mathbb{R}P^2)$. Next use the fact that $\mathbb{R}P^2$ can be built by attaching a 2-cell to $\mathbb{R}P^1 = S^1$, where the attaching map wraps S^1 around itself twice. That is, $\mathbb{R}P^2$ is the mapping cone for $S^1 \xrightarrow{2} S^1$. The Puppe sequence for this map looks like

$$S^1 \xrightarrow{2} S^1 \longrightarrow \mathbb{R}P^2 \longrightarrow S^2 \xrightarrow{2} S^2 \longrightarrow \cdots$$

hence we have an exact sequence

$$\leftarrow \cdots \leftarrow \widetilde{KO}(S^1) \leftarrow \widetilde{KO}(S^1) \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \widetilde{KO}(S^2) \leftarrow \widetilde{KO}(S^2) \leftarrow \cdots$$

Note that this is just the long exact sequence for the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$ in $\widetilde{KO}$-cohomology, where we are using the identification $\widetilde{KO}(S^2) = \widetilde{KO}^0(S^2) = \widetilde{KO}^{-1}(S^1)$.

We know $\widetilde{KO}(S^k)$ for all $k \geq 0$, so the above sequence becomes

$$\mathbb{Z}/2 \leftarrow \mathbb{Z}/2 \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \mathbb{Z}/2 \leftarrow \mathbb{Z}/2$$

where both maps $\mathbb{Z}/2 \leftarrow \mathbb{Z}/2$ are multiplication by 2, i.e. the 0 map. Hence we have a short exact sequence

$$0 \leftarrow \mathbb{Z}/2 \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \mathbb{Z}/2 \leftarrow 0, \tag{13.11}$$

and so either $\widetilde{KO}(\mathbb{R}P^2) = (\mathbb{Z}/2)^2$ or $\widetilde{KO}(\mathbb{R}P^2) = \mathbb{Z}/4$. It remains to decide which one.

The short exact sequence in (13.11) is really

$$0 \longleftarrow \widetilde{KO}(S^1) \xleftarrow{i^*} \widetilde{KO}(\mathbb{R}P^2) \xleftarrow{p^*} \widetilde{KO}(S^2) \longleftarrow 0.$$

We have previously seen that the generator of $\widetilde{KO}^0(S^1) = KO^{st}(S^1)$ corresponds to the Mobius bundle $[M]$, and the generator of $\widetilde{KO}(S^2) = KO^{st}(S^2)$ is $[\mathcal{O}(1)]$, the rank 2 bundle whose clutching map is the isomomorphism $S^1 \to SO(2)$. The image of $[\mathcal{O}(1)]$ in $\widetilde{KO}^0(\mathbb{R}P^2)$ is $p^*\mathcal{O}(1)$, where $p\colon \mathbb{R}P^2 \to S^2$ is the projection.

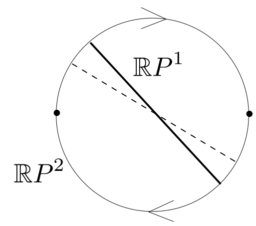
We happen to know one bundle on $\mathbb{R}P^2$, the tautological line bundle γ. When we restrict γ to $\mathbb{R}P^1$ we get M, and so $[\gamma]$ is a preimage for $[M]$ under i^*. We need to decide if $2[\gamma] = 0$ in $KO^{st}(\mathbb{R}P^2)$; if it is, then $\widetilde{KO}(\mathbb{R}P^2) \cong (\mathbb{Z}/2)^2$ and if it is not then $\widetilde{KO}(\mathbb{R}P^2) \cong \mathbb{Z}/4$. So the question becomes: is $\gamma \oplus \gamma$ stably trivial?

The answer turns out to be that $\gamma \oplus \gamma$ is *not* stably trivial; this is an elementary exercise using characteristic classes (Stiefel-Whitney classes), but we have not discussed such techniques yet—see Section 23.8 below for complete details. For now we will just accept this fact, and conclude that $\widetilde{KO}(\mathbb{R}P^2) \cong \mathbb{Z}/4$. Note that this calculation demonstrates an important principle to keep in mind: often the machinery of cohomology theories get you a long way, but not quite to the end, and one has to do some geometry to complete the calculation.

There is a better way to think about this calculation, and we can't resist pointing it out even though it won't make complete sense yet. But it ties in to intersection theory, which is our overarching theme in these notes. In our discussion above we used $KO^{st}(\mathbb{R}P^2)$ as our model for $\widetilde{KO}(\mathbb{R}P^2)$, but let us change perspective and use the model that is the kernel of $KO(\mathbb{R}P^2) \to KO(*)$, for some chosen basepoint. Recall that $[E]$ in $KO^{st}(\mathbb{R}P^2)$ corresponds to $[E] - \text{rank}(E)$ in $\widetilde{KO}(\mathbb{R}P^2)$; so the class we wrote as $[\gamma]$ is $[\gamma] - 1$ in the shifted perspective, and we need to decide if $2([\gamma] - 1) = 0$ in $KO(\mathbb{R}P^2)$. The element $1 - [\gamma]$ should be thought of as corresponding to a chain complex of vector bundles

$$0 \to \gamma \to 1 \to 0,$$

and thinking of it this way one finds that it plays the role of the K-theoretic fundamental class of the submanifold $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$. Then $(1 - [\gamma])^2$ represents the self-intersection product of $\mathbb{R}P^1$ inside $\mathbb{R}P^2$, which we know is a point by the standard geometric argument (shown in the picture below, depicting an $\mathbb{R}P^1$ and a small perturbation of it)):



In particular, the self-intersection is not empty. This translates to the statement that $(1-[\gamma])^2 \neq 0$. But

$$(1-[\gamma])^2 = 1 - 2[\gamma] + [\gamma^2] = 1 - 2[\gamma] + 1 = 2(1-[\gamma])$$

where we have used that $\gamma \otimes \gamma \cong \underline{1}$ (this is true for any real line bundle, over any base); so this explains why $2(1-[\gamma]) \neq 0$. Again, we understand this argument doesn't make much sense yet. We will come back to it in Section ?????. For the moment just get the idea that it is the intersection theory of submanifolds in $\mathbb{R}P^2$ that is ultimately forcing $\widetilde{KO}(\mathbb{R}P^2)$ to be $\mathbb{Z}/4$ rather than $(\mathbb{Z}/2)^2$.

Remark 13.12. It seems worth pointing out that in fact for every n one has $\widetilde{KO}(\mathbb{R}P^n) \cong \mathbb{Z}/2^k$ for a certain value k depending on n. We will return to this calculation (and complete it) in Section 32.

Exercise 13.13. It is a good idea for the reader to try his or her hand at similar calculations, to see how the machinery is working. Try calculating some of the groups below, at least for small values of n:

- $K(\mathbb{C}P^n)$ (reasonably easy)
- $KO(\mathbb{C}P^n)$ (a little harder)
- $K(\mathbb{R}P^n)$ (even harder)
- $KO(\mathbb{R}P^n)$ (hardest).

Don't worry if you can't completely determine some of the groups; just see how far the machinery takes you.

14. Vector fields on spheres

It is a classical problem to determine how many independent vector fields one can construct on a given sphere S^n. This problem was heavily studied throughout the 1940s and 1950s, and then finally solved by Adams in 1962 using K-theory. It is one of the great successes of generalized cohomology theories. In this section we discuss some background to the vector field problem. We will not tackle the solution until Section 33, when we have more tools at our disposal.

14.1. **The vector field problem.** Given a nonzero vector $u = (x, y)$ in $\mathbb{R}^2$, there is a formula for producing a (nonzero) vector that is orthogonal to u: namely, $(-y, x)$. However, there is no analog of this that works in $\mathbb{R}^3$. That is, there is no single formula that takes a vector in $\mathbb{R}^3$ and produces a (nonzero) orthogonal vector. If such a formula existed then it would give a nonvanishing vector field on S^2, and we know that such a thing does not exist by elementary topology.

Let us next consider what happens in $\mathbb{R}^4$. Given $u = (x_1, x_2, x_3, x_4)$, we can produce an orthogonal vector via the formula $(-x_2, x_1, -x_4, x_3)$. But of course this is not the only way to accomplish this: we can vary what pairs of coordinates we choose to flip. In fact, if we consider

$$v_1 = \begin{bmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -x_4 \\ -x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

then we find that v_1, v_2, and v_3 are not only orthogonal to u but they are orthogonal to each other as well. In particular, at each point of S^3 we have given an orthogonal basis for the tangent space.

We aim to study this problem for any $\mathbb{R}^n$. What is the maximum k for which there exist formulas for starting with $u \in \mathbb{R}^n$ and producing k orthogonal vectors, with u as the first of the set? The following gives a different phrasing for the same question:

Question 14.2. *On S^n, how many vectors fields $v_1, v_2, \dots, v_r$ can we find so that $v_1, v_2, \dots, v_r$ are linearly independent for each $x \in S^n$?*

Note that by the Gram-Schmidt process we can replace "linearly independent" by "orthonormal." If n is even, the answer is zero because there does not exist even a single nonvanishing vector field on an even sphere. To start to see what happens when n is odd, we look at a couple of more examples.

Let $u \in S^5$ have the standard coordinates. We notice that the vector $v_1 = (-x_2, x_1, -x_4, x_3, -x_6, x_5)$ is orthogonal to v. However, a little legwork shows that no other pattern of switching coordinates will produce a vector that is orthogonal to both u and v_1. Of course this does not mean that there isn't some more elaborate formula that would do the job, but it shows the limits of what we can do using our naive constructions.

For $v \in S^7$ we can divide the coordinates into the top four and the bottom four. Take the construction that worked for S^3 and repeat it simultaneously in the top and bottom coordinates—this yields a set of three orthonormal vector fields on S^7, given by the formulas

$$\begin{aligned} &(-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7), \\ &(-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6), \\ &(-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5). \end{aligned} \tag{14.2}$$

This idea generalizes at once to prove the following:

Proposition 14.3. *If there exist r (independent) vector fields on S^{n-1}, then there also exist r vector fields on S^{kn-1} for all k.*

For example, since there is one vector field on S^1 we also know that there is at least one vector field on S^{2k-1} for every k. Likewise, since there are three vector fields on S^3 we know that there are at least three vector fields on S^{4k-1} for every k.

We have constructed three vector fields on S^7, but one can actually make seven of them. This can be done via trial-and-error attempts at extending the patterns in (14.2), but there is a slicker way to accomplish this as well. Recall that S^3 is a Lie group, being the unit quaternions inside of $\mathbb{H}$. We can choose an orthonormal frame at the origin and then use the group structure to push this around to any point, thereby obtaining three independent vector fields; in other words, for any point $x \in S^3$ use the derivative of right-multiplication-by-x to transport our vectors in T_1S^3 to T_xS^3. The space S^7 is not quite a Lie group, but it still has a multiplication coming from being the set of unit octonions. The multiplication is not associative, but this is of no matter—the same argument works to construct 7 vector fields on S^7. Note that this immediately gives us 7 vectors fields on S^{15}, S^{23}, etc.

Based on the data so far, one would naturally guess that if $n = 2^r$ then there are $n-1$ vector fields on S^{n-1}. However, this guess turns out to fail already when $n = 16$ (and thereafter). To give a sense of how the numbers grow, we give a chart showing the maximum number of vector fields known to exist on low-dimensional spheres:

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$n-1$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
vf on S^{n-1}	1	3	1	7	1	3	1	8	1	3	1	7	1	3	1	9

Notice that we have explained how to construct the requisite number of vector fields until we get to S^{15}—there we know how to make seven of them, but the claim is that one can make one more. Once we know how to make eight on S^{15} we automatically know how to make eight on S^{31}, but the claim is again that one can construct one more.

Okay. Now that we have a basic sense of the problem let us explain the numerology behind the answer.

Definition 14.4. *If $n = m \cdot 2^{a+4b}$ where m is odd, then the **Hurwitz-Radon number** for n is $\rho(n) = 2^a + 8b - 1$.*

Theorem 14.5 (Hurwitz-Radon)**.** *There exist at least $\rho(n)$ independent vector fields on S^{n-1}.*

Consider $n = 32 = 2^5 = 2^{1+4\cdot 1}$. Using the definition, $a = b = 1$. Then $\rho(32) = 2^1 + 8(1) - 1 = 9$. That is, there are at least 9 vector fields on S^{31}. If $n = 1024 = 2^{10} = 2^{2+4\cdot 2}$ then $\rho(n) = 2^2 + 8 \cdot 2 - 1 = 19$; one can make 19 independent vector fields on S^{1023}. One should of course notice that these number are not going up very quickly.

We will prove the Hurwitz-Radon theorem by a slick, modern method using Clifford algebras. But it is worth pointing out that the theorem can be proven through very naive methods, too (it was proven in the 1920s). All of the Hurwitz-Radon vector fields follow the general patterns that we have seen, of switching pairs of coordinates and changing signs—one only has to find a way to organize the bookkeeping behind these patterns.

14.6. **Sums-of-squares formulas.** Hurwitz and Radon were not actually thinking about vector fields on spheres. They were instead considering an algebraic question about the existence of certain kinds of "composition formulas" for quadratic forms. For example, the following identity is easily checked:

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2.$$

Hurwitz and Radon were looking for more formulas such as this one, for larger numbers of variables:

Definition 14.7. *A **sum-of-squares formulas** of type* $[r,s,n]$ *is an identity*

$$(x_1^2 + x_2^2 + \ldots + x_r^2)(y_1^2 + y_2^2 + \ldots + y_s^2) = z_1^2 + z_2^2 + \ldots + z_n^2$$

in the polynomial ring $\mathbb{R}[x_1, \ldots, x_r, y_1, \ldots, y_s]$*, where each* z_i *is a bilinear expression in* x*'s and* y*'s.*

We will often just refer to an "$[r,s,n]$-formula", for brevity. For what values of r, s, and n does such a formula exist? This is currently an open question. There are three formulas that are easily produced, coming from the normed algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The multiplication is a bilinear pairing, and the identity $|xy|^2 = |x|^2|y|^2$ is the required sums-of-squares formula. These algebras give formulas of type $[2,2,2]$, $[4,4,4]$, and $[8,8,8]$. In a theorem from 1898 Hurwitz proved that these are the only normed algebras over the reals, and in doing so ruled out the existence of $[n,n,n]$-formulas for $n \notin \{1,2,4,8\}$. The question remained (and remains) about other types of formulas. See ???? for a detailed history of this problem.

Perhaps surprisingly, most of what is known about the non-existence of sums-of-squares formulas comes from topology. To phrase the question differently, we are looking for a function $\phi\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ such that $|\phi(x,y)|^2 = |x|^2 \cdot |y|^2$ for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. The bilinear expressions $z_1, \ldots, z_n$ are just the coordinates of $\phi(x,y)$.

Write $z = \phi(x,y) = \sum x_j A_j y$, where the A_j's are $n \times s$ matrices. The sum of squares formula says that $z^T z = (x^T x) \cdot (y^T y)$. But $z^T z = \sum_{i,j} (y^T A_j^T x_j)(x_i A_i y)$, hence

$$y^T \left(\sum_{i,j} x_i x_j A_j^T A_i \right) y = y^T \Big((x^T x) I \Big) y$$

for all y. These quadratic forms in y are equal only if $\sum_{i,j} x_i x_j A_j^T A_i = (x^T x) I = \sum x_i^2 I$, and this must hold for all x. Equating coefficients of the monomials in x, we find that

- $A_i^T A_i = I$ (that is, $A_i \in O_n$) for every i, and
- $A_j^T A_i + A_i^T A_j = 0$ for every $i \neq j$.

The case $s = n$ turns out to be significantly simpler to address than the general case. If $s = n$ we may set $B_i = A_i^{-1} A_i$. Then the conditions to satisfy become

- $B_i^T = -B_i$
- $B_i^2 = -I_n$
- $B_i B_j = -B_j B_i$ for all $i \neq j$.

Note that the first two conditions imply $B_i \in O_n$, and the first and third conditions imply $B_j^T B_i + B_i^T B_j = 0$. So by replacing the A's with the B's we have proven the following:

Corollary 14.8. *If an* $[r,n,n]$*-formula exists, then one exists where* $A_1 = I$ *and* $A_i^T = -A_i$*.*

In the setting of the corollary, the necessary conditions on the matrices $A_2, A_3, \ldots, A_r$ become that $A_i^2 = -I$ and $A_i A_j = -A_j A_i$.

Corollary 14.9. *If an* $[r,n,n]$*-formula exists, then there exist* $r-1$ *independent vector fields on* S^{n-1}*.*

Proof. If $y \in S^{n-1}$ then $\phi(e_i, y) \in S^{n-1}$ for $i = 1, 2, \dots, r$: this follows from the identity $|\phi(e_i, y)|^2 = |e_i|^2 \cdot |y|^2$. We also have that $\phi(e_1, y) = y$ since $A_1 = I$. We claim that $\phi(e_i, y) \perp \phi(e_j, y)$ if $i \neq j$. To see this, note that by the norm formula

$$|\phi(e_i + e_j, y)|^2 = |e_i + e_j|^2 \cdot |y|^2 = 2|y|^2.$$

On the other hand,

$$\begin{aligned} |\phi(e_i + e_j, y)|^2 &= |\phi(e_i, y) + \phi(e_j, y)|^2 \\ &= |\phi(e_i, y)|^2 + |\phi(e_j, y)|^2 + 2\phi(e_i, y) \cdot \phi(e_j, y) \\ &= 2|y|^2 + \phi(e_i, y) \cdot \phi(e_j, y). \end{aligned}$$

We conclude $\phi(e_i, y) \cdot \phi(e_j, y) = 0$. Therefore we have established that $\phi(e_2, -)$, $\phi(e_3, -), \dots, \phi(e_r, -)$ are orthonormal vector fields on S^{n-1}. □

14.10. **Clifford algebras.** We have seen that we get $r - 1$ independent vector fields on S^{n-1} if we have a sums-of-squares formula of type $[r, n, n]$. Having such a formula amounts to producing matrices $A_2, A_3, \dots, A_r \in O_n$ such that $A_i^2 = -I$ and $A_i A_j + A_j A_i = 0$ for $i \neq j$. If we disregard the condition that the matrices be orthogonal, we can encoded the latter two conditions by saying that we have a representation of a certain algebra:

Definition 14.11. *The Clifford algebra* Cl_k *is defined to be the quotient of the tensor algebra* $\mathbb{R}\langle e_1, \dots, e_k\rangle$ *by the relations* $e_i^2 = -1$ *and* $e_i e_j + e_j e_i = 0$ *for all* $i \neq j$.

The first few Clifford algebras are familiar: $\mathrm{Cl}_0 = \mathbb{R}$, $\mathrm{Cl}_1 = \mathbb{C}$, and $\mathrm{Cl}_2 = \mathbb{H}$. After this things become less familiar: for example, it turns out that $\mathrm{Cl}_3 = \mathbb{H} \times \mathbb{H}$ (we will see why in just a moment). It is somewhat of a miracle that it is possible to write down a precise description of *all* of the Clifford algebras, and *all* of their modules. Before doing this, let us be clear about *why* we are doing it:

Theorem 14.12. *An* $[r, n, n]$*-formula exists if and only if there exists a* Cl_{r-1}*-module structure on* $\mathbb{R}^n$*. Consequently, if there is a* Cl_{r-1}*-module structure on* $\mathbb{R}^n$ *then there are* $r - 1$ *independent vector fields on* S^{n-1}*.*

Before giving the proof, we need one simple fact. The collection of monomials $e_{i_1} \cdots e_{i_r}$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ give a vector space basis for Cl_k, which has size 2^k (note that we include the empty monomial, corresponding to 1, in the basis). This is an easy exercise.

Proof of Theorem 14.12. The forward direction is trivial: Given an $[r, n, n]$-formula, Corollary 14.8 gives us such a formula with $A_1 = I$. Then define a Cl_{r-1}-module structure on $\mathbb{R}^n$ by letting e_i act as multiplication by A_{i+1}, for $1 \leq i \leq r-1$.

Conversely, assume that Cl_{r-1} acts on $\mathbb{R}^n$. We can almost reverse the procedure of the previous paragraph, except that there is no guarantee that the e_i's act orthogonally on $\mathbb{R}^n$—and we need $A_i \in O_n$ to get an $[r, n, n]$-formula.

Equip $\mathbb{R}^n$ with a positive-definite inner product, denoted $x, y \mapsto x \cdot y$. This inner product probably has no compatibility with the Clifford-module structure. So define a new inner product on $\mathbb{R}^n$ by

$$\langle v, w\rangle = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq r-1} (e_I v) \cdot (e_I w),$$

where $e_I = e_{i_1}e_{i_2}\cdots e_{i_j}$ and the sum runs over all 2^{r-1} elements of the standard basis for Cl_{r-1}. Basically we are averaging out the dot product. Our inner product $\langle v, w\rangle$ is a symmetric bilinear form, and it is positive definite because the dot product is positive definite. It also has the property that it is invariant under the Clifford algebra: $\langle e_i v, e_i w\rangle = \langle v, w\rangle$ for all i.

Now let $v_1, \ldots, v_n$ be an orthonormal basis for $\mathbb{R}^n$ with respect to our new inner product. Let A_i be the matrix for e_i with respect to this basis. Then the A_i's are orthogonal matrices, and the relations $A_i^2 = -I$ and $A_iA_j + A_jA_i = 0$ are automatic because they are satisfied in Cl_{r-1}. In this way we obtain the desired $[r, n, n]$-formula. □

Remark 14.13. Most modern treatments of vector fields on spheres go straight to Clifford algebras and their modules, without ever talking about sums-of-squares formulas. It seems to us that the sums-of-squares material is an interesting part of this whole story, both for historical reasons and for its own sake.

From now on we can focus on the following question: For what values of n do we have a Cl_{r-1}-module structure on $\mathbb{R}^n$? This is the neatest part of the story, because on the face of things it doesn't seem like we have accomplished anything by shifting our perspective onto Clifford algebras. We have, after all, just rephrased the basic question. But a miracle now occurs, in that we can analyze all the Clifford algebras by a simple trick.

To do this part of the argument, we need a slight variant on our Clifford algebras. Given a real vector space V and a quadratic form $q\colon V \to \mathbb{R}$, define

$$\mathrm{Cl}(V, q) = T_{\mathbb{R}}(V)/\langle v \otimes v = q(v)\cdot 1 \mid v \in V\rangle.$$

For $\mathbb{R}^k$ with $q(x_1, \ldots, x_k) = -(x_1^2 + \cdots + x_k^2)$ this recovers the algebra Cl_k. For $q(x_1, \ldots, x_k) = x_1^2 + \cdots + x_k^2$ this gives a new algebra we will call Cl_k^-. It will be convenient to temporarily rename Cl_k as Cl_k^+. Of course there are other quadratic forms on $\mathbb{R}^k$, but these will be the only two we need for our present purposes.

The discussion that follows is based on the one by Haynes Miller that is given in [M]. This is by far the best treatment of Clifford algebras that I have found in the literature.

Proposition 14.14. *There are isomorphisms of algebras* $\mathrm{Cl}_k^{\pm} \cong \mathrm{Cl}_2^{\pm} \otimes_{\mathbb{R}} \mathrm{Cl}_{k-2}^{\mp}$.

Proof. One possible isomorphism sends $e_1 \mapsto e_1 \otimes 1$, $e_2 \mapsto e_2 \otimes 1$, and for $i > 2$ we send $e_i \mapsto e_1e_2 \otimes e_{i-2}$. We leave it as an exercise to check that this works.. □

In the analysis that follows we will write $A(n)$ for the algebra $M_{n\times n}(A)$, whenever A is an algebra. The following table gives a list of the first ten Clifford algebras:

We will now explain how to obtain the entries of this table.

Step 1: Rows 0–2.
We have already remarked that $\mathrm{Cl}_0^+ \cong \mathbb{R}$, $\mathrm{Cl}_1^+ \cong \mathbb{C}$, and $\mathrm{Cl}_2^+ \cong \mathbb{H}$. It is just as easy to see that $\mathrm{Cl}_0^- \cong \mathbb{R}$ and $\mathrm{Cl}_1^- \cong \mathbb{R}\times\mathbb{R}$. Finally, with a little more work we also have $\mathrm{Cl}_2^- \cong \mathbb{R}(2)$. To get this last isomorphism, note that Cl_2^- is generated by e_1 and e_2 subject to the relations $e_1^2 = -1$, $e_2^2 = -1$, and $e_1e_2 = -e_2e_1$. The conditions $e_i^2 = -1$ might make you think of reflections, and we can realize the second relation by using two reflections through lines ℓ_1 and ℓ_2 of $\mathbb{R}^2$ that have a 45 degree angle

TABLE 14.15. Clifford algebras

r	Cl_r^+	Cl_r^-
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \times \mathbb{R}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \times \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \times \mathbb{H}(2)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$
7	$\mathbb{R}(8) \times \mathbb{R}(8)$	$\mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$
9	$\mathbb{C}(16)$	$\mathbb{R}(16) \times \mathbb{R}(16)$
10	$\mathbb{H}(16)$	$\mathbb{R}(32)$

between them. We get an algebra homomorphism $\mathrm{Cl}_2^- \to \mathbb{R}(2)$ by sending e_i to the matrix for reflection in ℓ_i. As both the domain and codomain are four dimensional over $\mathbb{R}$, it is not hard to prove that the map is surjective and therefore gives an isomorphism.

Step 2: Rows 3–4.
Now using Proposition 14.14 we find

$$\mathrm{Cl}_3^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_1^- \cong \mathbb{H} \otimes_{\mathbb{R}} (\mathbb{R} \times \mathbb{R}) \cong \mathbb{H} \times \mathbb{H}$$

and

$$\mathrm{Cl}_4^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_2^- \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}(2) \cong \mathbb{H}(2).$$

We also get

$$\mathrm{Cl}_3^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_1^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2),$$

and

$$\mathrm{Cl}_4^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_2^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{H}(2).$$

The reader will note that this is fairly easy. Miller [M] describes this process as like the one of lacing up a shoe.

Step 3: Rows 5–6.
Continuing via the same methods, we now get

$$\mathrm{Cl}_5^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_3^- \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}(2) \cong (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})(2).$$

Here we need to know something new, namely that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$ (see Lemma 14.16 below). So then

$$\mathrm{Cl}_5^+ \cong \mathbb{C}(2)(2) \cong \mathbb{C}(4).$$

Likewise, we have

$$\mathrm{Cl}_6^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_4^- \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}(2) \cong (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H})(2)$$

and we again need to know a new fact: this time, that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$ (once again, see Lemma 14.16 below). Hence

$$\mathrm{Cl}_6^+ \cong \mathbb{R}(8).$$

On the other side, we have

$$\mathrm{Cl}_5^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_3^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} (\mathbb{H} \times \mathbb{H}) \cong \mathbb{H}(2) \times \mathbb{H}(2),$$

and

$$\mathrm{Cl}_6^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_4^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{H}(4).$$

Step 4: Rows 7–8.
Although this may be getting tedious, let's do it yet again. We have

$$\begin{aligned}\mathrm{Cl}_7^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_5^- \cong \mathbb{H} \otimes_{\mathbb{R}} (\mathbb{H}(2) \times \mathbb{H}(2)) &\cong (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H})(2) \times (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H})(2) \\ &\cong \mathbb{R}(8) \times \mathbb{R}(8),\end{aligned}$$

and

$$\mathrm{Cl}_8^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_6^- \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}(4) \cong (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H})(4) \cong \mathbb{R}(4)(4) \cong \mathbb{R}(16).$$

Also,

$$\mathrm{Cl}_7^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_5^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{C}(4) \cong \mathbb{C}(8)$$

and

$$\mathrm{Cl}_8^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_6^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{R}(8) \cong \mathbb{R}(16).$$

Step 5: The rest of the table.
We are almost ready to stop. Notice that at row eight the two columns of the table come back into juxtaposition: this is a magical fact! We next see that

$$\mathrm{Cl}_9^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_7^- \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}(8) \cong \mathbb{C}(16)$$

and

$$\mathrm{Cl}_{10}^+ \cong \mathrm{Cl}_2^+ \otimes_{\mathbb{R}} \mathrm{Cl}_8^- \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}(16) \cong \mathbb{H}(16).$$

Also,

$$\mathrm{Cl}_9^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_7^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} (\mathbb{R}(8) \times \mathbb{R}(8)) \cong \mathbb{R}(16) \times \mathbb{R}(16)$$

and

$$\mathrm{Cl}_{10}^- \cong \mathrm{Cl}_2^- \otimes_{\mathbb{R}} \mathrm{Cl}_8^+ \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{R}(16) \cong \mathbb{R}(32).$$

It is completely trivial to now prove by induction that

$$\mathrm{Cl}_{k+8}^+ \cong \mathrm{Cl}_k^+(16) \qquad \text{and} \qquad \mathrm{Cl}_{k+8}^- \cong \mathrm{Cl}_k^-(16).$$

So the table has a quasi-periodicity to it, repeating every eight rows but with an extra factor of (16) everywhere. We have successfully determined *all* the Clifford algebras!

The above analysis used two non-obvious isomorphisms, which we now explain in the following lemma:

Lemma 14.16. *There are isomorphisms* $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$ *and* $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$.

Proof. Define an algebra map $\phi\colon \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H})$ by sending $q \otimes z$ to the map $v \mapsto qvz$ and extending linearly. One readily checks that this is a map of algebras. Notice that the map $\phi(q \otimes z)$ is actually $\mathbb{C}$-linear, if we give $\mathbb{H}$ the right action of $\mathbb{C}$. So we actually have a map of algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(\mathbb{H}) \cong \mathbb{C}(2)$. Both the domain and targer are four-dimensional over $\mathbb{C}$, and it is not hard to prove that the map is surjective—hence, it is an isomorphism.

The proof of the second claim is similar. Define $\theta\colon \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H}) \cong \mathbb{R}(4)$ by sending $q \otimes u$ to the map $v \mapsto qv\overline{u}$. The conjugation is needed in order to get a map of algebras. ??? $\square$

14.17. **Modules over Clifford algebras.** Now that we know all the Clifford algebras, it is actually an easy process to determine all of their finitely-generated modules. We need three facts:

- If A is a division algebra then all modules over A are free;
- By Morita theory, the finitely-generated modules over $A(n)$ are in bijective correspondence with the finitely-generated modules over A. The bijection sends an A-module M to the $A(n)$-module M^n.
- If R and S are algebras then modules over $R \times S$ can all be written as $M \times N$ where M is an R-module and N is an S-module.

In the following table we list each Clifford algebra Cl_r^+ and the dimension of its smallest nonzero module.

TABLE 14.18. Dimensions of Clifford modules

r	Cl_r^+	Smallest dim. of a module over Cl_r^+
0	$\mathbb{R}$	1
1	$\mathbb{C}$	2
2	$\mathbb{H}$	4
3	$\mathbb{H} \times \mathbb{H}$	4
4	$\mathbb{H}(2)$	8
5	$\mathbb{C}(4)$	8
6	$\mathbb{R}(8)$	8
7	$\mathbb{R}(8) \times \mathbb{R}(8)$	8
8	$\mathbb{R}(16)$	16
9	$\mathbb{C}(16)$	32
10	$\mathbb{H}(16)$	64

Note that the third column has a quasi-periodicity, where row $k + 8$ is obtained from row k by multiplying by 16.

After all of this, we are ready to prove the Hurwitz-Radon theorem about constructing vector fields on spheres. Recall that if Cl_{r-1} acts on $\mathbb{R}^n$ then there are $r - 1$ independent vector fields on S^{n-1}. Going down the rows of the above table, we make the following deductions:

$$\begin{aligned}
&\mathrm{Cl}_1 \text{ acts on } \mathbb{R}^2, \text{ therefore we have 1 vector field on } S^1\\
&\mathrm{Cl}_2 \text{ acts on } \mathbb{R}^4, \text{ therefore we have 2 vector field on } S^3\\
&\mathrm{Cl}_3 \text{ acts on } \mathbb{R}^4, \text{ therefore we have 3 vector field on } S^3\\
&\mathrm{Cl}_4 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 4 vector field on } S^7\\
&\mathrm{Cl}_5 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 5 vector field on } S^7\\
&\mathrm{Cl}_6 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 6 vector field on } S^7\\
&\mathrm{Cl}_7 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 7 vector field on } S^7\\
&\mathrm{Cl}_8 \text{ acts on } \mathbb{R}^{16}, \text{ therefore we have 8 vector field on } S^{15}.
\end{aligned}$$

It is not hard to deduce the general pattern here. The key is knowing where the jumps in dimension occur, and then just doing bookkeeping. To this end, note that the smallest dimension of a nonzero module over Cl_r is $2^{\sigma(r)}$ where

$$\sigma(r) = \#\{s : 0 < s \leq r \text{ and } s \equiv 0, 1, 2, \text{ or } 4 \mod (8)\}$$

We know that we can construct r independent vector fields on $S^{2^{\sigma(r)}-1}$.

Proof of Theorem 14.5 (Hurwitz-Radon Theorem). First note that we know much more about Clifford modules than is indicated in Table 14.18. For each Clifford algebra Cl_r we know the complete list of all isomorphism classes of finitely-generated modules, and their dimensions are all *multiples* of the dimension listed in the table. This is important.

Given an $n \geq 1$, our job is to determine the largest r for which Cl_r acts on $\mathbb{R}^n$. We will then know that there are r vector fields on S^{n-1}. If we write $n = 2^u \cdot (\text{odd})$ it is clear from Table 14.18 and the previous paragraph that the only way Cl_r could act on $\mathbb{R}^n$ is if it actually acts on $\mathbb{R}^{2^u}$. Moreover, the quasi-periodicity in the table shows that if we add 4 to u then the largest r goes up by 8. It follows at once that if $u = a + 4b$ then the formula for the largest r is $8b+???$ where the missing expression just needs to be something that works for the values $a = 0, 1, 2, 3$. One readily finds that $r = 8b + 2^a - 1$ does the job.

So we know that there are $8b + 2^a - 1$ vector fields on S^{n-1}, where n has the form $(\text{odd}) \cdot 2^{a+4b}$. □

Remark 14.19 (First connection with KO^*)**.** Return to Table 14.18 and look at the column with the smallest dimensions of the modules. As one reads down the column, consider where the jumps in dimensions occur: we have "jump-jump-nothing-jump-nothing-nothing-nothing-jump," which then repeats. This is strangely reminiscent of the periodic sequence

$$\mathbb{Z}_2 \; \mathbb{Z}_2 \; 0 \; \mathbb{Z} \; 0 \; 0 \; 0 \; \mathbb{Z} \ldots$$

This is quite the coincidence, and must have been a source of much excitement when it was first noticed. We will eventually see, following [ABS], that there is a very direct connection between the groups KO^* and the module theory of the Clifford algebras. For now we leave it as an intriguing coincidence.

14.20. **Adams's Theorem.** So far we have done all this work just to construct collections of independent vector fields on spheres. The Hurwitz-Radon lower bound is classical, and was probably well-known in the 1940's. The natural question is, can one do any better? Is there a different construction that would yield *more* vector fields than we have managed to produce? People were actively working on this problem throughout the 1950's. Adams finally proved in 1962 [Ad2] that the Hurwitz-Radon bound was maximal, and he did this by using K-theory:

Theorem 14.21 (Adams)**.** *There do not exist $\rho(n) + 1$ independent vector fields on S^{n-1}.*

This is a difficult theorem, and it will be a long while before we are able to prove it. We are introducing it here largely to whet the reader's appetite. Note that it is far from being immediately clear how a cohomology theory would help one prove the result. There are several reductions one must make in the problem, but the first one we can explain without much effort:

Proposition 14.22. *If there are $r-1$ vector fields on S^{n-1} then the projection $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1} \to \mathbb{R}P^{un-1}/\mathbb{R}P^{un-2} \cong S^{un-1}$ has a section in the homotopy category, for every $u > \frac{2k-2}{n}$.*

The existence of a section in the homotopy category is something that can perhaps be contradicted by applying a suitable cohomology theory $E^*(-)$. See Exercise 14.24 below for a simple example.

We close this section by sketching the proof of Proposition 14.22. Define

$$V_k(\mathbb{R}^n) = \{(u_1, \dots, u_k) \,|\, u_i \in \mathbb{R}^n \text{ and } u_1, \dots, u_k \text{ are orthonormal}\}.$$

This is called the **Stiefel manifold** of k-frames in $\mathbb{R}^n$. Consider the map $p_1\colon V_k(\mathbb{R}^n) \to S^{n-1}$ which sends $(u_1, \dots, u_n) \mapsto u_1$. There exist r vector fields on S^{n-1} if and only if there is a section of $p_1\colon V_{r+1}(\mathbb{R}^n) \to S^{n-1}$.

We need a fact from basic topology, namely that there is a cell structure on $V_k(\mathbb{R}^n)$ where the cells look like

$$e^{i_1} \times \cdots \times e^{i_s}$$

with $n-k \le i_1 < i_2 < \cdots < i_s \le n-1$ and s is arbitrary. We will not prove this here: see Hatcher [Ha, Section 3.D] or Mosher-Tangora [MT, ???].

The cell structure looks like

$$\big[e^{n-k} \cup e^{n-k+1} \cup \cdots \cup e^{n-1}\big] \cup \big[(e^{n-k+1} \times e^{n-k}) \cup (e^{n-k+2} \times e^{n-k}) \cup \cdots\big] \cup \cdots$$

If $n-1 < (n-k+1)+(n-k)$ (these are the dimensions of the last cell in the first group and the first cell in the second group) then the $(n-1)$-skeleton just consists of the cells e^{n-k} through e^{n-1}. This looks like the top part of the cell structure for $\mathbb{R}P^{n-1}$, and indeed it is:

Proposition 14.23. *If $n+2 > 2k$ then the n-skeleton of our cell structure on $V_k(\mathbb{R}^n)$ is homeomorphic to $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$.*

Proof. See [MT, ???]. □

Proof of Proposition 14.22. If there exist $r-1$ vector fields on S^{n-1} there also exist $r-1$ vector fields on S^{un-1} for any u (see Proposition 14.3). Then $p_1\colon V_r(\mathbb{R}^{un}) \to S^{un-1}$ has a section s. By the cellular approximation theorem the map s is homotopic to a cellular map s'. So s' factors through the $(un-1)$-skeleton of $V_r(\mathbb{R}^{un})$, which is $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1}$ under the assumption that ??????. We have

$$S^{un-1} \xrightarrow{s'} \mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1} \xrightarrow{\pi} \mathbb{R}P^{un-1}/\mathbb{R}P^{un-2} \cong S^{un-1}.$$

The composition is $p_1 s'$, which is homotopic to $p_1 s = \text{id}$. □

Exercise 14.24. Use singular cohomology to prove that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-3} \to S^{n-1}$ does not have a section when n is odd. Deduce that an even sphere does not have a non-vanishing vector field (which you already knew).

Part 3. K-theory and geometry I

At this point we have seen that there exist cohomology theories $K^*(-)$ and $KO^*(-)$. We have not *proven* their existence, but we have seen that their existence falls out as a consequence of the Bott periodicity theorems $\Omega^2(\mathbb{Z}\times BU)\simeq\mathbb{Z}\times BU$ and $\Omega^8(\mathbb{Z}\times BO)\simeq\mathbb{Z}\times BO$. If the only cohomology theory you have even seen is singular cohomology, this will seem like an amazing thing: suddenly you know three times as many cohomology theories as you used to. But a deeper study reveals that cohomology theories are actually quite common—to be a little poetic about it, that they are as plentiful as grains of sand on the beach. What is rare, however, is to have cohomology theories with a close connection to geometry: and both K and KO belong to this (vaguely-defined) class. In the following sections we will begin to explore what this means.

To some extent we have a "geometric" understanding of $K^0(-)$ and $KO^0(-)$ in terms of Grothendieck groups of vector bundles. We also know that any $K^n(-)$ (or $KO^n(-)$) group can be shifted to a $K^0(-)$ group using the suspension isomorphism and Bott periodicity. One often hears a slogan like "The geometry behind K-theory lies in vector bundles". This slogan, however, doesn't really say very much; our aim will be to do better.

One way to encode geometry into a cohomology theory is via Thom classes for vector bundles. Such classes give rise to fundamental classes for submanifolds and a robust connection wth intersection theory. In the next section we begin our story by recalling how all of this works for singular cohomology.

15. The Thom isomorphism for singular cohomology

The theory of Thom classes begins with the cohomological approach to orientations. Recall that

$$H^*(\mathbb{R}^n,\mathbb{R}^n-0)\cong H^*(D^n,S^{n-1})\cong \tilde{H}^*(S^n)\cong\begin{cases}\mathbb{Z} & \text{if } *=n,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, an orientation on $\mathbb{R}^n$ determines a generator for $H^n(\mathbb{R}^n,\mathbb{R}^n-0)\cong\mathbb{Z}$. (For a review of how this correspondence works, see the proof of Lemma 16.2 in the next section).

Now consider a vector bundle $p\colon E\to B$ of rank n. Let $\zeta\colon B\to E$ be the zero section, and write $E-0$ as shorthand for $E-\mathrm{im}(\zeta)$. For any $x\in B$ let $F_x=p^{-1}(x)$. Then $H^n(F_x,F_x-0)\cong\mathbb{Z}$, and an orientation of the fiber gives a generator. We wish to consider the problem of giving compatible orientations for all the fibers at once; this can be addressed through the cohomology of the pair $(E,E-0)$.

For a neighborhood V of x, let $E_V=E|_V=p^{-1}(V)$. If E_V is trivial, then there is an isomorphism $E_V\cong V\times\mathbb{R}^n$, and $(E_V-0)\cong V\times(\mathbb{R}^n-0)$. Hence, $H^*(E_V,E_V-0)\cong H^*(V\times\mathbb{R}^n,V\times(\mathbb{R}^n-0))$. If V is contractible, this gives that

$$H^*(E_V,E_V-0)\cong H^*(\mathbb{R}^n,\mathbb{R}^n-0)\cong\begin{cases}\mathbb{Z} & \text{if } *=n,\\ 0 & \text{otherwise.}\end{cases}$$

Pick a generator $\mathcal{U}_V\in H^n(E_V,E_V-0)\cong\mathbb{Z}$. For all $x\in V$ the inclusion $j_x\colon(F_x,F_x-0)\hookrightarrow(E_V,E_V-0)$ gives a map $j_x^*\colon H^*(E_V,E_V-0)\to H^*(F_x,F_x-0)$. Since we are assuming that V is contractible, j_x^* is an isomorphism. So $\mathcal{U}_V$ gives rise to generators in $H^n(F_x, F_x - 0)$ for all $x \in V$. We think of $\mathcal{U}_V$ as orienting all of the fibers simultaneously.

Even when V is not contractible the conclusions of the last paragraph still hold. One has that $H^*(V \times \mathbb{R}^n, V \times (\mathbb{R}^n - 0)) \cong H^*(V) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$ by the Künneth Theorem, and so

$$H^i(V \times \mathbb{R}^n, V \times (\mathbb{R}^n - 0)) \cong H^{i-n}(V) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \begin{cases} H^{i-n}(V) & \text{if } i \geq n, \\ 0 & \text{if } i < n. \end{cases}$$

Let $\mathcal{U}_V \in H^n(E_V, E_V - 0)$ be an element that corresponds to $1 \in H^0(V)$ under the above isomorphism. Then one checks that $j_x^*(\mathcal{U}_V)$ is a generator for $H^n(F_x, F_x - 0)$ for every $x \in V$.

Next suppose that we have two open sets $V, W \subseteq B$, together with classes $\mathcal{U}_V \in H^n(E_V, E_V - 0)$ and $\mathcal{U}_W \in H^n(E_W, E_W - 0)$ that restrict to generators (orientations) on the fibers F_x for every $x \in V$ and every $x \in W$, respectively. We would like to require that these orientations match: so we require that the images of $\mathcal{U}_V$ and $\mathcal{U}_W$ in $H^n(E_{V\cap W}, E_{V\cap W} - 0)$ coincide. Consider the (relative) Mayer-Vietoris sequence:

$$\begin{array}{ccccc} & & & & H^{n-1}(E_{V\cap W}, E_{V\cap W} - 0) \\ & & & & \downarrow \\ H^n(E_{V\cap W}, E_{V\cap W} - 0) & \longleftarrow & \begin{array}{c} H^n(E_V, E_V - 0) \\ \bigoplus \\ H^n(E_W, E_W - 0) \end{array} & \longleftarrow & H^n(E_{V\cup W}, E_{V\cup W} - 0) \end{array}$$

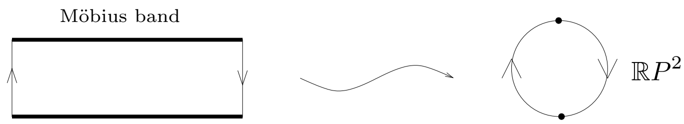
Under our requirement of compatibility between $\mathcal{U}_V$ and $\mathcal{U}_W$, the class $\mathcal{U}_V \oplus \mathcal{U}_W$ maps to zero; so it is the image of a class $\mathcal{U}_{V\cup W}$. Since $H^{n-1}(E_{V\cap W}, E_{V\cap W} - 0) = 0$, the class $\mathcal{U}_{V\cup W}$ is unique. Note that the Mayer-Vietories sequence also shows that $H^*(E_{V\cup W}, E_{V\cup W} - 0) = 0$ for $* < n$, which leaves us poised to inductively continue this argument. In other words, the argument shows that we may patch more and more $\mathcal{U}$-classes together, provided that they agree on the regions of overlap. This is the kind of behavior one would expect for orientation classes.

The above discussion suggests the following definition:

Definition 15.1. *Given a rank n bundle $E \to B$, a* ***Thom class*** *for E is an element $\mathcal{U}_E \in H^n(E, E - 0)$ such that for all $x \in B$, $j_x^*(\mathcal{U}_E)$ is a generator in $H^n(F_x, F_x - 0)$. (Here $j_x \colon F_x \hookrightarrow E$ is the inclusion of the fiber).*

There is no guarantee that a bundle has a Thom class. Indeed, consider the following example:

Example 15.2. Let $M \to S^1$ be the Möbius bundle. Take two contractible open subsets V and W of S^1, where $V \cup W = S^1$. We can choose a Thom class for $M|_V$, and one for $M|_W$, but the orientations won't line up correctly to give us a Thom class for M. In fact, notice that by homotopy invariance $H^*(M, M - 0)$ is the cohomology of the Möbius band relative to its boundary. But collapsing the boundary of the band gives an $\mathbb{R}P^2$



and we know $H^1(\mathbb{R}P^2) = 0$. So a Thom class cannot exist in this case.

If a bundle $E \to B$ has a Thom class then the bundle is called **orientable**. Said differently, an **orientation** on a vector bundle $E \to B$ is simply a choice of Thom class in $H^n(E, E-0; \mathbb{Z})$. One can readily prove that this notion of orientability agrees with other notions one may have encountered, and we leave this to the reader.

One can also talk about Thom classes with respect to the cohomology theories $H^*(-; R)$ for any ring R. Typically one only needs $R = \mathbb{Z}$ and $R = \mathbb{Z}/2$, however. In the latter case, note that any n-dimensional real vector space V has a *canonical* orientation in $H^n(V, V-0; \mathbb{Z}/2)$. It follows that local Thom classes always patch together to give global Thom classes, and so every vector bundle has a Thom class in $H^*(-; \mathbb{Z}/2)$.

Finally, note that we can repeat all that we have done for complex vector spaces and complex vector bundles. However, a complex vector space V of dimension n has a *canonical* orientation on its underlying real vector space, and therefore a canonical generator in $H^{2n}(V, V-0)$. Just as in the last paragraph, this implies that local Thom classes always patch together to give global Thom classes; so every complex vector bundle has a Thom class.

The following theorem summarizes what we have just learned:

Theorem 15.3.
(a) Every complex bundle $E \to B$ of rank n has a Thom class in $H^{2n}(E, E-0)$.
(b) Every real bundle $E \to B$ of rank n has a Thom class in $H^n(E, E-0; \mathbb{Z}/2)$.

The Mayer-Vietoris argument preceding Definition 15.1 shows that if $p\colon E \to B$ is a rank n orientable real vector bundle then $H^*(E, E-0)$ vanishes for $* < n$ and equals $\mathbb{Z}$ for $* = n$. A careful look at the argument reveals that it also gives a complete determination of the cohomology groups for $* > n$.

For any $z \in H^*(B)$, we may first apply p^* to obtain an element $p^*(z) \in H^*(E)$. We may then multiply by the Thom class $\mathcal{U}_E$ to obtain an element $p^*(z) \cup \mathcal{U}_E \in H^{*+n}(E, E-0)$. This gives a map $H^*(B) \to H^*(E, E-0)$ that increases degrees by n.

Theorem 15.4 (Thom Isomorphism Theorem)**.** *Suppose that $p\colon E \to B$ has a Thom class $\mathcal{U}_E \in H^*(E, E-0)$. Then the map $H^*(B) \to H^*(E, E-0)$ given by*

$$z \mapsto p^*(z) \cup \mathcal{U}_E$$

is an isomorphism of graded abelian groups that increases degrees by n.

Proof. If the bundle is trivial, then $E = B \times \mathbb{R}^n$, and $E - 0 = B \times (\mathbb{R}^n - 0)$. Here one just uses the suspension and Künneth isomorphisms to get

$$H^*(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0)) \cong H^{*-n}(B).$$

One readily checks that the map from the statement of the theorem gives the isomorphism.

For the case of a general bundle one uses Mayer-Vietoris and the Five Lemma to reduce to the case of trivial bundles. The argument is easy, but one can also look it up in [MS]. □

15.5. **Thom spaces.** The relative groups $H^*(E, E-0)$ coincide with the reduced cohomology groups of the mapping cone of the inclusion $E-0 \hookrightarrow E$. This mapping cone is sometimes called the **Thom space** of the bundle $E \to B$, although that name is more commonly applied to more geometric models that we will introduce next (the various models are all homotopy equivalent). For the most common model we require that the bundle have an inner product (see Section 8.20).

Definition 15.6. *Suppose that $E \to B$ is a bundle with an inner product. Define the* ***disk bundle*** *of E as $D(E) = \{v \in E \mid \langle v, v\rangle \leq 1\}$, and the* ***sphere bundle*** *of E as $S(E) = \{v \in E \mid \langle v, v\rangle = 1\}$.*

If E has rank n over each component of B, note that $D(E) \to B$ and $S(E) \to B$ are fiber bundles with fibers D^n and S^{n-1}, respectively. Note also that we have the following diagram:

$$\begin{array}{ccc} E-0 & \rightarrowtail & E \\ \simeq \uparrow & & \uparrow \simeq \\ S(E) & \rightarrowtail & D(E) \end{array}$$

This diagram shows that $E-0 \hookrightarrow E$ and $S(E) \hookrightarrow D(E)$ have weakly equivalent mapping cones. Unlike $E-0 \hookrightarrow E$, however, the map $S(E) \hookrightarrow D(E)$ is a cofibration (under the mild condition that X is cofibrant, say): so the mapping cone is weakly equivalent to the quotient $D(E)/S(E)$. This quotient is what is most commonly meant by the term 'Thom space':

Definition 15.7. *For a bundle $E \to B$ with inner product, the* ***Thom space*** *of E is* $\operatorname{Th} E = D(E)/S(E)$.

Remark 15.8. The notation B^E is also commonly used in the literature to denote the Thom space, although we will not use it in these notes.

Note that if B is compact then $\operatorname{Th} E$ is homeomorphic to the one-point compactification of the space E. To see this it is useful to first compactify all the fibers separately, which amounts to forming the pushout of $B \leftarrow S(E) \to D(E)$. The inclusion from B into the pushout P is the 'section at infinity', and the quotient P/B is readily seen to be the one-point compactification of E. But clearly the quotients P/B and $D(E)/S(E)$ are homeomorphic.

Example 15.9. We will show that $\operatorname{Th}(nL \to \mathbb{R}P^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}$, where L is the tautological line bundle. First, we define an isomorphism

$$\begin{array}{ccc} \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} & \xrightarrow{\cong} & nL^* \\ \pi \downarrow & \swarrow & \\ \mathbb{R}P^k. & & \end{array}$$

Consider $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^{n+k}$ as embedded via the last n coordinates. Take a point $\ell = [x_0 : \cdots : x_k : y_1 : \cdots : y_n] \in \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1}$, and note that at least one x_i is nonzero. The map $\pi \colon \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} \to \mathbb{R}P^k$ is defined to send ℓ to $[x_0 : \cdots : x_k]$.

Regard ℓ as a line in $\mathbb{R}^{n+k+1}$, and $\pi(\ell)$ as a line in $\mathbb{R}^{k+1}$. The formula $(x_0, \ldots, x_k) \mapsto y_1$ specifies a unique functional $\pi(\ell) \to \mathbb{R}$ (obtained by extending linearly). Likewise, we obtain n functionals on $\pi(\ell)$ via the formulas

$$(x_0, \ldots, x_k) \mapsto y_1, \qquad \ldots \qquad (x_0, \ldots, x_k) \mapsto y_n.$$

Note also that these functionals are independent of the choice of the homogeneous coordinates for ℓ: multiplying all the x_i's and y_j's by λ gives rise to the same functionals. We have therefore described a continuous map $\mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} \to nL^*$, and this is readily checked to be a homeomorphism.

Since the Thom space is the one-point compactification, we get that

$$\mathrm{Th}(nL^* \to \mathbb{R}P^k) \cong \widehat{(nL^*)} \cong (\mathbb{R}P^{n+k} - \mathbb{R}P^{n-1})^\wedge \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}.$$

We know by Corollary 8.23 that any real vector bundle over a paracompact space is isomorphic to its dual. So $nL^* \cong nL$, and we have shown that $\mathrm{Th}(nL \to \mathbb{R}P^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}$.

Remark 15.10. Note the case $n = 1$ in the above example: $\mathrm{Th}(L \to \mathbb{R}P^k) \cong \mathbb{R}P^{k+1}$.

Remark 15.11. A similar analysis to above shows that $\mathrm{Th}(nL^* \to \mathbb{C}P^k) \cong \mathbb{C}P^{n+k}/\mathbb{C}P^{n-1}$, but note that unlike the real case the dual is important here.

There is another approach to Thom spaces that does not require a metric for the bundle. If $E \to B$ is any vector bundle, let $\mathbb{P}(E) \to B$ be the corresponding bundle of projective spaces: the fiber of $\mathbb{P}(E) \to B$ over a point b is $\mathbb{P}(E_b)$. Another definition of Thom space is then

$$\mathrm{Th}\, E = \mathbb{P}(E \oplus \underline{1})/\mathbb{P}(E).$$

Note that this definition does not require a metric on the bundle.

To see that our definitions are equivalent, note that if V is a vector space then there is a canonical inclusion $V \hookrightarrow \mathbb{P}(V \oplus \mathbb{R})$ given by $v \mapsto \langle v \oplus 1 \rangle$. A little thought shows that we get a diagram

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{P}(V \oplus \mathbb{R}) \\ \downarrow & & \downarrow \\ \hat{V} & \xrightarrow{\cong} & \mathbb{P}(V \oplus \mathbb{R})/\mathbb{P}(V) \end{array}$$

where the bottom map is a homeomorphism. Extending this to the bundle setting, it is clear that the pushout of $B \longleftarrow \mathbb{P}(E) \to \mathbb{P}(E \oplus \underline{1})$ is the fiberwise one-point compactification of E. Then $\mathbb{P}(E \oplus \underline{1})/\mathbb{P}(E)$ is obtained by taking this fiberwise one-point compactification and collapsing the section at infinity: this clearly agrees with the other descriptions we have given of the Thom space.

It is sometimes useful to be able to connect the pairs $(\mathbb{P}(E \oplus \underline{1}), \mathbb{P}(E))$ and $(E, E - 0)$ in a way that doesn't make use of any metric. To do so, observe that every vector space V gives rise to a commutative diagram

$$\begin{array}{ccccc} V-0 & \rightarrowtail & \mathbb{P}(V\oplus\mathbb{R})-* & \xleftarrow{\sim} & \mathbb{P}(V) \\ \downarrow & & \downarrow & & \downarrow \\ V & \rightarrowtail & \mathbb{P}(V\oplus\mathbb{R}) & = & \mathbb{P}(V\oplus\mathbb{R}). \end{array}$$

Here $*\in\mathbb{P}(V\oplus\mathbb{R})$ is the line formed by the distinguished copy of $\mathbb{R}$, and $V\to\mathbb{P}(V\oplus\mathbb{R})$ is the map $v\mapsto\langle v\oplus 1\rangle$; all the other maps are the evident inclusions. Taking homotopy cofibers of the three columns, one gets a zig-zag of weak equivalences between the homotopy cofiber of $V-0\hookrightarrow V$ and the homotopy cofiber of $\mathbb{P}(V\oplus\mathbb{R})/\mathbb{P}(V)$. The latter is weakly equivalent to its cofiber, because $\mathbb{P}(V)\hookrightarrow\mathbb{P}(V\oplus\mathbb{R})$ is a cofibration.

Now consider a fiberwise version of the above diagram. If $E\to B$ is a real bundle then we have maps

$$\begin{array}{ccccc} E-0 & \rightarrowtail & \mathbb{P}(E\oplus\underline{1})-B & \xleftarrow{\sim} & \mathbb{P}(E) \\ \downarrow & & \downarrow & & \downarrow \\ E & \rightarrowtail & \mathbb{P}(E\oplus\underline{1}) & = & \mathbb{P}(E\oplus\underline{1}). \end{array}$$

The only difference worth noting is that $B\hookrightarrow\mathbb{P}(E\oplus\underline{1})$ is the evident section that in each fiber selects out the distinguished line determined by the trivial bundle $\underline{1}$. The left square is again a homotopy pushout square, and so taking homotopy cofibers of the columns gives a zig-zag of weak equivalences between the homotopy cofibers of $E-0\hookrightarrow E$ and $\mathbb{P}(E)\hookrightarrow\mathbb{P}(E\oplus\underline{1})$.

15.12. **Thom spaces for virtual bundles.** Thom spaces behave in a very simple way in relation to adding on trivial bundles:

Proposition 15.13. *For any real bundle $E\to X$ one has $\mathrm{Th}(E\oplus\underline{n})\cong\Sigma^n\,\mathrm{Th}(E)$. For a complex bundle $E\to X$ one has $\mathrm{Th}(E\oplus\underline{n})\cong\Sigma^{2n}\,\mathrm{Th}(E)$.*

Proof. We only prove the statement for real bundles, as the case of complex bundles works the same (and is even a consequence of the real case). Also, we will give the proof assuming the bundle has a metric, although the result is true in more generality. Note the isomorphisms

$$D(E\oplus\underline{n})\cong D(E)\times D^n,\qquad S(E\oplus\underline{n})\cong(S(E)\times D^n)\amalg_{S(E)\times S^{n-1}}(D(E)\times S^{n-1}).$$

From this one readily sees that

$$D(E\oplus\underline{n})/S(E\oplus\underline{n})\cong[D(E)/S(E)]\wedge[D^n/S^{n-1}]\cong\mathrm{Th}(E)\wedge S^n.$$

□

Proposition 15.13 allows one to make sense of Thom spaces for virtual bundles, provided that we use *spectra*. This material will only be needed briefly in the rest of the notes, but we include it here because these Thom spectra play a large role in modern algebraic topology.

Assume that X is compact and let $E\to X$ be a bundle. Then E embeds in some trivial bundle $\underline{N}$; let Q denote the quotient, so that we have $E\oplus Q\cong\underline{N}$. Assuming that $\mathrm{Th}(-E)$ had some meaning then we would expect

$$\mathrm{Th}(Q)=\mathrm{Th}(\underline{N}-E)=\Sigma^N\,\mathrm{Th}(-E).$$

This suggests the definition

$$\mathrm{Th}(-E) = \Sigma^{-N}\,\mathrm{Th}(Q),$$

where the negative suspension must of course be interpreted as taking place in a suitable category of spectra.

Our definition seems to depend on the choice of embedding $E \hookrightarrow \underline{N}$. To see that this dependence is an illusion, let $E \hookrightarrow \underline{N'}$ be another embedding and let Q' be the quotient. Then $\underline{N'} \oplus Q \cong Q' \oplus E \oplus Q \cong Q' \oplus \underline{N}$. On Thom spaces this gives $\Sigma^{N'}\,\mathrm{Th}(Q) \cong \Sigma^{N}\,\mathrm{Th}(Q')$, or $\Sigma^{-N}\,\mathrm{Th}(Q) \simeq \Sigma^{-N'}\,\mathrm{Th}(Q')$.

The above discussion can be extended to cover any element $\alpha \in KO(X)$. Write $\alpha = E - F$ for vector bundles E and F, and choose an embedding $F \hookrightarrow \underline{N}$. Let Q denote the quotient $\underline{N}/F$. Note that $\alpha + \underline{N} = (E - F) + (F + Q) = E + Q$. If $\mathrm{Th}(\alpha)$ makes sense then we would expect $\Sigma^{N}(\mathrm{Th}\,\alpha) \simeq \mathrm{Th}(\alpha + \underline{N}) \simeq \mathrm{Th}(E + Q)$, and so this suggests the definition

$$\mathrm{Th}(\alpha) = \Sigma^{-N}\,\mathrm{Th}(E \oplus Q).$$

Again, one readily checks that this does not depend on the choice of E, F, N, or the embedding $F \hookrightarrow \underline{N}$.

15.14. **An application to stunted projective spaces.** To demonstrate the usefulness of Thom spaces we give an application to periodicities amongst stunted projective spaces. This material will be needed later, in the solution of the vector fields on spheres problem.

Consider the space $\mathbb{R}P^{a+b}/\mathbb{R}P^{a}$. This has a cell structure with exactly b cells (not including the zero cell), in dimensions $a+1$ through $a+b$. The space $\mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}$ has a similar cell structure, although here the cells are in dimensions $a+1+r$ through $a+b+r$. The natural question arises: fixing a and b, what values of r (if any) satisfy

$$\Sigma^{r}[\mathbb{R}P^{a+b}/\mathbb{R}P^{a}] \simeq \mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}.$$

One can use singular cohomology and Steenrod operations to produce some necessary conditions here. For example, integral singular homology easily yields that if if $b \geq 2$ then r must be even. Use of Steenrod operations produces more stringent conditions (we leave this for the reader to think about).

We will use Thom spaces to provide some *sufficient* conditions for a stable homotopy equivalence between stunted projective spaces. We begin with a simple lemma:

Lemma 15.15. *The element* $\lambda = [L] - 1 \in \widetilde{KO}(\mathbb{R}P^{n})$ *satisfies* $\lambda^2 = -2\lambda$ *and* $\lambda^{n+1} = 0$. *Consequently,* $2^n\lambda = 0$.

Proof. L^2 is the trivial bundle, and this immediately yields $\lambda^2 = -2\lambda$. The second statement follows from the fact that $\mathbb{R}P^n$ may be covered by $n+1$ contractible sets $U_0, \ldots, U_n$. (If homogeneous coordinates are used on $\mathbb{R}P^n$, one may take U_i to be the open set $x_i \neq 0$). The element $\lambda \in KO(\mathbb{R}P^n, *)$ lifts to a class $\lambda_i \in KO(\mathbb{R}P^n, U_i)$, and therefore λ^{n+1} is the image of $\lambda_0\lambda_1\cdots\lambda_n$ under the natural map

$$KO(\mathbb{R}P^n, U_0 \cup \cdots \cup U_n) \to KO(\mathbb{R}P^n).$$

But since $\cup_i U_i = \mathbb{R}P^n$, the domain of the above map is zero; hence $\lambda^{n+1} = 0$.

Finally, since $\lambda^2 = -2\lambda$ it follows that $\lambda^e = (-2)^{e-1}\lambda$ for all e. In particular, $(-2)^n\lambda = \lambda^{n+1} = 0$. □

Proposition 15.16. *Let r be any positive integer such that $r([L]-1)=0$ in $\widetilde{KO}(\mathbb{R}P^{b-1})$. Then there is a stable homotopy equivalence*

$$\Sigma^r[\mathbb{R}P^{a+b}/\mathbb{R}P^a] \simeq \mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}.$$

Proof. The assumption that $r([L]-1)=0$ implies that $rL\oplus \underline{s} \cong \underline{r}\oplus\underline{s}$ for some $s\geq 0$. We have

$$\begin{aligned}
\mathbb{R}P^{a+b}/\mathbb{R}P^a &\cong \operatorname{Th}\begin{pmatrix} {\scriptstyle (a+1)L} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{b-1}} \end{pmatrix} \simeq \Sigma^{-r-s}\operatorname{Th}\begin{pmatrix} {\scriptstyle (a+1)L\oplus\underline{r+s}} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{b-1}} \end{pmatrix} \\
&\simeq \Sigma^{-r-s}\operatorname{Th}\begin{pmatrix} {\scriptstyle (a+1+r)L\oplus\underline{s}} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{b-1}} \end{pmatrix} \\
&\simeq \Sigma^{-r}\operatorname{Th}\begin{pmatrix} {\scriptstyle (a+1+r)L} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{b-1}} \end{pmatrix} \\
&\simeq \Sigma^{-r}\left[\mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}\right].
\end{aligned}$$

The first and last steps use the identification of stunted projective spaces with a corresponding Thom space—see Example 15.9 for this. ☐

Combining Lemma 15.15 and Proposition 15.16 we see that stunted projective spaces with b cells have a periodicity of 2^{b-1}:

$$\Sigma^{2^{b-1}}\left[\mathbb{R}P^{a+b}/\mathbb{R}P^a\right] \simeq \mathbb{R}P^{a+b+2^{b-1}}/\mathbb{R}P^{a+2^{b-1}}.$$

However, this is not the best result along these lines: we will get a better result by finding the exact order of $[L]-1$ in $\widetilde{KO}(\mathbb{R}P^{b-1})$. This was determined by Adams; see Theorem 32.14

16. Thom classes and intersection theory

In this section we will see how Thom classes give rise to fundamental classes for submanifolds, and we will develop the connection between products of such classes and intersection theory.

Let $E\to B$ be a real vector bundle of rank n. In general, E may not have a Thom class; and if it does have a Thom class, it actually has *two* Thom classes (since $H^n(E,E-0)\cong\mathbb{Z}$ by the Thom Isomorphism Theorem). The situation is familiar, as it matches the usual behavior of orientations. It is, of course, possible—and necessary!—to do geometry in a way that includes keeping track of orientations and computing signs according to whether orientations match up or not. But it is easier if we are in a situation where we don't have to keep track of quite so much, and there are two situations with that property: we can work always with mod 2 coefficients, or we can work in the setting of complex geometry. In either case we have canonical Thom classes all the time. In this section, and for most of the rest of these notes, we choose to work in the setting of complex bundles and complex geometry. But it is important to note that almost everything works verbatim for real bundles if we use $\mathbb{Z}/2$ coefficients, and that many things can be made to work for oriented real bundles if one is diligent enough about keeping track of signs.

For a rank n complex bundle $E\to B$ we have a canonical Thom class $\mathcal{U}_E\in H^{2n}(E,E-0)$. The following result gives two useful properties:

Proposition 16.1.

(a) *(Naturality) Suppose $E \to B$ is a rank n complex vector bundle, and $f\colon A \to B$. Consider the pullback*

$$\begin{array}{ccc} f^*E & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

Then $\bar{f}^*\colon H^{2n}(E, E-0) \to H^{2n}(f^*E, f^*E-0)$ sends $\mathcal{U}_E$ to $\mathcal{U}_{f^*E}$; that is, $\bar{f}^*(\mathcal{U}_E) = \mathcal{U}_{f^*E}$

(b) *(Multiplicativity) Suppose that $E \to B$ is a rank n vector bundle, and $E' \to B$ is a rank k vector bundle, with Thom classes $\mathcal{U}_E \in H^{2n}(E, E-0)$ and $\mathcal{U}_{E'} \in H^{2k}(E', E'-0)$. Then $\mathcal{U}_E \times \mathcal{U}_{E'} = \mathcal{U}_{E\oplus E'}$ in $H^{2n+2k}(E \times E', (E\times E')-0)$.*

Proof. Recall that the Thom class of a rank n complex bundle $E \to B$ is the unique class in $H^{2n}(E, E-0)$ that restricts to the canonical generator in $H^{2n}(F_x, F_x - 0)$ for every fiber F_x. Part (a) follows readily from this characterization. Using the same reasoning, part (b) is reduced to the case there B is a point; this is checked in the lemma below. □

Lemma 16.2. *Let V and W be two real vector spaces, of dimensions n and k, respectively. Assume given orientations on V and W, and let $V \oplus W$ have the product orientation. Let $U_V \in H^n(V, V-0)$, $U_W \in H^k(W, W-0)$, and $U_{V\oplus W} \in H^{n+k}(V\oplus W, (V\oplus W)-0)$ be the corresponding orientation classes. Then $U_{V\oplus W} = U_V \times U_W$.*

Proof. Let $v_1, \ldots, v_n$ be an oriented basis for V, and let $\sigma_V\colon \Delta^n \to V$ be the affine simplex whose ordered list of vertices is $0, v_1, \ldots, v_n$. Let σ_V^t denote any translate of σ that contains the origin of V in the interior. Then $[\sigma_V^t]$ is a generator for $H_n(V, V-0)$, and any relative cocycle in $C^*_{sing}(V, V-0)$ that evaluates to 1 on σ_V^t is a generator (in fact, the same generator) for $H^n(V, V-0)$. This is how an orientation of V determines a generator of $H^n(V, V-0)$.

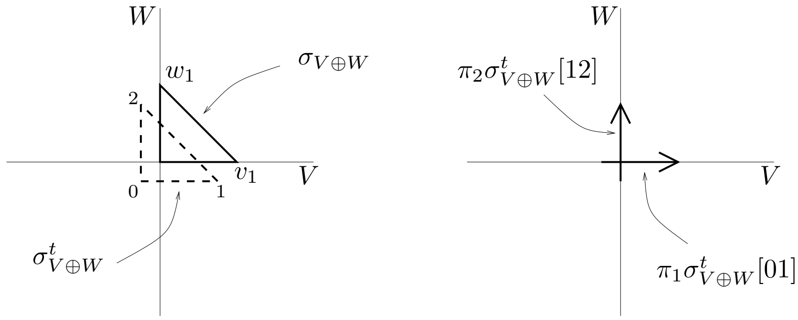
Now let $w_1, \ldots, w_k$ be an oriented basis for W. Let $\sigma_{V\oplus W}\colon \Delta^{n+k} \to V \oplus W$ be the affine simplex whose ordered list of vertices is $0, v_1, \ldots, v_n, w_1, \ldots, w_k$ (note that omitting 0 gives an oriented basis for $V \oplus W$). Again, let $\sigma^t_{V\oplus W}$ denote a translate of $\sigma_{V\oplus W}$ that contains the origin in its interior.

Recall that $U_V \times U_W = (\pi_1)^*(U_V) \cup (\pi_2^*)(U_W)$, where $\pi_1\colon V \times W \to V$ and $\pi_2\colon V\times W \to W$ are the two projections. The definition of the cup product gives

$$\begin{aligned}(U_V \times U_W)(\sigma^t_{V\oplus W}) &= (\pi_1^*U_V)(\sigma^t_{V\oplus W}[01\cdots n]) \cdot (\pi_2^*U_W)(\sigma^t_{V\oplus W}[n\cdots(n+k)]) \\ &= U_V(\pi_1 \circ \sigma^t_{V\oplus W}[01\cdots n]) \cdot U_W(\pi_2\circ\sigma^t_{V\oplus W}[n\cdots(n+k)]).\end{aligned}$$

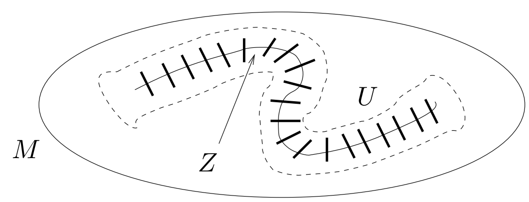
It is clear that $\pi_1 \circ \sigma^t_{V\oplus W}[01\cdots n]$ gives a simplex in the same homology class as σ_V^t, and so U_V evaluates to 1 on this simplex. Similarly, $\pi_2 \circ \sigma^t_{V\oplus W}[n\cdots(n+k)]$ gives a simplex in the same homology class as σ_W^t, and so U_W evaluates to 1 here. Since $1\cdot 1 = 1$, we see that $U_V \times U_W$ satisfies the defining property of $U_{V\oplus W}$.

Given that a picture is worth a thousand words, here is a picture showing what is happening in the smallest nontrivial case:



□

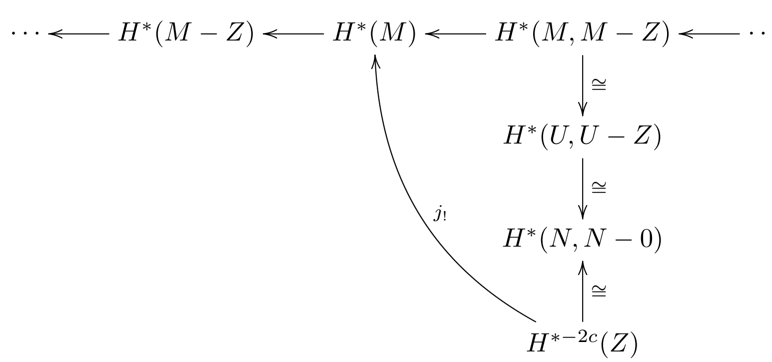
16.3. **Fundamental classes.** Next we use the Thom isomorphism to define fundamental classes for submanifolds. Let M be a complex manifold, and let Z be a regularly embedded submanifold of complex codimension c. By "regularly embedded" we mean that there exists a neighborhood U of Z and a homeomorphism $\phi\colon U \to N$ between U and the normal bundle $N = N_{M/Z}$, with the property that ϕ carries Z to the zero section of N. The neighborhood U is called a **tubular neighborhood** of Z. Keep in mind the following rough picture:



In the above situation we have that $H^*(U, U-Z) \cong H^*(N, N-0)$. Notice that $N \to Z$ is a complex bundle of rank c, with Thom class $\mathcal{U}_N \in H^{2c}(N, N-0)$, and so by the Thom Isomorphism we get $H^{i-2c}(Z) \cong H^i(N, N-0)$. Also, by excision one has $H^*(M, M-Z) \cong H^*(U, U-Z)$. So we have isomorphisms

$$H^{i-2c}(Z) \xrightarrow{Thom} H^i(N, N-0) \cong H^i(U, U-Z) \xleftarrow{\cong} H^i(M, M-Z).$$

Now consider the long exact sequence for the pair $(M, M-Z)$, but use the above isomorphisms to rewrite the relative groups $H^*(M, M-Z)$ and $H^{*-2c}(Z)$:



If $j\colon Z \hookrightarrow M$ is the inclusion, then the indicated composition in the above diagram is denoted $j_!$ and called a **pushforward map** or **Gysin map**. We can rewrite the long exact sequence to get the **Gysin sequence**, also called a **localization sequence** by algebraic geometers:

$$\cdots \longleftarrow H^i(M-Z) \longleftarrow H^i(M) \xleftarrow{j_!} H^{i-2c}(Z) \longleftarrow H^{i-1}(M-Z) \longleftarrow \cdots$$

Definition 16.4. *Let Z be a regularly embedded, codimension c submanifold of the complex manifold M. Let $j_!$ be the Gysin map described above, and take $1 \in H^0(Z)$. We define the* ***fundamental class*** *of Z to be $[Z]_M = j_!(1) \in H^{2c}(M)$. We also define the* ***relative fundamental class*** *$[Z]_{rel} \in H^{2c}(M, M-Z)$ to be the image of 1 under the chain of isomorphisms from $H^0(Z)$ to $H^{2c}(M, M-Z)$. Note that $j^*([Z]_{rel}) = [Z]$, where j^* denotes the induced map in cohomology associated to the inclusion $(M, \emptyset) \hookrightarrow (M, M-Z)$.*

On an intuitive level one should think of $[Z]$ as being the Poincaré Dual of the usual fundamental class of Z in $H_*(M)$. The point, however, is that we don't need to think through the hairiness of the Poincaré duality isomorphism; this has been replaced with the machinery of vector bundles and Thom classes.

One must of course prove a collection of basic results showing that the classes $[Z]$ really do behave as one would expect fundamental classes to behave, and have the expected ties with geometry. We will do a little of this, just enough to give the reader the idea that it is not hard. Before tackling this let us do the most trivial example:

Example 16.5.

(a) Check that the relative fundamental class of the origin in $\mathbb{C}^d$ is the canonical generator: i.e., $[0]_{rel} \in H^{2d}(\mathbb{C}^d, \mathbb{C}^d - 0)$ is the canonical generator provided by the complex orientation on $\mathbb{C}^d$.

(b) Let M be a d-dimensional complex manifold. If $a, b \in M$ are path-connected, verify that $[a] = [b]$. Hint: Reduce to the case where a and b belong to a common chart U of M, with $U \cong \mathbb{C}^d$. Let I be a line joining a and b inside of U, and consider the diagram

$$\begin{array}{ccccc} H^*(M, M-a) & \xrightarrow{\cong} & H^*(M, M-I) & \xleftarrow{\cong} & H^*(M, M-b) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ H^*(U, U-a) & \xrightarrow{\cong} & H^*(U, U-I) & \xleftarrow{\cong} & H^*(U, U-b). \end{array}$$

Using an argument similar to that in the proof of Lemma 16.2, show that $[a]_{rel,U}$ and $[b]_{rel,U}$ map to the same element in $H^*(U, U-I)$.

(c) Suppose M is compact and connected. Verify that if $a \in M$ then $[a] \in H^{2d}(M)$ is a generator. (Use that the map $H^{2d}(M) \to H^{2d}(M, M-a)$ is an isomorphism in this case).

The following theorem connects our fundamental classes to intersection theory. It is far from the most general statement along these lines, but it will suffice for our applications later in the text. The diligent reader will find that the proof readily generalizes to tackle more complicated situations, for example where the intersection is not discrete.

Theorem 16.6. *Let M be a connected complex manifold. Suppose that Z and W are regularly embedded submanifolds of M that intersect transversally in d points. Then*
(a) $[Z]_M \cup [W]_M = d[*]_M$
(b) $j^*([Z]_M) = d[*]_W$, *where* $j\colon W \hookrightarrow M$.

Proof. We begin by proving (a). Suppose that $\dim Z = k$ and $\dim W = \ell$, so that $\dim M = k + \ell$. Let $Z \cap W = \{p_1, \ldots, p_d\}$, and for each i let U_i be a Euclidean neighborhood of p_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Consider the following diagram:

$$\begin{array}{ccc}
H^k(M, M-Z) \otimes H^l(M, M-W) & \longrightarrow & H^k(M) \otimes H^l(M) \\
\downarrow{\scriptstyle \cup_{rel}} & & \downarrow{\scriptstyle \cup} \\
H^{k+l}(M, M-(Z\cap W)) & \longrightarrow & H^{k+l}(M) \\
\| & \nearrow & \\
H^{k+l}(M, M-\{p_1,\ldots,p_d\}) & & \\
\uparrow{\scriptstyle \cong} & & \\
\oplus_r H^{k+l}(M, M-\{p_r\}) & &
\end{array}$$

Since $[Z]$ and $[W]$ lift to relative classes $[Z]_{rel}$ and $[W]_{rel}$, it will suffice to show that if we take $[Z]_{rel} \cup [W]_{rel}$ and take its projection to the rth factor $H^{k+l}(M, M-\{p_r\})$ of the summand then we get $[p_r]_{rel}$. From this it will follow from the diagram that $[Z] \cup [W] = [p_1] + \ldots + [p_d]$ in $H^{k+l}(M)$. Since we have already seen in Exercise 16.5 that $[p_i] = [p_j]$ for any i and j, this will complete the proof of (a).

Next, fix an index r and consider the second diagram

$$\begin{array}{ccc}
H^k(U_r, U_r - Z) \otimes H^l(U_r, U_r - W) & \longleftarrow & H^k(M, M-Z) \otimes H^l(M, M-W) \\
\downarrow & & \downarrow \\
H^{k+l}(U_r, U_r - \{p_r\}) & \longleftarrow & H^{k+l}(M, M-\{p_1,\ldots,p_d\}) \\
& \nwarrow{\scriptstyle \cong} & \uparrow \\
& & H^{k+l}(M, M-\{p_r\}).
\end{array}$$

Thanks to this diagram, it is enough to replace M by U_r, Z by $Z \cap U_r$, and W by $W \cap U_r$, and to prove that $[Z]_{rel} \cup [W]_{rel} = [p_r]_{rel}$.

But now M is just $\mathbb{C}^{k+l}$. By choosing our neighborhood small enough, we can find local coordinates so that Z is just $\mathbb{C}^k$ and W is just $\mathbb{C}^l$, intersecting transversally at the origin. We need to compute $[\mathbb{C}^k]_{rel} \cup [\mathbb{C}^l]_{rel} \in H^{k+l}(\mathbb{C}^{k+l}, \mathbb{C}^{k+l} - 0)$. By writing $\mathbb{C}^{k+l} = \mathbb{C}^k \times \mathbb{C}^l$ one sees that $[\mathbb{C}^k]_{rel}$ coincides with the Thom class for the bundle $\underline{l} \to \mathbb{C}^k$. Likewise, $[\mathbb{C}^l]_{rel}$ coincides with the Thom class for the bundle $\underline{k} \to \mathbb{C}^l$. These are trivial bundles, so they are pulled back from $\mathbb{C}^l \to *$ and $\mathbb{C}^k \to *$ along the projection maps $\mathbb{C}^k \to *$ and $\mathbb{C}^l \to *$, respectively. In particular, by Proposition 16.1(a) we can write

$$[\mathbb{C}^k]_{rel} \cup [\mathbb{C}^l]_{rel} = \pi_1^*(\mathcal{U}_1) \cup \pi_2^*(\mathcal{U}_2)$$

where $\mathcal{U}_1 \in H^{2l}(\mathbb{C}^l, \mathbb{C}^l - 0)$ and $\mathcal{U}_2 \in H^{2k}(\mathbb{C}^k, \mathbb{C}^k - 0)$ are the canonical classes and $\pi_1 \colon \mathbb{C}^{k+l} \to \mathbb{C}^l$, $\pi_2 \colon \mathbb{C}^{k+l} \to \mathbb{C}^k$ are the projection maps. But $\pi_1^*(\mathcal{U}_1) \cup \pi_2^*(\mathcal{U}_2)$ is the external cross product $\mathcal{U}_1 \times \mathcal{U}_2$, and so Lemma 16.2 says that this is the same as the canonical generator in $H^{2k+2l}(\mathbb{C}^{k+l}, \mathbb{C}^{k+l} - 0)$. This canonical generator is $[0]_{rel}$, by Exercise 16.5(a). We have therefore shown that $[\mathbb{C}^k]_{rel} \cup [\mathbb{C}^l]_{rel} = [0]_{rel}$, and this completes the proof of (a).

The proof of (b) is very similar. One considers the diagram

$$\begin{array}{ccccc}
 & & H^k(W) & \xleftarrow{\;j^*\;} & H^k(M) \\
 & & \uparrow & & \uparrow \\
\oplus_r H^k(W, W - p_r) & \xrightarrow{\cong} & H^k(W, W - \{p_1, \dots, p_d\}) & \longleftarrow & H^k(M, M - Z) \\
 & & \uparrow & & \downarrow \\
 & & H^k(W \cap U_r, (W \cap U_r) - p_r) & \longleftarrow & H^k(U_r, U_r - p_r)
\end{array}$$

where r is an arbitrary choice of index. The top square implies that it suffices to show that the projection of $j^*([Z]_{rel})$ to $H^k(W, W - p_r)$ equals $[p_r]_{rel}$, for any choice of r. The bottom square then allows us to replace M by U_r and Z and W by $Z \cap U_r$ and $W \cap U_r$. That is, we are again reduced to the case where $M = \mathbb{C}^{k+l}$, $Z = \mathbb{C}^k$, and $W = \mathbb{C}^l$. Here we are considering the map

$$H^{2l}(\mathbb{C}^l, \mathbb{C}^l - 0) \xleftarrow{j^*} H^{2l}\big(\mathbb{C}^k \times \mathbb{C}^l, (\mathbb{C}^k \times \mathbb{C}^l) - (\mathbb{C}^k \times \{0\})\big)$$

and must show that the image of $[\mathbb{C}^k]_{rel}$ is the canonical generator in the target. But if we identity $\mathbb{C}^k \times \mathbb{C}^l$ with the bundle $\underline{l} \to \mathbb{C}^k$ then $[\mathbb{C}^k]_{rel}$ is just the Thom class $\mathcal{U}$, and the map j^* is restriction to the fiber over $0 \in \mathbb{C}^k$; so it becomes the canonical generator by definition of the Thom class. □

It is important to notice that for the most part the above proof used nothing special about singular cohomology—we only used the basic properties of Thom classes, together with generic properties that hold in *any* cohomology theory. In the proof of Lemma 16.2 we apparently used particular details about the definition of the cup product, but in fact what we needed could have been written in a way that doesn't reference the peculiar definition of the cup product at all. Indeed, we have the identifications $H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^k) = H^*(\mathbb{C}^k \times \mathbb{C}^{n-k}, \mathbb{C}^k \times (\mathbb{C}^{n-k} - 0)) = H^*(\mathbb{C}^{n-k}, \mathbb{C}^{n-k} - 0) = H^*(D^{2n-2k}, \partial D^{2n-2k}) \cong \tilde{H}^*(S^{2(n-k)})$ (for the second identification we use the map induced by projection $\mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$, and for third identification we use the induced map of any orientation-preserving embedding of the disk into $\mathbb{C}^{n-k}$). Similarly, we have a canonical identification $H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^{n-k}) = \tilde{H}^*(S^{2k})$. Considering the commutative diagram

$$\begin{array}{ccc}
H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^k) \otimes H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^{n-k}) & \xrightarrow{\;\mu\;} & H^*(\mathbb{C}^n, \mathbb{C}^n - 0) \\
\| & & \| \\
\tilde{H}^*(S^{2(n-k)}) \otimes \tilde{H}^*(S^{2k}) & \xrightarrow{\;\mu\;} & \tilde{H}^*(S^{2n})
\end{array}$$

where μ denotes our product, the property needed for the proof of Lemma 16.2 boils down to the requirement that

$$\sigma^{2(n-k)}(1) \otimes \sigma^{2k}(1) \xrightarrow{\mu} \sigma^{2n}(1).$$

In other words, the computation comes down to the fact that the product behaves well with respect to the suspension isomorphism.

Example 16.7. We will be content with the usual first example. Let $Z \hookrightarrow \mathbb{C}P^n$ be a codimension c complex submanifold. Then $[Z] \in H^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}$. A generator for this group is $[\mathbb{C}P^{n-c}]$, so $[Z] = d[\mathbb{C}P^{n-c}]$ for a unique integer d. This integer is called the **degree** of the submanifold Z. A generic, c-dimensional, linear subspace of $\mathbb{C}P^n$ will intersect Z transversally in finitely many points, say e of them. Theorem 16.6 gives that $[Z] \cup [\mathbb{C}P^c] = e[*]$, but we also have $d[\mathbb{C}P^{n-c}] \cup [\mathbb{C}P^c] = d[*]$ since $[\mathbb{C}P^{n-c}] \cup [\mathbb{C}P^c] = [*]$ (again by Theorem 16.6). So $d = e$, and this gives the geometric description of the degree: the number of intersection points with a generic linear subpace of complementary dimension.

The following result is the evident generalization of Theorem 16.6.

Theorem 16.8. *Let M be a connected complex manifold. Suppose that Z and W are regularly embedded submanifolds of M that intersect transversally. Then*
(a) $[Z]_M \cup [W]_M = [Z \cap W]$;
(b) $j^*([Z]_M) = [Z \cap W]$, where $j\colon W \hookrightarrow M$.

Outline of proof. We omit the details here, since the proof is largely similar to that of Theorem 16.6. For (a) use the relative fundamental classes $[Z]_{rel}$ and $[W]_{rel}$, and show that $[Z]_{rel} \cup [W]_{rel} = [Z \cap W]_{rel}$ in $H^*(M, M - (Z \cap W))$. For this, restrict to a tubular neighborhood and then show that both classes restrict to the canonical generators on the fibers of the normal bundle. For $[Z \cap W]_{rel}$ this is the definition, and for $[Z]_{rel} \cup [W]_{rel}$ this is a computation with the cup product. The proof of (b) is similar. □

16.9. **Topological intersection multiplicities.** We can now use our machinery to give a topological definition of intersection multiplicity. Suppose that Z and W are complex submanifolds of the complex manifold M, and that Z and W have an isolated point of intersection at p. Let U be a Euclidean neighborhood of p that contains no other points of $Z \cap W$. Consider the classes ????

17. Thom classes in K-theory

In the last section we saw how Thom classes for complex vector bundles give rise to cohomological fundamental classes for submanifolds, and we saw that these fundamental classes have the expected connections to geometry. The discussion was carried out in the case of singular cohomology, but very little specific information about this cohomology theory was actually used. In fact, once we showed that Thom classes existed everything else followed formally. So let us now generalize a bit:

Definition 17.1. *A **multiplicative generalized cohomology theory** is a cohomology theory $\mathcal{E}$ equipped with product maps*

$$\mathcal{E}^p(X, A) \otimes \mathcal{E}^q(Y, B) \to \mathcal{E}^{p+q}(X \times Y, X \times B \cup A \times Y)$$

has requirements:

(1) natural
(2) $\exists$ unit in $\mathcal{E}^0(pt, \emptyset) = \mathcal{E}^0(pt)$
(3) associative

(4) compatibility with δ, the connecting homomorphism

Let $\mathcal{E}$ be a multiplicative generalized cohomology theory.

Definition 17.2. *Let $E \to B$ be a rank n complex vector bundle. A* ***Thom class*** *for E is an element $\mathcal{U}_E \in \mathcal{E}^{2n}(E, E-0)$ such that $\forall x \in B$, $i^*(U_E) \in \mathcal{E}^{2n}(F_x, F_x \setminus 0) \cong \mathcal{E}^{2n}(\mathbb{C}^n, \mathbb{C}^n \setminus 0) \cong \mathcal{E}^{2n}(D^{2n}, \partial D^{2n}) \cong \tilde{\mathcal{E}^{2n}} \cong \tilde{\mathcal{E}}^0(S^0) = \mathcal{E}^0(pt)$, and the condition is $i^*(U_E)$ maps to $1 \in \mathcal{E}^0(pt)$*

Definition 17.3. *A* ***complex orientation*** *for $\mathcal{E}$ is a choice, for every rank n complex bundle $E \to B$, of a Thom class $\mathcal{U}_E \in \mathcal{E}^{2n}(E, E-0)$ such that*

(1) (Naturality) $\mathcal{U}_{f^*E} = f^*(\mathcal{U}_E)$ for every map $f\colon A \to B$;
(2) (Multiplicativity) $\mathcal{U}_{E\oplus E'} = \mathcal{U}_E \cdot \mathcal{U}_{E'}$

A given cohomology theory may or may not admit a complex orientation—most likely, it will not. The complex-orientable cohomology theories are a very special class. Note that once a complex orientation is provided one gets the Thom isomorphism, Gysin sequences, and fundamental classes for complex submanifolds just as before—as well as the same connections to intersection theory.

Our goal in this section is the following:

Theorem 17.4. *Complex K-theory admits a complex orientation.*

We will spend a long time exploring the geometric consequences of this, but let us go ahead and give one example right away. Let $Z \hookrightarrow \mathbb{C}P^n$ be a complex submanifold of codimension c. The above theorem implies that we have a fundamental class $[Z] \in K^{2c}(\mathbb{C}P^n)$, just as we did in the case of singular cohomology. Whereas $H^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}$ and only resulted in one integral invariant, we will find that $K^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$. This is a much larger group, and so there is suddenly the potential for detecting more information: the K-theoretic fundamental class $[Z]$ is an $(n+1)$-tuple of integers rather than just a single integer. Of course it might end up that all of these new invariants are just zero, or some algebraic function of the invariant we already had—we will have to do some computations to find out. But this demonstrates the general situation: K-theory has an inherent ability to detect more information than singular cohomology did.

To prove Theorem 17.4 we need to give a construction, for every rank n complex vector bundle $E \to B$, of a Thom class in $K^{2n}(E, E-0)$. By Bott periodicity this group is the same as $K^0(E, E-0)$. Our first step will be to develop some tools for producing elements in relative K-groups.

17.5. **Relative K-theory.** Let $A \hookrightarrow X$ be an inclusion of topological spaces. When we talked about algebraic K-theory back in Part 1, we defined the relative K-group $K^0(X, A)$ using quasi-isomorphism classes of chain complexes that were exact on A (Section 5.10). We will make a similar construction in the topological case, with some important differences.

Definition 17.6. *Let $\mathcal{F}(X, A)$ be the free abelian group on isomorphism classes of bounded chain complexes of vector bundles $E_\bullet$ on X that are exact on A (meaning that for every $x \in X$ the complex of vector spaces $(E_x)_\bullet$ is exact). Define $\mathcal{K}(X, A)$ to be the quotient of $\mathcal{F}(X, A)$ by the following relations:*

(1) $[E_\bullet \oplus F_\bullet] = [E_\bullet] + [F_\bullet]$;
(2) $[E_\bullet] = 0$ whenever $E_\bullet$ is exact on all of X;

(3) If $\mathcal{E}_\bullet$ is a boundex complex of vector bundles on $X \times I$ that is exact on $A \times I$ then $[\mathcal{E}|_{X\times 0}] = [\mathcal{E}|_{X\times 1}]$.

Note that pullback of vector bundles makes $\mathcal{K}(X, A)$ into a contravariant functor.

Relations (1) and (2) are familiar, although the reader might be surprised that (1) only deals with direct sums and not short exact sequences. We will say more about this in a moment. Let us first make some remarks on relation (3), since we have not dealt with relations of this type before.

If $d\colon E \to F$ is a map of vector bundles over X there is a clear, intuitive notion of *deformation* of d. One way to make this rigorous is to consider the subspace $\mathrm{VB}(E, F) \subseteq \mathcal{T}op(E, F)$ consisting of the vector bundle maps; then a deformation of d is just a continuous map $I \to \mathrm{VB}(E, F)$ that sends 0 to d. If $\pi\colon X \times I \to X$ is the projection, a little thought shows that the above notion of deformation is the same as a map of vector bundles $\pi^*E \to \pi^*F$ over $X \times I$ that restricts to d on $X \times \{0\}$.

Likewise, a deformation of a chain complex $E_\bullet$ over X can be thought of in two ways. One way involves a collection of deformations for all the maps of $E_\bullet$, having the property that at any given time t the maps in the deformation assemble into a chain complex. The other way is simply as a chain complex structure on the set of vector bundles $\{\pi^*E_i\}$. These notions are equivalent.

The point here is that if $(E_\bullet, d)$ is a given chain complex and d' is a deformation of the differential on d, then relation (3) implies that $[(E_\bullet, d)] = [(E_\bullet, d')]$. (To be precise, there is an exactness condition required for the deformation, namely that at every time t the differential d_t is exact on A). Moreover, if X is paracompact Hausdorff then by Corollary 11.2(b) every bundle on $X \times I$ is isomorphic to the pullback of a bundle from X; it follows that *every* relation from (3) can be recast in this form. That is to say, for paracompact Hausdorff spaces it is equivalent to replace (3) by

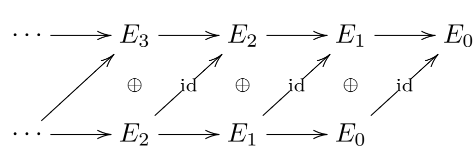
(3') $[(E_\bullet, d)] = [(E_\bullet, d')]$ for any bounded chain complex $(E_\bullet, d)$ and any deformation d' of d.

The following important lemma will help give a feel for the idea of deforming a chain complex:

Lemma 17.7. *Let $E_\bullet$ be a bounded complex of vector bundles on X that is exact on A. Then $[E_\bullet] = -[\Sigma E_\bullet]$ in $\mathcal{K}(X, A)$, where $\Sigma E_\bullet$ is the shifted complex having E_i in degree $i + 1$.*

Proof. First note that if $V_\bullet$ and $W_\bullet$ are exact complexes of vector spaces and $f\colon V_\bullet \to W_\bullet$ is any map, then the mapping cone Cf is still exact. This follows by the long exact sequence on homology. Consequently, if $E_\bullet \to F_\bullet$ is a map between complexes of vector bundles on X, each of which is exact on A, then the mapping cone is also exact on A.

Let C denote the mapping cone of the identity $\mathrm{id}\colon E_\bullet \to E_\bullet$. We depict this complex as follows:



The arrows depict the various components of the differentials in the mapping cone; recall that $d(a,b) = (da + \mathrm{id}(b), -db)$ for $(a,b) \in E_n \oplus E_{n-1}$, where we have written $\mathrm{id}(b)$ just to indicate the role of the original chain map.

Consider the deformation of C obtained by putting a t in front of all the diagonal arrows and letting $t \mapsto 0$. That is, C_t is the mapping cone for $t(\mathrm{id})\colon E_\bullet \to E_\bullet$. Then C_t is exact on A for every t, and when $t = 0$ we have $C_0 = E_\bullet \oplus \Sigma E_\bullet$. So $[C] = [C_1] = [C_0] = [E_\bullet] + [\Sigma E_\bullet]$ in $\mathcal{K}(X,A)$.

But C is exact on all of X, being the mapping cone of an identity map. So $[C] = 0$ in $\mathcal{K}(X,A)$, and hence $[E_\bullet] = -[\Sigma E_\bullet]$. □

Remark 17.8. The ideas used in the above proof immediately also give the following. Let $E_\bullet$ and $F_\bullet$ be complexes of vector bundles on X that are exact on A and let $f\colon E_\bullet \to F_\bullet$ be any map, with Cf denoting the mapping cone. Then $[Cf] = [F_\bullet] - [E_\bullet]$ in $\mathcal{K}(X,A)$.

Our next task is to analyze exact complexes, and see that just as in homological algebra they split up into basic pieces.

Definition 17.9. *An* ***elementary complex*** *is one of the form*

$$[0 \to \cdots \to 0 \to E \xrightarrow{\mathrm{id}} E \to 0 \to \cdots \to 0]$$

where E is a vector bundle on X and the E's occur in some dimensions i and $i+1$. Denote this complex as $D_i(E)$.

Proposition 17.10. *Let X be a paracompact Hausdorff space. If $E_\bullet$ is a bounded complex of vector bundles on X that is exact, then $E_\bullet$ is a direct sum of elementary complexes.*

Proof. The proof is really the same as in homological algebra. Assume without loss of generality that $E_i = 0$ for $i < 0$. Then $E_1 \to E_0$ is a surjection, so the kernel K_1 is a vector bundle by Proposition 9.3. By Proposition 9.2 the sequence $0 \to K_1 \to E_1 \to E_0 \to 0$ is split-exact, and a choice of splitting allows us to write $E_1 \cong K_1 \oplus Q_1$ where the composite $Q_1 \hookrightarrow E_1 \to E_0$ is an isomorphism. Noting that $E_2 \to E_1$ has image contained in K_1, the complex $E_\bullet$ splits as the direct sum of $D_0(E_0)$ and a complex that is zero in dimensions smaller than 1. Now continue inductively, replacing $E_\bullet$ with this smaller factor, until the nonzero degrees of $E_\bullet$ have been exhausted. □

Remark 17.11. Observe now that relation (2) of Definition 17.6 could be replaced with the relation that $[D_i(E)] = 0$ for any vector bundle E on X and any $i \in \mathbb{Z}$. This fact is sometimes useful.

The next result explains why we were able to forego short exact sequences in relation (1) from Definition 17.6.

Proposition 17.12. *Let X be paracompact and Hausdorff. Assume given a short exact sequence $0 \to E'_\bullet \to E_\bullet \to E''_\bullet \to 0$ of complexes of vector bundles, where each complex is exact on A. Then $[E_\bullet] = [E'_\bullet] + [E''_\bullet]$ in $\mathcal{K}(X,A)$.*

Proof. Let $C_\bullet$ be the mapping cone of $E'_\bullet \hookrightarrow E_\bullet$, and recall that there is a natural map $C_\bullet \twoheadrightarrow E''_\bullet$. Let $K_\bullet$ be the kernel, which is a chain complex of vector bundles by Proposition 9.3. Elementary homological algebra (applied in each fiber) shows that $K_\bullet$ is exact on X. By Lemma 17.13 below the inclusion $K_\bullet \hookrightarrow C_\bullet$ is split, and so $C_\bullet \cong K_\bullet \oplus E''_\bullet$. So $[C_\bullet] = [K_\bullet] + [E''_\bullet] = [E''_\bullet]$ in $\mathcal{K}(X, A)$. Yet Remark 17.8 gives $[C_\bullet] = [E_\bullet] - [E'_\bullet]$. □

Lemma 17.13. *Let X be a paracompact Hausdorff space. Let $j\colon K_\bullet \hookrightarrow C_\bullet$ be an inclusion between bounded complexes of vector bundles on X, and assume that $K_\bullet$ is exact. Then the map j admits a splitting $\chi\colon C_\bullet \to K_\bullet$.*

Proof. Without loss of generality assume that $K_i = 0 = C_i$ for $i < 0$. We start by writing $K = \oplus_{i=0}^N D(A_i)$ for some vector bundles $A_1, \ldots, A_N$ on X. The inclusion j looks as follows:

$$\begin{array}{ccccccc}
\vdots & & & & & & \vdots \\
\downarrow{\scriptstyle \mathrm{id}} & & & & & & \downarrow \\
A_2 & \oplus & A_1 & \longrightarrow & & & C_2 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow \\
& & A_1 & \oplus & A_0 & \longrightarrow & C_1 \\
& & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
& & & & A_0 & \longrightarrow & C_0
\end{array}$$

Starting at the bottom, choose a splitting χ_0 for the inclusion $A_0 \hookrightarrow C_0$, using Corollary 9.4. Likewise, choose a splitting α_1 for the inclusions $A_1 \hookrightarrow C_1/A_0$ (note that C_1/A_0 is a vector bundle by Proposition 9.3). Define $\chi_1\colon C_1 \to A_1 \oplus A_0$ to be the sum of $C_1 \to C_1/A_0 \xrightarrow{\alpha_1} A_1$ and $C_1 \to C_0 \xrightarrow{\chi_0} A_0$. It is readily checked that χ_1 is a splitting for j_1 and that $d\chi_1 = \chi_0 d$. Continue inductively to define χ at all levels. □

The groups $\mathcal{K}(X, A)$ are readily seen to be homotopy invariant constructions:

Proposition 17.14. *For any pair (X, A) the map $\pi^*\colon \mathcal{K}(X, A) \to \mathcal{K}(X \times I, A \times I)$ is an isomorphism.*

Proof. If $j_0, j_1\colon (X, A) \hookrightarrow (X \times I, A \times I)$ are the evident inclusions then it is clear that $j_0^* = j_1^*$. It then follows by category theory that homotopic maps $(X, A) \to (Y, B)$ induce the same map upon applying $\mathcal{K}(-, -)$. Consequently, if $f\colon (X, A) \to (Y, B)$ is part of a relative homotopy equivalence then it induces an isomorphism on $\mathcal{K}$-groups. Now just apply this to π. □

Before finishing with our basic exploration of the group $\mathcal{K}(X, A)$, let us note the following simple result:

Proposition 17.15. *For any compact Hausdorff space X there is an isomorphism $\mathcal{K}(X, \emptyset) \to K^0(X)$ given by the formula $[E_\bullet] \to \sum_i (-1)^i [E_i]$.*

Proof. It is immediate that the indicated formula gives a group homomorphism $\chi\colon \mathcal{K}(X, \emptyset) \to K^0(X)$; the only nontrivial part is verifying relation (3), but here one uses that if F is a vector bundle on $X \times I$ then $F|_{X\times 0} \cong F|_{X\times 1}$.

There is also the evident map $j\colon K^0(X) \to \mathcal{K}(X, \emptyset)$ sending a vector bundle E to the chain complex $E[0]$ consisting of E in degree 0 and zeros in all other degrees. Certainly $\chi \circ j = \mathrm{id}$.

If $E_\bullet$ is any chain complex of vector bundles on X then we may deform $E_\bullet$ to the complex with zero differentials, by putting a t in front of all the d maps and letting $t \mapsto 0$. So $[E_\bullet] = [(E_\bullet, d = 0)] = \sum_i [\Sigma^i E_i]$ in $\mathcal{K}(X, \emptyset)$. But by Lemma 17.7 we know $[\Sigma^i E_i] = (-1)^i [E_i]$. This proves that $j \circ \chi = \mathrm{id}$. □

Let $E_\bullet$ and $F_\bullet$ be bounded chain complexes of vector bundles on X. Let $E_\bullet \otimes F_\bullet$ denote the usual tensor product of chain complexes, giving another complex of vector bundles on X. In contrast to this, there is also an *external* tensor product. If $G_\bullet$ is a complex of vector bundles on a space Y, define

$$E_\bullet \hat{\otimes} G_\bullet = \pi_1^*(E_\bullet) \otimes \pi_2^*(G_\bullet)$$

where $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are the two projections. Note that if $\Delta \colon X \to X \times X$ is the diagonal map then $E_\bullet \otimes F_\bullet \cong \Delta^*(E_\bullet \hat{\otimes} F_\bullet)$.

The internal and external tensor products induce pairings on the $\mathcal{K}$-groups defined above, taking the form

$$\otimes \colon \mathcal{K}(X,A) \otimes \mathcal{K}(X,B) \to \mathcal{K}(X, A \cup B)$$

and

$$\hat{\otimes} \colon \mathcal{K}(X,A) \otimes \mathcal{K}(Y,B) \to \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)).$$

The main point is that if $V_\bullet$ and $W_\bullet$ are bounded exact sequences of vector spaces and $V_\bullet$ is exact, then $V_\bullet \otimes W_\bullet$ is exact. It follows that if $E_\bullet$ is exact on A and $F_\bullet$ is exact on B, then $E_\bullet \otimes F_\bullet$ is exact on $A \cup B$, with a similar analysis for the external case. Note again that the internal and external tensor products are connected by the formula

$$[E_\bullet] \otimes [F_\bullet] = \Delta^*\big([E_\bullet] \hat{\otimes} [F_\bullet]\big).$$

The following theorem is essentially due to Atiyah, Bott and Shapiro [ABS].

Theorem 17.16. *On the category of homotopically compact pairs, there is a unique natural transformation of functors* $\chi \colon \mathcal{K}(X,A) \to K^0(X,A)$ *such that when* $A = \emptyset$ *one has* $\chi(E_\bullet) = \sum_i (-1)^i [E_i]$. *In fact,* χ *is a natural isomorphism and is compatible with (external and internal) products in the sense that* $\chi(E_\bullet \otimes F_\bullet) = \chi(E_\bullet) \cdot \chi(F_\bullet)$.

The proof of Theorem 17.16 involves some technicalities that would be a distraction at this particular moment, so we postpone the proof until Section 19 below. See, in particular, Section 19.21 for the final proof.

17.17. **Koszul complexes.** Now that we know how to produce classes in relative K-theory, we will put this knowledge to good use.

Let V be a complex vector space of dimension n. For any $v \in V$ consider the chain complex

$$0 \longrightarrow \Lambda^0 V \xrightarrow{v\wedge -} \Lambda^1 V \xrightarrow{v\wedge -} \cdots \xrightarrow{v\wedge -} \Lambda^{n-1} V \xrightarrow{v\wedge -} \Lambda^n V \longrightarrow 0.$$

Denote this chain complex by $J_{V,v}$. It is easy to see that this is exact when $v \neq 0$: indeed, pick a basis $e_1, \ldots, e_n$ for V where $e_1 = v$, then use the usual induced basis for the exterior products. It is clear that if $e_1 \wedge \omega = 0$ then all the basis elements appearing in ω have an e_1 in them.

Exercise 17.18. Check that $J_{V,v} \otimes J_{W,w} \cong J_{V\oplus W, v\oplus w}$, and the isomorphism is canonical.

For various reasons we will need to consider the dual of $J_{V,v}$, which has the form

$$0 \to \Lambda^n V^* \xrightarrow{d_v} \Lambda^{n-1} V^* \xrightarrow{d_v} \cdots \xrightarrow{d_v} \Lambda^1 V^* \xrightarrow{d_v} \Lambda^0 V^* \to 0$$

We denote this by $J^*_{V,v}$, and this is called a **Koszul complex**. Here is a description of the differential:

Proposition 17.19. *Let $e_1, \cdots, e_n$ be a basis for V and write $v = \sum v_i e_i$. Let $e_1^*, \ldots, e_n^*$ be the dual basis for V^*. Then the differential in $J_{V,v}^*$ is given by*

$$d_v(e_{i_0}^* \wedge \cdots \wedge e_{i_k}^*) = \sum_{j=0}^{k} (-1)^j v_{i_j} e_{i_0}^* \wedge \cdots \wedge \widehat{e_{i_j}^*} \wedge \cdots \wedge e_{i_k}^*,$$

where the hat indicates that that term is omitted from the wedge.

Proof. Left to the reader. □

Example 17.20. Prove that $J_{V,v}$ and $J_{V,v}^*$ are isomorphic as chain complexes. (The isomorphism is not canonical, however.)

Recall that K-theory is largely about 'doing linear algebra fiberwise over a base space'. Anything canonical that we can do for vector spaces can be done for vector bundles as well. So let $E \to B$ be a rank n complex vector bundle, and let $s\colon B \to E$ be a section. We get a chain complex of vector bundles

$$0 \longrightarrow \Lambda^0 E \xrightarrow{s\wedge -} \Lambda^1 E \xrightarrow{s\wedge -} \cdots \xrightarrow{s\wedge -} \Lambda^{n-1} E \xrightarrow{s\wedge -} \Lambda^n E \longrightarrow 0$$

which we will denote $J_{E,s}$. For $x \in B$ this chain complex is exact over x provided that $s(x) \neq 0$. Thus it determines an element in $K^0(B, B - s^{-1}(0))$. We can just as well consider the dual complex, which also determines an element $[J_{E,s}^*] \in K^0(B, B - s^{-1}(0))$.

Now let V be a complex vector space of dimension n. Consider the vector bundle $\pi_1\colon V \times V \to V$, with section given by the diagonal map $\Delta\colon V \to V \times V$. Our Koszul complex $J_{V\times V,\Delta}^*$ is exact on $V - 0$, and so defines an element

$$\beta(V) = [J_{V\times V,\Delta}^*] \in K^0(V, V - 0).$$

Example 17.21. One readily checks that $\beta(\mathbb{C})$ is the complex

$$\begin{array}{ccc} \underline{1} & \xrightarrow{\cdot z} & 1 \\ & \searrow \quad \swarrow & \\ & \mathbb{C} & \end{array}$$

where the fiber over $z \in \mathbb{C}$ is the chain complex $0 \to \mathbb{C} \xrightarrow{z} \mathbb{C} \to 0$ (multiplication by z). The Koszul complex $\beta(\mathbb{C}^2)$ has the form

$$\begin{array}{ccccc} \underline{1} & \xrightarrow{A} & \underline{2} & \xrightarrow{B} & \underline{1}\,, \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{C}^2 & & \end{array}$$

where over a point $(z, w) \in \mathbb{C}^2$ we have

$$A = \begin{bmatrix} -w \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} z & w \end{bmatrix}.$$

Finally we look at $\beta(\mathbb{C}^3)$, which has the form

$$\underline{1} \xrightarrow{A} \underline{3} \xrightarrow{B} \underline{3} \xrightarrow{C} \underline{1}$$

where the fiber over $(z, w, u) \in \mathbb{C}^3$ has

$$A = \begin{bmatrix} u \\ -w \\ z \end{bmatrix}, \quad B = \begin{bmatrix} -w & -u & 0 \\ z & 0 & -u \\ 0 & z & w \end{bmatrix}, \quad C = \begin{bmatrix} z & w & u \end{bmatrix}.$$

Let us return to our element $\beta(V) \in K^0(V, V-0)$. If we pick a basis for V then we get isomorphisms

$$K^0(V, V-0) \cong K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^0(D^{2n}, \partial D^{2n}) \cong \widetilde{K}^0(S^{2n}) \cong \widetilde{K}^{-2n}(S^0) = K^{-2n}(pt).$$

Moreover, one checks that any two choices of basis for V give rise to the same isomorphism (essentially because a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ is orientation-preserving). So we may regard $\beta(V)$ as giving us an element of $K^{-2n}(pt)$. Using Exercise 17.18 we have $\beta(V \oplus W) = \beta(V) \cdot \beta(W)$.

When we first learned about K-theory as a cohomology theory, we set ourselves the goal of having explicit generators for $K^*(pt)$. We can now at least state the basic result:

Theorem 17.22.
(a) $K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^{-2n}(pt) \cong \mathbb{Z}$ *and* $\beta(\mathbb{C}^n) = (\beta(\mathbb{C}))^n$ *is a generator.*
(b) $K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$*, where* $\beta = \beta(\mathbb{C}) \in K^{-2}(pt)$*.*

The element $\beta = \beta(\mathbb{C}) \in K^{-2}(pt)$ is often called the **Bott element**, although sometimes this name is applied to $\beta^{-1} \in K^2(pt)$ instead. This theorem is best regarded as part of Bott periodicity. And just as for the periodicity theorem, we again postpone the proof in favor of moving forward and seeing how to use it.

Let $p\colon E \to B$ be a rank n complex vector bundle. Consider the pullback p^*E, which is $\pi_1\colon E \times_B E \to E$. This bundle has an evident section given by the diagonal map $\Delta\colon E \to E \times_B E$, and we may consider the Koszul complex with respect to this section. Since Δ is nonzero away from the zero-section of E, this gives us an element in $K^0(E, E-0)$: we define

$$\mathcal{U}_E = [J^*_{p^*E,\Delta}] \in K^0(E, E-0).$$

Note that if $x \in B$ and $j_x\colon F_x \hookrightarrow E$ is the inclusion of the fiber, it is completely obvious that $j_x^*(\mathcal{U}_E) = \beta(F_x) \in K^0(F_x, F_x - 0)$.

The element $\mathcal{U}_E$ is not quite our desired Thom class, since the Thom class is supposed to lie in $K^{2n}(E, E-0)$ rather than $K^0(E, E-0)$. Of course these groups are the same because of Bott periodicity. To be completely precise, we should define our Thom class to be $\mathcal{U}_E = \beta^{-n} \cdot [J^*_{p^*E,\Delta}]$. However, it is common practice to leave off the factors of β and just do constructions in K^0. We will often follow this practice, but sometimes we will put the factors of β back into the equations in order to emphasize a point. Hopefully this won't be too confusing.

17.23. **Koszul complexes in algebra.** Now that we have seen Koszul complexes in geometry it seems worthwhile to also see how they appear in algebra. They turn out to be very important tools in homological algebra.

Let R be a commutative ring, and let $x_1, \cdots, x_n \in R$. Define the Koszul complex $K(x_1, \cdots, x_n; R)$ to be the complex

$$0 \longrightarrow \Lambda^n R^n \xrightarrow{d} \Lambda^{n-1} R^n \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^2 R^n \xrightarrow{d} \Lambda^1 R^n \xrightarrow{d} \Lambda^0 R^n \longrightarrow 0,$$

where the differential d is given by

$$d(e_{i_0} \wedge \cdots \wedge e_{i_k}) = \sum_{j=0}^{k} (-1)^j x_{i_j} (e_{i_0} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}).$$

Note that d is the unique derivation such that $d(e_i) = x_i$. Define the **Koszul homology groups** as $H_*(x_1, \cdots, x_n; R) = H_*(K(x_1, \cdots, x_n; R))$. We will often abbreviate the sequence $x_1, \ldots, x_n$ to just $\underline{x}$, and write $K(\underline{x}; R)$ and so forth. It is easy to see that $H_0(\underline{x}; R) = R/(x_1, \ldots, x_n)$.

In some cases the Koszul complex $K(\underline{x}; R)$ is actually a resolution of $R/(x_1, \ldots, x_n)$, and this is perhaps the main reason it is useful. To explain when this occurs we need a new definition. The sequence $x_1, \ldots, x_n$ is said to be a **regular sequence** if x_i is a non-zero-divisor in $R/(x_1, \ldots, x_{i-1})$ for every $1 \leq i \leq n$ (in partcular, x_1 is a non-zero-divisor in R). For example, in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ the indeterminates $z_1, \ldots, z_n$ are a regular sequence.

Theorem 17.24. *Let $x_1, \ldots, x_n \in R$.*

(a) If $x_1, \ldots, x_n$ is a regular sequence, then $H_i(\underline{x}; R) = 0$ for all $i \geq 1$.
(b) Suppose R is local Noetherian and $x_1, \cdots, x_n \in m$, where m is the maximal ideal. Then $x_1, \ldots, x_n$ is a regular sequence if and only if $H_i(\underline{x}; R) = 0$ for all $i \geq 1$.

Proof. The subalgebra of $\Lambda^* R^n$ generated by $e_1, \ldots, e_{n-1}$ is a subcomplex of $K(x_1, \ldots, x_n; R)$, and is isomorphic to $K(x_1, \ldots, x_{n-1}; R)$. The quotient complex has a free basis consisting of wedge products that contain e_n; and in fact the process of 'wedging with e_n' gives an isomorphism between $K(x_1, \ldots, x_{n-1}; R)$ and this quotient complex that shifts degrees by one. We can summarize this by saying that there is a short exact sequence of chain complexes

$$0 \to K(x_1, \cdots, x_{n-1}; R) \hookrightarrow K(x_1, \cdots, x_n; R) \twoheadrightarrow \Sigma K(x_1, \cdots, x_{n-1}; R) \to 0.$$

Denote the sequence $x_1, \ldots, x_n$ by $\underline{x}$ and $x_1, \ldots, x_{n-1}$ by $\underline{x'}$.

Our short exact sequence induces a long exact sequence in homology groups:

$$\cdots \to H_i(\underline{x'}; R) \to H_i(\underline{x}; R) \to H_{i-1}(\underline{x'}; R) \xrightarrow{d} H_{i-1}(\underline{x'}; R) \to H_{i-1}(\underline{x}; R) \to \cdots$$

and one easily checks that the connecting homomorphism is multiplication by $\pm x_n$ (we leave this as an exercise).

Our proof of part (a) now proceeds by induction on the length of the sequence n. When $n = 1$ the Koszul complex is $0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow 0$, so $H_1(\underline{x}; R) = \operatorname{Ann}_R x_1 = 0$ since x_1 is a nonzerodivisor.

Now assume that we know part (a) for all regular sequences of length $n-1$. By the induction hypothesis and the above long exact sequence, it is easy to see that $H_i(\underline{x}; R) = 0$ for $i \geq 2$. So we only need to worry about $H_1(\underline{x}; R)$, for which we have

$$H_1(\underline{x'}; R) \to H_1(\underline{x}; R) \to H_0(\underline{x'}; R) \xrightarrow{\pm x_n} H_0(\underline{x'}; R) \to H_0(\underline{x}; R) \to 0$$

By induction $H_1(\underline{x}';R)=0$, and we know $H_0(\underline{x}';R)=R/(x_1,\dots,x_{n-1})$. Since x_n is a nonzerodivisor in this ring, the kernel of the map labelled $\pm x_n$ is zero—hence $H_1(\underline{x};R)=0$ as well. This completes the proof of (a).

For (b), the point is that the above argument is almost reversible. For $n=1$ the other direction works without any assumptions on R, because $H_1(x;R)=\mathrm{Ann}(x)$. So assume by induction that the result holds for sequences of length $n-1$. It follows from the long exact sequence we saw in part (a) that there are short exact sequences

$$0\to H_i(\underline{x}';R)/x_nH_i(\underline{x}';R)\to H_i(\underline{x};R)\to \mathrm{Ann}_{H_{i-1}(\underline{x}';R)}(x_n)\to 0.$$

The assumption that $H_i(\underline{x};R)=0$ implies that $x_nH_i(\underline{x}';R)=H_i(\underline{x}';R)$. But $x_n\in m$, so by Nakayama's Lemma this yields $H_i(\underline{x}';R)=0$. This holds for all $i\geq 1$, so induction gives that $\underline{x}'$ is a regular sequence. The assumption $H_i(\underline{x};R)=0$ also yields that x_n is a nonzerodivisor on $\mathrm{Ann}_{H_{i-1}}(\underline{x}';R)$; so for $i=1$ this says that x_n is a nonzerodivisor on $R/(x_1,\dots,x_{n-1})$. Hence, $\underline{x}$ is a regular sequence. □

We can use our knowledge of Koszul complexes to prove the Hilbert Syzygy Theorem:

Theorem 17.25 (Hilbert Syzygy Theorem)**.** *Let L be a field. Then every finite-generated module over $L[x_1,\dots,x_n]$ has a finite, projective resolution.*

Proof. We first prove the result in the graded case. Let $R=L[x_1,\cdots,x_n]$, and grade R by setting $\deg(x_i)=1$. Assume that M is a finitely-generated, graded R-module. We construct the so-called "minimal resolution" of M: Start by picking a minimal set of homogeneous generators $w_1,\dots,w_k$ for M. Define $F_0=R^k$, graded so that the ith generator has degree equal to $\deg(w_i)$. Let $d_0\colon F_0\to M$ be the map sending e_i to w_i, and let K_0 be the kernel. Then d_0 preserves degrees, so K_0 is again a graded module. Repeat this process to construct $F_1\twoheadrightarrow K_0$, let K_1 be the kernel, repeat to get $F_2\twoheadrightarrow K_1$, and so forth. This constructs a free resolution $F_\bullet\to M$ of the form

$$\cdots\to R^{b_2}\to R^{b_1}\to R^{b_0}\to M\to 0$$

We claim that the matrix for each differential has entries in the ideal $(x_1,\dots,x_n)$: this follows from the fact that at each stage we chose a *minimal* set of generators.

Next, form the complex $F_\bullet\otimes_R R/(x_1,\dots,x_n)$ and take homology. Tensoring with $R/(x_1,\dots,x_n)$ kills all the entries of the matrices and changes every R to an L; so we have

$$L^{b_i}\cong H_i(F_\bullet\otimes_R R/(x_1,\dots,x_n))=\mathrm{Tor}_i(M,R/(x_1,\dots,x_n)).$$

Now we use the fact that we can also compute Tor by resolving $R/(x_1,\dots,x_n)$ and tensoring with M. Yet $R/(x_1,\dots,x_n)$ is resolved by the Koszul complex, which has length n: so this immediately yields that $\mathrm{Tor}_i(M,R/(x_1,\dots,x_n))=0$ for $i>n$. It follows that $b_i=0$ for $i>n$, which says that $F_\bullet$ was actually a finite resolution.

Now we prove the general case, for modules that are not necessarily graded. Choose a presentation of the module

$$R^{b_1}\xrightarrow{A}R^{b_0}\twoheadrightarrow M$$

where A is a matrix with entries in R. Now introduce a new variable x_0 and homogenize A to $\tilde{A}$: that is, multiply factors of x_0 onto the monomials appearing in the entries of A so that all the entries have the same degree. Put

$S = L[x_0, \cdots, x_n] = R[x_0]$, and let $\widetilde{M}$ be the cokernel of $\tilde{A}$:

$$S^{b_1} \xrightarrow{\tilde{A}} S^{b_0} \longrightarrow \widetilde{M} \longrightarrow 0 \, .$$

Note that $\widetilde{M}$ is a graded module over S, and $\widetilde{M} \otimes S/(1-x_0) \cong M$.

What we have already proven in the graded case guarantees a finite $S-$free resolution $\widetilde{F}_\bullet \to \widetilde{M} \to 0$. Let $F_\bullet = \widetilde{F}_\bullet \otimes (S/(1-x_0))$. This is an R-free chain complex, and $H_0(F_\bullet) \cong M$. Note that $H_i(F) = \mathrm{Tor}_i^S(\widetilde{M}, S/(1-x_0))$, and the Tor-module can again also be computed by resolving $S/(1-x_0)$. We use the resolution $0 \to S \xrightarrow{1-x_0} S \to 0$ and immediately find that $H_i(F) = 0$ if $i \geq 2$. We also have that $H_1(F) \cong \mathrm{Ann}_{\widetilde{M}}(1-x_0)$, but such an annihilator is zero for any finitely-generated, graded module. So $F_\bullet \to M$ is a finite free resolution over R. □

Remark 17.26. In the above proof, the deduction of the general case from the graded case was taken from [E, Corollary 19.8].

18. The denouement: connecting algebra, topology, and geometry

Although we are far from the end of these notes, we have reached the point where we can finally explain Serre's definition of intersection multiplicities.

18.1. **The local index.** Suppose that $E_\bullet$ is a bounded chain complex of vector bundles on $\mathbb{C}^n$ that is exact on $\mathbb{C}^n - 0$. Then we get a class $[E_\bullet] \in K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^{-2n}(pt)$. But the Bott calculations say that this group is cyclic, generated by β^n. Thus $[E_\bullet] = d \cdot \beta^n$ for a unique integer d. We call this integer the **local index** of the complex $E_\bullet$, and we will denote it $\mathrm{ind}_0(E_\bullet)$. The natural question is: how do we compute this invariant from the data in $E_\bullet$?

I don't know a simple answer to this question, but the question becomes more manageable if we assume that the complex $E_\bullet$ is algebraic: that is, if we assume that each E_i is an algebraic vector bundle and the maps $E_i \to E_{i-1}$ are algebraic.

Theorem 18.2. *If $E_\bullet$ is a bounded complex of algebraic vector bundles on $\mathbb{C}^n$ that is exact on $\mathbb{C}^n - 0$, then the local index is given by*

$$\mathrm{ind}_0(E_\bullet) = \sum_i (-1)^i \dim_{\mathbb{C}} H_i(P_\bullet)$$

where $P_\bullet$ is a complex of finitely-generated, projective $\mathbb{C}[x_1, \ldots, x_n]$-modules such that $P_\bullet(\mathbb{C}) \cong E_\bullet$.

For the above statement, recall that $P \mapsto P(\mathbb{C})$ is the functor that associates to every projective $\mathbb{C}[x_1, \ldots, x_n]$-module the corresponding vector bundle over $\mathbb{C}^n$; see Section 10.6.

The proof of Theorem 18.2 comes down to a comparison between algebraic and topological K-theory groups. Once the machinery for this comparison is in place, the theorem follows by a simple computation. To set up this machinery we need to recall some ideas from Part 1 of these notes.

Let $R = \mathbb{C}[x_1, \ldots, x_n]/Q$ where Q is a prime ideal, and let $X = \operatorname{Spec} R$ be the corresponding algebraic variety. Let $Z \subseteq \operatorname{Spec} R$ be a Zariski closed set. We use the term "algebraic vector bundle on X" synonymously with "finitely-generated projective over R". If $P_\bullet$ is a complex of algebraic vector bundles on X and p is a point of X, we will say that $P_\bullet$ is **exact at *p*** if the localization $(P_\bullet)_p$ is exact as a complex of R_p-modules (where p is regarded as a prime ideal in R). This notion of exactness at first seems a bit different than what we used in the topological case: here we are considering exactness on germs of sections, whereas in the topological case we used exactness on fibers. In the algebraic world the notions are equivalent:

Lemma 18.3. *Let $P_\bullet$ be a bounded-below complex of finitely-generated projectives over a Noetherian ring R. Let $\mathcal{U} \subseteq \operatorname{Spec} R$ be closed under specialization (i.e., having the property that if $q_1 \in \mathcal{U}$ and $q_1 \subseteq q_2$ then $q_2 \in \mathcal{U}$). Then the following statements are equivalent:*
(1) $(P_\bullet)_q$ is exact for all primes $q \in \mathcal{U}$;
(2) $(P_\bullet)_m$ is exact for all maximal ideals $m \in \mathcal{U}$;
(3) $P_\bullet \otimes_R R/m$ is exact for all maximal ideals $m \in \mathcal{U}$.

Condition (c) is the fiberwise exactness condition, analagous to what we used in the toplogical case. The complex $P_\bullet \otimes_R R/m$ is the pullback of $P_\bullet$ along the map $\operatorname{Spec} R/m \to \operatorname{Spec} R$, and therefore represents the fiber of $P_\bullet$ over the geometric point $\operatorname{Spec} R/m$.

Proof of Lemma 18.3. Of course (1)⇒(2) is trivial, and (2)⇒(1) follows from the fact that $(P_\bullet)_q = \big[(P_\bullet)_m\big]_q$ for any maximal ideal $m \supseteq q$ (and the fact that a localization of an exact complex is still exact).

The direction (2)⇒(3) is also easy, since

$$P_\bullet \otimes_R R/m \cong (P_\bullet)_m \otimes_{R_m} R_m/mR_m. \tag{18.4}$$

The complex $(P_\bullet)_m$ is a bounded-below exact sequence of projectives, and hence is split-exact; so tensoring with any module still gives an exact sequence.

Finally, we must prove (3)⇒(2). Using the isomorphism of (18.4) it suffices to prove that if $Q_\bullet$ is a bounded-below complex of finitely-generated projectives over a Noetherian local ring R such that $Q_\bullet \otimes_R R/m$ is exact, then $Q_\bullet$ is exact. Without loss of generality assume that $Q_i = 0$ for $i < 0$, and let M be the cokernel of $Q_1 \to Q_0$. Since $Q_1 \to Q_0$ is surjective after tensoring with R/m, this implies that $M/mM = 0$. Nakayama's Lemma says that this can only happen in M=0. So $Q_1 \to Q_0$ is surjective and $H_0(Q) = 0$. Now choose a splitting for $Q_1 \to Q_0$, and use this to write $Q_\bullet$ as a direct sum of $Q_0 \xrightarrow{id} Q_0$ and a complex that vanishes in degrees smaller than 1. Apply the same argument as above to this smaller complex, and continue by induction.

To phrase the above argument slightly differently, over a local ring R one can always decompose $Q_\bullet = F_\bullet \oplus G_\bullet$ where $F_\bullet$ is split-exact and $G_\bullet$ is a complex where the matrices for all differentials have entries in m. The assumption that $Q_\bullet \otimes R/m$ is exact implies that $G_\bullet \otimes R/m$ is exact, but this can only happen if $G_\bullet = 0$. So we conclude $Q_\bullet = F_\bullet$, and hence $Q_\bullet$ is exact. □

Define $K^0_{alg}(X, X - Z)$ by taking the free abelian group on bounded chain complexes of algebraic vector bundles on X that are exact at every point in Z and quotienting by the following two relations:
(1) $[P_\bullet] = 0$ if $P_\bullet$ is exact on all of X, and
(2) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ for every short exact sequence of chain complexes $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

Note that $K^0_{alg}(X, X - Z)$ is exactly the same as the group denoted $K(R)_Z$ in Section 5.16; sometimes we will revert to that notation when we want to concentrate on the underlying algebraic perspective.

The assignment $P \mapsto P(\mathbb{C})$ from algebraic to topological vector bundles induces a map of abelian groups

$$\phi\colon K^0_{alg}(X, X-Z) \to K^0(X(\mathbb{C}), X(\mathbb{C}) - Z(\mathbb{C})).$$

Indeed we just have to note that relations (1) and (2) in the definition of $K^0_{alg}(X, X-Z)$ are preserved, but this is something that we know. To avoid cumbersome notation it will be convenient to write the target group of ϕ as $K^0_{top}(X, X-Z)$; it looks much more pleasant to write

$$\phi\colon K^0_{alg}(X, X-Z) \to K^0_{top}(X, X-Z).$$

Sometimes we will drop the "top" and just write $K^0(X, X-Z)$, but we will never drop the "alg".

If M is a finitely-generated R-module, recall that the **support** of M is

$$\operatorname{Supp} M = \{Q \subseteq R \,|\, Q \text{ is prime and } M_Q \neq 0\}.$$

This coincides with $V(\operatorname{Ann} M)$, namely the set of all primes containing $\operatorname{Ann} M$. In particular, $\operatorname{Supp} M$ is Zariski-closed. Let $G(X)_Z$ (or $G(R)_Z$) denote the Grothendieck group of finitely-generated R-modules M such that $\operatorname{Supp} M \subseteq Z$.

We have the usual Euler characteristic map $\chi\colon K^0_{alg}(X, X-Z) \to G(X)_Z$ that sends $[P_\bullet]$ to $\sum_i (-1)^i [H_i(P)]$. The following result should come as no surprise:

Theorem 18.5. *If R is regular, then $\chi\colon K^0_{alg}(X, X-Z) \to G(X)_Z$ is an isomorphism, for any $Z \subseteq \operatorname{Spec} R$.*

Proof. The inverse sends a class $[M]$ to the class $[P_\bullet]$ for any finite projective resolution $P_\bullet$ for M over R. The proof that this is well-defined, and that the maps are inverses, is exactly the same as for Theorem 2.10. □

Now let us restrict to the case where $R = \mathbb{C}[x_1, \ldots, x_n]$, so that X is affine n-space $A^n_{\mathbb{C}}$; we will just write $X = \mathbb{C}^n$ for convenience. Let $Z = \{0\} = V(x_1, \ldots, x_n)$ be the closed set consisting only of the origin. Our aim will be to calculate the group $K^0_{alg}(X, X-Z) = K^0_{alg}(\mathbb{C}^n, \mathbb{C}^n - 0)$ in this case.

Let $m = (x_1, \ldots, x_n)$. It is easy to see that the following conditions on an finitely-generated R-module M are equivalent:

(1) $\operatorname{Supp} M = \{m\}$;
(2) $\operatorname{Ann} M$ is contained in only one maximal ideal, namely m;
(3) $\operatorname{Rad}(\operatorname{Ann} M) = m$;
(4) M is killed by a power of m.

Assuming M satisfies these conditions, consider the finite filtration

$$M \supseteq mM \supseteq m^2 M \supseteq \cdots \supseteq m^k M \supseteq m^{k+1} M = 0.$$

Then in $G(R)_Z$ we have $[M] = \sum_{i=0}^k [m^i M / m^{i+1} M]$. But each quotient is a finite-dimensional R/m-vector space, so $[M]$ is just a multiple of $[R/m]$. This shows that $G(R)_Z$ is cyclic, generated by $[R/m]$. Moreover, each quotient $m^i M/m^{i+1} M$ is finite-dimensional as a $\mathbb{C}$-module (where the module structure is coming from $\mathbb{C} \subseteq R$). It follows that M is also finite-dimensional as a $\mathbb{C}$-module. Since dimension is additive it gives a function

$$\dim\colon G(R)_Z \to \mathbb{Z},$$

which is clearly surjective and hence an isomorphism.

Since $G(\mathbb{C}^n)_{\{0\}}$ is generated by $[R/m]$, it follows that $K^0_{alg}(\mathbb{C}^n, \mathbb{C}^n - 0)$ is generated by the Koszul complex $K(x_1, \ldots, x_n; R)$.

Now consider the following diagram:

(18.6)
$$\begin{array}{ccccc} K^0_{alg}(\mathbb{C}^n, \mathbb{C}^n - 0) & \xrightarrow{\phi} & K^0_{top}(\mathbb{C}^n, \mathbb{C}^n - 0) & \xrightarrow{\cong} & \mathbb{Z}\langle \beta^n \rangle \\ \cong \big\downarrow \chi & & & & \\ G(R)_{\{m\}} & & \xrightarrow[\cong]{\dim} & & \mathbb{Z}. \end{array}$$

We know by Bott's calculations that the target of ϕ is isomorphic to $\mathbb{Z}$ and is generated by the Koszul complex. Likewise, we have just see that the domain of ϕ is isomorphic to $\mathbb{Z}$ and is generated by the algebraic Koszul complex. Since ϕ clearly carries the algebraic Koszul complex to the topological one, ϕ is an isomorphism.

Proof of Theorem 18.2. Fill in diagram (18.6) with the map $\mathbb{Z}\langle \beta^n \rangle \to \mathbb{Z}$ that sends β^n to 1. The diagram then commutes, because one only has to check this on the Koszul complex that generates $K^0_{alg}(\mathbb{C}^n, \mathbb{C}^n - 0)$; and here it is obvious. The commutativity of this diagram is exactly the statement of Theorem 18.2. □

18.7. **Resolutions and fundamental classes.** Now we'll use these ideas to do something a bit more sophisticated. Let $Z \hookrightarrow \mathbb{C}^n$ be a closed algebraic subvariety: $Z = V(I)$ for some ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Assume that Z is smooth of codimension c. Then we have a relative fundamental class $[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$.

Let $P_\bullet$ be a bounded, projective resolution of $\mathbb{C}[x_1, \ldots, x_n]/I$ over $\mathbb{C}[x_1, \ldots, x_n]$. Note that if $Q \in \operatorname{Spec} \mathbb{C}^n$ then

$$Q \in Z \iff Q \supseteq I \iff (R/I)_Q \neq 0,$$

and so if $Q \notin Z$ then $(P_\bullet)_Q$ is exact. So $P_\bullet$ gives a class $[P_\bullet] \in K^0_{alg}(\mathbb{C}^n, \mathbb{C}^n - Z)$. Using our natural transformation $K^0_{alg}(\mathbb{C}^n, \mathbb{C}^n - Z) \to K^0(\mathbb{C}^n, \mathbb{C}^n - Z)$, we get a corresponding class $[P_\bullet]$ in relative topological K-theory. We can promote this to a class in relative K^{2c} by multiplying by β^{-c}. It is reasonable to expect this class to be related to $[Z]_{rel}$:

Theorem 18.8. *In the above situation we have* $[Z]_{rel} = \beta^{-c} \cdot [P_\bullet]$.

Note that the β^{-c} could be dropped if we regarded $[Z]_{rel}$ as a class in $K^0(\mathbb{C}^n, \mathbb{C}^n - Z)$ instead of $K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$.

Theorem 18.8 gives the main connection between K-theory and homological algebra: projective resolutions give fundamental classes in K-theory. To prove this theorem, recall that $[Z]_{rel}$ is defined by choosing a tubular neighborhood U of Z in $\mathbb{C}^n$, together with an isomorphism between U and the normal bundle $N = N_{\mathbb{C}^n/Z}$. The class $[Z]_{rel}$ is the unique class that restricts to the Thom class $\mathcal{U}_N$. This is all encoded in the following diagram:

$$\begin{array}{ccccc}
[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z) & \xleftarrow[\beta^{-c}]{\cong} & K^0(\mathbb{C}^n, \mathbb{C}^n - Z) & \xleftarrow{\phi} & K^0_{\mathrm{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z) \\
\text{excision} \downarrow \cong & & \downarrow & & \\
K^{2c}(U, U - Z) & \xleftarrow[\beta^{-c}]{} & K^0(U, U - Z) & & \\
\| & & \| & & \\
\mathcal{U}_N \in K^{2c}(N, N - 0) & \xleftarrow[\beta^{-c}]{} & K^0(N, N - 0) & \xleftarrow{\phi} & K^0_{alg}(N, N - 0).
\end{array}$$

Our goal is to take $[P_\bullet] \in K^0_{alg}(\mathbb{C}^n, \mathbb{C}^n - Z)$, push it across the top row and then down, and show that the image is $\mathcal{U}_N$. But note that $\mathcal{U}_N$ is algebraic—it is represented by the Koszul complex, which is entirely algebraic. So $\mathcal{U}_N$ lifts to a class in $K^0_{alg}(N, N - 0)$. In some sense the most natural idea for our proof would be to stay entirely in the right-most column, and to compare both $[P_\bullet]$ and $\mathcal{U}_N$ on the algebraic side of things. Of course one immediately sees the trouble, which is that the neighborhood U is *not* algebraic—and so we have a missing group in the third column, obstructing our proof. Our goal will be to give a clever way around this, using a technique from algebraic geometry called *deformation to the normal bundle*.

Remark 18.9. Before giving the next argument we need to give a brief review of blow-ups. Let X be a smooth variety and $A \hookrightarrow X$ a closed subvariety, which for convenience we assume to be smooth as well. The blow-up $\mathrm{Bl}_A(X)$ of X at A is an algebraic variety that topologically corresponds to removing A and then sewing in a copy of the projectived normal bundle in its place. That is, let V be a tubular neighborhood of A, with associated homeomorphism $V \cong N_{X/A}$. Then there is a homeomorphism

$$\mathrm{Bl}_A(X) \cong (X - A) \amalg_{(V-A)} \mathbb{P}(N).$$

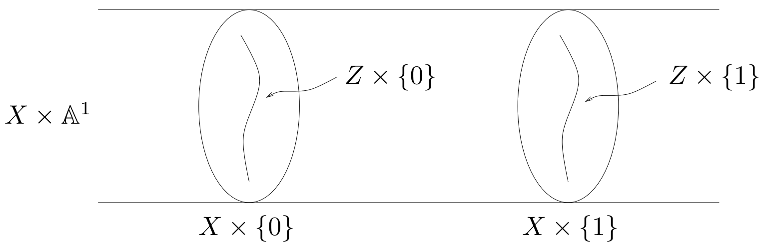
Here the map $V - A \to \mathbb{P}(N)$ is the map $N - 0 \to \mathbb{P}(N)$ that sends any nonzero element of a fiber F_a to the corresponding line it spans, regarded as an element of $\mathbb{P}(F_a)$. Observe that the pushout

$$\begin{array}{ccc}
\mathbb{P}(N) & \longrightarrow & \mathrm{Bl}_A(X) \\
\downarrow & & \vdots \\
A & \cdots\!\!> & X
\end{array}$$

is homeomorphic to X. If π denotes the map $\mathrm{Bl}_A(X) \to X$, note that $\pi^{-1}(X - A) \to X - A$ is an isomorphim, whereas for any point $a \in A$ the fiber $\pi^{-1}(a)$ is a projective space $\mathbb{C}P^{c-1}$ where c is the codimension of A in X. These are the main properties of blow-ups.

We are now ready to give the proof of our result:

Proof of Theorem 18.8. Write $\mathbb{C}^n = X$. The argument we will give doesn't use anything special about $\mathbb{C}^n$, and actually works for any smooth variety. We begin by considering $X \times \mathbb{A}^1$:



We will work with the blow-up of $X \times \mathbb{A}^1$ at the subvariety $Z \times \{0\}$:

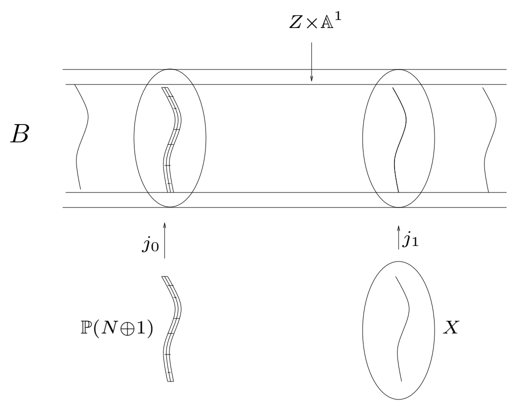
$$B = B\ell_{Z\times 0}(X \times \mathbb{C}).$$

Set $N = N_{X/Z}$ and $N' = N_{X\times\mathbb{A}^1/Z\times\{0\}}$. Note that $N' = N \oplus 1$. Topologically, we have a homeomorphism

$$B \cong \left[(X \times \mathbb{A}^1) - (Z \times \{0\})\right] \amalg_{(V-0)} \mathbb{P}(N')$$

where V is a tubular neighborhood of $Z \times \{0\}$ in $X \times \mathbb{A}^1$.

Let $\pi\colon B \to X \times \mathbb{A}^1$ be the blow-up map. Let $j_1\colon X \hookrightarrow B$ be the map $x \mapsto (x, 1)$. Let $j_0\colon \mathbb{P}(N \oplus 1) \hookrightarrow B$ be the inclusion into $\pi^{-1}(X \times \{0\})$. These can be visualized via the following schematic picture:



We claim that $\pi\colon B \to X \times \mathbb{A}^1$ has a section f over $Z \times \mathbb{A}^1$. The definition of this section is completely clear (and unique) on $Z \times (\mathbb{A}^1 - 0)$, the only sublety is the definition on $Z \times \{0\}$; but here we use the canonical section of $\mathbb{P}(N \oplus 1) \to Z$. A little effort shows that this gives a well-defined map $Z \times \mathbb{A}^1 \to B$, and it is clearly a section (indicated in the above picture).

Consider the following (non-commutative) diagram of pairs of spaces:

$$\begin{array}{ccc}
(X, X - Z) & \xrightarrow{j_1} & (B, B - (Z \times \mathbb{A}^1)) \\
\uparrow & & \\
(U, U - Z) & & \uparrow j_0 \\
\| & & \\
(N, N - 0) & \xrightarrow{i} & (\mathbb{P}(N \oplus 1), \mathbb{P}(N \oplus 1) - \mathbb{P}(1))
\end{array}$$

Here j_1 is the inclusion of the fiber of $B \to \mathbb{A}^1$ over 1. A little thought shows that this diagram commutes up to homotopy. Note that all of the maps are algebraic except for the inclusion $h\colon N \hookrightarrow X$ (we implicitly identify N with U here). Apply $K^0(-)$ to obtain the commutative diagram

$$\begin{array}{ccc} K^0(X, X-Z) & \xleftarrow{\ j_1^*\ } & K^0(B, B-(Z\times\mathbb{A}^1)) \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle j_0^*} \\ K^0(N, N-0) & \xleftarrow{\ i^*\ } & K^0(\mathbb{P}(N\oplus 1), \mathbb{P}(N\oplus 1)-\mathbb{P}(1)). \end{array} \tag{18.10}$$

The left vertical arrow is dotted only as a reminder that it is not algebraic. There is a similar diagram, without the dotted arrow, in which every $K^0(-)$ has been replaced with K^0_{alg}; and this new diagram maps to the one above.

Let $Q_\bullet$ be a resolution of $\mathcal{O}_{Z\times\mathbb{A}^1}$ by locally free $\mathcal{O}_B$-modules. Then we have the corresponding class $[Q_\bullet] \in K^0(B, B-(Z\times\mathbb{A}^1))$. We will show that

(1) $j_1^*(Q_\bullet)$ is a resolution of $\mathcal{O}_Z$ on X, and
(2) $(j_0 \circ i)^*(Q_\bullet)$ is a resolution of the structure sheaf of the zero-section on N.

From (1) it follows that there is a chain homotopy equivalence $j_1^*(Q_\bullet) \simeq P_\bullet$, where $P_\bullet$ is our chosen resolution of $\mathcal{O}_Z$ on X. Hence $j_1^*([Q_\bullet]) = [P_\bullet]$ in $K^0_{alg}(X, X-Z)$. From (2) it follows that there is a chain homotopy equivalence $(j_0\circ i)^*(Q_\bullet) \simeq J^*_{p^*N,\Delta}$, since both gives resolutions of the structure sheaf of the zero section on N. Hence $i^*(j_0^*([Q_\bullet])) = \mathcal{U}_N$ in $K^0_{alg}(N, N-0)$. Now push all of this into topological K^0 and use the commutativity of (18.10) to obtain that $h^*([P_\bullet]) = \mathcal{U}_N$. But h^* is an isomorphism, and $[Z]_{rel}$ was defined to be the unique class in $K^0(X, X-Z)$ that maps to $\mathcal{U}_N$ via h^*. So $[P_\bullet] = [Z]_{rel}$.

So the proof reduces to checking the algebraic facts (1) and (2). To do so, start with the following diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\ j_1\ } & B \\ \downarrow & & \downarrow \\ X\times\{1\} & \longrightarrow & X\times\mathbb{A}^1 \\ \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \mathbb{A}^1 \end{array} \tag{18.11}$$

where B_1 is the fiber of $B \to \mathbb{A}^1$ over 1 (note that $B_1 \cong X$). Every square in this diagram is a pullback, including the outer square (which is more of a rectangle). Applying j_1^* to $Q_\bullet$ amounts to pulling back along $\{1\} \hookrightarrow \mathbb{A}^1$ (or algebraically, tensoring over $\mathbb{C}[t]$ with $\mathbb{C}[t]/(t-1)$). But note that $B \to \mathbb{A}^1$ is flat—a map from a variety to a curve is flat as long as it is dominant, which this clearly is. So $\mathcal{O}_B$ is flat over $\mathbb{A}^1$, and it is trivial that the $\mathcal{O}_B$-module $\mathcal{O}_{Z\times\mathbb{A}^1}$ is flat over $\mathbb{A}^1$. So the complex $Q_\bullet \to \mathcal{O}_{Z\times\mathbb{A}^1} \to 0$ is an exact sequence of sheaves that are flat over $\mathbb{A}^1$, therefore the pullback to B_1 is still exact. This proves (1).

The proof of (2) is very similar. Consider the analog of diagram (18.11) built around the inclusions $B_0 \hookrightarrow B$ and $\{0\} \hookrightarrow \mathbb{A}^1$. We argue as above that the pullback of $Q_\bullet \to \mathcal{O}_{Z\times\mathbb{A}^1}$ to B_0 is still exact. Note that $B_0 = \mathrm{Bl}_Z(X) \amalg_{\mathbb{P}(N)} \mathbb{P}(N\oplus 1)$, with $\mathrm{Bl}_Z(X)$ a closed subscheme. So $B_0 - \mathrm{Bl}_Z(X) = \mathbb{P}(N\oplus 1) - \mathbb{P}(N) = N$. The bundle N is therefore an open subscheme of B_0, and of course the inclusion of an open subscheme is flat—so restricting further to N still preserves exactness. The last thing to check is that restricting $\mathcal{O}_{Z\times\mathbb{A}^1}$ to B_0 gives the structure sheaf for the canonical section of $\mathbb{P}(N\oplus 1)$ (which then corresponds to the zero section of N under $N\hookrightarrow \mathbb{P}(N\oplus 1)$)—but this is obvious. □

Remark 18.12. The technique used in the above proof, centering around the variety B, is called "deformation to the normal bundle". It was used extensively in papers by Fulton and Macpherson in the 1970s, and has a prominent role in the book [F]. The technique gives a substitute in algebraic geometry for the role played by tubular neighborhoods in topology.

As a consequence of Theorem 18.8 we can now obtain Serre's formula for intersection multiplicities:

Corollary 18.13. *Let Z and W be smooth, closed subvarieties of $\mathbb{C}^n$ such that $Z\cap W=\{0\}$. Then $i(Z,W;0)=\sum(-1)^i\dim\operatorname{Tor}_i(R/I,R/J)$, where $R=\mathbb{C}[x_1,\dots,x_n]$ and I and J are the ideals of functions vanishing on Z and W, respectively.*

Proof. Start with the relative fundamental classes $[Z]_{rel}\in K^{2c}(\mathbb{C}^n,\mathbb{C}^n-Z)$ and $[W]_{rel}\in K^{2d}(\mathbb{C}^n,\mathbb{C}^n-W)$, where c and d are the codimensions of Z and W inside of $\mathbb{C}^n$. Note that since 0 is an isolated point of intersection we must have $c+d\geq n$ (this is an algebraic lemma, see ????). There are in some sense two cases, depending on whether $c+d=n$ or $c+d>n$. In the former case, multiplying our fundamental classes together we get

$$[Z]_{rel}\cdot[W]_{rel}\in K^{2n}(\mathbb{C}^n,\mathbb{C}^n-(Z\cap W))=K^{2n}(\mathbb{C}^n,\mathbb{C}^n-0).$$

Note that $K^{2n}(\mathbb{C}^n,\mathbb{C}^n-0)\cong\mathbb{Z}$ and is generated by $[0]_{rel}$. The topological definition of $i(Z,W;0)$ is that it is the unique integer for which

$$[Z]_{rel}\cdot[W]_{rel}=i(Z,W;0)\cdot[0]_{rel} \tag{18.14}$$

This definition works in any complex-oriented cohomology theory.

If $c+d>n$ then it is clear that Z and W may be moved near 0 (in the topological setting) so that they do not intersect at all, and therefore $[Z]_{rel}\cdot[W]_{rel}=0$. Equation (18.14) is still a valid definition, it just yields that $i(Z,W;0)=0$ here.

The key to the proof is simply realizing that all of our constructions can be lifted back into K^0_{alg}. Let $P_\bullet\to R/I$ and $Q_\bullet\to R/J$ be bounded free resolutions. Then $[Z]_{rel}=\beta^{-c}\cdot[P_\bullet]$ and $[W]_{rel}=\beta^{-d}\cdot[Q_\bullet]$. So $[Z]_{rel}\cdot[W]_{rel}=\beta^{-c-d}\cdot[P.\otimes_R Q]\in K^{2(c+d)}(\mathbb{C}^n,\mathbb{C}^n-0)$. Recall that $K^0(\mathbb{C}^n,\mathbb{C}^n-0)\cong\mathbb{Z}$ and is generated by the Koszul complex J^*. Recall as well that $[0]_{rel}=\beta^{-n}\cdot[J^*]$, by definition. If we write $[P\otimes_R Q]=s[J^*]$ for $s\in\mathbb{Z}$, then we have the formula

$$s\cdot\beta^{-c-d}\cdot[J^*]=i(Z,W;0)\cdot\beta^{-n}\cdot[J^*].$$

If $c+d=n$ then the formula implies $s=i(Z,W;0)$. If $c+d\neq n$ then the only way the formula can be true is if both sides are zero, in which case $s=0=i(Z,W;0)$. So $s=i(Z,W;0)$ in either case.

To conclude the proof we just note that the Local Index Theorem (18.2) gives $s=\sum_i(-1)^i\dim_{\mathbb{C}}H_i(P\otimes_R Q)$, and $H_i(P\otimes Q)\cong\operatorname{Tor}_i(R/I,R/J)$. □

Exercise 18.15. Suppose that Z and W are smooth algebraic subvarieties of $\mathbb{C}^n$ such that $Z\cap W=\{p_1,\dots,p_d\}$.

(a) Choose a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(p_i) = 0$ for $i > 1$ and $f(p_1) \neq 0$, and let $S = R_f$. Let $U = \operatorname{Spec} R_f \subseteq \mathbb{C}^n$ be the corresponding Zariski open set. Convince yourself that it is reasonable to define the intersection multiplicity $i(Z, W; p_1)$ by the formula

$$[Z]_{rel,U} \cdot [W]_{rel,U} = i(Z, W; p_1) \cdot \beta^{-n} \cdot [p_1]_{rel,U}$$

where $[Z]_{rel,U}$ is the image of $[Z]_{rel}$ under $K^*(\mathbb{C}^n, \mathbb{C}^n - Z) \to K^*(U, U - Z)$, and similarly for $[W]_{rel,U}$ and $[p_1]_{rel,U}$. In particular, convince yourself that this is independent of the choice of f.

(b) Next, modify the proof of Corollary 18.13 to show that

$$i(Z, W; p_1) = \sum (-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)_f$$

where I and J are the ideals corresponding to Z and W.

(c) If M is a finitely-generated module over R_f such that $\operatorname{Supp} M = \{p\}$ (p a maximal ideal of R), prove that $M = M_p$. Deduce that

$$i(Z, W; p_1) = \sum (-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)_{p_1}.$$

Corollary 18.13 (and Exercise 18.15) in some sense brings to a close the main questions we raised at the beginning of these notes. We have now seen why Serre's alternating sum of Tor's definition of intersection multiplicity is the 'correct' one, and how this ties in to the study of K-theory.

19. More about relative K-theory

Our aim in this section is to revisit the issue of relative K-theory and give the proof of Theorem 17.16. The ideas behind the proof are interesting and have their own intrinsic appeal, but in fact they hardly ever resurface outside the confines of this one argument. This section should be regarded as giving some technical information which is not necessary for anything later in the notes.

Throughout this section (X, A) will be a finite CW-pair, unless otherwise noted. Recall the group $\mathcal{K}(X, A)$ introduced in Definition 17.6, made from bounded complexes of vector bundles on X that are exact on A. We take several steps aimed at analyzing these groups:

(1) Instead of studying $\mathcal{K}(X, A)$ directly we look instead at a certain set of equivalence classes of complexes. This set is a monoid whose group completion is $\mathcal{K}(X, A)$, but it has the property that the equivalence classes are a bit easier to get our hands on.

(2) Rather than consider all bounded complexes, we consider complexes which are concentrated in degrees between 0 and n, for some fixed $n \geq 1$. We prove that all choices of n give rise to the same theory; so in some sense the use of chain complexes is overkill, as it suffices to just look at complexes of length 1. (The use of all chain complexes makes for a more natural theory, however—for example, the tensor product of two complexes of length 1 is a complex of length 2, and can only be turned back into a complex of length 1 by a cumbersome process).

(3) Finally, and most importantly, we replace our consideration of chain complexes by that of a related but different construct. Namely, we consider $\mathbb{Z}$-graded collections of vector bundles $\{E_i\}$ together with an exact differential defined only over the set A (that is, a differential on $(E_\bullet)|_A$). Let us call such things

"A-relative chain complexes". It turns out that every A-relative chain complex may be extended to give an ordinary chain complex (that is exact on A), and the space of all possible extensions is contractible. So homotopically speaking there is no real difference between the theories obtained from the two notions. The A-relative chain complexes turn out to give a theory that is a bit easier to manipulate, essentially because one doesn't have to deal with extraneous data.

19.1. **Relative chain complexes.** Let $\mathcal{C}\mathrm{h}(X, A)$ denote the category whose objects are bounded chain complexes of vector bundles on X that are exact on A. A map in this category is simply a map of chain complexes of vector bundles. In contrast to this, let $\mathcal{C}\mathrm{h}(X, A)_A$ denote the category whose objects are collections $\{E_i\}$ of vector bundles on X, all but finitely-many of which are zero, together with maps $d\colon E_{i+1}|A \to E_i|A$ making the restriction $E_\bullet|_A$ into an exact chain complex of vector bundles on A. A map $E \to E'$ in $\mathcal{C}\mathrm{h}(X, A)_A$ is a collection of maps $E_i \to E_i'$ of vector bundles on X that commute with the maps d where defined (i.e., over the set A). An object in $\mathcal{C}\mathrm{h}(X, A)_A$ will be called an ***A*-relative chain complex**.

Note the difference between $\mathcal{C}\mathrm{h}(X, A)$ and $\mathcal{C}\mathrm{h}(X, A)_A$: in the former the differentials are defined on all of X, whereas in the latter they are only defined on A. Observe that there is an evident functor $\mathcal{C}\mathrm{h}(X, A) \to \mathcal{C}\mathrm{h}(X, A)_A$, which we will denote $E \mapsto E(A)$.

Say that two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$ are **homotopic** if there is an object $\mathcal{E}$ in $\mathcal{C}\mathrm{h}(X \times I, A \times I)$ together with isomorphisms $\mathcal{E}|_{X\times 0} \cong E_\bullet$ and $\mathcal{E}|_{X\times 1} \cong E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$. Write this relation as $E_\bullet \sim_h F_\bullet$. Likewise, define two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)_A$ to be homotopic if there is an object $\mathcal{E}$ in $\mathcal{C}\mathrm{h}(X \times I, A \times I)_{A\times I}$ together with isomorphisms $\mathcal{E}|_{X\times 0} \cong E_\bullet$ and $\mathcal{E}|_{X\times 1} \cong E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)_A$. In each of these two settings the notion of homotopy is readily seen to be an equivalence relation.

We will also need a second equivalence relation on chain complexes. Say that two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$ are **stably equivalent** if there exist elementary complexes (see Definition 17.9) $P_1, \dots, P_r, Q_1, \dots, Q_s$ in $\mathcal{C}\mathrm{h}(X, A)$ such that

$$E \oplus P_1 \oplus \cdots \oplus P_r \cong E' \oplus Q_1 \oplus \cdots \oplus Q_s.$$

Write this as $E_\bullet \sim_{st} E'_\bullet$. We define a similar equivalence relation on the objects of $\mathcal{C}\mathrm{h}(X, A)_A$.

Let $\mathcal{C}\mathrm{h}_n(X, A)$ be the full subcategory of $\mathcal{C}\mathrm{h}(X, A)$ consisting of chain complexes $E_\bullet$ such that $E_i = 0$ when $i \notin [0, n]$, and let $\mathcal{C}\mathrm{h}_n(X, A)_A$ be the analogous subcategory of A-relative complexes. It will be convenient to allow the index n to be ∞ here. Let $\mathcal{L}_n(X, A)$ and $\mathcal{L}_n(X, A)_A$ denote the sets of isomorphism classes of objects in these two categories.

Write $\mathcal{L}_n(X, A)^h$ for the equivalence classes of $\mathcal{L}_n(X, A)$ under the homotopy relation. Also, write $\mathcal{L}_n(X, A)^{st}$ for the equivalence classes of $\mathcal{L}_n(X, A)$ under the stable-equivalence relation. Finally, write $\mathcal{L}_n(X, A)^{h,st}$ for the equivalence classes generated by combination of these two types of equivalence. Note that we also have sets $\mathcal{L}_n(X, A)^h_A$, $\mathcal{L}_n(X, A)^{st}_A$, and $\mathcal{L}_n(X, A)^{h,st}_A$. The restriction functor $\mathcal{C}\mathrm{h}_n(X, A) \to \mathcal{C}\mathrm{h}_n(X, A)_A$ clearly induces maps of sets $\mathcal{L}_n(X, A)^{h,st} \to \mathcal{L}_n(X, A)^{h,st}_A$ and so forth.

Note that direct sum of complexes makes $\mathcal{L}_n(X, A)$ into a monoid, and similarly for $\mathcal{L}_n(X, A)^h$, $\mathcal{L}_n(X, A)^{st}$, and $\mathcal{L}_n(X, A)^{h,st}$. The same remark holds for the A-relative versions.

19.2. **Comparing complexes to A-relative complexes.** Our first important result is the following:

Proposition 19.3. *For any $n \geq 1$ the map $\mathcal{L}_n(X,A)^{h,st} \to \mathcal{L}_n(X,A)_A^{h,st}$ is a bijection.*

We start with a lemma:

Lemma 19.4. *The functor $\mathcal{C}h(X,A) \to \mathcal{C}h(X,A)_A$ is surjective on objects. Additionally, if E_1 and E_2 are objects in $\mathcal{C}h(X,A)_A$ such that $E_1(A) \cong E_2(A)$ then E_1 and E_2 are homotopic.*

Proof. Given a collection of complex vector spaces V_i, $i \in \mathbb{Z}$, let $\text{Ch-struct}(V) \subseteq \prod_i \text{Hom}(V_{i+1}, V_i)$ denote the collection of sequences $(d_i)_{i\in\mathbb{Z}}$ satisfying $d_i \circ d_{i+1} = 0$ for all i. Regard $\text{Ch-struct}(V)$ as a topological space by giving it the subspace topology. Note that if $d \in \text{Ch-struct}(V)$ then $t \cdot d \in \text{Ch-struct}(V)$ for any $t \in \mathbb{C}$, and letting $t \mapsto 0$ thereby gives a contracting homotopy showing that $\text{Ch-struct}(V)$ is contractible.

Now suppose that $\{E_i\}_{i\in\mathbb{Z}}$ is a collection of vector bundles on X having the property that only finitely many are nonzero. Let $\text{Ch-struct}(E) \to X$ be the evident map whose fiber over a point x is $\text{Ch-struct}(\{(E_i)_x\})$. The assumption that there are only finitely many E_i implies that each x has a neighborhood over which all the bundles are trivial, and from this it readily follows that $\text{Ch-struct}(E) \to X$ is a fiber bundle. As the fibers are contractible, it is also a weak homotopy equivalence.

Note that making $\{E_i\}$ into a chain complex is precisely the same as giving a section of $\text{Ch-struct}(E) \to X$. Likewise, equipping $\{E_i\}$ with a chain complex differential *over* A is the same as giving a section defined over A.

Suppose given an object E in $\mathcal{C}h(X,A)_A$. Consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & \text{Ch-struct}(E) \\ \downarrow & & \downarrow \simeq \\ X & \longrightarrow & X \end{array} \tag{19.5}$$

where the top horizontal map encodes the differentials on E. Since $A \to X$ is a cofibration and $\text{Ch-struct}(E) \to X$ is a trivial fibration, there is a lifting $X \to \text{Ch-struct}(E)$. This lifting precisely gives a chain complex structure on $\{E_i\}$, defined on all of X, that extends the one defined over A. This proves that the functor $E \mapsto E(A)$ is surjective on objects.

Next observe that if $f\colon Y \to X$ is any map then $\text{Ch-struct}(f^*E)$ is canonically identified with the pullback of $Y \to X \leftarrow \text{Ch-struct}(E)$. This is an easy exercise. Suppose that (E,s) and (E,s') are two preimages in $\mathcal{C}h(X,A)$ for the same object E in $\mathcal{C}h(X,A)_A$. Then s and s' correspond to two liftings in the square (19.5). Given this data, form the new diagram

$$\begin{array}{ccc} (X\times 0) \amalg_{(A\times I)} (X \times 1) & \longrightarrow & \text{Ch-struct}(E) \\ \downarrow & & \downarrow \simeq \\ X \times I & \xrightarrow{\ \pi\ } & X \end{array}$$

where $\pi\colon X \times I \to X$ is the projection, $A \times I \to \text{Ch-struct}(E)$ is the constant homotopy, and the top horizontal map equals s and s' on the two copies of X.

Once again, the diagram has a lifting. The resulting map $X \times I \to \text{Ch-struct}(E)$ corresponds to a section of $\text{Ch-struct}(\pi^*E) \to X \times I$, and so specifies a complex of vector bundles on $X \times I$. It is clear that the differentials are constant (with respect to 'time') on $A \times I$, so the complex lives in $\mathcal{C}h(X \times I, A \times I)$. By construction it restricts to the two liftings (E, s) and (E, s') at times 0 and 1.

Now suppose that E and F are objects in $\mathcal{C}h(X, A)$ and that $E(A) \cong F(A)$. So there are isomorphisms $f_i\colon E_i \to F_i$ of bundles over X which, when restricted to A, commute with the differentials d. Let d'_i be the composite

$$E_i \xrightarrow{f_i} F_i \xrightarrow{d} F_{i-1} \xrightarrow{f_{i-1}^{-1}} E_{i-1}.$$

Note that the d' maps give a chain complex structure on $\{E_i\}$; call this new chain complex E'. We have $E' \cong F$ as objects in $\mathcal{C}h(X, A)$. Observe that $d'|_A = d|_A$, and so $E(A) = E'(A)$. It follows by what we have already proven that E and E' are homotopic in $\mathcal{C}h(X, A)$. Since E' and F are (trivially) homotopic, transitivity gives that E is homotopic to F. □

Remark 19.6. The above proof was written in part to demonstrate the technique of translating a desired task into a lifting problem. This is a useful technique that we will need again later in this section. However, it is worth pointing out that in the particular cases from the above proof the lifting problems could be solved in a very concrete and simple way. If (Y, B) is a CW-pair and $E_\bullet$ is an object in $\mathcal{C}h(Y, B)_B$, we extend the differentials from B to all of Y by an induction over the cells of $Y - B$. If e^n is such a cell, we assume inductively that the differentials have been defined over the boundary. Identifying the cell with the disk D^n, points in the interior of the cell have the form tx for $x \in \partial D^n$ and $t \in [0, 1)$. Define the differential over tx to be t times the differential over x. This clearly gives the required extension.

Proof of Proposition 19.3. It remains to prove that $\mathcal{L}_n(X, A)^{h,st} \to \mathcal{L}_n(X, A)_A^{h,st}$ is injective. Let E and F be two objects in $\mathcal{C}h_n(X, A)$. It will suffice to prove that

(1) if $E(A) \sim_h F(A)$ then $E = F$ in $\mathcal{L}_n(X, A)^{h,st}$, and
(2) if $E(A) \sim_{st} F(A)$ then $E = F$ in $\mathcal{L}_n(X, A)^{h,st}$.

For suppose that $E(A)$ and $F(A)$ are identified in $\mathcal{L}_n(X, A)_A^{h,st}$. Then there is a finite chain of objects $\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_r$ in $\mathcal{C}h_n(X, A)_A$ such that $\tilde{E}_1 = E(A)$, $\tilde{E}_r = F(A)$, and for each i either $\tilde{E}_i \sim_h \tilde{E}_{i+1}$ or $\tilde{E}_i \sim_{st} \tilde{E}_{i+1}$. By Lemma 19.4 there are chain complexes $E_i \in \mathcal{C}h_n(X, A)$ such that $E_i(A) = \tilde{E}_i$, for each i. We can choose $E_1 = E$ and $E_r = F$, and we do so. By iterated applications of (1) and (2) we then know that $E_1, E_2, \ldots, E_r$ are all identified in $\mathcal{L}_n(X, A)^{h,st}$, and in particular E and F are identified. This is what we needed to prove.

We turn to the proofs of (1) and (2). Suppose that $E(A) \sim_h F(A)$. Then there exists an $\tilde{\mathcal{E}} \in \mathcal{C}h(X \times I, A \times I)_{A \times I}$ together with isomorphisms $\tilde{\mathcal{E}}|_{X \times 0} \cong E(A)$ and $\tilde{\mathcal{E}}|_{X \times 1} \cong F(A)$. By Lemma 19.4 there is an $\mathcal{E} \in \mathcal{C}h(X \times I, A \times I)$ such that $\mathcal{E}(A) = \tilde{\mathcal{E}}$. So $\mathcal{E}|_{X \times 0}$ and $\mathcal{E}|_{X \times 1}$ are homotopic. Moreover, we have $\mathcal{E}|_{X \times 0}(A) \cong E(A)$, and so by Lemma 19.4 $\mathcal{E}|_{X \times 0}$ is homotopic to E. The same reasoning gives that $\mathcal{E}|_{X \times 1}$ is homotopic to F. By transitivity, E is homotopic to F.

Finally, suppose that $E(A) \sim_{st} F(A)$. So there exist elementary complexes $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_s$ such that $E(A) \oplus \bigoplus_i P_i \cong F(A) \oplus \bigoplus_j Q_j$. Let $P = \oplus_i P_i$ and $Q = \oplus_j Q_j$, and observe that these belong to $\mathcal{C}h_n(X, A)$ (i.e, the differentials are defined over all of X). Then $(E\oplus P)(A)\cong(F\oplus Q)(A)$, and so Lemma 19.4 tells us that $E\oplus P$ and $F\oplus Q$ are homotopic. Consequently, E and F are identified in $\mathcal{L}_n(X,A)^{h,st}$. □

19.7. **Where we are headed.** The group $\mathcal{K}(X,A)$ is readily checked to be the group completion of the monoid $\mathcal{L}_\infty(X,A)^{h,st}$ (use Proposition 17.10). By Proposition 19.3 this is the same as the group-completion of $\mathcal{L}_\infty(X,A)^{h,st}_A$, and we will now focus entirely on this latter object.

Note that $\mathcal{L}_\infty(X,A)^{h,st}_A$ is the colimit of the directed system

$$\mathcal{L}_1(X,A)^{h,st}_A\longrightarrow\mathcal{L}_2(X,A)^{h,st}_A\longrightarrow\mathcal{L}_3(X,A)^{h,st}_A\longrightarrow\cdots$$

Our next task will be reduce the study of $\mathcal{L}_\infty(X,A)^{h,st}_A$ to the much more accessible object $\mathcal{L}_1(X,A)^{h,st}_A$. We will show that the maps $\mathcal{L}_n(X,A)^{h,st}_A\to\mathcal{L}_{n+1}(X,A)^{h,st}_A$ are bijective for all $n\geq 1$. This will be based on a strange technique for folding the top group of an exact complex two degrees lower down, to construct a new complex that happens to also be exact. We pause to describe the algebra underlying this technique.

19.8. **An unusual construction from homological algebra.** Let V be an exact complex of vector spaces and assume that $V_i=0$ for $i>n$. (The complex could actually consist of projectives over some ring, but let us stick with the simpler setting). Since the complex is exact there exists a contracting homotopy: maps $e\colon V_i\to V_{i+1}$ such that $de+ed=\mathrm{id}$. Let ΓV be the following chain complex, concentrated in degrees smaller than n and agreeing with V in degrees smaller than $n-2$:

$$\begin{array}{l}0\longrightarrow V_{n-1}\longrightarrow V_{n-2}\oplus V_n\longrightarrow V_{n-3}\longrightarrow V_{n-4}\longrightarrow\cdots\\ \qquad x\longmapsto(dx,ex)\\ \qquad\qquad (a,b)\longmapsto da.\end{array}$$

It is an elementary exercise to prove that ΓV is exact, but this will also follow directly from the two decompositions we produce next.

For any vector space W and any $k\in\mathbb{Z}$, write $D_k(W)$ for the chain complex consisting of W in degrees k and $k+1$, where the differential is the identity.

Returning to our chain complex V with contracting homotopy e, note that for $x\in V_n$ one has $ed(x)=x$. Using this, we can write down a natural chain map $V\to D_{n-1}(V_n)$ that is the identity in degree n and the map $e\colon V_{n-1}\to V_n$ in degree $n-1$. Let $\widetilde{\Gamma}V$ be the desuspension of the mapping cone of this map; specifically, $\widetilde{\Gamma}V$ is the following chain complex, concentrated in dimensions at most n and agreeing with V in dimensions smaller than $n-2$:

$$\begin{array}{l}0\longrightarrow V_n\longrightarrow V_{n-1}\oplus V_n\longrightarrow V_{n-2}\oplus V_n\longrightarrow V_{n-3}\longrightarrow V_{n-4}\longrightarrow\cdots\\ \qquad x\longmapsto(dx,-x)\qquad\qquad (x,y)\longmapsto dx\\ \qquad\qquad (a,b)\longmapsto(da,ea+b)\end{array}$$

There is a canonical inclusion $D_{n-2}(V_n)\hookrightarrow\widetilde{\Gamma}V$, and the quotient is V. Moreover, this inclusion has a canonical splitting χ defined by $\chi(a,b)=ea+b$ for $(a,b)\in$

$V_{n-1} \oplus V_n = (\widetilde{\Gamma} V)_{n-1}$ and $\chi(a,b) = b$ for $(a,b) \in V_{n-2} \oplus V_n = (\widetilde{\Gamma} V)_{n-2}$. This splitting gives an isomorphism $\widetilde{\Gamma} V \cong D_{n-2}(V_n) \oplus V$.

Notice as well that there is an evident map $D_{n-1}(V_n) \hookrightarrow \widetilde{\Gamma} V$: in degree n this equals the identity and in degree $n-1$ it equals the differential of $\widetilde{\Gamma} V$. The cokernel of this inclusion is precisely ΓV. Moreover, there is again a canonical splitting for the inclusion: in degree n it is equal to the identity, and in degree $n-1$ it is the negation of the projection map $V_{n-1} \oplus V_n \to V_n$. This splitting gives $\widetilde{\Gamma} V \cong D_{n-1}(V_n) \oplus \Gamma V$.

Summarizing, we have produced two split-exact sequences

$$0 \to D_{n-2}(V_n) \to \widetilde{\Gamma} V \to V \to 0 \quad \text{and} \quad 0 \to D_{n-1}(V_n) \to \widetilde{\Gamma} V \to \Gamma V \to 0$$

and these induce isomorphisms

$$\widetilde{\Gamma} V \cong D_{n-2}(V_n) \oplus V \quad \text{and} \quad \widetilde{\Gamma} V \cong D_{n-1}(V_n) \oplus \Gamma V. \tag{19.9}$$

An important point is that the maps in these short exact sequences, their splittings, and therefore the induced isomorphisms in (19.9) are all canonical in the pair (V, e).

By the way, notice that it follows immediately from the isomorphisms in (19.9) that the homology groups of ΓV and V coincide; therefore ΓV is exact.

The constructions from above depended on the choice of contracting homotopy e. As one last remark before getting back to topology, let us consider the space of *all* contracting homotopies on an arbitrary chain complex V. Denote this space as $\text{contr-h}(V) \subseteq \prod_i \text{Hom}(V_i, V_{i+1})$; an element of $\text{contr-h}(V)$ is a collection of maps $\{e_i : V_i \to V_{i+1}\}$ satisfying $de + ed = \text{id}$. Of course this space might be empty, but we claim that it is either empty of contractible. To see this, recall the internal Hom-complex $\underline{\text{Hom}}(V,V)$. In dimension k this is $\prod_i \text{Hom}(V_i, V_{i+k})$, and given a collection $\{\alpha_i : V_i \to V_{i+k}\}$ the differential is the collection of maps $\{d \circ \alpha_i - (-1)^k \alpha_{i-1} \circ d\}$. A contracting homotopy for V is just an element $e \in \underline{\text{Hom}}(V,V)_1$ satisfying $de = \text{id}$, and the space of contracting homotopies is just $d^{-1}(\text{id})$. If this space is nonempty then it is homeomorphic to the space of 1-cycles in $\underline{\text{Hom}}(V,V)$, which is a vector space and hence contractible.

Remark 19.10. The Γ- and $\widetilde{\Gamma}$-constructions used in this section seem to have first appeared in [Do]. Notice that very little about the contracting homotopy e was ever used—in fact, all we really needed was the component of e in the top dimension, the map $e : V_{n-1} \to V_n$. And all that was important about this map was that it was a splitting for the differential $d_n : V_n \to V_{n-1}$. Rather than use the space of contracting homotopies as a parameter space, we could have used the (simpler) space of splittings for d_n. The reader can check that this is again an affine space, homeomorphic to the vector space of all maps $f : V_{n-1} \to V_n$ such that $fd = 0$; in particular, this parameter space is again contractible.

We have used the space of contracting homotopies because this approach generalizes a bit more easily to the situation of algebraic K-theory. See [FH] and [D3] for the importance of these contracting homotopies.

19.11. **Back to topology.** Let $E_\bullet$ be a bounded chain complex of vector bundles on a space Z, and assume that $E_\bullet$ is exact. One can prove by brute force that $E_\bullet$ has a contracting homotopy, by successively splitting off bundles starting in the bottom degree of the complex (as in the proof of Proposition 17.10). But as another argument, consider the map $\text{contr-h}(E) \to Z$ whose fiber is the space of contracting homotopies for $(E_\bullet)_z$. It is easy to see that contr-h$(E) \to Z$ is a fiber bundle, and our remarks in the last section show that the fibers are contractible. If Z is a CW-complex then a lift is guaranteed in the diagram

$$\begin{array}{ccc} & & \text{contr-h}(E) \\ & & \downarrow \simeq \\ Z & \xrightarrow{\text{id}} & Z, \end{array}$$

and this lift precisely gives a contracting homotopy for $E_\bullet$. Moreover, if e and e' are two liftings then there is a homotopy $Z \times I \to$ contr-h(E) between them because the diagram

$$\begin{array}{ccc} (Z \times 0) \amalg (Z \times 1) & \xrightarrow{e \amalg e'} & \text{contr-h}(E) \\ \downarrow & & \downarrow \simeq \\ Z \times I & \xrightarrow{\pi} & Z, \end{array}$$

admits a lifting. This is all we will need, but it is worth observing that one can say even more here: the space of all liftings, which is the space of contracting homotopies on $E_\bullet$, is contractible.

If e is a chosen contracting homotopy for $E_\bullet$ then we can form the associated chain complex ΓE by repeating the construction from Section 19.8 but in the bundle setting. This is a new chain complex of vector bundles that is still exact on Z. This construction of course depends on the choice of contracting homotopy e, and so we should probably write $\Gamma_e E$. But since any two choices for e are homotopic, it follows that ΓE is well-defined up to homotopy.

We will use the above construction to prove the following:

Proposition 19.12. *For any $n \geq 2$ the map $j\colon \mathcal{L}_{n-1}(X,A)_A^{h,st} \to \mathcal{L}_n(X,A)_A^{h,st}$ is a bijection.*

Proof. Let $E_\bullet$ be a chain complex in $\mathcal{L}_n(X,A)_A$, where $n \geq 2$. By the preceding considerations there exists a contracting homotopy for $(E_\bullet)|_A$. Using such a contraction e we can form $\Gamma_e E$, which is an object in $\mathcal{L}_{n-1}(X,A)_A$. Different choices for e give rise to homotopic complexes, so we get a well-defined function

$$\Gamma\colon \mathcal{L}_n(X,A)_A \to \mathcal{L}_{n-1}(X,A)_A^{h,st}.$$

It is an elementary exercise to see that if $E \sim_{st} E'$ then $\Gamma E \sim_{st} \Gamma E'$, and that if $E \sim_h E'$ then $\Gamma E \sim_h \Gamma E'$. So we actually get

$$\Gamma\colon \mathcal{L}_n(X,A)_A^{h,st} \to \mathcal{L}_{n-1}(X,A)_A^{h,st}.$$

If j denotes the map $\mathcal{L}_{n-1}(X,A)^{h,st} \to \mathcal{L}_n(X,A)_A^{h,st}$ induced by inclusion, it is trivial that $\Gamma j = \text{id}$. We claim that $j\Gamma = \text{id}$ as well, thus establishing that j is a bijection. This claim almost follows directly from the canonical isomorphisms (19.9), except there is an important step we must fill in. We would like to say that these isomorphisms globalize to give

(19.13) $$\widetilde{\Gamma} E \cong D_{n-2}(E_n) \oplus E \quad \text{and} \quad \widetilde{\Gamma} E \cong D_{n-1}(E_n) \oplus \Gamma E.$$

This is certainly true if we restrict all the chain complexes to the subspace A. However, to give an isomorphism in $\mathcal{C}h(X,A)_A$ we actually need to give a collection of isomorphisms for bundles **over X** (they are only required to commute with the differentials over A, however). So we must verify that in each degree the isomorphisms from (19.9) globalize not just to A but to X. Those isomorphisms were obtained from split short exact sequences, so we must check that all the maps involved can be extended over X. But we have formulas for all of these maps, and most of them are inclusions of a summand or projections onto a summand—these obviously extend to all of X. The one exception is in one of the splittings, where we used the map $\chi\colon E_{n-1}\oplus E_n \to E_n$ given on fibers by $(a,b)\mapsto ea+b$ where e was part of the given contracting homotopy. Since e is only defined on A, this does not automatically make sense on all of X. However, the particular map e we are using in this formula is a section over A of the bundle $\underline{\mathrm{Hom}}(E_{n-1},E_n)\to X$. The diagram

$$\begin{array}{ccc} A & \longrightarrow & \underline{\mathrm{Hom}}(E_{n-1},E_n) \\ \big\downarrow & & \big\downarrow \simeq \\ X & \xrightarrow{\ \mathrm{id}\ } & X \end{array}$$

must have a lifting, and this gives an extension of e to a bundle map $\tilde{e}\colon E_{n-1}\to E_n$ defined on all of X. The formula $(a,b)\mapsto \tilde{e}a+b$ then gives the desired splitting that works on all of X. Note: it is important here that the splitting is only required to commute with the differentials on A, since this is all we have guaranteed.

To summarize, we have indeed justified the isomorphisms in (19.13). These imply that ΓE and E represent the same class in $\mathcal{L}_n(X,A)_A^{h,st}$. In other words, we have proven that $j\circ\Gamma = \mathrm{id}$, and so Γ is a two-sided inverse for j. □

Remark 19.14 (Atiyah's proof)**.** Atiyah proves a version of Proposition 19.12 in [At1, Chapter 2.6]; the argument orginally comes from [ABS]. We will explain the basic ways his proof differs from ours, and why these differences are important.

Let $E_\bullet$ be a chain complex in $\mathcal{C}\mathrm{h}_n(X,A)_A$. There is a canonical map $D_{n-1}(E_n)\to E_\bullet$ which in degree n equals the identity and in degree $n-1$ equals the differential $E_n\to E_{n-1}$. Let $E'_\bullet = E_\bullet \oplus D_{n-2}(E_n)$ and consider the composition

$$D_{n-1}(E_n)\to E_\bullet \hookrightarrow E'_\bullet.$$

The map in degree $n-1$ is $d\oplus 0\colon E_n\to E_{n-1}\oplus E_n$, which is defined only on A. Atiyah shows via a lifting argument that this can be extended to a monomorphism of bundles on X. Let Q be the quotient, and observe that by Proposition 9.2 the sequence $0\to E_n\to E'_n\to Q\to 0$ is a split-exact sequence of bundles over X. A choice of splitting $\chi\colon E'_n\to E_n$ then shows that $E'_\bullet$ is the direct sum of $D_{n-1}(E_n)$ and a complex

$$0\to Q\to E_{n-2}\oplus E_n\to E_{n-3}\to E_{n-4}\to\cdots \tag{19.15}$$

This last complex lies in $\mathcal{C}\mathrm{h}_{n-1}(X,A)_A$, and it represents the same class as E in $\mathcal{L}_n(X,A)_A^{h,st}$. This shows that $j\colon \mathcal{L}_{n-1}(X,A)_A^{h,st}\to\mathcal{L}_n(X,A)_A^{h,st}$ is surjective.

This argument does not give an inverse for j, however. The complex in (19.15) depends on a choice (the extension of a certain map to all of X), and so it is not clear how to use this construction to make an inverse for j.

In our argument we gave a construction ΓE that **did not depend** on choosing any such extensions to X. Such extensions did appear, but only in the isomorphisms showing that our ΓE had the correct properties. By pushing these choices into the maps rather than the objects, we were able to write down an explicit inverse for j.

In Atiyah's case he found a clever way around his problem, by instead constructing a map $\mathcal{L}_n(X,A)_A^{h,st} \to \mathcal{L}_1(X,A)_A^{h,st}$ that is an inverse to the appropriate composition of j's. This is clearly enough to deduce injectivity of all the j maps. Atiyah's construction proceeds by choosing Hermitian inner products on all of the bundles E_i, and then letting $\alpha\colon E_i \to E_{i+1}$ be the adjoint of the differential. The map $d+\alpha\colon \bigoplus_{i \text{ odd}} E_i \to \bigoplus_{i \text{ even}} E_i$ is seen to lie in $\mathcal{C}\mathrm{h}_1(X,A)_A$, and it clearly has the desired properties. This element seems to again depend on choices, namely the choice of inner products; but the space of all such choices is contractible, and so one indeed gets a well-defined element of $\mathcal{L}_1(X,A)_A^{h,st}$. The exercises below will give you enough information to fill in the details of this approach.

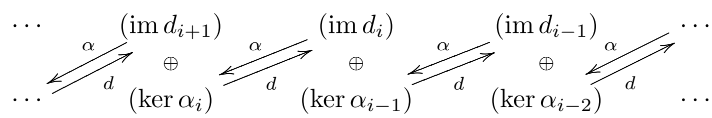
The disappointing aspect of Atiyah's argument is that it does not work in the related context of algebraic K-theory. In that setting one cannot play a corresponding game with inner products. In contrast, our argument with contracting homotopies does generalize. See ????.

Exercise 19.16. Assume V is an exact chain complex of real vector spaces, and that $V_i = 0$ for $i < 0$ and for $i > n$. Choose an inner product on each V_i, and let $\alpha_i\colon V_i \to V_{i+1}$ be the adjoint of $d_{i+1}\colon V_{i+1} \to V_i$. That is, for each $x \in V_i$ and $y \in V_{i+1}$ one has $\langle \alpha x, y\rangle = \langle x, dy\rangle$.

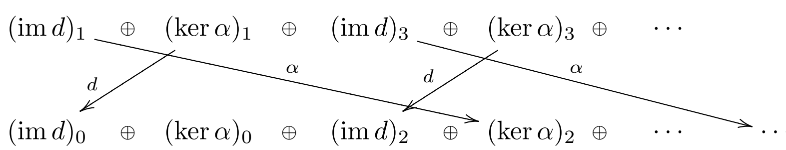
(a) Prove that $\alpha^2 = 0$, and so (V,α) is a cochain complex. Observe that this is isomorphic to the dual complex V^*, and therefore is exact.

(b) For each i prove that $\ker\alpha_i$ is orthogonal to $\operatorname{im} d_{i+1}$ inside of V_i. As a corollary, deduce that d restricts to an isomorphism $\ker\alpha_i \to \operatorname{im} d_{i-1}$ and α restricts to an isomorphism $\operatorname{im} d_{i-1} \to \ker\alpha_i$. Produce an example when $n=1$ showing that these isomorphisms need not be inverses.

(c) Use a dimension count to show $V_i = (\operatorname{im} d_{i+1}) \oplus (\ker\alpha_i)$, for each i. The following picture shows V decomposing into a direct sum of length 1 complexes:



(d) Let $V_{odd} = \oplus_{i \text{ odd}} V_i$ and $V_{ev} = \oplus_{i \text{ even}} V_i$. Observe that $d+\alpha\colon V_{odd} \to V_{ev}$ is an isomorphism, by the following diagram:



Note: Here we have written $(\operatorname{im} d)_i$ for the component of $\operatorname{im} d$ contained in degree i; i.e., $(\operatorname{im} d)_i = \operatorname{im} d_{i+1}$, but $(\ker\alpha)_i = \ker\alpha_i$.

(e) Prove an analog of these results for exact complexes of $\mathbb{C}$-vector spaces, in which one chooses Hermitian inner products on all of the V_i's.

Exercise 19.17. Let IP_n denote the space of all inner products on $\mathbb{R}^n$. Note that this may be identified with the space of all positive-definite, symmetric $n\times n$ matrices, which we topologize as a subspace of $M_{n\times n}(\mathbb{R})$. We will prove that IP_n is contractible, for all n.

(a) Prove $\mathrm{IP}_1 \cong \mathbb{R}_{>0}$.

(b) Consider the subspace $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$, and let H^n denote the upper half-space $\{x \in \mathbb{R}^n \,|\, x_n > 0\}$. Prove that $\mathrm{IP}_n \cong \mathrm{IP}_{n-1} \times H^n$, and deduce that IP_n is contractible for all $n \geq 1$.
(c) Use a similar line of argument to show that the space of Hermitian inner products on $\mathbb{C}^n$ is contractible. The analog of H^n is the space $(\mathbb{C}^n - \mathbb{C}^{n-1})/S^1$, where S^1 is the group of unit complex numbers acting via scalar multiplication on $\mathbb{C}^n$. As part of your argument you will have to show that this orbit space is contractible.

19.18. **Another interlude on where we are headed.** Recall that we are trying to produce a natural map of groups $\mathcal{K}(X, A) \to K^0(X, A)$. The following reductions have been made:

- Produce a natural map of monoids $\mathcal{L}_\infty(X, A)^{h,st} \to K^0(X, A)$;
- Produce a natural map of monoids $\mathcal{L}_\infty(X, A)^{h,st}_A \to K^0(X, A)$;
- Produce a collection of natural monoid maps $\mathcal{L}_n(X, A)^{h,st}_A \to K^0(X, A)$ that are compatible as n changes;
- Produce a natural map of monoids $\mathcal{L}_1(X, A)^{h,st}_A \to K^0(X, A)$.

In all of these cases we will call the natural map an **Euler characteristic**. In the next section we produce an Euler characteristic on $\mathcal{C}\mathrm{h}_1(X, A)_A$ and show that it is the unique one satisfying some evident desirable properties.

19.19. **The difference bundle construction.** Consider the space $X \amalg_A X$ together with the two evident inclusions $i_1, i_2 \colon X \to X \amalg_A X$. We will write X_1 and X_2 for the images of X under these two maps. The inclusion of pairs $(X_1, A) \to (X \amalg_A X, X_2)$ induces an excision isomorphism $i_1^* \colon K^0(X \amalg_A X, X_2) \to K^0(X_1, A)$. The fold map $X \amalg_A X \to X$ shows that the long exact sequence for the pair $(X \amalg_A X, X_2)$ breaks up into split short exact sequences, and in particular we get

$$\begin{array}{ccc} 0 \longrightarrow & K^0(X \amalg_A X, X_2) & \longrightarrow K^0(X \amalg_A X) \xrightarrow{i_2^*} K^0(X_2) \longrightarrow 0. \\ & \cong \Big\downarrow i_1^* & \\ & K^0(X, A) & \end{array}$$

This diagram shows that we can produce elements of $K^0(X, A)$ by producing elements in the kernel of i_2^*.

Suppose $0 \to E_1 \xrightarrow{d} E_0 \to 0$ is an object in $\mathcal{C}\mathrm{h}_1(X, A)_A$. Produce a bundle on $X \amalg_A X$ by taking $E_1 \to X_1$ and $E_0 \to X_2$ and gluing them together along the isomorphism d defined over A (see Corollary 8.17). Call this new bundle $\widetilde{E} \to X \amalg_A X$. Likewise, by gluing two copies of E_0 along the identity map we produce a bundle $\widetilde{E}_0$ on $X \amalg_A X$. Set $\alpha(E) = [\widetilde{E}_0] - [\widetilde{E}] \in K^0(X \amalg_A X)$ and notice that $i_2^*(\alpha_E) = [E_0] - [E_0] = 0$. Let $d(E) = i_1^*(\alpha(E)) \in K^0(X, A)$. This element is called the **difference bundle** corresponding to the complex E. It is immediate from the definition that the image of $d(E)$ in $K^0(X)$ is $[E_0] - [E_1]$.

We must argue that if $E \sim_h E'$ then $d(E) = d(E')$. But this follows from the homotopy invariance of K-theory; to see this, suppose $\mathcal{E} \in \mathcal{C}\mathrm{h}_1(X \times I, A \times I)_A$ and that there are isomorphisms $\mathcal{E}|_{X \times 0} \cong E$ and $\mathcal{E}|_{X \times 1} \cong E'$. Then $d(\mathcal{E}) \in K^0(X \times I, A \times I)$, and it is immediate that $j_0^* d(\mathcal{E}) = d(E)$ and $j_1^* d(\mathcal{E}) = d(E')$ where $j_0, j_1 \colon X \hookrightarrow X \times I$ are the inclusions of the two ends. But homotopy invariance of K-theory gives that $j_0^* = j_1^*$, hence $d(E) = d(E')$.

We must also argue that if $E \sim_{st} E'$ then $d(E) = d(E')$. This reduces to showing that if Q is the complex $0 \to F \xrightarrow{\text{id}} F \to 0$ then $d(E \oplus Q) = d(E)$. But there is an isomorphism of bundles $\widetilde{E \oplus Q} \cong \widetilde{E} \oplus \nabla^* Q$ where $\nabla\colon X \amalg_A X \to X$ is the projection, and also $\widetilde{E_0 \oplus Q} \cong \widetilde{E}_0 \oplus \nabla^* Q$. We now simply observe that

$$\alpha(E \oplus Q) = [\widetilde{E}_0 \oplus \nabla^* Q] - [\widetilde{E} \oplus \nabla^* Q] = [\widetilde{E}_0] - [\widetilde{E}] = \alpha(E),$$

which immediately yields $d(E \oplus Q) = d(E)$.

We have now shown that the difference bundle construction gives a map $\chi_1\colon \mathcal{L}_1(X, A)_A^{h,st} \to K^0(X, A)$. It is clearly natural in the pair (X, A), and also a map of monoids.

Proposition 19.20. *The map* $\chi_1\colon \mathcal{L}_1(X, A)_A^{h,st} \to K^0(X, A)$ *is an isomorphism for every finite CW-pair* (X, A).

Proof. The proof proceeds in three steps.
Step 1: $A = \emptyset$. This step is trivial.
Step 2: $A = *$. Here we consider the diagram

$$\begin{CD}
0 @>>> \mathcal{L}_1(X,*)_*^{h,st} @>\alpha>> \mathcal{L}_1(X,\emptyset)_\emptyset^{h,st} @>\beta>> \mathcal{L}_1(*,\emptyset)_\emptyset^{h,st} @>>> 0 \\
@. @V\chi VV @VV\chi V @VV\chi V \\
0 @>>> K^0(X,*) @>>> K^0(X) @>>> K^0(*) @>>> 0
\end{CD}$$

(note that the objects on the top row are *a priori* only monoids, not groups). The bottom row is exact, and the middle and right vertical maps are isomorphisms by Step 1. It will suffice to show that the top row is exact, since then the left vertical map is also an isomorphism.

For the remainder of this step let $x \in X$ denote the basepoint.

Surjectivity of β is trivial (and also not really needed for the proof). For exactness at the middle spot, note that $\mathcal{L}_1(*, \emptyset)_\emptyset^{h,st} \cong \mathbb{Z}$ via $(E_1, E_0) \to \operatorname{rank} E_0 - \operatorname{rank} E_1$. So if $E_\bullet$ is in the kernel of β then $\operatorname{rank} E_1 = \operatorname{rank} E_0$, and certainly there exists an isomorphism $(E_1)_x \cong (E_0)_x$. Such an isomorphism determines an element of $\mathcal{L}_1(X, *)_*^{h,st}$ that is a preimage for $E_\bullet$.

Finally, let $E_\bullet \in \mathcal{C}h(X, *)_*$ and assume that $E_\bullet$ is in the kernel of α. Using the isomorphism $\mathcal{L}_1(X, \emptyset)_\emptyset^{h,st} \cong K^0(X)$, this means that $[E_1] = [E_0]$ in $K^0(X)$. So there exists a trivial vector bundle Q such that $E_1 \oplus Q \cong E_0 \oplus Q$. Adding $\text{id}\colon Q \to Q$ to $E_\bullet$ and using this isomorphism, we may assume that $E_\bullet$ satisfies $E_1 = E_0$. Our isomorphism of fibers over x is then an element $\sigma \in \operatorname{Aut}((E_0)_x) \cong GL_n(\mathbb{C})$ for $n = \operatorname{rank}(E_0)$. As $GL_n(\mathbb{C})$ is connected, choose a path between σ and the identity element. Consider the lifting square

$$\begin{CD}
(X \times 0) \cup (x \times I) @>>> \underline{\operatorname{Aut}}(E_0) \\
@V\simeq VV @VVV \\
X \times I @>\pi>> X
\end{CD}$$

where the top map is the identity section on $X \times 0$ and the chosen path from σ to the identity on $\{x\} \times I$. The object $\underline{\operatorname{Aut}}(E_0) \to X$ is the fiber bundle whose fiber over a point z is $\operatorname{Aut}((E_0)_z$. Since the left vertical map in the square is a trivial cofibration, the square admits a lifting. Such a lifting is an isomorphism $\phi\colon E_0 \to E_0$ that restricts to σ on x. The diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi} & E_0 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ E_0 & \xrightarrow{\mathrm{id}} & E_0 \end{array}$$

is an isomorphism between $E_\bullet$ and $E_0 \xrightarrow{\mathrm{id}} E_0$, thereby showing that $E_\bullet$ represents 0 in $\mathcal{L}_1(X,*)_*^{h,st}$.

Step 3: General case. First consider the square

$$\begin{array}{ccc} \mathcal{L}_1(X,A)_A^{h,st} & \xrightarrow{\chi} & K^0(X,A) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \cong} \\ \mathcal{L}_1(X/A,*)_A^{h,st} & \xrightarrow[\cong]{} & K^0(X/A,*) \end{array}$$

induced by the map of pairs $\pi\colon (X,A) \to (X/A,*)$. The bottom horizontal map is an isomorphism by Step 1, and the right vertical map is an isomorphism by excision. It follows that the map π^* is injective. If we prove π^* is surjective then it is an isomorphism, and therefore χ is also an isomorphism.

Suppose given a chain complex $0 \to E_1 \to E_0 \to 0$ in $\mathcal{C}h(X,A)_A$. Since X is compact, there is a bundle Q such that $E_1 \oplus Q$ is trivial. By adding $\mathrm{id}\colon Q \to Q$ to the complex $E_\bullet$, we may assume that E_1 is trivial. Choose a trivialization $\phi\colon E_1 \xrightarrow{\cong} X \times \mathbb{C}^n$ and consider the following diagrams:

$$\begin{array}{ccccc} \mathbb{C}^n & \longleftarrow & (E_1)|_A & \longrightarrow & E_1 \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & A & \longrightarrow & X \end{array} \qquad \begin{array}{ccccc} \mathbb{C}^n & \longleftarrow & (E_0)|_A & \longrightarrow & E_0 \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & A & \longrightarrow & X. \end{array}$$

Here $(E_1)|_A \to \mathbb{C}^n$ is the composite of ϕ with the projection onto $\mathbb{C}^n$, and $(E_0)|_A \to \mathbb{C}^n$ is obtained by precomposing the former map with the inverse of the given isomorphism $(E_1)|_A \cong (E_0)|_A$. Let F_1 and F_0 be the pushouts of the top rows of the two diagrams. These come with maps $F_1 \to X/A$ and $F_2 \to X/A$ which are readily checked to be vector bundles. Moreover, there are natural maps $E_1 \to \pi^*F_1$ and $E_0 \to \pi^*F_0$, and these are isomorphisms of vector bundles.

The fibers of $F_1 \to X/A$ and $F_2 \to X/A$ over the basepoint are canonically identified with $\mathbb{C}^n$, and so the identity map on these fibers yields a relative chain complex $0 \to F_1 \to F_0 \to 0$ in $\mathcal{C}h(X/A,*)_*$. It is an easy check that the diagram

$$\begin{array}{ccc} (E_1)|_A & \longrightarrow & (E_0)|_A \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ (\pi^*F_1)|_A & \longrightarrow & (\pi^*F_0)|_A \end{array}$$

is commutative. So our original chain complex $E_\bullet$ is isomorphic to $\pi^*(F_\bullet)$. This proves surjectivity of π^* in the original diagram. $\square$

19.21. **Wrapping things up.** We can now bring together all the work in this section to prove the main results.

Proposition 19.22. *Fix $1 \le n \le \infty$. There is a unique natural transformation $\chi_n \colon \mathcal{L}_n(X,A)^{h,st}_A \to K^0(X,A)$ such that when $A=\emptyset$ one has $\chi(E_\bullet) = \sum_i (-1)^i [E_i]$. Moreover, the map χ_n is an isomorphism for every pair (X,A).*

Proof. We have already established the existence of χ_1 and seen that it is a natural isomorphism. Define χ_n via the zig-zag

$$\mathcal{L}_n(X,A)^{h,st}_A \xleftarrow{j} \mathcal{L}_1(X,A)^{h,st}_A \xrightarrow{\chi_1} K^0(X,A).$$

using the fact that j is an isomorphism by Proposition 19.12. Since j is natural, χ_n is also natural.

For uniqueness, note first that a natural transformation χ'_n determines a corresponding natural tranformation χ'_1 by precomposing with j. Since j is an isomorphism, it will be sufficient to prove that $\chi'_1 = \chi_1$. For this, first note that exactly the same proof as for Proposition 19.20 shows that χ'_1 is a natural isomorphism. This allows us to consider the composite $\eta = \chi'_1 \circ (\chi_1)^{-1}$, which is a natural isomorphism $\eta \colon K^0(X,A) \to K^0(X,A)$. The fact that χ_1 and χ'_1 agree on $K^0(X,\emptyset)$ for any X shows that η is the identity on these groups. Now one merely observes the natural isomorphism $\pi^* \colon K^0(X/A,*) \to K^0(X,A)$ and the natural short exact sequence

$$0 \to K^0(X/A,*) \to K^0(X/A) \to K^0(*) \to 0.$$

It follows at once that a natural isomorphism η that is the identity on $K^0(X)$ for all X is also equal to the identity on $K^0(X,A)$ for all pairs (X,A). So $\eta = \mathrm{id}$, thus $\chi_1 = \chi'_1$, and therefore $\chi_n = \chi'_n$. □

Since the natural maps $\mathcal{L}_n(X,A)^{h,st} \to \mathcal{L}_n(X,A)^{h,st}_A$ are isomorphisms by Proposition 19.3, the above proposition also gives a uniquely determined natural transformation $\mathcal{L}_n(X,A)^{h,st} \to K^0(X,A)$. We will also call this χ_n, by abuse.

Recall that $\mathcal{K}(X,A)$ is the group completion of $\mathcal{L}_\infty(X,A)^{h,st}$. But it follows as a consequence of Proposition 19.22 above that $\mathcal{L}_\infty(X,A)^{h,st}$ is already a group, hence the two are canonically identified. This proves the following:

Corollary 19.23. *The evident map $\mathcal{L}_\infty(X,A)^{h,st} \to \mathcal{K}(X,A)$ is an isomorphism.*

Corollary 19.24. *There is a unique natural map $\chi \colon \mathcal{K}(X,A) \to K^0(X,A)$ having the property that for $A=\emptyset$ one has $\chi([E_\bullet]) = \sum_i (-1)^i [E_i]$. The map χ is a natural isomorphism.*

Proof. Immediate from the preceding results. □

As we have observed before, if $E_\bullet \in \mathcal{C}h(X,A)$ and $F_\bullet \in \mathcal{C}h(Y,B)$ then $E_\bullet \hat{\otimes} F_\bullet \in \mathcal{C}h(X \times Y, (A \times Y) \cup (X \times B))$. This is readily seen to induce pairings

$$\mu \colon \mathcal{K}(X,A) \otimes \mathcal{K}(Y,B) \to \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)).$$

Corollary 19.25. *For any finite CW-pairs (X,A) and (Y,B) the diagram*

$$\begin{array}{ccc}
\mathcal{K}(X,A) \otimes \mathcal{K}(Y,B) & \xrightarrow{\mu} & \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)) \\
\downarrow{\scriptstyle \chi \otimes \chi} & & \downarrow{\scriptstyle \chi} \\
K^0(X,A) \otimes K^0(Y,B) & \longrightarrow & K^0(X \times Y, (A \times Y) \cup (X \times B))
\end{array}$$

is commutative, where the bottom horizontal map is the product on K-theory.

Proof. It is trivial to check that the diagram commutes when $A = B = \emptyset$. The general case follows formally from this one using naturality. First check that it works for $A = B = *$ using the diagram

$$\begin{array}{ccccc} K^0(X,*) \otimes K^0(Y,*) & \longrightarrow & K^0(X \times Y, (X \times *) \cup (* \times Y)) & \xleftarrow{\cong} & K^0(X \wedge Y, *) \\ \cong \downarrow & & & \searrow & \downarrow \\ K^0(X) \otimes K^0(Y) & & \longrightarrow & & K^0(X \times Y) \end{array}$$

(and the similar one for $\mathcal{K}(-,-)$). It is important that the indicated maps are injections, but that they are so is an easy argument. Finally, deduce the general case using the diagrams

$$\begin{array}{ccccc} K^0(X,A) \otimes K^0(Y,B) & \longrightarrow & K^0(X \times Y, (A \times Y) \cup (X \times B)) & \xleftarrow{\cong} & K^0(X/A \wedge Y/B, *) \\ \uparrow \cong & & & \swarrow \cong & \\ K^0(X/A,*) \otimes K^0(Y/B,*) & \longrightarrow & K^0(X/A \times Y/B, X/A \vee Y/B). & & \end{array}$$

∎

Finally, we note that the above results for finite CW-pairs (X, A) automatically extend to homotopically compact pairs. If (X, A) is homotopically compact then choose a finite CW-pair (X', A') and a map $(X', A') \to (X, A)$ that is a weak homotopy equivalence on each factor. Define $\chi \colon \mathcal{K}(X, A) \to K^0(X, A)$ via the diagram

$$\begin{array}{ccc} \mathcal{K}(X', A') & \xleftarrow{\cong} & \mathcal{K}(X, A) \\ \cong \downarrow & & \vdots \\ K^0(X', A') & \xleftarrow[\cong]{} & K^0(X, A). \end{array}$$

A straightforward argument using Proposition 17.14 and Proposition C.3 shows that this does not depend on the choice of (X', A'). Everything that we have proven about $\mathcal{K}(-,-) \to K^0(-,-)$ for finite CW-pairs now immediately follows for homotopically compact pairs as well.

Remark 19.26. Note that we have now proven Theorem 17.16, since that result is just a compilation of Corollaries 19.24 and 19.25.

Part 4. K-theory and geometry II

In the last several sections we developed the basic connections between K-theory and geometry. We have seen that K-theory is a complex-oriented cohomology theory, and we understand "geometric" representatives for the Thom classes and fundamental classes that come with such a theory; in this case "geometric" means that we can write down explicit chain complexes of vector bundles representing the classes. In the following sections our aim is to further explore this general area: now that the basic picture is in place, where does it take us? The topics we cover are somewhat of a hodgepodge, but in some sense they they all revolve around the exploration of fundamental classes.

20. $K^*(\mathbb{C}P^n)$ and the K-theoretic analog of the degree

If $Z \hookrightarrow \mathbb{C}P^n$ is a complex submanifold then it has a fundamental class $[Z]$ in $H^*(\mathbb{C}P^n)$. Knowing this fundamental class comes down to knowing a single integer, called the degree of Z. The geometric interpretation of the degree is that it equals the number of points of intersection between Z and a generically chosen linear subspace of complementary dimension. In this section we will repeat this line of investigation but replacing H^* with K^*. So we must compute $K^*(\mathbb{C}P^n)$ and investigate what information is encoded in the fundamental class $[Z]_K$. We will find that knowing $[Z]_K$ amounts to knowing *several* integers (not just one); while we can give methods for computing these, their geometric interpretation is somewhat mysterious.

20.1. **Calculation of $K^*(\mathbb{C}P^n)$.** We begin with the following easy lemma:

Lemma 20.2. *Let E be any multiplicative cohomology theory. If $x_1, \dots, x_{n+1} \in \tilde{E}^*(\mathbb{C}P^n)$, then $x_1 \cdots x_{n+1} = 0$.*

Proof. The key is just that $\mathbb{C}P^n$ can be covered by $n+1$ contractible sets. To be explicit, let $U_i = \{[z_0 : \dots : z_n] \in \mathbb{C}P^n \,|\, z_i \neq 0\}$. Then U_i is open in $\mathbb{C}P^n$ and is homeomorphic to $\mathbb{C}^n$.

Choose our basepoint of $\mathbb{C}P^n$ to be $[1 : 1 : \cdots : 1]$ (or any other point in the intersection of all the U_i's). The contractibility of U_i implies that $E^*(\mathbb{C}P^n, U_i) \to E^*(\mathbb{C}P^n, *)$ is an isomorphism. So we may lift each x_i to a class $\tilde{x}_i \in E^*(\mathbb{C}P^n, U_i)$.

It now follows that $\tilde{x}_1\tilde{x}_2 \cdots \tilde{x}_{n+1}$ lifts $x_1 \dots x_{n+1}$ in the map

$$E^*(\mathbb{C}P^n, U_1 \cup \dots \cup U_{n+1}) \to E^*(\mathbb{C}P^n, *).$$

Since $U_1 \cup \dots \cup U_{n+1} = \mathbb{C}P^n$, the domain is the zero group. So $x_1 \dots x_{n+1} = 0$. □

Recall that $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ where x is a generator in degree 2. It is not hard to see that we may take $x = [\mathbb{C}P^{n-1}]$. This calculation works for any complex-oriented cohomology theory:

Proposition 20.3. *Let E be a complex-oriented cohomology theory. There is an isomorphism of rings*

$$E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$$

sending x^i to $[\mathbb{C}P^{n-i}]$.

Proof. Consider the reduced Gysin sequence for the submanifold $\mathbb{C}P^{n-1} \overset{j}{\hookrightarrow} \mathbb{C}P^n$:

$$\begin{array}{ccccc} \cdots \leftarrow \tilde{E}^k(\mathbb{C}P^n - \mathbb{C}P^{n-1}) & \longleftarrow & \tilde{E}^k(\mathbb{C}P^n) & & \\ & & \uparrow{\scriptstyle j_!} & & \\ & & E^{k-2}(\mathbb{C}P^{n-1}) & \leftarrow & \tilde{E}^{k-1}(\mathbb{C}P^n - \mathbb{C}P^{n-1}) \leftarrow \cdots \end{array}$$

Let $x = [\mathbb{C}P^{n-1}] = j_!(1) \in \tilde{E}^2(\mathbb{C}P^n)$. By Lemma 20.2 we know $x^{n+1} = 0$, so we get a map $E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$. We will show that this map is an isomorphism via induction on n. The case $n = 0$ is trivial.

Note that $x^2 = [\mathbb{C}P^{n-2}]$ by intersection theory, and more generally $x^i = [\mathbb{C}P^{n-i}]$.

The spaces $\mathbb{C}P^n - \mathbb{C}P^{n-1}$ are homeomorphic to $\mathbb{C}^n$ and hence contractible. So the reduced Gysin sequence considered above breaks up into a collection of isomorphisms

$$j_!\colon E^{k-2}(\mathbb{C}P^{n-1}) \xrightarrow{\cong} \tilde{E}^k(\mathbb{C}P^n).$$

Taking all k's together, $j_!$ is a map of $E^*(pt)$-modules and therefore an isomorphism of such modules. By induction $E^*(\mathbb{C}P^{n-1})$ is a free $E^*(pt)$-module generated by the classes $1 = [\mathbb{C}P^{n-1}]$, $[\mathbb{C}P^{n-2}]$, $[\mathbb{C}P^{n-3}], \ldots, [\mathbb{C}P^0]$. Since the pushforward $j_!$ sends $[\mathbb{C}P^{n-i}]$ to $[\mathbb{C}P^{n-i}]$, we conclude that $\tilde{E}^*(\mathbb{C}P^n)$ is a free $E^*(pt)$-module on $[\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$. If we add 1 to this collection then we get a free basis for $E^*(\mathbb{C}P^n)$ over $E^*(pt)$. This proves that our map $E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$ is an isomorphism. □

In the case of complex K-theory, we can rephrase the above result as saying that $K^{odd}(\mathbb{C}P^n) = 0$ and $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$, where $X = \beta \cdot [\mathbb{C}P^{n-1}]$. In particular, note that additively we have $K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$ with free basis consisting of the powers of X. Ignoring powers of the Bott element at usual, we can write this free basis as $[\mathbb{C}P^n], [\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$.

20.4. **Fundamental classes in $K^*(\mathbb{C}P^n)$.** Now let $Z \hookrightarrow \mathbb{C}P^n$ be a complex, closed submanifold of codimension c. Recall that if we consider fundamental classes in singular cohomology then we have $[Z] = d \cdot [\mathbb{C}P^{n-c}]$ for a unique integer d that is called the **degree** of Z. Geometrically, d is the number of points on intersection of Z with a generically chosen copy of $\mathbb{C}P^c$.

We can now play this same game in the context of K-theory. We have a fundamental class $[Z]_K \in K^0(\mathbb{C}P^n)$, and we can therefore write

$$\begin{aligned} [Z]_K &= d_0 \cdot [\mathbb{C}P^n] + d_1 \cdot [\mathbb{C}P^{n-1}] + \cdots + d_n[\mathbb{C}P^0] \\ &= d_0 \cdot 1 + d_1 X + d_2 X^2 + \cdots + d_n X^n \end{aligned}$$

for unique integers d_i. These integers are topological invariants of the embedding $Z \hookrightarrow \mathbb{C}P^n$; our goal will be to explore them further. Multiply the above equation by X^n to obtain

$$d_0 X^n = [Z]_K \cdot X^n = [Z]_K \cdot [\mathbb{C}P^0] = [Z \cap \mathbb{C}P^0]$$

where in the last term we mean the intersection of Z with a generically chosen copy of $\mathbb{C}P^0$. But as long as Z is not all of $\mathbb{C}P^n$, this generic intersection will be empty—so if Z is codimension at least one then $d_0 = 0$. We then repeat this argument, but multiplying by X^{n-1} instead of X^n: we get

$$d_1 X^n = [Z]_K \cdot [\mathbb{C}P^1] = [Z \cap \mathbb{C}P^1].$$

Again, if the codimension of Z is at least 2 then the $\mathbb{C}P^1$ can be moved so that it doesn't intersect Z at all, hence $d_1 = 0$. Continuing this argument we find that

$$0 = d_0 = d_1 = \cdots = d_{c-1} \qquad \text{and} \qquad d_c = \deg(Z).$$

The last equality follows because Z intersects a generic $\mathbb{C}P^c$ in $\deg(Z)$ many points. The situation can be summarized as follows: the first non-vanishing d_i coincides with the classical degree of Z, but there is the possibility of the higher d_i's being nonzero. This is what we will investigate next.

Remark 20.5. Notice that what we have done so far works in any complex-oriented cohomology theory E^*. The fundamental class $[Z]$ can be written as a linear combination of the classes $[\mathbb{C}P^{n-i}]$ with coefficients from $E^*(pt)$. The first c of these coefficients vanish, until one gets to the coefficient of $[\mathbb{C}P^{n-c}]$—which must be equal to $\deg(Z)$. After this things become interesting, in the sense that one has invariants that are not detected in singular cohomology.

To proceed further with our analysis of the higher d_i's in K-theory, we need to connect our fundamental classes with the vector bundle description of K-theory:

Lemma 20.6. *In $K^0(\mathbb{C}P^n)$ one has $[\mathbb{C}P^{n-1}] = 1 - L$ where $L \to \mathbb{C}P^n$ is the tautological line bundle. Consequently, $[\mathbb{C}P^{n-k}] = (1-L)^k$ for all k.*

Proof. We give two explanations. For the first, consider the map of vector bundles $f\colon L \to 1$ defined as follows: in the fiber over $x = [x_0 : x_1 : \cdots : x_n]$ we send $(x_0, \ldots, x_n)$ to x_0. Note that this is well-defined, since multiplying all the x_i's by a scalar λ yields the same homomorphism $L_x \to \mathbb{C}$.

The map f is exact on all fibers except those where $x_0 = 0$. The complex $0 \to L \to 1 \to 0$ is a resolution of the structure sheaf for $\mathbb{C}P^{n-1}$, and hence $1 - L$ represents the associated fundamental class in K-theory by Theorem 18.8.

Our second explanation takes place entirely in the topological world. The key fact is that the normal bundle to $\mathbb{C}P^{n-1}$ inside $\mathbb{C}P^n$ is L^*. Let U be a tubular neighborhood of $\mathbb{C}P^{n-1}$, and consider the chain of isomorphisms

$$K^0(\mathbb{C}P^n, \mathbb{C}P^n - \mathbb{C}P^{n-1}) \xrightarrow{\cong} K^0(U, U - \mathbb{C}P^{n-1}) \cong K^0(N, N-0).$$

The relative fundamental class $[\mathbb{C}P^{n-1}]_{rel}$ is the unique class that maps to the Thom class $\mathcal{U}_N$ under the above isomorphisms. But N is a line bundle, and recall that the Thom class is then $[J^*] = [\pi^* N^* \to 1]$ where $\pi\colon N \to \mathbb{C}P^{n-1}$. A little thought shows that the complex $0 \to \pi^*N^* \to 1 \to 0$ is exactly the restriction of $0 \to L \to 1 \to 0$ on $\mathbb{C}P^n$. So this latter complex represents $[\mathbb{C}P^{n-1}]_{rel}$, and therefore $1 - L$ equals $[\mathbb{C}P^{n-1}]$. □

Example 20.7. Note that it follows from the above lemma that $(1-L)^{n+1} = 0$ in $K^0(\mathbb{C}P^n)$. This relation comes up in many contexts, and it is useful to have a different perspective on it. Let $R = \mathbb{C}[x_0, \ldots, x_n]$ and consider the Koszul complex for the regular sequence $x_0, \ldots, x_n$. It has the form

$$0 \to R(-n-1) \to \cdots \to R(-2)^{\binom{n+1}{2}} \to R(-1)^{n+1} \to R \to 0,$$

and we know this complex is exact. It therefore gives a corresponding exact sequence of vector bundles on $\mathbb{C}P^n$, which tells us that

$$0 = 1 - (n+1)L + \tbinom{n+1}{2}L^2 - \cdots + (-1)^{n+1}L^{n+1}$$

in $K^0(\mathbb{C}P^n)$. Of course the expression on the right is precisely $(1-L)^{n+1}$.

We next compute the K-theoretic fundamental classes in a couple of simple examples:

Example 20.8. Let $Z = V(f) \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Let $R = \mathbb{C}[x_0, \ldots, x_n]$. The coordinate ring of Z is $R/(f)$, which has the short free resolution given by

$$0 \to R(-d) \xrightarrow{\cdot f} R \to R/(f) \to 0.$$

So $[Z] = 1 - L^d$ in $K^0(\mathbb{C}P^n)$. To write $[Z]$ as a linear combination of the $[\mathbb{C}P^{n-i}]$'s we need to write $1 - L^d$ in terms of powers of $1 - L$. This is easy, of course:

$$\begin{aligned}[Z] = 1 - L^d = 1 - [1 - (1 - L)]^d &= 1 - (1 - X)^d \\ &= 1 - [1 - dX + \tbinom{d}{2}X^2 - \cdots + (-1)^d X^d] \\ &= dX - \tbinom{d}{2}X^2 + \tbinom{d}{3}X^3 - \cdots\end{aligned}$$

So we find that the higher d_i's for a hypersurface are all just (up to sign) binomial coefficients of d. This is somewhat disappointing, as we are not seeing new topological invariants—it is just the degree over and over again, encoded in different ways. [This is not actually not a surprise: the degree is known to be the *only* topological invariant for embedded hypersurfaces].

Things become more interesting in the next example:

Example 20.9. Consider $Z = V(f, g)$ where f, g is a regular sequence of homogeneous elements in $R = \mathbb{C}[x_0, \ldots, x_n]$. Let $d = \deg(g)$ and $e = \deg(g)$. Because f, g is a regular sequence, $R/(f, g)$ is resolved by the Koszul complex:

$$0 \to R(-d-e) \to R(-d) \oplus R(-e) \to R \to R/(f, g) \to 0.$$

We can now calculate

$$\begin{aligned}[Z]_K &= \Big[\, 0 \longrightarrow L^{d+e} \longrightarrow L^d \oplus L^e \longrightarrow 1 \longrightarrow 0 \,\Big] \\ &= 1 - (L^d + L^e) + L^{d+e} \\ &= 1 - (1 - X)^d - (1 - X)^e + (1 - X)^{d+e} \\ &= \Big[\tbinom{d+e}{2} - \tbinom{d}{2} - \tbinom{e}{2}\Big]X^2 - \Big[\tbinom{d+e}{3} - \tbinom{d}{3} - \tbinom{e}{3}\Big]X^3 + \cdots \\ &= deX^2 + \tfrac{1}{2}de(2 - d - e)X^3 + \ldots.\end{aligned}$$

The classical degree of Z is de, but our 'higher invariants' now see more than just this number. In fact, knowing the expansion of $[Z]_K$ as a linear combination of the X^i's implies that we know de and $de(2-d-e)$, which implies that we know $d+e$. But if we know de and $d+e$ then we know the polynomial $(\xi-d)(\xi-e) = \xi^2-(d+e)\xi+de$, which means we know its roots. So knowing the expansion of $[Z]_K$ is the same as knowing d and e. This example shows that the K-theoretic fundamental class sees more topological information than the singular cohomology fundamental class does.

Now that we have seen these simple examples we can return to our main question. Given $Z \hookrightarrow \mathbb{C}P^n$ of codimension c, how does one compute the d_i's in the equation

$$[Z]_K = (\deg Z)[\mathbb{C}P^{n-c}] + d_{c+1}[\mathbb{C}P^{n-c-1}] + d_{c+2}[\mathbb{C}P^{n-c-2}] + \cdots$$

And what do these d_i's mean in terms of geometry? We will soon see that one answer is given by the Hilbert polynomial.

20.10. **Review of the Hilbert polynomial.** Let $R = \mathbb{C}[x_0, \dots, x_n]$, and regard this as a graded ring where each x_i has degree one. If M is a graded R-module write M_s for the graded piece in degree s. The **Poincaré series** of M is the formal power series

$$P_M(\xi) = \sum_{s=-\infty}^{\infty} (\dim M_s)\xi^s \in \mathbb{Z}[[\xi]]$$

(defined if M is finitely-generated over R). Note that this is evidently an additive invariant of finitely-generated, graded modules: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence then clearly $P_M(\xi) = P_{M'}(\xi) + P_{M''}(\xi)$. We may therefore regard the Poincaré series as a map of abelian groups

$$P \colon G_{grd}(R) \to \mathbb{Z}[[\xi]]$$

where $G_{grd}(R)$ is the Grothendieck group of finitely-generated *graded* modules over R.

We will calculate the Poincaré series of each $R(-k)$, but for this we need the following useful calculation:

Lemma 20.11. *The number of monomials of degree d in the variables $z_1, \dots, z_n$ is equal to $\binom{d+n-1}{d}$.*

Proof. Monomials are in bijective correspondence with patterns of "dashes and slashes" that look like

$$- - -/- -//- - - -/- \,.$$

The above pattern corresponds to the monomial $z_1^3 z_2^2 z_4^4 z_5$, and from this the general form of the bijection should be clear. Monomials of degree d will correspond to patterns with d dashes, and if there are n variables then there will be $n-1$ slashes. So we need to count patterns where there will be $d+n-1$ total symbols, of which d are dashes: the number of these is $\binom{d+n-1}{d}$. □

It is an immediate consequence that $P_R(\xi) = \sum_{s\geq 0} \binom{s+n}{n}\xi^s$. For $R(-k)$ we simply shift the coefficients and obtain

$$P_{R(-k)}(\xi) = \sum_{s\geq k} \binom{s+n-k}{n}\xi^s = \xi^k P_R(\xi).$$

The power series $\{\xi^k P_R(\xi) \,|\, k \in \mathbb{Z}\}$ are obviously linearly independent over $\mathbb{Q}$, which shows that $G_{grd}(R)$ has infinite rank as an abelian group.

Proposition 20.12. *$G_{grd}(R)$ is isomorphic to $\mathbb{Z}^\infty$, with free basis the set of rank one, free modules $\{[R(-d)] \,|\, d \in \mathbb{Z}\}$.*

Proof. The key is the Hilbert Syzygy Theorem. Consider the diagram

$$\begin{array}{ccc} \mathbb{Z}^\infty & \longrightarrow & G_{grd}(R) \\ & \searrow & \downarrow{\scriptstyle P} \\ & & \mathbb{Z}[[\xi]] \end{array}$$

where the horizontal map sends the ith basis element to $[R(-i)]$ and the diagonal map is the composite. We have already seen that this composite is injective, because there is no $\mathbb{Z}$-linear relation amongst the images of the basis elements. So $\mathbb{Z}^\infty \to G_{grd}(R)$ is injective, and it only remains to prove surjectivity. But if M is any finitely-generated, graded R-module then the Syzygy Theorem guarantees a finite, graded, free resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

So $[M] = \sum_i (-1)^i [F_i]$, and each F_i is a sum of elements $[R(-k)]$. This proves surjectivity. □

The Hilbert polynomial is a variant of the Poincaré series that keeps track of less information. At first this might seem to be a bad thing, but we will see that what it does is give us a closer connection to geometry and topology. Here is the main result that gets things started:

Proposition 20.13. *Let M be a finitely-generated module M over R. Then there exists a unique polynomial $H_M(s) \in \mathbb{Q}[s]$ that agrees with the function $s \mapsto \dim M_s$ for $s \gg 0$. One has $\deg H_M(s) \leq n$. The polynomial $H_M(s)$ is called the **Hilbert polynomial** of M.*

Proof. Consider first the case $M = R$. A basis for M_s consists of all monomials in $x_0, \ldots, x_n$ of degree s, which Lemma 20.11 calculates to be $\binom{n+s}{s} = \binom{n+s}{n}$. This is a polynomial in s of degree n. Next consider $M = R(-k)$. The function $s \mapsto M_s$ is zero for $s < k$, and then for $s \geq k$ it coincides with $\binom{n+s-k}{n}$. This is again a polynomial in s of degree n.

Finally, consider the case of a general M. By the Hilbert Syzygy Theorem M has a finite, graded, free resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

It follows that

$$\dim M_s = \dim(F_0)_s - \dim(F_1)_s + \cdots + (-1)^n \dim(F_n)_s.$$

But each F_i is a direct sum of finitely-many $R(-k)$'s, and so $s \mapsto \dim(F_i)_s$ has been shown to agree with a polynomial in s of degree at most n, for $s \gg 0$. The desired result follows at once. □

The following calculation was given in the above proof, but we record it below because it comes up so often:

Corollary 20.14. *When $R = \mathbb{C}[x_0, \ldots, x_n]$ and $k \in \mathbb{Z}$ the Hilbert polynomial for $R(-k)$ is $\binom{s+n-k}{n}$.*

Example 20.15. Consider a hypersurface $M = R/(f)$, where $f \in R$ is homogeneous of degree d. We then have the resolution

$$0 \longrightarrow R(-d) \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0$$

from which we find

$$H_{R/(f)} = H_R - H_{R(-d)} = \binom{s+n}{n} - \binom{s+n-d}{n} = \frac{(s+n)\ldots(s+1)}{n!} - \frac{(s+n-d)\ldots(s-d+1)}{n!}.$$

Note that the two binomial coefficients have leading terms $s^n/n!$, which therefore cancel. The coefficient of s^{n-1} is

$$\frac{1}{n!} \cdot \left[\frac{n(n+1)}{2} - \frac{(n-2d+1)n}{2} \right] = \frac{d}{(n-1)!}.$$

Note that the degree of $H_{R/(f)}$ is one less than the Krull dimension of $R/(f)$, and the leading coefficient is the degree of f (the geometric degree of the hypersurface) divided by $n!$. These are general facts that hold for any module: the degree of H_M is one less than the Krull dimension of M, and the leading coefficient times $(\deg H_M)!$ is always an integer—this integer is called the **multiplicity** of the module M. Proofs can be found in most commutative algebra texts.

The Hilbert polynomial is an additive invariant of finitely-generated modules: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence then clearly $H_M(s) = H_{M'}(s) + H_{M''}(s)$. We may therefore regard the Hilbert polynomial as a map of abelian groups

$$\text{Hilb}\colon G_{grd}(R) \to \mathbb{Q}[s].$$

This map is clearly not injective, however: it kills any module that is finite-dimensional as a $\mathbb{C}$-vector space. The subgroup of $G_{grd}(R)$ generated by such modules is $A = \langle [\mathbb{C}(-d)] \,|\, d \in \mathbb{Z} \rangle$. We may regard Hilb as a map

$$\text{Hilb}\colon G_{grd}(R)/A \to \mathbb{Q}[s].$$

The domain of this map is calculated as follows:

Proposition 20.16. *The group $G_{grd}(R)/A$ is isomorphic to $\mathbb{Z}^{n+1}$, with free basis $R, R(-1), R(-2), \dots, R(-n)$. The map* Hilb *is an injection.*

Proof. Let B be the subgroup of $G_{grd}(R)/A$ generated by $[R], [R(-1)], \dots, [R(-n)]$. We will show that B is equal to the whole group.

The module $\mathbb{C}$ is resolved by the Koszul complex $K(x_0, \dots, x_n; R)$. This gives the relation in $G_{grd}(R)$ of

$$[\mathbb{C}] = [R] - (n+1)[R(-1)] + \tbinom{n+1}{2}[R(-2)] + \cdots + (-1)^{n+1}[R(-n-1)].$$

In $G_{grd}(R)/A$ we therefore have $[R(-n-1)] \in B$. For $d > 0$ we can tensor the Koszul complex with $R(-d)$ and then apply the same argument to find that $[R(-n-1-d)] \in \langle [R], [R(-1)], \dots, R(-n-d)] \rangle$. By induction we therefore have that $[R(-k)] \in B$ for every $k \geq n+1$.

A similar induction works for $d < 0$ to show that $[R(-d)] \in B$ for all $d \in \mathbb{Z}$. In other words, $B = G_{grd}(R)/A$.

Now consider the map $\mathbb{Z}^{n+1} \to G_{grd}(R)/A$ that sends the ith basis element e_i to $[R(-i)]$ for $0 \leq i \leq n$. We have just shown that this map is surjective. Consider, then, the composite

$$\mathbb{Z}^{n+1} \longrightarrow G_{grd}(R)/A \xrightarrow{\text{Hilb}} \mathbb{Q}[s].$$

The images of our basis elements are the polynomials

$$\tbinom{s+n}{n},\ \tbinom{s+n-1}{n},\ \tbinom{s+n-2}{n},\ \dots,\ \tbinom{s}{n}.$$

Evaluating these polynomials at $s = 0$ gives the sequence $1, 0, 0, \dots, 0$. Evaluating at $s = 1$ gives $n+1, 1, 0, 0, \dots, 0$, and so forth. For $s = i$ the ith polynomial in the list evaluates to 1 and all the subsequent polynomials evaluate to 0. This proves that these polynomials are linearly independent over $\mathbb{Z}$, hence the above composite map is injective. So $\mathbb{Z}^{n+1} \to G_{grd}(R)/A$ is injective, and therefore is an isomorphism. Moreover, Hilb is injective. $\square$

The reader might notice that the $\mathbb{Z}^{n+1}$ in the above result is really 'the same' as the group $K^0(\mathbb{C}P^n)$. This is the basis for what we do in the next section.

Example 20.17. We will not need this, but it is a cute fact: Prove that the image of Hilb: $G(R)/A \to \mathbb{Q}[s]$ equals the set of polynomials $f(s) \in \mathbb{Q}[s]$ having the property that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. That is, the image consists of all rational polynomials that take integer values on integers.

20.18. **K-theory and the Hilbert polynomial.** We now explain how the Hilbert polynomial encodes the same information as the K-theoretic fundamental class.

Given $Z \hookrightarrow \mathbb{C}P^n$ a smooth subvariety of codimension c, recall that we have the fundamental class $[Z] \in K^0(\mathbb{C}P^n)$ and that we may write

$$[Z] = d_c[\mathbb{C}P^{n-c}] + d_{c+1}[\mathbb{C}P^{n-c-1}] + \cdots + d_n[\mathbb{C}P^0].$$

We know that $d_c = \deg(Z)$, and our goal is to understand how to compute the higher d_i's.

Recall that $K^0(\mathbb{C}P^n)$ is spanned by the classes $[\mathbb{C}P^{n-k}] = (1-L)^k$ for $k = 0, 1, \ldots, n$. Evidently one can also use the basis $1, L, L^2, \ldots, L^n$. We next introduce an algebraic analogue of $K^0(\mathbb{C}P^n)$. Let $R = \mathbb{C}[x_0, \ldots, x_n]$. Take the Grothendieck group $K^0_{grd}(R)$ of finitely-generated, graded, projective R-modules (or equivalently, chain complexes of such modules) and quotient by the subgroup $\tilde{A}$ generated by all complexes $K(x_0, \ldots, x_n; R) \otimes R(-d)$ for $d \in \mathbb{Z}$. We obtain a diagram

$$\begin{array}{ccc} K^0_{\mathrm{grd}}(R)/\tilde{A} & \xrightarrow[\cong]{\phi} & K^0(\mathbb{C}P^n) \\ \downarrow{\scriptstyle\cong} & & \\ G_{grd}(R)/A & \xrightarrow{\mathrm{Hilb}} & \mathbb{Q}[s] \end{array} \tag{20.19}$$

where the vertical map is our usual 'Poincaré Duality' isomorphism and ϕ sends $[R(-d)]$ to L^d. The map ϕ is an isomorphism by inspection: we have computed $G_{grd}(R)/A$ and $K^0(\mathbb{C}P^n)$, both are $\mathbb{Z}^{n+1}$, and ϕ clearly maps a basis to a basis.

Given $Z = V(I)$, we know that its fundamental class $[Z] \in K^0(\mathbb{C}P^n)$ is represented by a finite, free resolution $F_\bullet \to R/I \to 0$. This resolution (or the alternating sum of its terms) lifts to $K^0_{grd}(R)/\tilde{A}$, and pushing this around the diagram into $\mathbb{Q}[s]$ just gives us $\mathrm{Hilb}_{R/I}(s)$. So the above diagram shows that knowing $\mathrm{Hilb}_{R/I}(s)$ is the same as knowing $[Z]$.

To say something more specific here, consider first the case $Z = \mathbb{C}P^{n-k}$. Then $R/I = R/(x_0, \ldots, x_{k-1})$, which as a graded ring is just $\mathbb{C}[x_k, \ldots, x_n]$. The Hilbert polynomial is then

$$\mathrm{Hilb}_{\mathbb{C}P^{n-k}}(s) = \tbinom{s+n-k}{n-k}.$$

So pushing our basis $[\mathbb{C}P^n], [\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$ around diagram (20.19) into $\mathbb{Q}[s]$ yields the polynomials

$$\tbinom{s+n}{n},\ \tbinom{s+n-1}{n-1},\ \tbinom{s+n-2}{n-2},\ \ldots,\ \tbinom{s}{0}.$$

If $Z = V(I) \hookrightarrow \mathbb{C}P^n$ is now arbitrary, then writing

$$\mathrm{Hilb}_{R/I}(s) = d_0\tbinom{s+n}{n} + d_1\tbinom{s+n-1}{n-1} + d_2\tbinom{s+n-2}{n-2} + \ldots$$

implies that $[Z] = d_0[\mathbb{C}P^n] + d_1[\mathbb{C}P^{n-1}] + d_2[\mathbb{C}P^{n-2}] + \ldots$. In other words, one obtains the expansion of $[Z]$ as a linear combination of the $[\mathbb{C}P^{n-i}]$'s by writing $\mathrm{Hilb}_Z(s)$ as a linear combination of the above-listed binomial functions.

We have shown how to calculate the d_i's from the ideal I of equations defining Z: decompose the Hilbert polynomial into a sum of terms $\binom{s+n-k}{k}$, and take the resulting coefficients. This is still not exactly a 'geometric' description of the d_i's, although it is a description that takes place in the domain of algebraic geometry. We will get another perspective on this material via the Todd genus and the Grothendieck-Riemann-Roch Theorem, studied in Section 25. See especially Section 25.28.

21. Interlude on the calculus of finite differences

If $f \in \mathbb{Q}[t]$ let Δf be the polynomial given by

$$(\Delta f)(t) = f(t+1) - f(t).$$

For example, $\Delta(t^2) = (t+1)^2 - t^2 = 2t+1$, and $\Delta(t^3) = (t+1)^3 - t^3 = 3t^2+3t+1$. Note that $\Delta\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$ is a linear map, and that it lowers degrees by one. We call Δ the **finite difference** operator, and we regard it as an analog of the familiar differentiation operator $D\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$.

The opposite of differentiation is integration, and there is an analogous operator that is the opposite of Δ. If $f \in \mathbb{Q}[t]$ define Sf to be the function given by

$$(Sf)(t) = f(0) + f(1) + f(2) + \cdots + f(t-1).$$

For example,

$$S(t) = 0 + 1 + 2 + \cdots + (t-1) = \binom{t}{2} = \tfrac{t(t-1)}{2}$$

and

$$S(t^2) = 0^2 + 1^2 + \cdots + (t-1)^2 = \tfrac{(t-1)t(2t-1)}{6}.$$

One should think of the formula for $S(t)$ has giving a finite Riemann sum, based on intervals of width 1. It is not immediately clear that Sf is always a polynomial, nor that it raises degrees by one, but we will prove these things shortly. The following two facts are easy, though:

$$\Delta S(f) = f \qquad\qquad D\textstyle\int(f) = f$$

$$S\Delta(f) = f - f(0) \qquad\qquad \textstyle\int D(f) = f - f(0).$$

We have written each identity paired with the corresponding identity for classical differentiation/integration. Note that, like the usual integral, Sf will always be a polynomial that has zero as its constant term, since $(Sf)(0) = 0$ by definition.

Derivatives and integrals of polynomials are easy to compute because their values on the basic polynomials t^n are very simple. In fact, the point is really that operators D and $\int$ act very simply on the sequence of polynomials

$$1, \quad t, \quad \tfrac{t^2}{2}, \quad \tfrac{t^3}{3!}, \quad \tfrac{t^4}{4!}, \quad \ldots$$

Of course D carries each polynomial in the sequence to the preceding one, and $\int$ carries each polynomial to the subsequent one. In contrast, the operators Δ and S are *not* very well-behaved on this sequence. It is better to use the sequence

$$1, \quad S(1), \quad S^2(1), \quad S^3(1), \quad \ldots$$

so let us compute these. It is easy to see that $S(1) = t$, and we previously saw that $S^2(1) = S(t) = \binom{t}{2}$. It follows from the following useful lemma that $S^k(1) = \binom{t}{k}$, where the binomial coefficient stands for the polynomial $\frac{1}{k!}t(t-1)(t-2)\cdots(t-k+1)$.

Lemma 21.1. *For any $d, k \in \mathbb{Z}$ one has*

$$\Delta\binom{t+d}{k} = \binom{t+d}{k-1}, \qquad S\binom{t}{k} = \binom{t}{k+1}.$$

Proof. The statement about Δ follows from Pascal's Identity. The statement about S is the identity

$$\binom{0}{k} + \binom{1}{k} + \cdots + \binom{t-1}{k} = \binom{t}{k+1}.$$

To prove this, imagine t slots labelled $1 - -t$ where we are to place $k+1$ asterisks:

$$\_\_ \ \underline{*} \ \_\_ \ \underline{*} \ \underline{*} \ \_\_ \ \underline{*}$$

If the leftmost asterisk is in spot i, there are $\binom{t-i}{k}$ ways to place the remaining asterisks. Summing over $i \in [1, t]$ yields the desired formula. □

The sequence of polynomials

$$1 = \binom{t}{0}, \quad \binom{t}{1}, \quad \binom{t}{2}, \quad \binom{t}{3}, \quad \ldots$$

is clearly a basis for $\mathbb{Q}[t]$, as the kth term has degree equal to k. In relation to the Δ and S operators, this basis plays the role classically taken by the polynomials $\frac{t^d}{d!}$. It is now clear that S applied to a polynomial of degree d yields a polynomial of degree $d+1$: a polynomial of degree d is a linear combination of $\binom{t}{d}, \binom{t}{d-1}, \ldots, \binom{t}{0}$ with the coefficient on the first term nonzero. Applying S changes each $\binom{t}{k}$ to $\binom{t}{k+1}$, and it is clear that this yields a polynomial of degree $d+1$.

For another example of the use of this binomial basis, note the following analog of the Taylor–Maclaurin explansion for writing polynomials in this basis:

Proposition 21.2. *Let $f \in \mathbb{Q}[t]$. Then*

$$f = \sum_{k=0}^{\infty} (\Delta^k f)(0) \cdot \binom{t}{k}.$$

Proof. We know $f = \sum_{k=0}^{N} a_k \binom{t}{k}$ for some N and some values $a_k \in \mathbb{Q}$. Plugging in $t = 0$ immediately gives $f(0) = a_0$. Now apply Δ to get $\Delta f = \sum_{k=0}^{N-1} a_{k+1}\binom{t}{k}$ and again plug in $t = 0$: this yields $(\Delta f)(0) = a_1$. Continue. □

Exercise 21.3. Let $\mathbb{Q}[t]_{int} \subseteq \mathbb{Q}[t]$ be the set of all polynomials $f(t)$ such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. Note that $\mathbb{Q}[t]_{int}$ is stable under Δ and S, and use this to prove that $\mathbb{Q}[t]_{int}$ is the $\mathbb{Z}$-linear span of the polynomials $\binom{t}{k}$, $k \geq 0$.

In contrast to the large number of similarities of the pair of operators (Δ, S) to $(D, \int)$, there is an important difference when it comes to the product rule. Of course we have $D(fg) = (Df)g + f(Dg)$, but one readily checks that this does **not** work for Δ. The correct rule is as follows:

Lemma 21.4. *For any $f, g \in \mathbb{Q}[t]$ one has $\Delta(fg) = (\Delta f)g + f(\Delta g) + (\Delta f)(\Delta g)$.*

Proof. A simple calculation, left to the reader. □

In the present section we will not have much use for this product rule, but it is a very important formula whose significance will become larger in subsequent sections.

21.5. **Translating between** (Δ, S) **and** $(D, \int)$**.** If $f \in \mathbb{Q}[t]$ and h is any integer (or even better, a formal variable) we have the Taylor fomula

$$f(t+h) = \sum_{k=0}^{\infty} f^{(k)}(t) \cdot \tfrac{h^k}{k!} = \sum_{k=0}^{\infty} (D^k f)(t) \cdot \tfrac{h^k}{k!}.$$

Note that the sum is really finite, since large enough derivatives of f are all zero. Taking $h = 1$ and rearranging somewhat we get

$$f(t+1) = \Big(\sum_{k=0}^{\infty} \tfrac{D^k}{k!}\Big) f = (e^D) f.$$

Let us be clear about what this means. The expression $\sum_{k=0}^{\infty} \frac{D^k}{k!}$ makes perfect sense as an *operator*, since evaluating this at any fixed polynomial gives a well-defined answer. It is sensible to denote this operator as e^D.

Taking one more step, we can write $(\Delta f)(t) = f(t+1) - f(t) = (e^D - 1) f$ where 1 denotes the identity operator. Or even more compactly,

$$\Delta = e^D - 1$$

is an identity of operators on $\mathbb{Q}[t]$. This identity gives us Δ as a linear combination of iterates of D.

We are next going to cook up a similar formula for the operator S, but this is a little harder. One does not wish for a formula using higher and higher powers of $\int$, as these operators become more and more complicated. Instead, we say to ourselves that S is very close to being an inverse for D (it is a right inverse, but not a left inverse). If it *were* an inverse we might want to write $S = \frac{1}{e^D - 1}$, but it is difficult to make sense of the latter expression as an operator—the trouble is that $e^D - 1$ has no constant term, otherwise we could expand as a power series. While this didn't work, we *can* make sense of the operator

$$\mathcal{B} = \tfrac{D}{e^D - 1}. \tag{21.6}$$

By this we mean write $\frac{x}{e^x - 1} = \sum a_k x^k = 1 - \frac{x}{2} + \frac{x^2}{12} - \cdots$ as a formal power series, and set $\mathcal{B}$ equal to $\sum_{k=0}^{\infty} a_k D^k$. It follows purely formally that $\Delta \mathcal{B} f = Df$ for any polynomial f, and therefore that

$$\mathcal{B} f - (\mathcal{B} f)(0) = SDf$$

(because $S(Df)$ will not have a constant term). Replacing f by $\int f$ in the above formula, we get

$$Sf = SD(\textstyle\int f) = \mathcal{B}(\int f) - (\mathcal{B} \int f)(0).$$

Note that $\mathcal{B}(\int f)$ is equal to $\int f - \frac{1}{2} f + \frac{1}{12} Df - \cdots$. So this is as computable as the series $\frac{x}{e^x - 1}$; the coefficients of this series are related to the Bernoulli numbers (see Appendix A). We have

$$\mathcal{B} = \sum_{k=0}^{\infty} \tfrac{B_k}{k!} D^k$$

where the B_k's are the Benoulli numbers.

21.7. **Sums of powers and Bernoulli numbers.** The (Δ, S) pair of operators is useful in a variety of mathematical situations. In a moment we will see an application to determining K-theoretic fundamental classes, but let us first look at a non-topological example. Almost every math student has seen the formulas

$$1+2+\cdots+n=\tfrac{n(n+1)}{2} \quad \text{and} \quad 1^2+2^2+\cdots+n^2=\tfrac{n(n+1)(2n+1)}{6}.$$

The proof of such formulas by mathematical induction is a common exercise in elementary proof courses. It turns out that there are also formulas for higher powers, of the form

$$1^k+2^k+\cdots+n^k=p_k(n)$$

where $p_k(n)$ is a polynomial of degree $k+1$. How does one discover the appropriate polynomials? This seems to have first been done by Jacob Bernoulli, the coefficients in these polynomials being closely related to what are now called Bernoulli numbers.

Clearly we may rephrase the problem as that of computing $S(t^k)$, the exact connection being $S(t^k)=p_k(t-1)$. In the last section we developed a formula for S in terms of the usual derivative and integral operators, and we will now use that; but here it is easiest to use it in the form

$$SD(f)=\mathcal{B}f-(\mathcal{B}f)(0)=\left(\sum_{j=0}^{\infty}\tfrac{B_j}{j!}D^j\right)f-(\text{constant term of preceding expression}).$$

We obtain

$$\begin{aligned} S(t^k)=SD\left(\tfrac{t^{k+1}}{k+1}\right)&=\tfrac{1}{k+1}\sum_{j=0}^{\infty}\tfrac{B_j}{j!}D^j(t^{k+1})-(\text{constant term})\\ &=\tfrac{1}{k+1}\sum_{j=0}^{k}\tfrac{B_j}{j!}(k+1)(k)(k-1)\cdots(k+2-j)t^{k+1-j}\\ &=\tfrac{1}{k+1}\sum_{j=0}^{k}B_j\cdot\tbinom{k+1}{j}t^{k+1-j}.\end{aligned}$$

Note that the last expression is a polynomial in t, with all of its coefficients clearly visible.

Finally, recalling that $S(t^k)=p_k(t-1)$ we conclude that

$$1^k+2^k+\cdots+n^k=\tfrac{1}{k+1}\sum_{j=0}^{k}B_j\cdot\tbinom{k+1}{j}(n+1)^{k+1-j}.$$

Exercise 21.8. Check that the above formula gives the familiar identities in the cases $k=1$ and $k=2$, and then see what it gives when $k=3$. Compare the above formula to (A.3).

21.9. **Back to K-theoretic fundamental classes.** Let $Z\hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface defined by the homogeneous ideal $I\subseteq \mathbb{C}[x_0,\ldots,x_n]$. We saw in Section 20.18 that the coefficients in

$$[Z]_K=a_0[\mathbb{C}P^0]_K+a_1[\mathbb{C}P^1]_K+a_2[\mathbb{C}P^2]_K+\cdots$$

are the same as the coefficients in

$$\text{(21.10)} \qquad \text{Hilb}_Z(t)=a_0\tbinom{n+t}{n}+a_1\tbinom{n+t-1}{n-1}+a_2\tbinom{n+t-2}{n-2}+\cdots$$

The sequence of polynomials $\binom{n+t}{n}, \binom{n+t-1}{n-1}, \binom{n+t-2}{n-2}, \dots$ is not quite the standard basis of binomial coefficients we used above, but it is close. Here is a small lemma about expanding polynomials in this basis (compare to Proposition 21.2):

Lemma 21.11. *If $f \in \mathbb{Q}[t]$ then $f = \sum_{k=0}^{\infty} (\Delta^k f)(-k-1) \cdot \binom{t+k}{k}$.*

Proof. The collection of polynomials $\binom{t+k}{k}$ for $k \geq 0$ is clearly a basis for $\mathbb{Q}[t]$. Write $f = \sum c_k \binom{t+k}{k}$. Plugging in $t = -1$ immediately gives $f(-1) = c_0$. Apply Δ to both sides to get $\Delta f = \sum a_k \binom{t+k}{k-1} = a_1 + a_2 \binom{t+2}{1} + \cdots$. Now plugging in $t = -2$ makes all the expressions vanish except the first, so $a_1 = (\Delta f)(-2)$. Continue in this way. □

The following corollary is immediate, by applying the above lemma to (21.10):

Corollary 21.12. *We have $[Z]_K = \sum_i a_i [\mathbb{C}P^i]_K$ where the coefficients are given by $a_i = (\Delta^{n-i} \operatorname{Hilb}_Z)(-n+i-1)$.*

Example 21.13. Let $Z \hookrightarrow \mathbb{C}P^n$ be a hypersurface of degree d, and write $Z = V(f)$. Then $R/(f)$ is resolved by $0 \to R(-d) \to R$ where the map is mutiplication by f. So

$$\operatorname{Hilb}_Z = \operatorname{Hilb}_R - \operatorname{Hilb}_{R(-d)} = \binom{n+t}{n} - \binom{n+t-d}{n}.$$

But then

$$\Delta^{n-i} \operatorname{Hilb}_Z = \binom{n+t}{n-(n-i)} - \binom{n+t-d}{n-(n-i)} = \binom{n+t}{i} - \binom{n+t-d}{i}$$

and

$$a_i = \binom{i-1}{i} - \binom{i-1-d}{i} = \binom{i-1}{i} + (-1)^{i+1}\binom{d}{i}$$

where in the last step we have used the identity $\binom{-s}{r} = (-1)^r r+s-1r$. The expression $\binom{i-1}{i}$ is nonzero only when $i = 0$, so that we get

$$a_i = \begin{cases} 0 & \text{if } i = 0 \\ (-1)^{i+1}\binom{d}{i} & \text{if } i > 0. \end{cases}$$

This of course agrees with what we found in Example 20.8.

Just as in the last example, in practice Hilbert polynomials are often computed by first having a free resolution of R/I. Let us look at what happens in general here, so let the (finite) free resolution be

$$\cdots \oplus_j R(-d_{2j}) \to \oplus_i R(-d_{1i}) \to \oplus_e R(-d_{0e}) \to R/I \to 0.$$

Then the Hilbert series is given by

$$\operatorname{Hilb}_Z(t) = \sum_{k=0}^{\infty} (-1)^k \sum_j \binom{t+n-d_{kj}}{n}.$$

Then

$$a_i = (\Delta^{n-i} \operatorname{Hilb}_Z)(-n+i-1) = \sum_{k=0}^{\infty} (-1)^k \sum_j \binom{i-1-d_{kj}}{i} = (-1)^i \cdot \sum_{k=0}^{\infty} (-1)^k \sum_j \binom{d_{kj}}{i}.$$

Example 21.14. Let $Z \hookrightarrow \mathbb{C}P^n$ be a complete intersection where the degrees of the defining equations are d and e. Then R/I is resolved by the Koszul complex, which looks like $0 \to R(-d-e) \to R(-d) \oplus R(-e) \to R \to R/I \to 0$. We conclude that $a_i = \binom{d}{i} + \binom{e}{i} - \binom{d+e}{i}$.

Recall that if the codimension of $Z \hookrightarrow \mathbb{C}P^n$ is equal to c then we have $a_i = 0$ for $i < c$ and $a_c = \deg(Z)$. So any free resolution of R/I must satisfy the identities

$$\sum_k (-1)^k \sum_j \tbinom{d_{kj}}{i} = \begin{cases} 0 & \text{for } i < c \\ \deg(Z) & \text{for } i = c. \end{cases}$$

In this way we obtain topological conditions on what free resolutions can look like.

Example 21.15. Show that the above conditions on the free resolution are equivalent to

$$\sum_k (-1)^k \sum_j d_{kj}^i = \begin{cases} 0 & \text{for } i < c \\ \deg(Z) \cdot i! & \text{for } i = c. \end{cases}$$

21.16. **A digression on the Riemann zeta function.** If $Z \hookrightarrow \mathbb{C}P^n$ then the values of the polynomial $\operatorname{Hilb}_Z(t)$ only have an *a priori* significance for $t \gg 0$; recall that these values represent the dimensions of the graded pieces of the coordinate ring of Z. Yet, we saw in Corollary 21.12 that the values of $\operatorname{Hilb}_Z(t)$ at certain *negative* integers (encoded as the value of a particular finite difference $\Delta^? \operatorname{Hilb}_Z$) are equal to some naturally-occurring topological invariants of Z. The fact that these negative values have any significance at all is a bit surprising. This situation is somewhat reminiscent of one involving the Riemann zeta function, that coincidentally (or not) is also related to the story of the (Δ, S) operators. We are going to take a moment and talk about this, because of the feeling that it might be related to topology in a way that no one really understands yet.

Recall that Riemann's $\zeta(s)$ is defined for $Re(s) > 1$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Standard results from analysis show that this sum converges for $Re(s) > 1$, and defines an analytic function in that range. It is a non-obvious fact that $\zeta(s)$ can be analytically continued to the punctured plane $\mathbb{C} - \{1\}$. The values on negative numbers turn out to be computable and are related to the Bernoulli numbers. We will give an entirely non-rigorous treatment of this computation; despite its failure to actually make sense, it is nevertheless somewhat intriguing.

If $f \in \mathbb{Q}[t]$ then we have seen that $e^D f$ makes sense and is equal to the polynomial $f(t+1)$. It readily follows that $e^{nD} f = f(t+n)$ for any integer $n \geq 0$. Now write

$$f(t) + f(t+1) + f(t+2) + \cdots = [I + e^D + e^{2D} + e^{3D} + \cdots]f = \left[\tfrac{1}{I - e^D}\right] f.$$

Of course none of the terms in the above identities make any sense, but let us pretend for a moment that this is not a problem. Replacing f by Df we can then write

$$Df(t) + Df(t+1) + \cdots = \left[\tfrac{D}{I - e^D}\right] f = -\mathcal{B} f$$

where $\mathcal{B}$ is the Bernoulli operator of (21.6). Evaluating at $t = 0$ we would obtain

$$Df(0) + Df(1) + D^2 f(2) + \cdots = -(\mathcal{B} f)(0).$$

Let us next try to apply this fanciful formula to compute $\zeta(-n) = 1^n + 2^n + 3^n + 4^n + \cdots$. We want $Df(t) = t^n$, and so should take $f(t) = \frac{t^{n+1}}{n+1}$. The above formula then suggests that

$$\zeta(-n) = -\mathcal{B}\big(\tfrac{t^{n+1}}{n+1}\big)(0) = -\left[\sum_{k=0}^{\infty} \tfrac{B_k}{k!} D^k\big(\tfrac{t^{n+1}}{n+1}\big)\right](0) = -\tfrac{B_{n+1}}{(n+1)!}\cdot\tfrac{(n+1)!}{n+1} = -\tfrac{B_{n+1}}{n+1}.$$

Amazingly, this is the correct answer—the same value can be deduced by rigorous arguments from complex analysis. The challenge is to find some explanation for why this wacky argument actually leads to something correct.

22. The Euler class

There is a general principle in algebraic topology that all rational cohomology theories detect the same information. The information is not necessarily detected in the same *way*, however, and this makes it hard to formulate the principle precisely. But here is a nice example of it. If $Z \hookrightarrow \mathbb{C}P^n$ is a complex submanifold then we have seen that knowing $[Z] \in K^0(\mathbb{C}P^n)$ is the same as knowing the integers d_i for which $[Z] = \sum d_i[\mathbb{C}P^{n-i}]$. Since $K^0(\mathbb{C}P^n)$ is free abelian, there is no loss of information in regarding this equation as taking place in $K^0(\mathbb{C}P^n)\otimes\mathbb{Q}$. By the above-mentioned principle, the numbers d_i should be able to be detected in rational singular cohomology. The Grothendieck-Riemann-Roch Theorem tells us how to do this, and that will be our next main goal.

To understand Riemann-Roch we need to first understand characteristic classes. I will give a very brief treatment, spread over the next two sections. For a more in-depth treatment I suggest the book [MS].

The present section deals with the Euler class, which is in some sense the most "primary" of characteristic classes. We discuss two versions: Euler classes in singular cohomology and Euler classes in K-theory.

22.1. **The Euler class for a vector bundle.** We will start with a geometric treatment that is lacking in rigor but shows the basic ideas, and then I will give a more rigorous treatment. Don't worry about verifying all the details in the following, just get the basic idea.

Start with a bundle $E \to B$ of rank k, where B is a smooth manifold of dimension n. Let ζ denote the zero-section. We will try to construct something like an intersection-product $\zeta\cdot\zeta$. To do this, we let s be a section of E that is a "small-perturbation" of ζ, chosen so that s and ζ intersect as little as possible. A good example to keep in mind is the Möbius bundle, shown below with two deformations of the zero section:

The zero locus $s^{-1}(0) \subseteq \mathrm{im}(\zeta)$ may, under good conditions, be given the structure of a cycle—part of this involves assigning multiplicities to the components in a certain way. The "good conditions" are that the bundle must be orientable for multiplicites in $\mathbb{Z}$, whereas for any bundle one may assign multiplicities in $\mathbb{Z}/2$. The dimension of this cycle is $\dim\zeta + \dim\zeta - \dim E$, which is $n+n-(n+k) = n-k$. This cycle clearly depends on the choice of s, but a different choice of s gives a homologous cycle. So the associated class in homology is independent of our choices, and is an invariant of E. We call it the **homology Euler class** of E: for orientable bundles we have

$$e_H(E) = \zeta\cdot\zeta = s^{-1}(0) \in H_{n-k}(B),$$

whereas for arbitrary bundles we have $e_H(E) \in H_{n-k}(B;\mathbb{Z}/2)$. The sections s used here are usually referred to as "generic sections" of E.

The following are easy properties of the Euler class construction:

(1) if E has a nonvanishing section, the $e_H(E) = 0$, and
(2) $e_H(E\oplus F) = e_H(E)\cdot e_H(F)$ (where $\cdot$ is the intersection product).
(3) If L_1 and L_2 are line bundles on B then $e_H(L_1\otimes L_2) = e_H(L_1) + e_H(L_2)$.

For (1) we simply note that if σ is a nonvanishing section then the deformation $t \mapsto t\sigma$ (for $t\in[0,1]$) allows us to regard s as a deformation of the zero-section. Taking $s = \epsilon\sigma$ for small ϵ, the vanishing locus of s is the same as the vanishing locus of σ—which is the empytset. So $e_H(E) = 0$.

For (2), if s is a generic section of E and s' is a generic section of F then $s\oplus s'$ is a generic section of $E\oplus F$. The vanishing locus of $s\oplus s'$ is the intersection of the vanishing loci of s and s'.

For (3), if s_1 and s_2 are generic sections of L_1 and L_2 then $s_1\otimes s_2$ is a generic section of $L_1\otimes L_2$. But $s_1\otimes s_2$ vanishes at points in B where *either* s_1 or s_2 vanish. So the vanishing locus of $s_1\otimes s_2$ is the union of the vanishing loci of s_1 and s_2.

Example 22.2.

(a) For the Möbius bundle $M \to S^1$ we have $e_H(M) = [*] \in H_0(S^1;\mathbb{Z}/2)$, as is clearly depicted in the pictures above. Note the necessity of $\mathbb{Z}/2$-coefficients here.
(b) If B is an orientable, smooth manifold of dimension d, then $e_H(TB) \in H_0(B)$. So $e_H(TB)$ is a multiple of $[*]$, and this multiple is precisely the Euler characteristic $\chi(B)$: a section of TB is just a vector field on B, and so this is the classical statement that a generic vector field on B vanishes at precisely $\chi(B)$ points. This connection with the Euler characteristic is why e_H is called the Euler class.
(c) Let $L \to \mathbb{R}P^n$ be the tautological line bundle, and recall that $L\cong L^*$. We examine L^* instead, since it is easier to write down formulas for sections. Generically choose a tuple $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ and consider the section s_α whose value over $x = [x_0 : \cdots : x_n]$ is the functional sending $(x_0,\dots,x_n)$ to $\alpha_0x_0+\cdots+\alpha_nx_n$. The vanishing locus for this section is a linear subspace of $\mathbb{R}P^n$, and of course we know that all such things are homotopic. So $e_H(\gamma) = [\mathbb{R}P^{n-1}]$.
(d) A similar analysis allows one to calculare $e_H(L^*)$ where $L\to\mathbb{C}P^n$ is the tautological line bundle, but here one must be careful about getting the signs correct. It is clear enough that $e_H(L^*) = \pm[\mathbb{C}P^{n-1}]$, but determining the sign takes some thought. ?????
(e) Normal bundle of $Z \hookrightarrow \mathbb{C}P^n$????

22.3. **Cohomology version.** Let $E \to B$ be an orientable real bundle of rank k; that is, a bundle with a Thom class $\mathcal{U}_E \in H^k(E, E-0)$. Note that B need no longer be a manifold. Let $\zeta\colon B\to E$ denote the zero section, as usual. We may interpret ζ as a map of pairs $(B,\emptyset) \to (E, E-0)$, so that pulling back along ζ gives cohomology class $\zeta^k(\mathcal{U}_E) \in H^*(B)$. Define the **(cohomology) Euler class** of E to be

$$e^H(E) = \zeta^*(\mathcal{U}_E).$$

The main properties of the Euler class are as follows:

Proposition 22.4. *Let $E \to B$ be an oriented real bundle. Then*

(a) If E has a nonzero section then $e^H(E) = 0$.

(b) $e^H(E \oplus F) = e^H(E) \cup e^H(F)$ for any oriented bundle $F \to B$.

(c) For any map $f\colon Y \to B$ one has $e^H(f^*E) = f^*(e^H(E))$ (naturality under pullbacks).

Proof. Properties (b) and (c) follow from the corresponding properties of Thom classes. To see (a), let s be a nonzero section. Consider the homotopy $H\colon I \times B \to E$ given by $H(t,b) = t \cdot s(b)$. This can be regarded as a homotopy between maps of pairs $(B,\emptyset) \to (E, E-0)$. It follows that $e^H(E) = \zeta^*(\mathcal{U}_E) = s^*(\mathcal{U}_E)$. But s factors through $E - 0$, and so $s^*(\mathcal{U}_E) = 0$. □

Note that analogs of the above properties are all true for non-orientable bundles as long as one uses $\mathbb{Z}/2$-coefficients everywhere.

Example 22.5.

(1) For the Möbius bundle $M \to S^1$ we have $e^H(M) = [*] \in H^1(S^1;\mathbb{Z}/2)$. To prove this note that the Thom space of M is $D(M)/S(M) \cong \mathbb{R}P^2$, and the zero section $\zeta\colon S^1 \to \mathrm{Th}(M)$ is just a typical embedding of $\mathbb{R}P^1$ into $\mathbb{R}P^2$. The Thom class $\mathcal{U}_M \in H^1(\mathrm{Th}\, M;\mathbb{Z}/2) = H^1(\mathbb{R}P^2;\mathbb{Z}/2)$ is the unique nonzero class, and we know restricting along $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$ sends this class to the unique nonzero class in $H^1(\mathbb{R}P^1;\mathbb{Z}/2)$.

(2) Let $\gamma \to \mathbb{R}P^n$ be the tautological bundle. If $j \hookrightarrow \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ is the inclusion, then $j^*\gamma \cong M$. So naturality gives $j^*e^H(\gamma) = e^H(M)$. But $j^*\colon H^1(\mathbb{R}P^n;\mathbb{Z}/2) \to H^1(\mathbb{R}P^1;\mathbb{Z}/2)$ is an isomorphism, and so it follows from (1) that $e^H(\gamma)$ must be the unique nonzero class in $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$.

(3) Let M be a smooth, oriented manifold of dimension n. Then the tangent bundle $TM \to M$ is a rank n oriented bundle. In this case, $e^H(TM) \in H^n(M;\mathbb{Z}) = \mathbb{Z}\langle[*]\rangle$ and so the problem is to determine the integer d for which $e^H(TM) = d[*]$. It is a fact from geometric topology ???? that there is a vector field $s\colon M \to TM$ with a finite number of vanishing points, and that when counted with appropriate signs this number is $\chi(M)$. Let $A = s^{-1}(0) = \{p_1,\dots,p_r\} \subseteq M$. The deformation $t \mapsto ts$ shows that s is homotopic to the zero section ζ, so that we get the following diagram

$$H^n(M) \xleftarrow{j^*} H^n(M, M-\{p_1,\dots,p_r\}) \xleftarrow{s^*} H^n(TM, TM-0)$$

where the composite map is $\zeta^* = s^*$. As is typical in these arguments, we next use that $H^n(M, M-\{p_1,\dots,p_r\}) \cong \oplus_i H^n(M, M-p_i)$ and that the orientation of M gives canonical generators $[*] \in H^n(M)$ and $[p_i] \in H^n(M, M-p_i)$. The map j^* sends each $[p_i]$ to $[*]$, and so it is really just a fold map $\mathbb{Z}^r \to \mathbb{Z}$. It remains to see how the Thom class $\mathcal{U}_{TM}$ maps to the canonical generators in $H^n(M, M-p_i)$ under s^*, but this is a local problem—by working through the definitions one sees that $\mathcal{U}_{TM} \mapsto d_i[p_i]$ where d_i is the local index of the vector field at p_i. We finally obtain that $e^H(TM) = (d_1 + \cdots + d_r)[*]$, where $d_1 + \cdots + d_r$ is the sum of the local indices and therefore equal to $\chi(M)$.

The following property of the Euler class is also useful:

Proposition 22.6. *Let M be an oriented manifold and let $j\colon X \hookrightarrow M$ be a regularly embedded, oriented submanifold. Let $N_{M/X}$ be the normal bundle. Then $e(N_{M/X}) = j^*([X])$.*

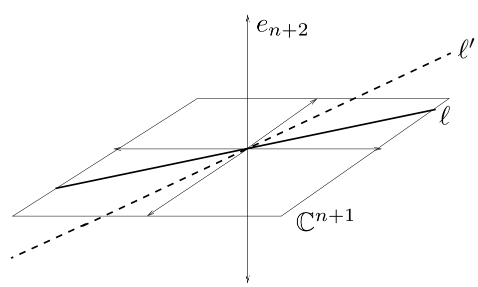
Proof. Intuitively the result should make sense, since both $e(N_{M/X})$ and $j^*([X])$ are modelled by the intersection product of X with itself inside of M. To give a rigorous proof, let U be a tubular neighborhood of X in M and $U \cong N$ be a regular homeomorphism. Note that the zero section $\zeta\colon X \hookrightarrow N$ corresponds with the inclusion $j\colon X \hookrightarrow U$ under this isomorphism.

Let c be the codimension of X in M. Consider the commutative diagram

$$\begin{array}{ccccccccc}
H^0(X) & \longrightarrow & H^c(N, N-0) & \xrightarrow{\cong} & H^c(U, U-0) & \xleftarrow{\cong} & H^c(M, M-X) & \longrightarrow & H^c(M) \\
 & & & \searrow^{\zeta^*} & \downarrow & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\
 & & & & H^c(X) & = & H^c(X) & = & H^c(X).
\end{array}$$

The image of 1 across the top row is $[X]$, with $\mathcal{U}_N$ being an intermediate value in the composite. The image of $\mathcal{U}_N$ under ζ^* is $e(N)$, and so the diagram immediately yields $e(N) = j^*[X]$. □

Example 22.7. Consider the usual embedding $j\colon \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$. We claim that the normal bundle is $L^* \to \mathbb{C}P^n$, the dual of the tautological line bundle. The proof is that a linear functional ϕ on the line $\ell \subseteq \mathbb{C}^{n+1}$ determines a "nearby" line $\ell' = \{(x, \phi(x))\, x \in \ell\}$, as shown below:



By Proposition 22.6 we find that $e(L^*) = j^*([\mathbb{C}P^n])$, and we know the latter is $[\mathbb{C}P^{n-1}]$ by intersection theory. We have shown that $e(L^* \to \mathbb{C}P^n) = [\mathbb{C}P^{n-1}]$.

We next wish to give a formula for the Euler class of a tensor product of line bundles. At the moment this might seem to be of limited interest, but it turns out to be very significant. We need to be careful about what context we are working in, however. All orientable real line bundles are trivial (one can write down an evident nonvanishing section), and so using the integral Euler class in this context doesn't lead to anything interesting. So we should work with mod 2 Euler classes and arbitrary real line bundles. Alternatively, if we use *complex* line bundles then they are automatically oriented and then we can indeed use the integral Euler class (of the underlying real plane bundle). So we really get two parallel results, one for the real and one for the complex case:

Proposition 22.8. *If L_1 and L_2 are real line bundles on B then the mod 2 Euler class satisfies*

$$e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2).$$

Likewise, if L_1 and L_2 are complex line bundles on B then the integral Euler class satisfies the same formula.

Proof. The proofs of the two parts are basically identical; we will do the complex case. Let $L \to \mathbb{C}P^\infty$ denote the tautological line bundle, and consider the bundle $\pi_1^*(L) \otimes \pi_2^*(L) \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, where $\pi_1, \pi_2 \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ are the two projections. Since L is the universal example of a line bundle, $\pi_1^*(L) \otimes \pi_2^*(L)$ is the universal example of a tensor product of line bundles. Write $E = \pi_1^*(L) \otimes \pi_2^*(L)$, for short. There is a classifying map $f \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ for E, giving a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & L \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{f} & \mathbb{C}P^\infty. \end{array}$$

By naturality $e^H(E) = f^*(e^H(L)) = f^*(x)$ where $x \in H^2(\mathbb{C}P^\infty)$ is the canonical generator.

If $*$ is a chosen basepoint in $\mathbb{C}P^\infty$ then observe that the diagram

$$\begin{array}{ccccc} \mathbb{C}P^\infty \times \{*\} & & \xrightarrow{\quad\mathrm{id}\quad} & & \\ & \searrow^{j_2} & & & \\ & & \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{f} & \mathbb{C}P^\infty \\ & \nearrow_{j_1} & & & \\ \{*\} \times \mathbb{C}P^\infty & & \xrightarrow{\quad\mathrm{id}\quad} & & \end{array}$$

commutes up to homotopy. This is because fj_1 classifies $j_1^*(E)$, and this bundle is clearly isomorphic to L; similarly for fj_2.

Note that $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is the free abelian group generated by $x \otimes 1$ and $1 \otimes x$. So $f^*(x) = k(x \otimes 1) + l(1 \otimes x)$ for some integers k and l. The above homotopy commutative diagram forces $k = l = 1$. So we have proven that

$$e^H(E) = e^H(L) \otimes 1 + 1 \otimes e^H(L).$$

Now let L_1 and L_2 be two complex line bundles on a space B. There are maps $g_1, g_2 \colon B \to \mathbb{C}P^\infty$ such that $L_1 = g_1^*(L)$ and $L_2 = g_2^*(L)$. Then $L_1 \otimes L_2 = \gamma^*(E)$, where γ is the composite

$$B \xrightarrow{\Delta} B \times B \xrightarrow{g_1 \times g_2} \mathbb{C}P^\infty \times \mathbb{C}P^\infty.$$

We obtain

$$\begin{aligned} e^H(L_1 \otimes L_2) = \gamma^*(e^H(E)) &= \gamma^*\big(e^H(L) \otimes 1 + 1 \otimes e^H(L)\big) \\ &= \Delta^*\big(e^H(L_1) \otimes 1 + 1 \otimes e^H(L_2)\big) = e^H(L_1) + e^H(L_2). \end{aligned}$$

□

Remark 22.9. One should note that the above proof is not geometric—this turns out to be important. Rather, the proof in some sense proceeds by showing that there is really not much choice for what $e^H(L_1 \otimes L_2)$ could be, given how small $H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is; there is in fact only *one* possibility. We will shortly see that replacing H by other cohomology theories—ones with "more room", so to speak—allows for more to happen here.

Corollary 22.10. *Let $L \to \mathbb{C}P^n$ be the tautological line bundle. Then $e^H(L^*) = [\mathbb{C}P^{n-1}]$ and so*

$$e^H((L^*)^{\otimes k}) = k[\mathbb{C}P^{n-1}] \quad \textit{and} \quad e^H(L^{\otimes k}) = -k[\mathbb{C}P^{n-1}].$$

Proof. The first statement was proven in Example 22.7. All of the other statements follow directly from the first via Proposition 22.8. For $e^H(L)$ use that $L \otimes L^* \cong \underline{1}$ and so $0 = e^H(\underline{1}) = e^H(L) + e^H(L^*)$. □

A nice consequence of all of the above work is that for complex line bundles the Euler class gives a complete invariant:

Corollary 22.11. *Let L_1, L_2 be two complex line bundles over a space X. If $e(L_1) = e(L_2)$ then $L_1 \cong L_2$.*

Proof. Let $f_1, f_2 \colon X \to \mathbb{C}P^\infty$ be classifying maps for the two line bundles: so $f_1^*L \cong L_1$ and $f_2^*L \cong L_2$. Then $e(L_1) = f_1^*e(L)$ and $e(L_2) = f_2^*(e(L))$, by naturality of the Euler class. Our assumption is therefore equivalent to $f_1^*(e(L)) = f_2^*(e(L))$.

But $\mathbb{C}P^\infty$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, and so $[X, \mathbb{C}P^\infty]$ is naturally isomorphic to $H^2(X)$ via the map $f \mapsto f^*(z)$ where z is a chosen generator of $H^2(\mathbb{C}P^\infty)$. Corollary 22.10 gives that $e(L)$ is such a generator, so the fact that $f_1^*(e(L)) = f_2^*(e(L))$ implies that f_1 is homotopic to f_2. But this implies that L_1 is isomorphic to L_2. □

The following easy corollary will be needed often:

Corollary 22.12. *Let $j \colon Z \hookrightarrow \mathbb{C}P^n$ be a degree d hypersurface. Then the normal bundle is isomorphic to $j^*\big((L^*)^{\otimes d}\big)$.*

Proof. By Proposition 22.6 the Euler class of the normal bundle is $e(N) = j^*([Z])$. But we know $[Z] = d[\mathbb{C}P^{n-1}] = e((L^*)^{\otimes d})$, and so $j^*[Z]$ is the Euler class of $j^*((L^*)^{\otimes d})$. Now use Corollary 22.11. □

22.13. **Euler classes in K-theory.** Let $E \xrightarrow{\pi} X$ be a $\mathbb{C}$-bundle of rank k. We have a Thom class $\mathcal{U}_E \in K^0(E, E-0)$, and so we can mimic the above construction and define

$$e^K(E) = \zeta^*(\mathcal{U}_E) \in K^0(X).$$

This is the K**-theory Euler class** of E.

Let us unravel the above definition a bit. First, recall that $\mathcal{U}_E$ is the complex

$$[\, 0 \longrightarrow \textstyle\bigwedge^0(\pi^*E) \xrightarrow{\Delta\wedge -} \bigwedge^1(\pi^*E) \longrightarrow \cdots \longrightarrow \bigwedge^k(\pi^*E) \longrightarrow 0.\,]^*$$

where Δ is the usual diagonal section as shown in the following diagram:

$$\begin{array}{ccccccc} E & = & \zeta^*(\pi^*E) & \longrightarrow & \pi^*E & \longrightarrow & E \\ & & \downarrow & & \Delta \uparrow\downarrow & & \downarrow \\ & & X & \xrightarrow{\zeta} & E & \xrightarrow{\pi} & X. \end{array}$$

Note that $\zeta^*(\pi^*(E)) = E$ because $\pi \circ \zeta = \mathrm{id}$. Next let us look at $\zeta^*(\mathcal{U}_E)$. For each j one has $\zeta^*(\bigwedge^j(\pi^*E)) = \bigwedge^j(E)$, and the restriction of Δ to the image of ζ is just the zero section! So the maps in the complex $\zeta^*(\mathcal{U}_E)$ are all zero. In other words, $\zeta^*(\mathcal{U}_E)$ is the Koszul complex for the zero section on E. Therefore

$$\begin{aligned}\zeta^*(\mathcal{U}_E) &= J^*_{E,0}\\ &= [\, 0 \longrightarrow \textstyle\bigwedge^0 E \xrightarrow{0} \bigwedge^1 E \xrightarrow{0} \cdots \longrightarrow \bigwedge^k E \longrightarrow 0\,]^*\\ &= \textstyle\sum(-1)^i[\bigwedge^i(E^*)].\end{aligned}$$

We see immediately that if E has a nonzero section then $e^K(E) = 0$. Indeed, if s is the nonzero section then we can deform ζ to s, and likewise deform $J^*_{E,\zeta}$ to $J^*_{E,s}$. But this latter complex is exact, and so represents zero in $K^0(X)$.

The analogs of properties (b) and (c) from Proposition 22.4 also hold, as these are simple consequences of corresponding properties of Thom classes. What about the analog of Proposition 22.8? If $L \to X$ is a complex line bundle then we have

$$e^K(L) = 1 - L^*.$$

So $e^K(L_1 \otimes L_2) = 1 - L_1^* L_2^*$, which is visibly *not* the same as $e^K(L_1) + e^K(L_2)$. Indeed, one can check the following more complicated formula:

Proposition 22.14. *Let L_1 and L_2 be complex line bundles on a space X. Then*

$$e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2).$$

Proof. We simply observe that $1 - L_1^*L_2^* = (1-L_1^*) + (1-L_2^*) - (1-L_1^*)(1-L_2^*)$. □

Remark 22.15. The difference between how e^H and e^K behave on tensor products of line bundles turns out to have much more significance than one might expect. In some sense it ends up accounting for all of the differences between H and K, at least in terms of how they encode geometry. See Section 27 for more discussion.

We end this section with some detailed computations of K-theoretic Euler classes:

Example 22.16. Let T be the complex tangent bundle to $\mathbb{C}P^n$. Our goal will be to compute $e^K(T)$ from first-principles. Let L denote the tautological line bundle over $\mathbb{C}P^n$. Note that L sits inside of the trivial bundle $\underline{n+1}$ in the evident way (if l is a line in $\mathbb{C}^{n+1}$, then points on l are defacto points in $\mathbb{C}^{n+1}$). Let $L^\perp$ be the orthogonal complement to L relative to the usual Hermitian metric on $\mathbb{C}^{n+1}$. So we have a short exact sequence of bundles

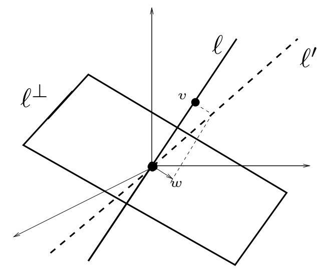
$$0 \to L \hookrightarrow \underline{n+1} \to L^\perp \to 0. \tag{22.17}$$

Since $\mathbb{C}P^n$ is compact this sequence is split, and hence $L \oplus L^\perp \cong \underline{n+1}$.

The basis of our computation is the following geometric fact:

$$T \cong \underline{\mathrm{Hom}}(L, L^\perp) \tag{22.18}$$

where $\underline{\mathrm{Hom}}(L, L^\perp)$ is the bundle over $\mathbb{C}P^n$ whose fiber over a point x is the vector space of linear maps $L_x \to L_x^\perp$. To understand this isomorphism, if ℓ is a point in $\mathbb{C}P^n$ then think of the tangent space T_ℓ as giving "local directions" for moving to all nearby points around ℓ. The following picture shows the line ℓ in $\mathbb{C}^{n+1}$ together with its orthogonal complement $\ell^\perp$ and a "nearby line" ℓ':



Note that ℓ' determines a linear map $\ell \to \ell^{\perp}$ as shown in the picture: a vector $v \in \ell$ is sent to the unique vector $w \in \ell^{\perp}$ such that $v + w \in \ell'$. This makes sense as long as ℓ' is not orthogonal to ℓ, which will be fine for all nearbly lines. We clearly get a bijection between $\mathrm{Hom}(\ell, \ell^{\perp})$ and a certain neighborhood of ℓ in $\mathbb{C}P^n$, and it is not hard to extrapolate from this to the isomorphism (22.18).

Now take the short exact sequence of (22.17) and apply $\underline{\mathrm{Hom}}(L, -)$ to get the short exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(L, L) \longrightarrow \underline{\mathrm{Hom}}(L, \underline{n+1}) \longrightarrow \underline{\mathrm{Hom}}(L, L^{\perp}) \longrightarrow 0.$$

(To see that this sequence is exact, just check it on fibers—there it is obvious, because we are just dealing with vector spaces.) For any line bundle $\mathcal{L}$ one has the identity $\underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{L}) = \underline{1}$: for a one-dimensional vector space V the map $\mathbb{C} \to \mathrm{Hom}(V, V)$ mapping 1 to the identity is a canonical isomorphism, so we can do this fiberwise. Using this, together with the identification $T \cong \underline{\mathrm{Hom}}(L, L^{\perp})$, the above short exact sequence can be written as

$$0 \to \underline{1} \to (n+1)L^* \to T \to 0.$$

Since there must be a splitting, $\underline{1} \oplus T \cong (n+1)L^*$. Dualizing, we obtain $\underline{1} \oplus T^* \cong (n+1)L$

Recall that $e^K(T) = \sum_i (-1)^i [\bigwedge^i (T^*)]$. We will compute $\bigwedge^i (1 \oplus T^*)$ and then extract formulas for $[\bigwedge^i (T^*)]$.

Of course $\bigwedge^0 \underline{1} = \bigwedge^1 \underline{1} = \underline{1}$ and $\bigwedge^j \underline{1} = 0$ for $j \geq 2$. This allows us to calculate

$$\begin{aligned} \textstyle\bigwedge^j (\underline{1} \oplus T^*) &= (\textstyle\bigwedge^0 \underline{1} \otimes \bigwedge^j T^*) \oplus (\bigwedge^1 \underline{1} \otimes \bigwedge^{j-1} T^*) \\ &= \textstyle\bigwedge^j T^* \oplus \bigwedge^{j-1} T^*. \end{aligned}$$

On the other hand, $\bigwedge^j (\underline{1} \oplus T^*) = \bigwedge^j ((n+1)L) = \binom{n+1}{j}(L^{\otimes j})$. So for every j we have

$$[\textstyle\bigwedge^j (T^*)] = \binom{n+1}{j} L^{\otimes j} - [\bigwedge^{j-1}(T^*)]$$

in K-theory.

The evident recursion now gives that

$$\begin{aligned} [T^*] &= (n+1)[L] - [\underline{1}] \\ [\textstyle\bigwedge^2 T^*] &= \binom{n+1}{2}[L^{\otimes 2}] - (n+1)[L] + [\underline{1}] \\ [\textstyle\bigwedge^3 T^*] &= \binom{n+1}{3}[L^{\otimes 3}] - \binom{n+1}{2}[L^{\otimes 2}] + (n+1)[L] - [\underline{1}], \end{aligned}$$

and so on. The general formula, obtained by an easy induction, is

$$[\bigwedge^j T^*] = \sum_{k=0}^{j} (-1)^{k+j} \tbinom{n+1}{k} [L^{\otimes k}].$$

This now lets us calculate

$$\begin{aligned} e^K(T) &= (-1)^n \left[\tbinom{n+1}{1}[L]^n - 2\tbinom{n+1}{2}[L]^{n-1} + 3\tbinom{n+1}{3}[L]^{n-2} - \dots \right] \\ &= (-1)^n (n+1) \left[[L]^n - \tbinom{n}{1}[L]^{n-1} + \tbinom{n}{2}[L]^{n-2} - \dots \right] \\ &= (n+1)(1-[L])^n. \end{aligned}$$

Recall that $(1-[L])^n = [*]$, and so we have determined that

$$e^K(T) = (n+1)[*].$$

Let us now confess that the result of this computation is not unexpected. Indeed, we saw previously that $e^H(TM) = \chi(M)[*]$ for any smooth manifold M, and in fact the same is true in any complex-oriented cohomology theory (by essentially the same proof). The "$n+1$" in our formula for $e^K(T)$ is just $\chi(\mathbb{C}P^n)$. But note that we computed this without writing down anything remotely resembling a cell structure! In fact, the only geometry in the calculation was in the fact $T \cong \underline{\mathrm{Hom}}(L, L^\perp)$; everything else was some simple linear algebra and then basic algebraic manipulation. It is useful to remember the overall theme of K-theory: do linear algebra fiberwise over a base space X, and see what this tells you about the topology of X. Our computation of $e^K(T_{\mathbb{C}P^n})$ gives an example of this.

The calculation in the above example is a little clunky. One way to streamline it is to introduce the formal power series

$$\lambda_t(E) = \sum_{i=0}^{\infty} t^i [\bigwedge^i(E)] = 1 + t[E] + t^2[\bigwedge^2(E)] + \cdots$$

which we regard as living in the ring $K(X)[[t]]$, where $E \to X$ was a vector bundle. Notice that this is actually a polynomial in t, since the exterior powers vanish beyond the rank of E; we consider it as a power series because in that context it has a multiplicative inverse, which we will shortly need.

If L is a line bundle then $\lambda_t(L) = 1 + t[L]$. Also, the formula $\bigwedge^k(E \oplus F) = \bigoplus_{i+j=k} \bigwedge^i(E) \otimes \bigwedge^j(F)$ yields the nice relation

$$\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F).$$

This is what ultimately simplifies our calculations. Finally, notice that the K-theoretic Euler characteristic can be written as $e^K(E) = \lambda_t(E^*)|_{t=-1}$.

Returning to the above calculation, the starting point for the algebra was the bundle isomorphism $\underline{1} \oplus T^* \cong (n+1)L$. Applying λ_t we obtain

$$\lambda_t(\underline{1})\lambda_t(T^*) = \lambda_t(\underline{1} \oplus T^*) = \lambda_t((n+1)L^*) = (\lambda_t(L))^{n+1}.$$

But $\lambda_t(\underline{1}) = 1 + t$ and $\lambda_t(L) = 1 + t[L]$, so we can write

$$\lambda_t(T^*) = \tfrac{(1+t[L])^{n+1}}{1+t}, \qquad \text{or} \qquad e^K(T) = \tfrac{(1+t[L])^{n+1}}{1+t}\Big|_{t=-1}$$

(and here is where we are using that our power series have multiplicative inverses). Our task is to expand the formula for $e^K(T)$ into powers of t, and then to set $t = -1$. The trick is to do this in a clever way.

We are going to ultimately want to write $e^K(T)$ in terms of powers of $(1-L)$, as they give our usual basis for $K^*(\mathbb{C}P^n)$. The trick is to do this *before* plugging in $t=-1$, rather than afterwards. Regard L as a formal variable and consider $f(L)=(1+tL)^{n+1}/(1+t)$, regarded as a formal power series in two variables (but where we are choosing not to write t in the inputs of f). Let us expand this in powers of $(L-1)$ via the usual Taylor series:

$$f(L)=f(1)+f'(1)(L-1)+\tfrac{f''(1)}{2}(L-1)^2+\cdots$$

It is simple to compute that the coefficient of $(L-1)^k$ is $\binom{n+1}{k}(1+t)^{n-k+1}t^k$, and so we obtain

$$\lambda_t(T^*)=\sum_{k=0}^{n+1}\tbinom{n+1}{k}(1+t)^{n-k}t^k(L-1)^k.$$

Notice that the substitution $t=-1$ will make the summands vanish for k smaller than n, and that the term $k=n+1$ vanishes because $(L-1)^{n+1}=0$ in $K(\mathbb{C}P^n)$. So we quickly find that

$$e^K(T)=\lambda_t(T^*)|_{t=-1}=\tbinom{n+1}{n}t^n(L-1)^n=(n+1)(1-L)^n=(n+1)[*].$$

Example 22.19 (Euler characteristic of a hypersurface)**.** Using the λ_t operators introduced above, we will attempt a harder computation of an Euler class. Let $j\colon Z\hookrightarrow\mathbb{C}P^n$ be a smooth hypersurface of degree d. Our goal is to compute the Euler characteristic $\chi(Z)$. If T_Z and N_Z denote the tangent and normal bundles, respectively, then $T_Z\oplus N_Z\cong j^*T_{\mathbb{C}P^n}$. We know from Corollary 22.12 that $N_Z\cong j^*\mathcal{O}(d)$, so we have

$$\underline{1}\oplus T_Z\oplus j^*\mathcal{O}(d)\cong\underline{1}\oplus j^*T_{\mathbb{C}P^n}\cong(n+1)j^*\mathcal{O}(1).$$

Taking duals and applying λ_t, we obtain

$$\lambda_t(\underline{1})\cdot\lambda_t(T_Z^*)\cdot\lambda_t(j^*L^d)=\lambda_t(j^*L)^{n+1}.$$

Let $X=j^*L$. Then we can write

$$\lambda_t(T^*)=\tfrac{(1+tX)^{n+1}}{(1+t)(1+tX^d)}.$$

We wish to compute the Euler class $e^K(T)=\lambda_t(T^*)|_{t=-1}$ and then write it in the form $(???)\cdot[*]$, in which case the mystery number in parentheses will be $\chi(Z)$. The trick is again to expand in powers of $(X-1)$, since the powers of $1-L$ are our standard generators for $K^0(\mathbb{C}P^n)$.

Consider the power series

$$f(X)=\tfrac{(1+tX)^{n+1}}{(1+t)(1+tX^d)}=f(t,1)+f'(1)(X-1)+\tfrac{f''(1)}{2}(X-1)^2+\cdots$$

Note that $(1-L)^{n+1}$ is zero in $K^0(\mathbb{C}P^n)$ and therefore $(1-X)^{n+1}$ vanishes in $K^0(Z)$. Even better, $(1-L)^n=[*]$ in $K^0(\mathbb{C}P^n)$ and since $j^*[*]=0$ by intersection theory it follows that $(1-X)^n=0$ in $K^0(X)$. So we don't care about any terms in the above series beyond $(X-1)^{n-1}$. Let us also note at this point that $(1-L)^{n-1}=[\mathbb{C}P^1]$ in $K^0(\mathbb{C}P^n)$, and therefore $(1-X)^{n-1}=j^*\big((1-L)^{n-1}\big)=j^*([\mathbb{C}P^1])=d[*]$; the last equality holds by intersection theory, since a generic $\mathbb{C}P^1$ will intersect Z in exactly d points.

Before tackling the general calculation let us do the first example, where $n = 2$. Here $f(X) = \frac{(1+tX)^3}{(1+t)(1+tX^d)}$ and we only need the first two terms of the series. Clearly $f(1) = 1 + t$ and an easy calculation gives

$$f'(X) = \tfrac{1}{1+t} \cdot \tfrac{(1+tX)^2}{(1+tX^d)^2} \Big[3t(1 + tX^d) - (1 + tX)tdX^{d-1}\Big]$$

so that

$$f'(1) = (3 - d)t.$$

Putting everything together,

$$f(X) = (1 + t) + (3 - d)t(X - 1)$$

and so

$$e^K(T_Z) = f(X)|_{t=-1} = 0 + (d - 3)(X - 1) = (3 - d)(1 - X) = (d - 3) \cdot d[*].$$

We conclude that $\chi(Z) = d(3 - d)$.

For larger n here is how things are going to work. First, we will calculate the derivatives $f^{(k)}(1)$ for $0 \leq k \leq n - 1$, and then substitute $t = -1$ into all of them. It will turn out (but is far from obvious) that the resulting expressions vanish for $k < n - 1$, so that

$$\begin{aligned} e^K(T_Z) &= f^{(n-1)}(1)|_{t=-1} \cdot (X - 1)^{n-1} \\ &= f^{(n-1)}(1)|_{t=-1} \cdot (-1)^{n-1} \cdot (1 - X)^{n-1} \\ &= f^{(n-1)}(1)|_{t=-1} \cdot (-1)^{n-1} \cdot d[*]. \end{aligned}$$

The conclusion will then be that

$$\chi(Z) = f^{(n-1)}(1)|_{t=-1} \cdot (-1)^{n-1} \cdot d.$$

Notice that everything comes down to computing the expressions $f^{(k)}(1)|_{t=-1}$, which is a purely algebraic problem. Unfortunately we cannot *first* plug in $t = -1$, since our formula for f has a $1 + t$ in the denominator; we have to first do the hard work of writing f as a polynomial in t before plugging in. At first glance this work looks very hairy! Already the formula for $f'(X)$ was quite complicated, and it only gets worse for the higher derivatives. The reader might wish to carry this out by brute force for $n = 3$, to get a feel for the difficulties.

We are going to sketch the completion of the calculations for the above example, but before diving into that we need to make a confession. It is possible to compute $\chi(Z)$ by doing a similar kind of calculation using singular cohomology instead of K-theory, and in that setting the algebra turns out to be much *easier*! There is a trade-off, which is that the computation cannot be done merely with Euler classes—one needs the complete theory of Chern classes, to be developed in the next section. See Example 23.7 for the computation of $\chi(Z)$ in that context. This situation is fairly typical of the relationship between K-theory and singular cohomology. For calculations that can be done in either theory, usually the singular cohomology version will involve simpler algebra, but more advanced geometric techniques; the K-theory version will involve more advanced algebra, but one needs less geometry. In some sense we saw this phenomenon already in the case of intersection multiplicities.

The completion of our calculation will proceed via the following steps:

(a) Suppose that $f(w) \in \mathbb{Q}[w]$ has degree k. Then there is an identity of formal power series

$$f(0) - tf(1) + t^2 f(2) - \cdots = \tfrac{u(t)}{(1+t)^{k+1}}$$

for a unique polynomial $u(t)$. Moreover, the degree of $u(t)$ is at most k and $u(-1)$ is $k!$ times the leading coefficient of f.

To justify the above claim, write $A_f = \sum_{k\geq 0}(-1)^k f(k)t^k$. Note the formula $(1+t)A_f = f(0) - tA_{\Delta f}$ where Δ is the finite difference operator from Section 21. Check the claim is true when $\deg f = 0$, and then do an induction on the degree. (Extra credit: Find a formula for $u(t)$ in terms of the numbers $\Delta^k f(0)$).

(b) By collecting terms notice that

$$\begin{aligned} \tfrac{1}{1+tX^d} &= 1 - tX^d + t^2X^{2d} - \cdots \\ &= 1 - t((X-1)+1)^d + t^2((X-1)+1)^{2d} - \cdots \\ &= \tfrac{1}{1+t} + (X-1)\Gamma_1 + (X-2)\Gamma_2 + \cdots \end{aligned}$$

where

$$\Gamma_k = -t\tbinom{d}{k} + t^2\tbinom{2d}{k} - t^3\tbinom{3d}{k} + \cdots$$

It follows from (a) that $\Gamma_k = \frac{u_k(t)}{(1+t)^{k+2}}$ where $u_k(t)$ is a polynomial such that $u(-1) = d^k$.

(c) Set $f_r = \frac{(1+tX)^r}{(1+t)(1+tX^d)}$. Notice the recursion relation

$$f_{r+1} = (1+tX)f_r = (1+t((X-1)+1))f_r = (1+t)f_r + (X-1)tf_r.$$

So if we know the expansion of f_r in terms of powers of $X-1$, it is easy to get the expansion of f_{r+1}. In the following table, row r shows the terms in the expansion for f_r (starting with f_0):

r	$(X-1)^0$	$(X-1)^1$	$(X-1)^2$	$(X-1)^3$	$\cdots$
0	$\frac{1}{(1+t)^2}$	$\frac{u_1}{(1+t)^3}$	$\frac{u_2}{(1+t)^3}$	$\frac{u_3}{(1+t)^3}$	$\cdots$
1	$\frac{1}{1+t}$	$\frac{u_1+t}{(1+t)^2}$	$\frac{u_2+tu_1}{(1+t)^3}$	$\frac{u_3+tu_2}{(1+t)^3}$	$\cdots$
2	1	$\frac{u_1+2t}{(1+t)}$	$\frac{u_2+2tu_1+t^2}{(1+t)^2}$	$\frac{u_3+2tu_2+t^2u_1}{(1+t)^3}$	$\cdots$
3	$1+t$	u_1+3t	$\frac{u_2+3tu_1+3t^2}{1+t}$	$\frac{u_3+3tu_2+2t^2u_1+t^3}{(1+t)^2}$	$\cdots$
4	$(1+t)^2$	$(1+t)(u_1+3t)+t(1+t)$	$u_2+4tu_1+6t^2$	$\cdots$	$\cdots$

(c) Ignoring the terrible-looking formulas, one important thing is evident from the table: in the column for $(X-1)^r$, after row $r+2$ we are getting polynomial multiples of $1+t$. So these entries will all vanish when we specialize to $t=-1$, and we see that in $f_r|_{t=-1}$ the first nonzero coefficient appears in the $(X-1)^{r-2}$ term. Our job is to calculate this coefficient. In order to do so, notice that it suffices to just look at all the numerators in the table; the denominators can basically be ignored. To this end, let $g_r(W)$ be the generating function for the numerators in row r of the table, specialized to $t=-1$. For example, $g_0(W) = 1 + dW + d^2W^3 + \cdots$. The recursion relation for these numerators is

$$g_{r+1} = g_r - Wg_r = (1-W)g_r.$$

So of course we will have

$$g_r = (1-W)^r g_0 = (1-W)^r \cdot \tfrac{1}{1-dW} = (1-W)^r \cdot (1 + dW + d^2W^2 + \cdots).$$

The number we are looking for is the coefficient of W^{r-2} in this power series, and it is a simple matter to compute it. The desired coeffient is

$$d^{r-2} - d^{r-3}\tbinom{r}{1} + d^{r-4}\tbinom{r}{2} - \cdots + (-1)^{r-2}\tbinom{r}{r-2}.$$

(d) Now putting everything together, we have proven that

$$\begin{aligned}\chi(Z_d \hookrightarrow \mathbb{C}P^n) &= (-1)^{n+1} d\Big[d^{n-1} - d^{n-2}\tbinom{n+1}{1} + \cdots + (-1)^{n-1}\tbinom{n+1}{n-1}\Big] \\ &= (-1)^{n+1} d \cdot \Big[\tfrac{(d-1)^{n+1} - (-1)^{n+1}((n+1)d-1)}{d^2}\Big] \\ &= \tfrac{(1-d)^{n+1} - ((n+1)d-1)}{d}.\end{aligned}$$

None of the above formulas are particularly pleasant to look at, and they are difficult to remember. I like to encode the formula in a different way. Let ℓ denote the formal "lowering operator" that sends d^s to d^{s-1}, for each s. Then we may write

$$\chi(Z_d \hookrightarrow \mathbb{C}P^n) = d \cdot (\ell - I)^{n+1}(d^{n-1})$$

where I is the identity operator. For example,

$$\chi(Z_d \hookrightarrow \mathbb{C}P^3) = d \cdot (\ell^4 - 4\ell^3 + 6\ell^2 - 4\ell + I)(d^2) = d \cdot (6 - 4d^2 + d^3).$$

23. Chern classes

Fix a certain collection of vector bundles. A *characteristic class* for this collection assigns to each vector bundle $E \to X$ a cohomology class $b(E)$ belonging to some cohomology theory; the assignment is required to be natural. We have seen essentially two examples so far: for the collection of oriented, rank k vector bundles, we have the Euler classes e^H and e^K.

The Chern classes are characteristic classes for *complex* vector bundles, that generalize the Euler class in a certain way. Like the Euler class, they have close ties to geometry. Also like the Euler class, there are versions of Chern classes in both singular cohomology and K-theory—indeed, there are versions in any complex-oriented cohomology theory.

In this section we begin with a purely geometric look at the Chern classes, where we again forego all attempts at rigor. Afterwards we will pursue a more rigorous approach, which can even be done axiomatically.

23.1. **Geometric Chern classes in homology.** Let B be a complex manifold of dimension n, and let $E \to B$ be a complex vector bundle of rank k. If s is a generic section of E, then the locus where s vanishes gives a cycle in B that carries the Euler class $e_H(E) \in H_{2(n-k)}(B)$. This homology class will now be renamed as $C_k(E)$ and called the kth homology Chern class of E.

Now let s_1 and s_2 be *two* generic sections, chosen so that $s_1(x)$ and $s_2(x)$ are linearly independent on as large a subset of B as possible. We can now look at the degeneracy locus

$$D(s_1, s_2) = \{b \in B \mid s_1(b) \text{ and } s_2(b) \text{ are linearly dependent}\}.$$

Again, for generically chosen s_1 and s_2 this gives a cycle on B whose associated homology class is independent of any choices. The homology class lies in dimension $2(n-k-1)$, and we call it the $(k-1)$st homology Chern class $C_{k-1}(E)$.

At this point it is clear how to continue. For each j in the range $1 \le j \le k$, let $s_1, \dots, s_j$ be sections generically chosen to be as maximally linearly independent as possible. Consider the degeneracy locus

$$D(s_1, \dots, s_j) = \{b \in B \mid s_1(b), \dots, s_j(b) \text{ are linearly dependent}\},$$

which determines a homology class $C_{k-j+1} \in H_{2(n-k-j+1)}(B)$.

These homology classes can be thought of as the primary obstructions to splitting off a trivial bundle. More precisely, if E contains a trivial bundle of rank s then $0 = C_k(E) = C_{k-1}(E) = \cdots = C_{k-s+1}(E)$. This is clear, as by working inside the trivial subbundle we can choose our "generic sections" so that they are linearly independent everywhere.

23.2. **Chern classes in singular cohomology.** We adopt an axiomatic approach. For any complex vector bundle $E \to X$ the Chern classes are cohomology classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ for $0 \le i < \infty$ satisfying the following properties:

(1) $c_0(E) = 1$
(2) $c_i(E) = 0$ if $i > \text{rank } E$
(3) $c_i(f^*E) = f^*c_i(E)$ (naturality under pullback)
(4) The Whitney Formula: $c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E)c_{k-i}(F)$, for any k.
(5) $c_1(L^* \to \mathbb{C}P^n) = e^H(L^*) = [\mathbb{C}P^{n-1}]$, where $L \to \mathbb{C}P^n$ is the tautological line bundle.

Remark 23.3. The Whitney Formula can be written in a more convenient way using the *total Chern class*, namely

$$c(E) = c_0(E) + c_1(E) + c_2(E) + \cdots \in H^*(X)$$

Then the Whitney Formula becomes $c(E \oplus F) = c(E) \cdot c(F)$.

Note that if $E \to X$ is a trivial bundle then $c_i(E) = 0$ for $i > 0$. Indeed, E is the pullback of a bundle on a point: $E \cong \pi^*(\mathbb{C}^n)$ where $\pi\colon X \to *$ and $n = \text{rank}(E)$. One has $c_i(\mathbb{C}^n \to *) = 0$ for $i > 0$ because a point has no cohomology in positive degrees; the fact that $c_i(E) = 0$ then follows from naturality.

Before showing the existence of the Chern classes, let us show that they are uniquely characterized by the above properties:

Proposition 23.4. *There is at most one collection of characteristic classes satisfying properties (1)–(5) above.*

Proof. Let η be the tautological k-plane bundle on $\text{Gr}_k(\mathbb{C}^\infty)$. Consider the diagram

$$\begin{array}{ccc} \pi_1^*(L) \oplus \pi_2^*(L) \oplus \cdots \oplus \pi_k^*(L) & \dashrightarrow & \eta \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty & \xrightarrow{S} & \text{Gr}_k(\mathbb{C}^\infty \times \cdots \times \mathbb{C}^\infty) = \text{Gr}_k(\mathbb{C}^\infty) \end{array}$$

with the obvious maps. Note that there are k copies of $\mathbb{C}P^\infty$ and $\mathbb{C}^\infty$ in the bottom row. Also, $\pi_i : \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the ith projection map.

This diagram is a pullback diagram. Hence,

$$S^*(\eta) = (\pi_1^* L) \oplus \cdots \oplus (\pi_k^* L).$$

Applying cohomology to the map on the bottom row gives

$$H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \xleftarrow{S^*} H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) .$$

By the Künneth Theorem,

$$H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \cong H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x_1, \dots, x_k]$$

where $x_i = \pi_i^*(x)$ with $x \in H^2(\mathbb{C}P^\infty)$ being the canonical generator. There is an evident action on $\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$ by the symmetric group Σ_k. This action descends in cohomology to give the statement

$$\begin{aligned} H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty)^{\Sigma_k} &\cong \Big[H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty)\Big]^{\Sigma_k} \\ &= \mathbb{Z}[x_1, \dots, x_k]^{\Sigma_k} \\ &= \mathbb{Z}[\sigma_1, \dots, \sigma_k] \end{aligned}$$

where σ_i is the ith elementary symmetric function in the x_i's.

Recall that $[X, \mathrm{Gr}_k(\mathbb{C}^\infty)] \simeq \mathrm{Vect}_k(X)$. Under this bijection, S corresponds to the bundle $E = \oplus_i \pi_i^*(L)$. If $\alpha \in \Sigma_k$ then the map $S \circ \alpha$ corresponds to the direct sum of $\pi_i^*(L)$'s but where the sum is taken in a different order. Since this is isomorphic to the original bundle E, is must be that S and $S \circ \alpha$ are homotopic; in particular, they induced the same map on cohomology. Since this holds for all α, it follows that S^* lands inside the Σ_k invariants. That is, S^* can be regarded as a map

$$H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \xrightarrow{S^*} \Big[H^*(\mathbb{C}P^\infty)^{\otimes k}\Big]^{\Sigma_k} = \mathbb{Z}[\sigma_1, \dots, \sigma_k].$$

It is a theorem that the above map S^* is an isomorphism. We will not take the time to prove this, but the idea is simple enough. The Schubert cell decomposition of $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has all cells in even dimensions, and hence the coboundary maps are all zero; this computes $H^*(\mathrm{Gr}_k(\mathbb{C}^\infty))$ additively, and one readily checks that the groups have the same ranks as in $\mathbb{Z}[\sigma_1, \dots, \sigma_k]$. ????

Using the Whitney Formula (iteratively), we can see that

$$\begin{aligned} c_i(\pi_1^* L \oplus \cdots \oplus \pi_k^* L) &= \sum_\beta c_1(\pi_{\beta(1)}^*(L)) \cdot c_1(\pi_{\beta(2)}^*(L)) \dots c_1(\pi_{\beta(i)}^*(L)) \\ &= \sum_\beta \pi_{\beta(1)}^*(x) \cdot \pi_{\beta(2)}^*(x) \dots \pi_{\beta(i)}^*(x) \end{aligned}$$

where the sum ranges over strictly-increasing maps $\beta\colon \{1, \dots, i\} \to \{1, \dots, k\}$, and in the second sum $x = c_1(L) \in H^2(\mathbb{C}P^\infty)$. But note that if we write $x_j = \pi_j^*(x)$ then the second sum is simply the elementary symmetric function σ_i in the x_j's.

It follows from the above that $c_i(\eta)$ is the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i under S^*.

Finally, suppose that $E \to X$ is any complex vector bundle, say of rank k. Then there is a map $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ and an isomorphism $f^*\eta \cong E$. It follows that $c_i(E) = f^*(c_i(\eta))$.

To complete the proof, assume that c_* and c'_* are two sets of characteristic classes satisfying properties (1)–(5). Then $c_i(\eta)$ and $c'_i(\eta)$ must agree, for they each must be the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i. It then follows from naturality that $c_i(E) = c_i'(E)$ for all bundles E. $\square$

By examining the above proof, one finds that we can *define* the Chern classes in the following way. First, when $\eta \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is the tautological bundle then define $c_i(\eta)$ to be the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i under S^*. Second, for an arbitrary bundle $E \to X$ let $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ be a classifying map and define $c_i(E) = f^*(c_i(\eta))$.

Remark 23.5. For a bundle $E \to X$ one can also define K-theoretic Chern classes $c_i^K(E) \in K^0(X)$ (or really in $K^{2i}(X)$, but this is the same by periodicity). We will not pursue this at the moment, but see ????

Example 23.6. Consider the tangent bundle $T = T\mathbb{C}P^n \to \mathbb{C}P^n$. We saw in Example 22.16 that $1 \oplus T \cong (n+1)L^*$. Then by the Whitney Formula,

$$c(T) = c(1) \cdot c(T) = c(1 \oplus T) = c(L^*)^{n+1} = (1 + [\mathbb{C}P^{n-1}])^{n+1}.$$

Therefore,

$$c_i(T) = \tbinom{n+1}{i}[\mathbb{C}P^{n-i}] = \tbinom{n+1}{i}x^i$$

where $x \in H^2(\mathbb{C}P^n)$ is the canonical generator $[\mathbb{C}P^{n-1}]$. Note that the Euler class is $e(T) = c_n(T) = (n+1)x^n = (n+1)[*]$, and so this again calculates that $\chi(\mathbb{C}P^n) = n+1$.

Example 23.7. Consider a hypersurface $j\colon Z \hookrightarrow \mathbb{C}P^n$ of degree d. Recall from Corollary 22.12 that the normal bundle of this inclusion is $j^*\mathcal{O}(d)$, and we know $T_Z \oplus N_Z \cong j^*T_{\mathbb{C}P^n}$. Then applying total Chern classes we get

$$c(T_Z) \cdot c(N_Z) = j^*c(T_{\mathbb{C}P^n}).$$

But above we calculated that $c(T_{\mathbb{C}P^n}) = (1+x)^{n+1}$, and $c(N_Z) = c(j^*\mathcal{O}(d)) = j^*(c(\mathcal{O}_d)) = 1 + d(j^*x)$. Let $z = j^*x$, so that we have

$$c(T_Z) = \tfrac{(1+z)^{n+1}}{1+dz} = (1 + (n+1)z + \tbinom{n+1}{2}z^2 + \cdots) \cdot (1 - dz + d^2z^2 - \cdots).$$

We can compute $\chi(Z)$ by finding the top Chern class (the Euler class), which in this case is $c_{n-1}(T_Z)$. A direct computation shows that

$$c_{n-1}(T_Z) = z^{n-1} \cdot \big(\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots\big).$$

Finally, we need to remember that $x^{n-1} = [\mathbb{C}P^1]$ in $H^*(\mathbb{C}P^n)$ and therefore $z^{n-1} = j^*(x^{n-1}) = d[*]$, since a generic line intersects Z in d distinct points. So we have

$$c_{n-1}(T_Z) = [*] \cdot d \cdot (\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots\big),$$

thereby yielding

$$\chi(Z) = d \cdot \Big(\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots\Big).$$

For example, a degree d hypersurface in $\mathbb{C}P^2$ has $\chi(Z) = d \cdot (3-d)$ and a degree d hypersurface in $\mathbb{C}P^3$ has $\chi(Z) = d \cdot (6 - 4d + d^2)$.

23.8. **Stiefel-Whitney classes.** One can repeat almost all of our above work in the setting of real vector bundles, but using $\mathbb{Z}/2$ coefficients everywhere. The analogs of the Chern classes in this setting are called **Stiefel-Whitney classes**. If $E \to X$ is a real vector bundle then the Stiefel-Whitney classes are cohomology classes $w_i(E) \in H^i(X;\mathbb{Z}/2)$, $0 \leq i < \infty$, satisfying the evident analogs of the axioms in Section 23.2. Geometrically, these are Poincaré Duals of certain cycles determined by degeneracy loci, just as in the complex case. In terms of our development, most things go through verbatim but the one exception is the computation $H^*(\mathrm{Gr}_k(\mathbb{R}^\infty);\mathbb{Z}/2) \cong \mathbb{Z}/2[\sigma_1,\dots,\sigma_k]$. In the complex case this was fairly easy, because the standard cell structure on $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has cells only in even dimensions. This is of course not true for $\mathrm{Gr}_k(\mathbb{R}^\infty)$, and so one must work a bit harder here. We will not give details; see [MS].

Just as for the Chern classes, we will write $w(E)$ for the total Stiefel-Whitney class $1 + w_1(E) + w_2(E) + \cdots$.

Example 23.9. Here is an example where we can use Stiefel-Whitney classes to solve a problem that appeared earlier in these notes. Let $\gamma \to \mathbb{R}P^n$ denote the tautological line bundle, and recall that once upon a time we needed to know whether $\gamma \oplus \gamma$ was stably trivial. This came up (for $n = 2$) in Section 13.10 during the course of trying to compute $KO(\mathbb{R}P^2)$.

If $(\gamma \oplus \gamma) \oplus \underline{N} \cong \underline{N+2}$ then applying total Stiefel-Whitney classes gives

$$1 = w(\underline{N+2}) = w(\gamma \oplus \gamma \oplus \underline{N}) = w(\gamma) \cdot w(\gamma) \cdot w(\underline{N}) = w(\gamma)^2.$$

But γ is a line bundle, so $w_i(\gamma) = 0$ for $i > 1$ and $w_1(\gamma)$ is the mod 2 Euler class, which we have previously computed to be the generator x on $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$. So $w(\gamma) = 1 + x$ and therefore $w(\gamma)^2 = 1 + x^2$. As long as $n \geq 2$ this is not equal to 1, and hence $\gamma \oplus \gamma$ cannot be stably trivial.

Example 23.10. Let T be the tangent bundle of $\mathbb{R}P^n$. Just as in Example 22.16 there is an isomorphism $\underline{1} \oplus T \cong (n+1)L^*$, where $L \to \mathbb{R}P^n$ is the tautological line bundle. But since we are now in the case of real bundles, $L \cong L^*$ by Corollary 8.23; so we will usually write $\underline{1} \oplus T \cong (n+1)L$. One of course finds that

$$w(T) = w(1 \oplus T) = w((n+1)L) = w(L)^{n+1} = (1+x)^{n+1},$$

similarly to the complex case.

24. Comparing K-theory and singular cohomology

We have seen that singular cohomology and K-theory both encode geometry in similar ways: they have Thom classes, Euler classes, fundamental classes for submanifolds, etc. They can both be used to compute intersection multiplicities. One might hope for a natural transformation from one to the other, that allows one to directly compare what is happening in each theory. Our goal in this section is to construct such a natural transformation, with some caveats which we will discover along the way.

Let us imagine that we have a natural transformation $\phi\colon K^*(-) \to H^*(-)$, and that this is a ring homomorphism. Note first that ϕ cannot preserve the gradings, for $\beta \in K^{-2}(pt)$ is a unit whereas there is no unit in $H^{-2}(pt)$. We can fix this by formally adjoining a unit to H^*: let $H^*[t,t^{-1}]$ be the cohomology theory $X \mapsto H^*(X)[t,t^{-1}]$, where t is given degree -2. Then we can ask for a natural ring homomorphism $\phi\colon K^*(-) \to H^*(-)[t, t^{-1}]$. Restricting to $* = 0$ would give a natural ring homomorphism

$$\phi\colon K^0(-) \to H^{ev}(-) = \oplus_i H^{2i}(-).$$

We will investigate what this map can look like.

If $L \to X$ is a complex line bundle then we have the element $e^K(L) \in K^0(X)$, in some sense representing the intersection of the zero-section with itself. One's first guess would be that ϕ should send $e^K(L)$ to $e^H(L)$, as the latter represents the same 'geometry' inside of H^*. However, this hypothesis is not compatible with ϕ being a ring homomorphism. Recall that

$$e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2),$$

whereas

$$e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2).$$

These formulas are incompatible.

So it cannot be that ϕ sends $e^K(L)$ to $e^H(L)$. However, it is *guaranteed* that $e^K(L)$ must be sent to *some* algebraic expression involving $e^H(L)$. Indeed, this is obviously so for the tautological bundle $L \to \mathbb{C}P^\infty$, since $e^H(L)$ is a generator of $H^2(\mathbb{C}P^\infty)$ and everything else in $H^*(\mathbb{C}P^\infty)$ is a polynomial in this generator; the case for general line bundles then follows from naturality.

So we know that we will have ϕ send $e^K(L) \mapsto f(e^H(L))$ where $f(x) \in \mathbb{Z}[[x]]$. Note that when X is compact then sufficiently large powers of $e^H(L)$ will be zero, so in practice $f(e^H(L))$ is really just a *polynomial* in $e^H(L)$. Using a power series allows us to treat all spaces X at once, without assuming some uniform bound on their dimensions.

Let $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$ be the expansion for f. Note that if $L \to X$ is a trivial bundle then both $e^K(L)$ and $e^H(L)$ are zero, and from this it follows that $\alpha_0 = 0$. Next note that if ϕ is a ring homomorphism then we must have

$$\begin{aligned} f(e^H(L_1) + e^H(L_2)) &= f(e^H(L_1 \otimes L_2)) \\ &= \phi(e^K(L_1 \otimes L_2)) \\ &= \phi(e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2)) \\ &= f(e^H(L_1)) + f(e^H(L_2)) - f(e^H(L_1))f(e^H(L_2)). \end{aligned}$$

This suggests that we're looking for $f(x) \in \mathbb{Z}[[x]]$ such that

$$f(a+b) = f(a) + f(b) - f(a)f(b). \tag{24.1}$$

We can take two approaches to determine the coefficients of such an f.

Approach 1. Substitute $f(x) = \sum_i \alpha_i x^i$ into (24.1) to get

$$\text{LHS} = \alpha_0 + \alpha_1(a+b) + \alpha_2(a+b)^2 + \dots$$

and

$$\begin{aligned} RHS = [\alpha_0 + \alpha_1 a + \alpha_2 a^2 + \dots] &+ [\alpha_0 + \alpha_1 b + \alpha_2 b^2 + \dots] \\ &- [\alpha_0 + \alpha_1 a + \alpha_2 a^2 + \dots][\alpha_0 + \alpha_1 b + \alpha_2 b^2 + \dots]. \end{aligned}$$

By expanding and equating coefficients, we can determine the coefficients α_i. The first equation is $\alpha_0^2 = \alpha_0$, and since $\alpha_0 \neq 0$ this means $\alpha_0 = 1$. It turns out there is no equation determining α_1, but looking at the coefficient of ab^{n-1} yields

$n\alpha_n = -\alpha_1\alpha_{n-1}$, or $\alpha_n = -\frac{\alpha_1\alpha_{n-1}}{n}$. So by induction $\alpha_n = (-1)^{n-1}\frac{\alpha_1^n}{n!}$. Note, in particular, this last equation: it shows that f cannot have integral coefficients, as we were orginally guessing! So we can only make things work if the target of ϕ is $H^{ev}(-;\mathbb{Q})$.

We have been led to the conclusion $f(x) = 1 - e^{-\alpha_1 x}$, and the reader may readily check that this does indeed yield a power series $f(x)$ satisfying (24.1).

Approach 2. In case you don't like the "equating coefficients" approach, one can also use some basic tools from differential equations to determine f. Recall that we want $f(a+b) = f(a)+f(b)-f(a)f(b)$. Define functions g and h by $g(a,b) = f(a+b)$ and $h(a,b) = f(a) + f(b) - f(a)f(b)$. The partial derivatives are readily computed to be

$$\frac{\partial g}{\partial a}(a,b) = f'(a+b) \quad \text{and} \quad \frac{\partial h}{\partial a}(a,b) = f'(a) - f'(a)f(b).$$

If $g(a,b)$ and $h(a,b)$ are the same function then the above partial derivatives are the same, so that $f'(a+b) = f'(a) - f'(a)f(b)$. Evaluating at $a = 0$ gives the ODE

$$f'(b) = f'(0)[1 - f(b)].$$

Setting $y = f(x)$, this becomes the separable ODE

$$\frac{dy}{dx} = f'(0)(1-y), \qquad \text{or} \qquad \frac{dy}{1-y} = f'(0)\,dx.$$

Integrating both sides yields

$$-\ln|1-y| = f'(0)x + C, \qquad \text{or} \qquad y = 1 - De^{-f'(0)x}$$

where C and D are constants. Since $f'(0) = \alpha_1$, we will write this solution as $f(x) = y = 1 - De^{-\alpha_1 x}$. We did lose some information in the differentiation process, so let's make sure this works by plugging this formula back into (24.1). We get

$$1 - De^{-\alpha_1(a+b)} = [1 - De^{-\alpha_1 a}] + [1 - De^{-\alpha_1 b}] - [1 - De^{-\alpha_1 a}][1 - De^{-\alpha_1 b}],$$

which reduces to

$$De^{-\alpha_1(a+b)} = D^2 e^{-\alpha_1(a+b)}.$$

This implies $D = D^2$, so $D = 0$ or $D = 1$. The case $D = 0$ is uninteresting to us (it corresponds to $f(x) = 1$, and we have already noted that the constant term must be zero for our application). So $D = 1$ and $f(x) = 1 - e^{-\alpha_1 x}$.

We now comment on the fact that α_1 seems to be able to take on any value whatsoever. Note that the presence of the grading on $H^*(X)$ immediately gives rise to a collection of endomorphisms on this theory. Indeed, for any $n \in \mathbb{Z}$ write $\psi_n \colon H^*(X) \to H^*(X)$ for the function that multiplies each $H^i(X)$ by n^i. This is clearly a ring homomorphism, and if we are using rational coefficients then it is even an isomorphism (provided $n \neq 0$). Note that with rational coefficients we actually have maps ψ_q for any $q \in \mathbb{Q}$.

So if we have a natural transformation $\phi\colon K^*(-) \to H^*(-;\mathbb{Q})[t,t^{-1}]$ we can compose it with the natural automorphisms ψ_q to makes lots of other natural transformations. We see that such a ϕ is far from unique. If we had a ϕ whose associated power series f was $f(x) = 1 - e^{-\alpha_1 x}$, then composing with ϕ_q gives one with associated power series $1 - e^{-q\alpha_1 x}$. This is why α_1 could not be explicitly determined.

We can turn these observations around and use them to our advantage. Since we can always compose with a ψ_q, we might as well do so in a way that simplifies things as much as possible. In particular, if we have a ϕ with associated power series $f(x) = 1 - e^{-\alpha_1 x}$ then we can compose with $\psi_{\alpha_1^{-1}}$ to get one with power series $1 - e^{-x}$. We might as well do this, to simplify matters.

Let us summarize what has happened so far. We knew that ϕ, if it exists, must send $e^K(L)$ to some power series in $e^H(L)$, for any line bundle $L \to X$. The different equations for $e(L_1 \otimes L_2)$ in K-theory versus singular cohomology then forced what this power series must be: $\phi(e^K(L)) = 1 - e^{-\alpha_1 x}|_{x=e^H(L)}$, for some $\alpha_1 \in \mathbb{Q}$. We then saw that we might as well assume $\alpha_1 = 1$, since by composing with a certain "trivial" automorphism one can arrange for this.

So now we are looking at an imagined natural transformation ϕ that sends $e^K(L)$ to $1 - e^{-x}|_{x=e^H(L)}$ for any line bundle $L \to X$. Recall that $e^K(L) = 1 - L^*$, and so $\phi(L^*) = e^{-x}|_{x=e^H(L)}$. But $e^H(L^*) = -e^H(L)$ (use that $L \otimes L^* \cong \underline{1}$, and so $e^H(L) + e^H(L^*) = e^H(\underline{1}) = 0$). So we have $\phi(L^*) = e^x|_{x=e^H(L^*)} = e^{c_1(L^*)}$. Since this must hold for any line bundle L, we might as well just write it as

$$\phi(L) = e^{c_1(L)}. \tag{24.2}$$

We next claim that ϕ is completely determined by formula (24.2). Recall the inclusion $j\colon (\mathbb{C}P^\infty)^{\times k} \hookrightarrow \mathrm{Gr}_k(\mathbb{C}^\infty)$, and consider the diagram

$$\begin{array}{ccc} K^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) & \xleftarrow{\ j^*\ } & K^0(\mathrm{Gr}_k(\mathbb{C}^\infty)) \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi} \\ H^{ev}(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty; \mathbb{Q}) & \xleftarrow{\ j^*\ } & H^{ev}(\mathrm{Gr}_k(\mathbb{C}^\infty); \mathbb{Q}). \end{array}$$

We know that $j^*\eta = \pi_1^*(L) \oplus \cdots \oplus \pi_k^*(L)$, and therefore we see that

$$j^*(\phi(\eta)) = \phi(j^*\eta) = \sum_{i=1}^k \phi(\pi_i^*(L)) = \sum_{i=1}^k \pi_i^*(\phi(L)) = \sum_{i=1}^k e^{\pi_i^*(c_1(L))}.$$

Clearly this expression is invariant under the action of Σ_k, and we have said previously that j^* maps its domain isomorphically onto the subring of Σ_k-invariants. Thus, $\phi(\eta)$ is determined by this formula.

Let $x_i = \pi_i^*(c_1(L))$. The power sum $x_1^r + \cdots + x_k^r$ can be written uniquely as a polynomial $S_r(\sigma_1, \ldots, \sigma_k)$ in the elementary symmetric functions of the x_i's. Here S_r is called the **rth Newton polynomial**; see Appendix B for a review of these. The first few Newton polynomials are

$$S_1 = \sigma_1, \quad S_2 = \sigma_1^2 - 2\sigma_2, \quad S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

If $E \to X$ is a vector bundle then define $s_r(E) = S_r(c_1(E), \ldots, c_k(E)) \in H^{2r}(X)$, where $k = \operatorname{rank} E$. This is a characteristic class for bundles, but it doesn't seem to have a common name. We have seen that

$$\phi(\eta) = \sum_{k=0}^\infty \frac{1}{k!} s_k(\eta).$$

But if $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is a classifying map for E then the commutative diagram

$$\begin{array}{ccc} K^0(X) & \xleftarrow{\;f^*\;} & K^0(\mathrm{Gr}_k(\mathbb{C}^\infty)) \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ H^{ex}(X;\mathbb{Q}) & \xleftarrow{\;f^*\;} & H^{ev}(\mathrm{Gr}_k(\mathbb{C}^\infty);\mathbb{Q}) \end{array}$$

gives

$$\phi(E) = \phi(f^*(\eta)) = f^*(\phi(\eta)) = f^*\Big(\sum_{k=0}^{\infty} s_k(\eta)\Big) = \sum_{k=0}^{\infty} s_k(f^*\eta) = \sum_{k=0}^{\infty} s_k(E).$$

We have, at this point, reasoned as follows. IF there is a natural transformation of rings $\phi\colon K^0(-) \to H^{ev}(-;\mathbb{Q})[t,t^{-1}]$ THEN there is one that is given by the above formula. One can turn this around, by starting with the above formula and proving that it is a natural transformation of rings. This is not hard, and we will leave it to the reader. This natural transformation is called the **Chern character**, and is usually denoted $\mathrm{ch}\colon K^0(-) \to H^{ev}(-,\mathbb{Q})$. The defining formula is

$$\mathrm{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} s_k(E).$$

Since ch is a natural transformation, it of course maps $\widetilde{K}^0(X)$ into $\tilde{H}^{ev}(X;\mathbb{Q})$. Replacing X with ΣX and shifting indices, we get

$$\mathrm{ch}\colon K^1(X) \to H^{odd}(X;\mathbb{Q}).$$

By periodicity we might as well regard the Chern character as giving maps

$$\mathrm{ch}\colon K^n(X) \to \oplus_p H^{n+2p}(X;\mathbb{Q}).$$

Perhaps more reasonably, we can regard ch as a map of graded rings $K^*(X) \to H^*(X)[t,t^{-1}]$ where t has degree 2.

Theorem 24.3. *The induced maps $K^n(X) \otimes \mathbb{Q} \to \oplus_p H^{n+2p}(X;\mathbb{Q})$ are isomorphisms, for all CW-complexes X.*

Proof. One checks this for spheres by brute force calculation. Then use the cellular filtration, long exact sequences for a pair, and the Five Lemma to deduce the result for finite CW-complexes. Pass to arbitrary CW-complexes by taking a direct limit. □

Our definition of $\mathrm{ch}\colon K^*(X) \to H^*(X;\mathbb{Q})$ extends in a unique way to a natural transformation defined on pairs; that is, it uniquely determines maps $\mathrm{ch}\colon K^*(X,A) \to H^*(X,A;\mathbb{Q})$. To see this, recall the space $X \amalg_A X$ where X_1 and X_2 denote the two copies of X. We saw in Section 19.19 that we have a natural diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^*(X \amalg_A X, X_2) & \longrightarrow & K^*(X \amalg_A X) & \longrightarrow & K(X_2) & \longrightarrow & 0 \\ & & \cong\downarrow & & & & & & \\ & & K^*(X_1,A) & & & & & & \end{array}$$

where the top row is split short-exact. There is a similar diagram involving $H^*(-;\mathbb{Q})$, or any cohomology theory for that matter. The ch maps defined on pairs

$(Y,\emptyset)$ then uniquely determines a map $\mathrm{ch}\colon K^*(X\amalg_A X, X_2)\to H^*(X\amalg_A X, X_2;\mathbb{Q})$, and therefore a map $\mathrm{ch}\colon K^*(X,A)\to H^*(X,A;\mathbb{Q})$.

This raises the following interesting question. Suppose $E_\bullet$ is a complex of vector bundles on X that is exact on A. Then $\mathrm{ch}([E_\bullet])$ is a well-defined element of $H^*(X,A;\mathbb{Q})$. How does one describe what this element is? We will return to this question in the future. ?????

The existence of the Chern character, as a multiplicative natural transformation between cohomology theories, immediately has an interesting and unexpected consequence:

Proposition 24.4. *For any $n\geq 1$, the image of $\mathrm{ch}\colon \widetilde{K}^0(S^{2n})\to \widetilde{H}^{ev}(S^{2n};\mathbb{Q})$ is precisely $\widetilde{H}^{ev}(S^{2n};\mathbb{Z})$. Consequently, if X is any $(2n-1)$-connected space and $E\to X$ is a complex vector bundle then $c_n(E)\in H^{2n}(X;\mathbb{Z})$ is a multiple of $(n-1)!$.*

Proof. For the first statement recall that $\widetilde{K}^0(S^{2n})$ is generated by $\beta^{\times n}$ where $\beta = 1-L\in \widetilde{K}^0(S^2)$. We can compute that $\mathrm{ch}(\beta)=1-\mathrm{ch}(L)=1-(1+c_1(L))=-c_1(L)$, but this is a generator of $H^2(S^2;\mathbb{Z})$. Multiplicativity of the Chern character gives

$$\mathrm{ch}(\beta^{\times n}) = \big(\mathrm{ch}(\beta)\big)^{\times n} = c_1(L)^{\times n},$$

but the nth external product of a generator for $H^2(S^2;\mathbb{Z})$ gives a generator of $H^{2n}(S^{2n};\mathbb{Z})$. This completes the proof of the first statement.

Note that if $E\to S^{2n}$ is a complex vector bundle then $\mathrm{ch}(E)=\frac{1}{n!}\cdot s_n(E)$. The Newton identities from Lemma B.1 show that $s_n(E)=(-1)^{n+1}n\cdot c_n(E)$, since in this case $c_1(E),\ldots,c_{n-1}(E)$ must all vanish. The fact that ch takes its image in $H^{2n}(S^{2n};\mathbb{Z})$ then shows that $\frac{nc_n(E)}{n!}$ is integral; that is, $c_n(E)$ is a multiple by $(n-1)!$.

Finally, let X be any $(2n-1)$-connected space. Replacing X by a weakly equivalent space, we can assume X has a cell structure with no cells of degree smaller than $2n$; that is, the $2n$-skeleton is a wedge of $2n$-spheres. Consider the cofiber sequence $\vee S^{2n}\hookrightarrow X\to Q$ where C is the cofiber, and the induced maps in cohomology:

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & \oplus\widetilde{K}^0(S^{2n}) & \longleftarrow & \widetilde{K}^0(X) & \longleftarrow & \widetilde{K}^0(C) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \\
\cdots & \longleftarrow & \oplus\widetilde{H}^{ev}(S^{2n};\mathbb{Q}) & \xleftarrow{\ j^*\ } & \widetilde{H}^{ev}(X;\mathbb{Q}) & \longleftarrow & \widetilde{H}^{ev}(C;\mathbb{Q}) & \longleftarrow & \cdots
\end{array}$$

We know by the commutativity of the diagram that $j^*(\mathrm{ch}_n(E))$ lies in $\oplus\widetilde{H}^{ev}(S^{2n};\mathbb{Z})$. But since $H^{2n}(C;\mathbb{Q})=0$ the map j^* is injective in degree $2n$, and it is easy to see that $(j^*)^{-1}(H^{2n}(\vee S^{2n};\mathbb{Z}))=H^{2n}(X;\mathbb{Z})$ (if a cellular $2n$-cochain takes integral values on all of the $2n$-cells, it is integral). So $\mathrm{ch}_n(E)$ is an integral class. The same computation as in the previous paragraph shows that $\mathrm{ch}_n(E)=\pm\frac{c_n(E)}{(n-1)!}$, and this completes the proof. □

The space $\mathbb{C}P^1$ is a complex manifold whose underlying topological manifold is S^2. Can any other spheres be given the structure of a complex manifold? Clearly this is only interesting for the even spheres. A simple corollary of the previous result rules out almost all possibilities:

Corollary 24.5. *If $n \geq 4$ then there is no complex structure on S^{2n}. Even more, there is no complex vector bundle whose underlying real bundle is the tangent bundle $T_{S^{2n}}$.*

Proof. The second statement clearly implies the first. Let $T = T_{S^{2n}}$ and suppose that T has a complex structure. By Proposition 24.4 we know that $c_n(T)$ is a multiple of $(n-1)!$ in $H^{2n}(S^{2n};\mathbb{Z})$. But $c_n(T)$ is the Euler class of the underlying real bundle, and therefore it is twice a generator since $\chi(S^{2n}) = 2$. This implies that $\frac{2}{(n-1)!}$ is an integer, which clearly cannot happen if $n \geq 4$. □

Remark 24.6. It is also know that S^4 is not a complex manifold; we will give a proof in Example 25.16 below, using the Todd genus. Whether or not S^6 admits the structure of complex manifold is an open problem.

We close this section with an example showing how the Chern character can help us carry out the calculation of K-groups. This example will play an important role when we study the Atiyah-Hirzebruch spectral sequence.

Example 24.7. Recall that $\mathbb{C}P^2$ is the mapping cone on the Hopf map $\eta\colon S^3 \to S^2$. Since the suspension of η is 2-torsion ($\pi_4(S^3) \cong \mathbb{Z}/2$), a choice of null-homotopy for $\eta \circ 2$ gives a map $f\colon \Sigma^3\mathbb{R}P^2 \to S^3$ which coincides with η when restricted to the bottom cell. Let X be the cofiber of f; this is a cell complex with a 3-cell, a 5-cell, and a 6-cell. The 5-skeleton of X is $\Sigma\mathbb{C}P^2$. Our goal will be to compute the groups $\widetilde{K}^*(X)$. [Note: This choice of X, which seemingly has come out of nowhere, is motivated by the fact that this is in some sense the smallest space for which $\widetilde{K}^*(X)$ and $\widetilde{H}^*(X)$ have different orders—see Remark 29.19 for a deeper perspective.]

There are two cofiber sequences that we can exploit: $S^3 \hookrightarrow X \to \Sigma^4\mathbb{R}P^2$ and $\Sigma\mathbb{C}P^2 \hookrightarrow X \to S^6$. We leave it to the reader to compute that $\widetilde{K}^0(\mathbb{R}P^2) = \mathbb{Z}/2$ and $K^1(\mathbb{R}P^2) = 0$, using that $\mathbb{R}P^2$ is the cofiber of $2\colon S^1 \to S^1$. Using this, the first cofiber sequence gives

$$0 = \widetilde{K}^0(S^3) \longleftarrow \widetilde{K}^0(X) \longleftarrow \mathbb{Z}/2 \longleftarrow \mathbb{Z} \longleftarrow K^1(X) \longleftarrow 0.$$

Note that we immediately deduce $K^1(X) \cong \mathbb{Z}$, and $\widetilde{K}^0(X)$ is either 0 or $\mathbb{Z}/2$. However, it is not clear how to analyze the map $\mathbb{Z} \to \mathbb{Z}/2$. The second cofiber sequence gives

$$0 \longleftarrow \widetilde{K}^0(X) \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z}^2 \longleftarrow K^1(X) \longleftarrow 0$$

where the map $\mathbb{Z}^2 \to \mathbb{Z}$ is the connecting homomorphism $\delta\colon \widetilde{K}^{-1}(\Sigma\mathbb{C}P^2) \to \widetilde{K}^0(S^6)$. Again, we are left with the task of determining this map; agreement with the previous (partial) calculation demands that the cokernel either be 0 or $\mathbb{Z}/2$, and we need to determine which one. The good news is that because the domain and target are both torsion-free, there is a chance that the Chern character will give us the information we need. We will examine the commutative square

$$\begin{array}{ccccccc} \mathbb{Z}^2 & \xrightarrow{\cong} & \widetilde{K}^{-1}(\Sigma\mathbb{C}P^2) & \xrightarrow{\delta_K} & \widetilde{K}^0(S^6) & \xrightarrow{\cong} & \mathbb{Z} \\ & & \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} & & \\ & & \widetilde{H}^{odd}(\Sigma\mathbb{C}P^2;\mathbb{Q}) & \xrightarrow{\delta_H} & \widetilde{H}^{ev}(S^6;\mathbb{Q}). & & \end{array}$$

The two Chern characters are injective because they are rational isomorphisms and the domains are torsion-free.

Recall that $K^0(\mathbb{C}P^2) = \mathbb{Z}[Y]/(Y^3)$ where $Y = 1 - L$. Let u be the standard generator for $H^2(\mathbb{C}P^2)$, so that $c_1(L) = -u$. We have

$$ch(Y) = 1 - \mathrm{ch}(L) = 1 - (1 - u + \tfrac{u^2}{2}) = u - \tfrac{u^2}{2}, \qquad \mathrm{ch}(Y^2) = \mathrm{ch}(Y)^2 = u^2.$$

Write Y_1 and Y_2 for the suspensions of Y and Y^2, lying in $\widetilde{K}^1(\Sigma\mathbb{C}P^2)$; likewise, write u_1 and u_2 for the suspensions of u and u^2 in $H^*(\Sigma\mathbb{C}P^2;\mathbb{Z})$. Compatibility of the Chern character with suspension shows that

$$\mathrm{ch}(Y_1) = u_1 - \tfrac{u_2}{2}, \qquad \mathrm{ch}(Y_2) = u_2.$$

We must next compute the images of these classes under δ_H. But this is easy from the long exact sequence for $\Sigma\mathbb{C}P^2 \hookrightarrow X \to S^6$: one finds that $\delta(u_1) = 0$ and $\delta(u_2)$ is twice a generator in $H^6(S^6)$. So the subgroup $\langle \delta_H(\mathrm{ch}(Y_1)), \delta_H(\mathrm{ch}(Y_2))\rangle \subseteq H^*(S^6;\mathbb{Q})$ equals the subgroup $H^6(S^6;\mathbb{Z}) \subseteq H^6(S^6;\mathbb{Q})$. Finally, recall from Proposition 24.4 that the image of $\mathrm{ch}\colon \widetilde{K}^0(S^6) \to H^{ev}(S^6;\mathbb{Q})$ is also equal to $H^6(S^6;\mathbb{Z})$. It follows that δ_K is surjective, and so the cokernel of δ_K is zero. This completes our calculation: $\widetilde{K}^0(X) = 0$.

To appreciate the significance of this example, note that $\widetilde{H}^{ev}(X) \cong \mathbb{Z}/2$ (concentrated in degree 6) and $\widetilde{H}^{odd}(X) \cong \mathbb{Z}$ (concentrated in degree 3). The corresponding K-groups are $\widetilde{K}^0(X) \cong 0$ and $\widetilde{K}^0(X) \cong \mathbb{Z}$. It is a general fact that all torsion-free summands in $H^*(X)$ will also appaear in $K^*(X)$, as this follows from Theorem 24.3. But the present example demonstrates that the torsion subgroups of $H^*(X)$ and $K^*(X)$ can be quite different.

25. The Grothendieck-Riemann-Roch Theorem

As we present it here, the Grothendieck-Riemann-Roch (GRR) Theorem really has two components: one that is purely topological, and one that is algebro-geometric. The topological part is a comparison between the complex-oriented structures on K-theory and singular cohomology, and gives precise formulas for how they line up under the Chern character. From the perspective that we have adopted in these notes, this topological GRR theorem is fairly easy. The algebro-geometric component, on the other hand, is of a somewhat different nature; in our presentation it is a comparison between algebraic and topological K-theory, showing that certain topologically-defined maps are compatible with purely algebraic ones that at first glance appear quite different. This second part of the GRR theorem lets us see that certain algebraic constructions actually give topological invariants; the first part leads to precise (although often complex) topological formulas for these invariants.

In the present section we discuss the topological GRR theorem and some of its consequences. The next section will deal with the algebro-geometric version.

25.1. **The Todd class.** We have seen that for a line bundle $L \to X$ one has

$$\mathrm{ch}(e^K(L)) = (1 - e^{-x})|_{x=e^H(L)}.$$

The power series $1-e^{-x}$ is a multiple of x, which means that the right-hand-side can be written as $e^H(L)$ multiplied by a 'correction factor':

$$\mathrm{ch}(e^K(L)) = e^H(L)\cdot\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=e^H(L)}.$$

It is useful to have a name for this correction factor; for historical reasons, the name is actually attached to its inverse. We define the **Todd class** of L to be

$$\begin{aligned}\mathrm{Td}(L) = \Big(\frac{x}{1-e^{-x}}\Big)\Big|_{x=c_1(L)} &= \big(1-\tfrac{x}{2}+\tfrac{x^2}{6}-\tfrac{x^3}{24}+\cdots\big)\Big|_{x=c_1(L)}\\ &= \Big(1+\tfrac{x}{2}+\tfrac{x^2}{12}-\tfrac{x^4}{720}+\cdots\Big)\Big|_{x=c_1(L).}\end{aligned}$$

The coefficients in this power series are related to Bernoulli numbers, and we refer the reader to Appendix A for a review of the basics about these. The Bernoulli numbers are defined by $\frac{x}{e^x-1}=\sum_i \frac{B_i}{i!}x^i$, and so we have

$$\mathrm{Td}(L)=\sum_i (-1)^i\tfrac{B_i}{i!}c_1(L)^i.$$

Next observe that if $E\to X$ is a sum of line bundles $L_1\oplus\cdots\oplus L_k$ then

$$\begin{aligned}\mathrm{ch}(e^K(E)) &= \mathrm{ch}(e^K(L_1)\cdots e^K(L_k))\\ &= \mathrm{ch}(e^K(L_1))\cdots\mathrm{ch}(e^K(L_k))\\ &= [e^H(L_1)\cdots e^H(L_k)]\cdot\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(L_1)}\cdots\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(L_k)}\\ &= e^H(E)\cdot\prod_i\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(L_i)}.\end{aligned}$$

It therefore makes sense to define $\mathrm{Td}(E)$ to be the inverse of the product in the final formula. More generally, if E is a bundle of rank k then the Todd class of E is

$$\mathrm{Td}(E)=\prod_{i=1}^k\Big(\frac{x_i}{1-e^{-x_i}}\Big)$$

where $c_i(E)=\sigma_i(x_1,\ldots,x_k)$. In other words, take the expression on the right and write each homogeneous piece as a polynomial in the elementary symmetric functions. Then replace those symmetric functions with the Chern classes of E, and one gets the Todd class. For example, if $\operatorname{rank}E=2$ then we would expand

$$\big(1+\tfrac{x}{2}+\tfrac{x^2}{12}-\tfrac{x^4}{720}+\cdots\big)\cdot\big(1+\tfrac{y}{2}+\tfrac{y^2}{12}-\tfrac{y^4}{720}+\cdots\big)$$

to get

$$1+\tfrac12(x+y)+\tfrac1{12}(x^2+y^2)+\tfrac14xy+\tfrac{x^2y+xy^2}{24}-\tfrac{x^4+y^4}{720}+\tfrac{x^2y^2}{144}+\cdots$$

and then write this as

$$1+\tfrac12\sigma_1+\tfrac1{12}(\sigma_1^2-2\sigma_2)+\tfrac14\sigma_2+\tfrac1{24}(\sigma_1\sigma_2)+\tfrac1{720}(-\sigma_1^4+4\sigma_1^2\sigma_2+3\sigma_2^4).$$

Then replace each σ_i with $c_i(E)$ to get the formula for $\mathrm{Td}(E)$.

The first few terms of the Todd class of an arbitrary bundle are

$$\mathrm{Td}(E)=1+\frac{c_1}{2}+\frac{c_1^2+c_2}{12}+\frac{c_1c_2}{24}+\frac{-c_1^4+4c_1^2c_2-c_4+c_1c_3+3c_2^2}{720}+\cdots \tag{25.2}$$

The homogeneous components of this series are called the **Todd polynomials**. The fourth Todd polynomial, seen as the last term in the formula above, gives a sign of the growing complexity—particularly in the size of the denominators.

We will have more to say about computing the Todd class in Section 25.17 below. But let us now turn to the study of how it measures the comparison between K-theory and singular cohomology. As we have remarked above, one should think of the Todd class as a 'correction factor'. The most basic formula where it enters is

$$\operatorname{ch}(e^K(E)) = e^H(E) \cdot \operatorname{Td}(E)^{-1}.$$

In our above analysis we showed this when E is a sum of line bundles, but the general case readily follows from this one using the splitting principle. A very similar formula, which actually implies the above one, is the following:

Proposition 25.3. *Let $E \to X$ be a complex vector bundle. Then*

$$\operatorname{ch}(\mathcal{U}_E^K) = \mathcal{U}_E^H \cdot \operatorname{Td}(E)^{-1}.$$

That is to say, applying the Chern character to a K-theoretic Thom class does not quite give the H-theoretic Thom class—one needs the Todd class correction factor. Note, by the way, that it does not matter whether we write $\mathcal{U}_E^H \cdot \operatorname{Td}(E)^{-1}$ or $\operatorname{Td}(E)^{-1} \cdot \mathcal{U}_E^H$ in the above result, since both the Thom class and the Todd class are concentrated in even degrees.

Proof. The proof has four steps:

Step 1: If the result is true for sums of line bundles, it is true for all bundles.
Step 2: If the result is true for line bundles, it is true for all sums of line bundles.
Step 3: If the result is true for the tautological line bundle over $\mathbb{C}P^\infty$, it is true for all line bundles.
Step 4: The result is true for the tautological line bundle $L \to \mathbb{C}P^\infty$.

Step 1 is a direct consequence of the splitting principle. Indeed, if $E \to X$ is a line bundle then choose a map $p\colon \widetilde{X} \to X$ such that p^*E is a sum of line bundles and such that p^* induces monomorphisms in both singular cohomology and K-theory. If $\widetilde{E} = p^*E$, the claim follows at once from the commutative square

$$\begin{array}{ccc} K^0(E, E-0) & \xrightarrow{\text{ch}} & H^*(E, E-0) \\ \downarrow & & \downarrow \\ K^0(\widetilde{E}, \widetilde{E}-0) & \xrightarrow{\text{ch}} & H^*(\widetilde{E}, \widetilde{E}-0). \end{array}$$

Step 2 follows from the fact that $\mathcal{U}_{L_1 \oplus L_2 \oplus \cdots \oplus L_r} = \mathcal{U}_{L_1} \otimes \mathcal{U}_{L_2} \otimes \cdots \otimes \mathcal{U}_{L_r}$ and the fact that ch is multiplicative. Step 3 follows at once from naturality and the fact that every line bundle is pulled back from the tautological line bundle.

So we are reduced to Step 4, which is a calculation. Consider the zero section $\zeta\colon \mathbb{C}P^\infty \hookrightarrow L$ and the composite of natural maps

$$H^*(L, L-0) \xrightarrow{j^*} H^*(L) \xrightarrow{\zeta^*} H^*(\mathbb{C}P^\infty).$$

Recall that this composite sends $\mathcal{U}_L$ to the Euler class $e^H(L)$. The map ζ^* is an isomorphism by homotopy invariance, and the map j^* is also an isomorphism: the latter follows from the long exact sequence for the pair $(L, L-0)$ together with the fact that as spaces $L - 0 \cong \mathbb{C}^\infty - 0$ and is therefore contractible.

Consider the two elements $\mathrm{ch}(\mathcal{U}_L^K)$ and $\mathcal{U}_L^H \cdot \mathrm{Td}(L)^{-1}$ in $H^*(L, L-0)$. Applying the composite $\zeta^* \circ j^*$ sends the first to $\mathrm{ch}(e^K(L))$, by naturality. Likewise, the second is sent to $e^H(L) \cdot \mathrm{Td}(L)^{-1}$. We have already computed that these two images are the same (indeed, this is how we started off this section); since $\zeta^* \circ j^*$ is an isomorphism this means $\mathrm{ch}(\mathcal{U}_L^K) = \mathcal{U}_L^H \cdot \mathrm{Td}(L)^{-1}$. □

25.4. **The Grothendieck-Riemann-Roch Theorem for embeddings.** Let $j\colon X \hookrightarrow Y$ be an embedding of complex manifolds of codimension c. We have seen that one can construct a push-forward map $j_!\colon K^*(X) \to K^{*+2c}(Y)$ and likewise in any complex-oriented cohomology theory (for example, in H^*). We will take advantage of Bott periodicity to write $j_!$ as a map $K^0(X) \to K^0(Y)$.

Consider the square

$$\begin{array}{ccc} K^0(X) & \xrightarrow{j_!} & K^0(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(X;\mathbb{Q}) & \xrightarrow{j_!} & H^{ev}(Y;\mathbb{Q}). \end{array}$$

This square does not commute; while this might seem strange, the point is just that the $j_!$ maps are defined using Thom classes and ch doesn't preserve these. But since ch almost preserves Thom classes, up to a correction factor, it follows that the above square almost commutes—up to the same factor. The precise result is as follows:

Proposition 25.5. *Let $j\colon X \hookrightarrow Y$ be an embedding of complex manifolds. Then for any $\alpha \in K^0(X)$ one has*

$$\mathrm{ch}(j_!\alpha) = j_!(\mathrm{Td}(N_{Y/X})^{-1} \cdot \mathrm{ch}(\alpha)).$$

Proof. Simply consider the diagram

$$\begin{array}{ccccccc} K^0(X) & \longrightarrow & K^0(N, N-0) & \xleftarrow{\cong} & K^0(Y, Y-X) & \longrightarrow & K^0(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(X) & \longrightarrow & H^{ev}(N, N-0) & \xleftarrow{\cong} & H^{ev}(Y, Y-X) & \longrightarrow & H^{ev}(Y) \end{array}$$

where all singular cohomology groups have rational coefficients. The left horizontal arrows are the Thom isomorphism maps, and so the leftmost square does not commute; but the other two squares do. The compositions across the two rows are the pushforward maps $j_!$ in K-theory and singular cohomology, respectively. The desired result is now an easy application of Proposition 25.3. □

The next result records how the Chern character behaves on fundamental classes:

Corollary 25.6. *If $j\colon X \hookrightarrow Y$ is an embedding of complex manifolds then $\mathrm{ch}([X]_K) = j_!(\mathrm{Td}(N_{Y/X})^{-1}) = [X]_H +$ (higher degree terms).*

Proof. Recall that $[X]_K = j_!(1)$, and so the first equality is just Proposition 25.5 applied to $\alpha = 1$. The second equality then follows directly from the fact that the Todd class of a bundle has the form $1 +$ higher degree terms together with $j_!(1) = [X]_H$. □

Example 25.7. Let $j\colon Z \hookrightarrow \mathbb{C}P^n$ be a hypersurface of degree d, and consider the GRR square

$$\begin{array}{ccc} K^0(Z) & \xrightarrow{j_!} & K^0(\mathbb{C}P^n) \\ \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(Z;\mathbb{Q}) & \xrightarrow{j_!} & H^{ev}(\mathbb{C}P^n;\mathbb{Q}). \end{array}$$

Recall that $K^0(\mathbb{C}P^n) = \mathbb{Z}[y]/(y^{n+1})$ where $y = 1 - L = [\mathbb{C}P^{n-1}]_K$, and that $H^*(\mathbb{C}P^n;\mathbb{Q}) = \mathbb{Q}[x]/(x^{n+1})$ where $x = [\mathbb{C}P^{n-1}]_H$. One has

$$\mathrm{ch}(y) = \mathrm{ch}(1-L) = \mathrm{ch}(1) - \mathrm{ch}(L) = 1 - e^{c_1(L)} = 1 - e^{-x}.$$

We will determine a formula for $[Z]_K$ by using the GRR statement $\mathrm{ch}([Z]_K) = j_!(\mathrm{Td}(N)^{-1})$. The normal bundle is $N = j^*\big((L^*)^{\otimes d}\big)$. So

$$\mathrm{Td}(N)^{-1} = j^*\left(\frac{1-e^{-dx}}{dx}\right) = j^*\big(1 - \tfrac{dx}{2} + \tfrac{d^2x^2}{6} - \tfrac{d^3x^3}{24} + \cdots\big).$$

Recall that for any α one has $j_!(j^*(\alpha)) = j_!(j^*(\alpha)\cdot 1) = \alpha \cdot j_!(1)$, and of course we know that $j_!(1) = dx$. We therefore conclude that

$$\mathrm{ch}([Z]_K) = dx \cdot \big(1 - \tfrac{dx}{2} + \tfrac{d^2x^2}{6} - \tfrac{d^3x^3}{24} + \cdots\big) = 1 - e^{-dx}.$$

We can now work backwards to determine $[Z]_K$. If we write

$$[Z]_K = a_1 y + a_2 y^2 + \cdots + a_n y^n$$

then

$$1 - e^{-dx} = \mathrm{ch}([Z]_K) = a_1(1-e^{-x}) + a_2(1-e^{-x})^2 + \cdots + a_n(1-e^{-x})^n.$$

Let $\alpha = e^{-x}$; then to determine the a_i's we need to expand $1 - \alpha^d$ in terms of powers of $1-\alpha$. To do this, simply write

$$1 - \alpha^d = 1 - (1-(1-\alpha))^d = \sum_{k=1}^{d} (-1)^{k-1}\tbinom{d}{k}(1-\alpha)^k.$$

We conclude $[Z]_K = dy - \binom{d}{2}y^2 + \binom{d}{3}y^3 - \cdots$.

Of course, we have seen this calculation before in a slightly different form—see Example 20.8. But notice that GRR allowed us to carry it through without knowing anything about $[Z]_K$, whereas before we relied on the connection between K-theoretic fundamental classes and resolutions (and our ability to write down an appropriate resolution in this case).

25.8. **The general GRR theorem and some applications.** One can produce a version of the Grothendieck-Riemann-Roch theorem that works for arbitrary maps $f\colon X \to Y$ between compact, complex manifolds, not just embeddings. To do this, we first must extend our definition of pushforward maps. Note that for large enough N there is an embedding $j\colon X \hookrightarrow \mathbb{C}^N$, and therefore the map f can be factored as

$$\begin{array}{ccc} & & Y \times \mathbb{C}^N \\ & \overset{f\times j}{\nearrow} & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

where π is projection onto the first factor. Let $B \subseteq \mathbb{C}^N$ be a large disk that contains $j(X)$. Now consider the composition

$$K^0(X) \xrightarrow{j_!} K^0(Y \times \mathbb{C}^N, Y \times (\mathbb{C}^N - B)) \cong \widetilde{K}^0(Y_+ \wedge S^{2N}) \cong K^{-2N}(Y).$$

Define this composite to be $f_!$. It requires some checking to see that this is independent of the choice of factorization of f.

Example 25.9. It is interesting to take $Y = *$. Then the pushforward $f_!$ is a map $K^0(X) \to K^{-2d}(*) \cong \mathbb{Z}$, so $f_!(1)$ gives an integer-valued invariant of the complex manifold X. It is called the **Todd genus** of X, and we will denote it Td-genus(X).

Example 25.10. We can duplicate the above definition of $f_!$ in any complex-oriented cohomology theory, and therefore we get an associated genus for complex manifolds (taking values in the coefficient ring of the theory). For singular cohomology let us call this the ***H*-genus**.

Note that if $f\colon X \to Y$ then $f_!$ is a map

$$f_!\colon H^i(X) \to H^{i+2(\dim Y - \dim X)}(Y).$$

If Y is a point then $f_!$ sends $H^i(X)$ to $H^{i-2\dim X}(pt)$, and so this is the zero map unless $i = 2\dim X$. In that dimension the cohomology of X is $\mathbb{Z}$, generated by $[*]$. But recall that $[*] = j_!(1)$ for any inclusion $j\colon * \hookrightarrow X$, and so

$$f_!([*]) = f_!(j_!(1)) = (f \circ j)_!(1) = \mathrm{id}_!(1) = 1.$$

We have therefore shown that $f_!\colon H^*(X) \to H^*(pt)$ sends an element $\alpha \in H^*(X)$ to the coefficient of $[*]$ appearing in its $2\dim(X)$-dimensional homogeneous piece. Usually it will be convenient to just say "$f_!(\alpha)$ is the top-dimensional piece of α".

As far as the H-genus is concerned, recall that it equals $f_!(1)$ for $f\colon X \to *$. But this will be zero unless $X = *$, in which case it is 1. So

$$H\text{-genus}(X) = \begin{cases} 0 & \text{if } \dim X > 0, \\ \#X & \text{if } \dim X = 0. \end{cases}$$

This is somewhat of a silly invariant, but it is what the theory gives us.

For the proof of our general version of GRR we will need to know the Todd genus of $\mathbb{C}P^n$, so let us compute this next:

Example 25.11 (Todd genus of $\mathbb{C}P^n$)**.** Recall that $\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \cdot \mathrm{Td}(F)$, and that $1 \oplus T_{\mathbb{C}P^n} \cong (n+1)L^*$. So

$$\mathrm{Td}(T_{\mathbb{C}P^n}) = \mathrm{Td}(T_{\mathbb{C}P^n}) \cdot \mathrm{Td}(1) = \mathrm{Td}\big((n+1)L^*\big) = \big[\mathrm{Td}(L^*)\big]^{n+1}.$$

Recall that $c_1(L^*) = [\mathbb{C}P^{n-1}] \in H^2(\mathbb{C}P^n)$. Call this generator x, for short. Then

$$\mathrm{Td}(T_{\mathbb{C}P^n}) = \Big(\frac{x}{1-e^{-x}}\Big)^{n+1} = \Big(1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots\Big)^{n+1}.$$

Let's look at some examples. When $n = 1$ we have $x^2 = 0$ and so $\mathrm{Td}(T_{\mathbb{C}P^1}) = 1 + x$. When $n = 2$ we have $x^3 = 0$ and

$$\mathrm{Td}(T_{\mathbb{C}P^2}) = \Big(1 + \tfrac{x}{2} + \tfrac{x^2}{12}\Big)^3 = 1 + \tfrac{3}{2}x + x^2.$$

Finally, for $n = 3$ we have $x^4 = 0$ and

$$\mathrm{Td}(T_{\mathbb{C}P^3}) = \Big(1 + \tfrac{x}{2} + \tfrac{x^2}{12}\Big)^4 = 1 + 2x + \tfrac{11}{6}x^2 + x^3.$$

One discernible pattern in these polynomials is that the leading coefficient is always 1. This is an amusing exercise that we leave to the reader. It shows that Td-genus$(\mathbb{C}P^n) = 1$ for all n.

Exercise 25.12. Complete the above example by proving that the coefficient of x^n in $\left(\frac{x}{1-e^{-x}}\right)^{n+1}$ is equal to 1, for all n. One method is to interpret the coefficient as a residue:

$$\mathrm{Res}_{x=0}\left(\tfrac{1}{(1-e^{-x})^{n+1}}\right) = \tfrac{1}{2\pi i}\int_C \tfrac{1}{(1-e^{-x})^{n+1}}\,dx$$

where C is a small counterclockwise circle around the origin. Use the substitution $z = 1 - e^{-x}$ to convert this to a different residue that is easily computed. You will need to convince yourself that the mapping $x \mapsto 1 - e^{-x}$ takes C to another small counterclockwise loop around the ogigin.

Exercise 25.13. Let $Z \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Show that

$$\text{Td-genus}(Z) = d - \tbinom{d}{2} + \tbinom{d}{3} - \cdots + (-1)^n\tbinom{d}{n}.$$

Conclude that if $d \leq n$ then Td-genus$(Z) = 1$.

We now state the general GRR theorem:

Theorem 25.14 (Grothendieck-Riemann-Roch, full version)**.** *Let X and Y be compact, complex manifolds and let $f\colon X \to Y$ be a map. Then in the square*

$$\begin{array}{ccc} K^0(X) & \xrightarrow{f_!} & K^0(Y) \\ \big\downarrow{\scriptstyle \mathrm{ch}} & & \big\downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(X;\mathbb{Q}) & \xrightarrow{f_!} & H^{ev}(Y;\mathbb{Q}). \end{array}$$

one has

$$\mathrm{ch}(f_!\alpha)\cdot \mathrm{Td}(T_Y) = f_!(\mathrm{ch}(\alpha)\cdot \mathrm{Td}(T_X))$$

for all $\alpha \in K^0(X)$, where T_X and T_Y are the tangent bundles of X and Y.

Exercise 25.15. Check that when $X \hookrightarrow Y$ is an embedding then the above version of GRR is equivalent to the version given in Proposition 25.5.

Proof of Theorem 25.14. The proof is via the steps listed below. We will outline arguments in each case, but leave some of the details to the reader.

Step 1: The result is true when f is an embedding.
Step 2: The result is true when f is $\mathbb{C}P^n \to *$.
Step 3: The result is true when f is the projection $Y \times \mathbb{C}P^n \to Y$, for any compact complex manifold Y.
Step 4: The result is true in general.

Step 1 was handled in Proposition 25.5. Step 2 is just a computation, where one computes both sides of the GRR formula and sees that they are the same. Use that $K^0(\mathbb{C}P^n)$ is generated by the classes $[\mathbb{C}P^i]$. For the left side of GRR use that $[\mathbb{C}P^i]$ is mapped to 1 via $f_!$, as Td-genus$(\mathbb{C}P^i) = 1$. For the right side use that $\mathrm{ch}([\mathbb{C}P^i]) = (1-e^{-x})^{n-i}$ and compute that the coefficient of x^n in the series $(1-e^{-x})^{n-i}\cdot\left(\frac{x}{1-e^{-x}}\right)^{n+1}$ is equal to 1. For this final piece use a method similar to what we did in Exercise 25.12 above.

For Step 3 consider the product map $K^0(Y) \times K^0(\mathbb{C}P^N) \to K^0(Y \times \mathbb{C}P^N)$. We claim that this is an isomorphism, for any CW-complex Y. Indeed, consider the functors $(X, A) \mapsto K^*(X, A) \otimes K^0(\mathbb{C}P^N)$ and $(X, A) \mapsto K^*(X \times \mathbb{C}P^N, A \times \mathbb{C}P^N)$. Our product map gives a natural transformation from the first to the second, and both functors are generalized cohomology theories (in the second case this is automatic, but in the first case this uses that $K^0(\mathbb{C}P^N)$ is free and therefore flat). One readily checks that the comparison map is an isomorphism when $(X, A) = (pt, \emptyset)$, and so it follows that it is an isomorphism for all CW-pairs (X, A).

To complete Step 3 it now suffices to verify the GRR formula on classes of the form $\alpha = (p_1)^*(\beta) \cdot (p_2)^*(\gamma)$ where $\beta \in K^0(Y)$, $\gamma \in K^0(\mathbb{C}P^N)$, and p_1 and p_2 are the projections of $Y \times \mathbb{C}P^N$ onto Y and $\mathbb{C}P^N$, respectively. Use the diagram

$$\begin{array}{ccc} Y \times \mathbb{C}P^N & \xrightarrow{p_2} & \mathbb{C}P^N \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ Y & \xrightarrow{\pi_2} & * \end{array}$$

and the formulas

$$(p_1)_!\big[p_1^*\beta \cdot p_2^*\gamma\big] = \beta \cdot (p_1)_!(p_2^*\gamma) = \beta \cdot \pi_2^*\big((\pi_1)_!\gamma\big)$$

(as well as the analog of this in singular cohomology), together with Step 2.

Finally, for Step 4 factor $f\colon X \to Y$ as $X \xrightarrow{j} Y \times \mathbb{C}P^N \xrightarrow{\pi} Y$ where j is an embedding and π is projection. Use the diagram

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{j_!} & K^0(Y \times \mathbb{C}P^N) & \xrightarrow{\pi_!} & K^0(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(X; \mathbb{Q}) & \xrightarrow{j_!} & H^*(Y \times \mathbb{C}P^N; \mathbb{Q}) & \xrightarrow{\pi_!} & H^*(Y; \mathbb{Q}), \end{array}$$

where the horizontal composites are $f_!$. We established GRR for the two squares, by Steps 1 and 3. Deduce the general GRR by putting these two squares together. □

To see one example of GRR, consider the case $Y = *$. Here the GRR theorem says $\mathrm{ch}(f_!(1)) = f_!(\mathrm{Td}(T_X))$. We will compute both sides independently, and then see what information this theorem is giving. On the left-hand-side, $f_!(1) = \text{Td-genus}(X) \cdot 1 \in K^0(*)$ and so $\mathrm{ch}(f_!(1)) = \text{Td-genus}(X) \cdot 1 \in H^0(pt; \mathbb{Q})$.

To analyze the right-hand-side we recall from Example 25.10 above that $f_!\colon H^{ev}(X) \to H^{ev}(pt)$ sends a class α to its top-dimensional piece (the component in dimension $2 \dim X$). So GRR says that

$$\text{Td-genus}(X) \cdot [*] = \text{top-dimensional piece of } \mathrm{Td}(T_X).$$

One of the surprises here is that the right-hand-side is not *a priori* an integer multiple of $[*]$: recall that the definition of the Todd class contains complicated denominators. The resulting integrality conditions can lead to some nonexistence results in topology, as demonstrated in the following example.

Example 25.16. We claim that there is no complex manifold whose underlying topological manifold is S^4; said differently, the space S^4 cannot be given a complex structure. If S^4 were a complex manifold then it would have a Todd genus, which we know will be an integer. But GRR tells us that the Todd genus is also the top-dimensional component of $\mathrm{Td}(T)$, where T denotes the complex tangent bundle of our fictitious complex manifold. But $c_1(T) = 0$ because $H^2(S^4) = 0$, and $c_2(T) = 2[*]$ because $c_2(T)$ is the Euler class and $\chi(S^4) = 2$. Plugging into (25.2) we find that $\mathrm{Td}(T) = 1 + \frac{c_2}{12}$ and so the top-dimensional piece is $\frac{1}{6}$. As this is not an integer, we have arrived at a contradiction.

A similar argument shows that S^{4n} is not a complex manifold for any n, although to follow through with this we will need to get better at computing terms in the Todd class. We return to this problem in Proposition 25.23 below.

25.17. **Computing the Todd class.** Let $a_1, a_2, \dots$ be indeterminates and write $Q(x) = 1 + a_1x + a_2x^3 + \cdots$. Let

$$Q(\underline{x}) = Q(x_1, \dots, x_n) = Q(x_1)Q(x_2)\cdots Q(x_n)$$

where the x_i's are formal variables of degree 1. This gives us a power series that is invariant under permutations of the x_i's, and so it may be written as a power series in the elementary symmetric functions $\sigma_i = \sigma_i(x_1, \dots, x_n)$, $1 \le i \le n$. Our goal will be to give a formula, in terms of the a_i's, for the coefficient of any given monomial $m = \sigma_1^{m_1} \cdots \sigma_n^{m_n}$.

Notice that the degree of m is $m_1 + 2m_2 + \cdots + nm_n$; call this number N. The coefficient of m in $Q(\underline{x})$ will not involve any terms a_i for $i > N$, so we might as well just assume that $a_i = 0$ for $i > N$. In this case write

$$Q(x) = 1 + a_1x + \cdots + a_Nx^N = (1 + t_1x)(1 + t_2x)\cdots(1 + t_Nx^N)$$

as a formal factorization of $Q(x)$ (or if you like, we are working in the algebraic closure of $\mathbb{Q}(a_1, \dots, a_n)$). Then

$$Q(\underline{x}) = \prod_{i=1}^{N} Q(x_i) = \prod_{i=1}^{N}\prod_{j=1}^{N}(1 + t_jx_i).$$

The evident next step is to reverse the order of the products, so let

$$Q_j = \prod_{i=1}^{N}(1 + t_jx_i) = 1 + t_j\sigma_1 + t_j^2\sigma_2 + \cdots + t_j^N\sigma_N$$

and observe that $Q(\underline{x}) = \prod_j Q_j$. Consider the process of multiplying out all factors in

$$(1 + \sigma_1t_1 + \cdots + \sigma_Nt_1^N) \cdot (1 + \sigma_1t_2 + \cdots + \sigma_Nt_2^N)\cdots(1 + \sigma_1t_N + \cdots + \sigma_Nt_N^N).$$

The first few terms are

$$1 + \sigma_1[t_1] + \sigma_2[t_1^2] + \sigma_1^2[t_1t_2] + \sigma_3[t_1^3] + \sigma_1\sigma_2[t_1t_2^2] + \cdots$$

where the bracket notation means to sum the terms in the Σ_N-orbit of the monomial inside the brackets (see Appendix B). Each of these brackets is a polynomial in the t_i's that is invariant under the symmetric group, and therefore can be written (in a unique way) as a polynomial in the a_i's. These are the desired coefficients of $Q(\underline{x})$. The general result, whose proof has basically just been given, is the following:

Proposition 25.18. *The coefficient of $\sigma_1^{m_1}\cdots\sigma_n^{m_n}$ in $Q(\underline{x})$ is*

$$[t_1t_2\cdots t_{m_1}t_{m_1+1}^2\cdots t_{m_1+m_2}^2t_{m_1+m_2+1}^3\cdots t_{m_1+m_2+m_3}^3\cdots t_{m_1+\cdots+m_n}^n].$$

The bracketed expression in the above result looks horrible, but it is simpler than it looks. The subscripts can basically be ignored. The idea is to write down a product of powers of the t_i's where no index i appears more than once and where the number of t_i's raised to the kth power is m_k. For example, here are a few σ-monomials and their associated coefficients:

$$\sigma_4 : [t_1^4], \qquad \sigma_1\sigma_3 : [t_1 t_2^3], \qquad \sigma_2\sigma_3^2 : [t_1^2 t_2^3 t_3^3], \qquad \sigma_1\sigma_4^2\sigma_6 : [t_1 t_2^4 t_3^4 t_4^6].$$

For Proposition 25.18 to be useful one has to write the bracketed expression as a polynomial in the elementary symmetric functions $\sigma_i(t) = a_i$. This is, of course, an unpleasant process. One case where it is not *so* bad is for the power sum $[t_1^n]$, since here we have the Newton polynomials S_n described in Appendix B.

Corollary 25.19. *For any $k \geq 1$, the coefficient of σ_k in $Q(\underline{x})$ is $S_k(a_1, \dots, a_k)$, where S_k is the kth Newton polynomial. By Proposition B.3 this is also equal to the two expressions*

$$(-1)^k \cdot \Big[\textit{coeff. of } x^k \textit{ in } 1 - x\frac{d}{dx}\big(\log Q(x)\big)\ \Big] = (-1)^{k-1} \cdot \Big[\textit{coeff. of } x^{k-1} \textit{ in } \frac{Q'(x)}{Q(x)}\Big].$$

Now let us specialize to $Q(x) = \frac{x}{1-e^{-x}} = \sum_i (-1)^i \frac{B_i}{i!} x^i$. Then writing $Q(x_1, \dots, x_n)$ as a power series in the elementary symmetric functions exactly yields an expression for the Todd class of a rank n vector bundle in terms of its Chern classes. Let us apply Corollary 25.19 to this situation; to do so we must compute $Q'(x)/Q(x)$. This is easy enough:

$$Q'(x) = \tfrac{1}{1-e^{-x}} - \tfrac{xe^{-x}}{(1-e^{-x})^2} = \tfrac{1}{x}Q(x) - \tfrac{e^{-x}}{1-e^{-x}}Q(x) = \big(\tfrac{1}{x} + 1 - \tfrac{1}{1-e^{-x}}\big) \cdot Q(x)$$

and so

$$1 - x\tfrac{Q'(x)}{Q(x)} = 1 - x\big(\tfrac{1}{x} + 1 - \tfrac{1}{1-e^{-x}}\big) = -x + Q(x).$$

Specializing Corollary 25.19 to the present situation now gives:

Corollary 25.20. *Let $E \to X$ be a rank n vector bundle. Then for any $2 \leq k \leq n$, the coefficient of c_k in the formula for $\mathrm{Td}(E)$ is equal to $\frac{B_k}{k!}$, whereas the coefficient of c_1 is $-\frac{B_1}{1!} = \frac{1}{2}$.*

Remark 25.21. Note that the above calculation reveals an interesting property of the coefficients of $\frac{x}{1-e^{-x}}$: when you put them into the Newton polynomials S_k the output is unaltered, at least for $k \geq 2$. That is, if $\frac{x}{1-e^{-x}} = \sum_i a_i x^i$ then $S_k(a_1, \dots, a_k) = a_k$ for $k \geq 2$. For example, $S_4 = a_1^4 - 4a_1^2 + 4a_1a_3 + 2a_2^a - 4a_4$ and the first few coeffcients of $\frac{x}{1-e^{-x}}$ are $\frac{1}{2}$, $\frac{1}{12}$, 0, and $-\frac{1}{720}$. Some grade-school arithmetic checks that indeed

$$S_4(\tfrac{1}{2}, \tfrac{1}{12}, 0, -\tfrac{1}{720}) = -\tfrac{1}{720}.$$

But this is hardly obvious from just looking at the formula for S_4.

25.22. **An application of GRR.** The following proposition is (mostly) weaker than what we proved back in Corollary 24.5. Still, we offer it as a sample application of the Todd genus.

Proposition 25.23. *No sphere S^{4n} admits the structure of a complex manifold.*

Proof. Assume S^{4n} is a complex manifold. Then it has a Todd genus, which is necessarily an integer by definition. This will be the coefficient of $[*]$ in the top component of the Todd class $\mathrm{Td}(T)$, where T denotes the complex tangent bundle to S^{4n}. To compute this Todd class directly from its definition, first note that $c_i(T) = 0$ for $i < 2n$, because $H^{2i}(S^{4n}) = 0$ in this range. We must also have $c_n(T) = e(T) = 2[*]$, because the Euler characteristic of an even sphere equals 2. These facts let us easily write down $\mathrm{Td}(T_{S^{4n}})$. Let us consider some examples of this.

When $n = 1$ we would have $\mathrm{Td}(T_{S^4}) = 1 + \frac{c_2}{12}$. The Todd genus of S^4 would then be $\frac{2}{12}$, which is not an integer. When $n = 2$ we would have $\mathrm{Td}(T_{S^8}) = 1 - \frac{c_4}{720}$, and so the Todd genus of S^8 would be $-\frac{2}{720}$; again, not an integer. These examples give the general idea, and the denominators only get worse as n gets larger.

To be specific, one will have $\mathrm{Td}_{S^{4n}} = 1 + Mc_{2n}$ where M is a mystery number that must be computed from the definition of the Todd class. It is a consequence of Corollary 25.20 that $M = \frac{B_{2n}}{(2n)!}$. The Todd genus of S^{4n} will then be $2 \cdot B_{2n}/(2n)!$. By Theorem A.5 the number 3 divides the lowest-terms-denominator of B_{2n}, and so this expression cannot be an integer. This is our contradiction. □

Remark 25.24. Notice why our proof of Proposition 25.23 does not extend to cover spheres S^{4n+2}: the Chern class c_{2n+1} does not appear by itself in the formula for the Todd class, because the odd Bernoulli numbers are zero. If S^{4n+2} has a complex structure one can conclude that its Todd genus is zero, but this by itself does not produce a contradiction.

25.25. **The arithmetic genus.** We now discuss the problem of computing the Todd genus for smooth algebraic subvarieties $Z \hookrightarrow \mathbb{C}P^n$. We will see that it can be described entirely in terms of algebro-geometric data. This material foreshadows much of what we do in Section 26.

Let $p\colon \mathbb{C}P^n \to *$ and $q\colon Z \to *$ be the squash maps, and consider the composition

$$K^0(Z) \xrightarrow{j_!} K^0(\mathbb{C}P^n) \xrightarrow{p_!} K^0(pt).$$

The composite is $q_!$ and therefore sends 1 to $\text{Td-genus}(Z) \cdot [*]$. On the other hand, if we write

$$j_!(1) = [Z] = a_{n-1}[\mathbb{C}P^{n-1}] + a_{n-2}[\mathbb{C}P^{n-2}] + \cdots + a_0[\mathbb{C}P^0]$$

then since $p_!([\mathbb{C}P^{n-i}]) = \text{Td-genus}(\mathbb{C}P^{n-i}) \cdot [*] = [*]$ we have

$$p_!(j_!(1)) = (a_{n-1} + a_{n-2} + \cdots + a_0)[*].$$

So $\text{Td-genus}(Z) = \sum_i a_i$.

Recall that knowing $[Z]$ is the same as knowing the Hilbert polynomial of Z. We wish to ask the question: how can the Todd genus be extracted from the Hilbert polynomial? To answer this, start by recalling the diagram

$$\begin{array}{ccc} K^0_{alg}(\mathbb{C}P^n) & \xrightarrow[\cong]{\phi} & K^0(\mathbb{C}P^n) \\ \cong\downarrow & & \\ G_{grd}(\mathbb{C}[x_0, \ldots, x_n])/\langle[\mathbb{C}]\rangle & \xrightarrow{\text{Hilb}} & \mathbb{Q}[s] \end{array}$$

from Section 20.18. The image of the function Hilb is the $\mathbb{Z}$-submodule of $\mathbb{Q}[s]$ generated by $\binom{s+n}{n}, \binom{s+n-1}{n-1}, \ldots, \binom{s}{0}$. If one takes $[\mathbb{C}P^{n-i}] \in K^0(\mathbb{C}P^n)$ and pushes it around the diagram, we have seen in Section 20.18 that the corresponding Hilbert polynomial is $\binom{s+n-i}{n-i}$.

The Todd genus can be thought of as the unique function $K^0(\mathbb{C}P^n) \to \mathbb{Z}$ sending all the classes $[\mathbb{C}P^{n-i}]$ to 1. We look for a similar function $\mathrm{im}(\mathrm{Hilb}) \to \mathbb{Z}$ that sends $\binom{s+n-i}{n-i}$ to 1, for all i. A moment's thought shows that the map "evaluate at $s = 0$" has this property. We have therefore proven the following:

Proposition 25.26. *Let $Z \hookrightarrow \mathbb{C}P^n$ be an algebraic subvariety. Then the Todd genus of Z is* $\mathrm{Hilb}_Z(0)$.

In algebraic geometry, the invariant $\mathrm{Hilb}_Z(0)$ is sometimes called the **arithmetic genus**. So we have proven that the arithmetic genus and Todd genus coincide.

Remark 25.27. Many authors use the term *arithmetic genus* for the invariant $(-1)^{\dim Z}(\mathrm{Hilb}_Z(0) - 1)$. This is the definition in both [H] and [GH], for example. Obviously the two definitions carry the same information, and the difference between them is only a matter of "normalization". The invariant $\mathrm{Hilb}_Z(0)$ is sometimes called the *Hirzebruch genus*, or the *holomorphic Euler characteristic* (see Section ??? below for more information about this).

25.28. **Fundamental classes and the Todd genus.** Again let $Z \hookrightarrow \mathbb{C}P^n$ be a complex submanifold of codimension c and consider the fundamental class $[Z] \in K^0(\mathbb{C}P^n)$. Write

$$[Z] = a_{n-c}[\mathbb{C}P^{n-c}] + a_{n-c-1}[\mathbb{C}P^{n-c-1}] + \cdots + a_0[\mathbb{C}P^0].$$

We have seen that a_{n-c} is the degree of Z, which has a simple geometric interpretation: it is the number of intersection points of Z with a generic linear subspace of dimension c. But the question remains as to how to give a geometric interpretation for the other a_i's. We will now explain how the Todd genus gives an answer this (although perhaps not an entirely satisfactory one).

As we saw in the last section, $\text{Td-genus}(Z) = \sum_{0\le i} a_i$. But note that multiplying the equation for $[Z]$ by $[\mathbb{C}P^{n-1}]$ gives $[Z]\cdot[\mathbb{C}P^{n-1}] = a_{n-c}[\mathbb{C}P^{n-c-1}] + \cdots + a_1[\mathbb{C}P^0]$ and therefore

$$\text{Td-genus}(Z \cap \mathbb{C}P^{n-1}) = \sum_{1\le i} a_i.$$

Here $Z \cap \mathbb{C}P^{n-1}$ indicates the intersection of Z with a generic hyperplane in $\mathbb{C}P^n$.

Likewise we have $[Z] \cdot [\mathbb{C}P^{n-j}] = a_{n-c}[\mathbb{C}P^{n-c-j}] + \cdots + a_j[\mathbb{C}P^0]$, and hence

$$\text{Td-genus}(Z \cap \mathbb{C}P^{n-j}) = \sum_{j\le i} a_i.$$

So the partial sums $\sum_{j\le i} a_i$ for $j = 0, 1, \ldots, n-c$ are the same as the Todd genera of $Z, Z \cap \mathbb{C}P^{n-1}, \ldots, Z \cap \mathbb{C}P^c$. (This gives another explanation for why a_{n-c} is the degree of Z). We immediately obtain the formulas

$$a_i = \text{Td-genus}(Z \cap \mathbb{C}P^{n-i}) - \text{Td-genus}(Z \cap \mathbb{C}P^{n-i-1}), \tag{25.29}$$

where again the intersections are interpreted to be generic. This is our desired geometric description of the a_i's. Note, however, that whether or not this is indeed "geometric" depends on whether one feels that this adjective applies to the Todd genus.

26. The algebro-geometric GRR theorem

Let X be a compact complex manifold and $E \to X$ a complex vector bundle. If π denotes the projection $X \to *$ then we get an element $\pi_!([E]) \in K^0(pt) = \mathbb{Z}$. This gives an integer-valued invariant of the bundle E, which we will call the **Todd number** of E. We will write it as

$$\text{Td-num}_X(E) = \pi_!([E]).$$

The topological GRR theorem identifies this number as $\Theta_X(\mathrm{Td}(T_X) \cdot \mathrm{ch}(E))$, and so we can calculate it in terms of the Chern classes of E and T_X. Note that the Todd number of the trivial bundle $\underline{1}$ is the Todd *genus* of X.

If X is an algebraic variety and $E \to X$ is an algebraic vector bundle then there is another way to compute the Todd number of E, in terms of algebro-geometric invariants. This identification of invariants is an example of the algebro-geometric GRR theorem. Although the theorem covers far more than just the Todd number, we will concentrate on this special case before stating the more general result.

26.1. **Sheaf cohomology.** As we saw in ???? the algebraic vector bundle $E \to X$ gives rise to an associated coherent sheaf on the Zariski space X_{Zar}. We also call this sheaf E, by abuse. Modern algebraic geometry shows how to obtain sheaf cohomology groups $H^i(X; E)$. The general theory is technical (although not incredibly hard), and would take too long to recount here; the level of abstraction and technicality is roughly comparable to that of singular cohomology. But just as in the latter case, there are methods for *computing* the sheaf cohomology groups that do not require the high-tech definitions.

Suppose we have a Zariski open cover $\{U_\alpha\}$ of X with the property that each U_α is affine, and moreover assume that each iterated intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ is affine (for each $k \geq 1$). Write $\Gamma(U_\alpha, E)$ for the algebraic sections of E defined over U_α. Then we may form the Čech complex

$$0 \to \bigoplus_\alpha \Gamma(U_\alpha, E) \to \bigoplus_{\alpha_1, \alpha_2} \Gamma(U_{\alpha_1} \cap U_{\alpha_2}, E) \to \cdots$$

and the sheaf cohomology group $H^i(X; E)$ is just isomorphic to the ith cohomology group of this complex.

Example 26.2 (Cohomology of $\mathcal{O}(k)$ on $\mathbb{C}P^n$)**.** Let $x_0, \ldots, x_{n+1}$ be homogeneous coordinates on $\mathbb{C}P^n$, and for each $0 \leq j \leq n$ let $U_j \subseteq \mathbb{C}P^n$ be the open subscheme defined by $x_j \neq 0$. Write $U_{j_1 \cdots j_r} = U_{j_1} \cap \cdots \cap U_{j_r}$, and note that all of these are affine. Indeed, U_j is the spectrum of $\mathbb{C}[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}]$, $U_{j,k}$ is the spectrum of the localization of this ring at x_k/x_j, and so forth.

Let $S = \mathbb{C}[x_0, \ldots, x_n]$, regarded as a graded ring where all x_i's have degree 1. Let $R_j = \Gamma(U_j, \mathcal{O}) = \mathbb{C}[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}]$. This is the degree zero homogeneous piece of the localization S_{x_j}. Further, observe that each $\mathcal{O}(k)$ is trivializable over U_j, and so $\Gamma(U_j, \mathcal{O}(k))$ will be a free R_j-module of rank 1. We can identify $\Gamma(U_j, \mathcal{O}(k))$ with the degree k homogeneous component of the localization S_{x_j}; that is, it is the $\mathbb{C}$-linear span of all monomials $x_0^{a_0} \cdots x_n^{a_n}$ where $a_j \in \mathbb{Z}$ and all other $a_i \geq 0$. This is readily checked to coincide with the cyclic R_j-module $x_j^k R_j$.

The analogs of the above facts work for any open set $U_{\underline{j}} = U_{j_1} \cap \cdots \cap U_{j_r}$. The sections $\Gamma(U_{\underline{j}}, \mathcal{O}(k))$ form the degree k homogeneous component of the ring

$S_{x_{j_1}\cdots x_{j_r}}$. We want to examine the Čech complex $\check{C}(U_\bullet, \mathcal{O}(k))$ for each value of k, but it is more convenient to take the direct sum over all values for k and consider them all at once.

For a collection of indices $\sigma \subseteq \{0, \dots, n\}$ let S_σ be the localization of S at the element $\prod_{i\in\sigma} x_i$. Then consider the augmented Čech complex

$$C^\bullet: \quad 0 \to S \to \oplus_i S_i \to \oplus_{i<j} S_{ij} \to \cdots \to S_{01\cdots n} \to 0$$

where the S is in degree -1 and where the differentials are all induced by the inclusions $S_\sigma \hookrightarrow S_{\sigma'}$ for $\sigma \subseteq \sigma'$. That is to say, if we have a tuple $\alpha = (\alpha_\sigma \in S_\sigma)_{\#\sigma=r}$ then $d\alpha$ is the tuple whose value at $\sigma' = \{i_0, \cdots, i_r\}$ (with the entries ordered from least to greatest) is

$$(d\alpha)_{\sigma'} = \sum_{k=0}^{r} (-1)^k \alpha_{i_0\cdots\widehat{i_k}\cdots i_r}.$$

The Čech complex for computing cohomology is obtained from $C^\bullet$ by omitting the S in degree -1, but we will quickly see why it is convenient to have that S around.

It is easy to compute the cohomology group $H^n(C^\bullet)$. The ring $S_{01\cdots\widehat{i}\cdots n}$ is generated as a vector space by monomials $x_0^{a_0}\cdots x_n^{a_n}$ where $a_i \geq 0$. So the image of $C^{n-1} \to C^n$ is the span of all monomials where *some* a_i is nonnegative. The monomials that are not in the image have the form $x_0^{-1}\cdots x_n^{-1}\cdot(x_0^{-b_0}\cdots x_n^{-b_n})$ where all $b_i \geq 0$. So $H^n(C^\bullet)$ only has terms in degree $k \leq -(n+1)$, and in such a degree the group is isomorphic to $S^{-k-(n+1)}(\mathbb{C}^{n+1})$. This computation will be subsumed by the more general one in the next paragraph, but it is useful to see this particular case by itself.

To compute the cohomology of $C^\bullet$ in all dimensions it is useful to regard S, and each of its localizations, as multigraded by the group $\mathbb{Z}^{n+1}$ (by the multidegrees of monomials). The maps in the complex preserve this multigrading, so we might as well look at one multidegree $\underline{a} = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ at a time. Let $\tau = \{i_0, \dots, i_u\}$ be the complete list of indices for which $a_i < 0$. Note that S_σ is zero in multidegree $\underline{a}$ unless $\sigma \supseteq \tau$: that is, we will only have monomials of multidegree $\underline{a}$ if we have inverted the x_i's for $i \in \tau$. It is not hard to check that $C^\bullet_{\underline{a}}$ (the portion of $C^\bullet$ in multidegree $\underline{a}$) coincides with the augmented simplicial chain complex for $\Delta^{n-\#\tau}$ with coefficients in $\mathbb{C}$. The point is that the rings S_σ that are nonzero in degree $\underline{a}$ correspond to precisely those σ that contain τ, and these correspond in turn to subsets of $\{0, 1, \dots, n+1\} - \tau$. Subsets of $\{0, 1, \dots, n+1\} - \tau$ also index the simplices of $\Delta^{n-\#\tau}$, and we leave it to the reader to verify that the complexes do indeed coincide.

The augmented simplicial cochain complex for $\Delta^{n-\#\tau}$ has zero cohomology except in one extreme case—for when $\#\tau = n+1$ we have the augmented cochain complex of the emptyset, and this has a single $\mathbb{Z}$ in its cohomology. This corresponds to those multidegrees $\underline{a}$ in which all $a_i < 0$; for these the total degree satisfies $\sum a_i \leq -(n+1)$.

We have seen that $C^\bullet$ is exact except in cohomological degree n, and there we get a single copy of $\mathbb{C}$ in every multidegree $\underline{a}$ for which $\sum a_i \leq -(n+1)$. So for a fixed integral degree $k \leq -(n+1)$ the kth homogeneous component of $H^n(C)$ is the $\mathbb{C}$-linear span of monomials $\underline{x}^{\underline{a}}$ where all $a_i < 0$ and $\sum a_i = k$. This is what we saw earlier in the argument as well, and it gives us that $H^n(C)_k \cong S^{-k-(n+1)}(\mathbb{C}^{n+1})$.

Finally, let us turn to our original Čech complex by removing the S from degree -1 of $C^\bullet$. In doing so we introduce homology in degree 0, and the graded homology groups exactly coincide with the homogeneous components of S. That is, the kth graded piece of H^0 is isomorphic to $S^k(\mathbb{C}^{n+1})$.

We have now proven that

$$H^i(\mathbb{C}P^n; \mathcal{O}(k)) \cong \begin{cases} S^k(\mathbb{C}^{n+1}) & \text{if } i = 0, \\ S^{-k-(n+1)}(\mathbb{C}^{n+1}) & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

The following table shows these sheaf cohomology groups. Note that these vanish except for $0 \le i \le n$. In the table we write $\mathcal{O}_k$ instead of $\mathcal{O}(k)$, for typographical reasons, and we write $V = \mathbb{C}^{n+1}$. The dth symmetric power of V is denoted S^dV; note that this is isomorphic to the space of degree d homogeneous polynomials in $x_0, \dots, x_n$.

Table 26.2. Cohomology groups $H^i(\mathbb{C}P^n; \mathcal{O}(k))$

i	$\mathcal{O}_{-(n+3)}$	$\mathcal{O}_{-(n+2)}$	$\mathcal{O}_{-(n+1)}$	$\mathcal{O}_{-n}$	$\cdots$	$\mathcal{O}_{-1}$	$\mathcal{O}$	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$
0	0	0	0	0	$\cdots$	0	$\mathbb{C}$	S^1V	S^2V	S^3V
1	0	0	0	0	$\cdots$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	0	0	0	0	$\cdots$	0	0	0	0	0
n	S^2V	S^1V	$\mathbb{C}$	0	$\cdots$	0	0	0	0	0

26.3. **Sheaf cohomology and the Todd number.** When X is a projective variety the sheaf cohomology groups we introduced in the last section turn out to have the following properties:

- They are finite-dimensional over $\mathbb{C}$;
- $H^i(X; E)$ vanishes when $i > \dim X$;
- A short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ gives rise to a long exact sequence of sheaf cohomology groups.

The first two properties allow us to define the sheaf-theoretic Euler characteristic

$$\chi(X; E) = \sum_i (-1)^i \dim H^i(X; E).$$

and the third property yields that $\chi(X; E) = \chi(X; E') + \chi(X; E'')$. So $\chi(X; -)$ gives a homomorphism $K^0_{alg}(X) \to \mathbb{Z}$. It will not come as a surprise that this agrees with the topologically-defined pushforward map $\pi_!$:

Proposition 26.4. *If X is a projective algebraic variety and $E \to X$ is an algebraic vector bundle then*

$$\text{Td-num}_X(E) = \chi(X; E).$$

This gives us our algebro-geometric interpretation of the Todd number. We postpone the proof for the moment, prefering to obtain this as a corollary of our general GRR theorem. But let us at least check the proposition in the important example of $\mathbb{C}P^n$. Recall that $K^0_{alg}(\mathbb{C}P^n)$ is the free abelian group generated by $[\mathcal{O}], [\mathcal{O}(1)], [\mathcal{O}(2)], \dots, [\mathcal{O}(n)]$; so we can verify the result for all algebraic vector bundles E by checking it for these particular $n+1$ cases. Luckily we have already computed the vector spaces $H^i(\mathbb{C}P^n;\mathcal{O}(k))$. From Table 26.1 we find that

$$\chi(\mathbb{C}P^n;\mathcal{O}(k)) = \begin{cases} \dim S^k\mathbb{C}^{n+1} & \text{if } k \geq 0, \\ 0 & \text{if } -n \leq k < 0, \\ (-1)^n \dim S^{-k-(n+1)}\mathbb{C}^{n+1} & \text{if } k < -n. \end{cases}$$

We leave the reader to check that all three cases in the above formula can be unified into the simple statement $\chi(\mathbb{C}P^n;\mathcal{O}(k)) = \binom{n+k}{n}$.

It remains to compute the Todd number of $\mathcal{O}(k)$. We use the by-now-familiar technique from Exercise 25.12:

$$\begin{aligned}\Theta_X\Big[\mathrm{Td}(T_X)\cdot \mathrm{ch}(\mathcal{O}(k))\Big] = \Theta_n\Big[\big(\tfrac{x}{1-e^{-x}}\big)^{n+1}\cdot e^{kx}\Big] &= \mathrm{Res}_{x=0}\big(\tfrac{1}{1-e^{-x}}\big)^{n+1}\cdot e^{kx}\,dx \\ &= \mathrm{Res}_{z=0}\big(\tfrac{1}{z^{n+1}}\cdot\tfrac{1}{(1-z)^{k+1}}\,dz\big) \\ &= \Theta_n\big((1-z)^{-(k+1)}\big) \\ &= (-1)^n\tbinom{-(k+1)}{n} \\ &= \tbinom{n+k}{n}.\end{aligned}$$

Note that the substitution $z = 1-e^{-x}$ was used for the third equality.

27. Formal group laws and complex-oriented cohomology theories

28. Algebraic cycles on complex varieties

Note: The material in this section requires the Atiyah-Hirzebruch spectral sequence from Section 29 below.

Let X be a smooth, projective algebraic variety over $\mathbb{C}$. Every smooth subvariety Z of codimension q has a fundamental class $[Z] \in H^{2q}(X;\mathbb{Z})$. In fact the smoothness of Z is not needed here: *every* subvariety $Z \hookrightarrow X$ of codimension q has such a fundamental class. This can be proven either by using resolution of singularieties or by more naive methods—we will explain below.

Define $H^{2q}_{alg}(X;\mathbb{Z}) \subseteq H^{2q}(X;\mathbb{Z})$ to be the subgroup generated by the fundamental classes of all algebraic subvarieties. How large are these "algebraic" parts of the even cohomology groups? Are there examples of varieties X for which the algebraic part does not equal everything?

The answer to the latter question is provided by Hodge theory: yes, there do exist varieties X where not all of the even cohomology is algebraic. Hodge theory gives a decomposition of the cohomology groups $H^n(X;\mathbb{C}) \cong \oplus_{p+q=n}H^{p,q}(X)$, and the algebraic classes all lie in the $H^{i,i}$ pieces. The (even-dimensional) cohomology is entirely algebraic only when $H^{p,q}=0$ for $p \neq q$ and $p+q$ even. But it is known that for an elliptic curve E one has $H^{1,0} = \mathbb{C} = H^{0,1}$, and so for $E\times E$ one gets $H^{2,0}(E\times E) \cong \mathbb{C} \cong H^{0,2}(E\times E)$. So not all of $H^2(E\times E)$ is algebraic.

The problem with Hodge theory is that it cannot see any torsion classes, as the coefficients of the cohomology groups need to be $\mathbb{C}$. Could it be true that torsion cohomology classes are always algebraic? A classical theorem of Lefschetz, reproved by Hodge, says that this holds for classes in H^2. But for higher cohomology groups the answer is again no, and the first proof was given by Atiyah and Hirzebruch [AH2]. There are three basic components to their proof:

(1) Every algebraic class must survive the Atiyah-Hirzebruch spectral sequence. The vanishing of the differentials therefore gives a sequence of obstructions for a given cohomology class to be algebraic.
(2) The differentials in the spectral sequence can be analyzed in terms of cohomology operations. On a p^e-torsion class the first nonzero differential is d_{2p-1} and coincides with the operation $-\beta P^1$.
(3) A clever construction of Serre's shows how to obtain smooth algebraic varieties whose cohomology contains that of BG through a range of dimensions. Using this, one readily finds smooth varieties with even-dimensional, p-torsion cohomology classes for which βP^1 does not vanish; such a class cannot be algebraic.

The Atiyah-Hirzebruch proof is no longer the most efficient way to obtain conditions for torsion classes to be algebraic. Resolution of singularities shows that every algebraic class is actually a pushforward of the fundamental class of a manifold—i.e., every algebraic class lifts into complex cobordism. In the 1950s Thom had already obtained some necessary conditions for such a lifting to exist, in terms of Steenrod operations. Via this method K-theory is not needed at all, and moreover Thom's theory yields a stronger set of conditions. Note that resolution of singularities was not provided by Hironaka until 1964, and so of course was not available at the time of [AH2].

Despite the modern shortcomings of Atiyah and Hirzebruch's method, we will spend this section describing it in detail. It sheds some light on the relationship between K-theory and singular cohomology, and also offers some neat observations about algebraic varieties.

Remark 28.1. We should mention that [AH2] treats the case of *analytic* cycles in addition to algebraic cycles. The proofs are essentially the same, with one or two key differences. We will not cover the material on analytic cycles here.

Remark 28.2. The modern Hodge conjecture states that any class $\alpha \in H^{2n}(X;\mathbb{Q})$ whose image in $H^{2n}(X;\mathbb{C})$ lies in the Hodge group $H^{n,n}(X)$ is necessarily algebraic—that is, it lies in the subgroup $H^{2n}(X;\mathbb{Q})_{alg}$. When Hodge originally raised this question he did not explicitly specify rational coefficients. Since any torsion class in $H^{2n}(X;\mathbb{Z})$ would map to zero in $H^{2n}(X;\mathbb{C})$, and therefore lie inside $H^{n,n}(X)$, an integral version of the Hodge conjecture would imply that all torsion classes are algebraic. One of the main points of [AH2] was to demonstrate that this integral form of the Hodge conjecture does not hold.

28.3. **Fundamental classes for subvarieties.** If X is a smooth algebraic variety and $Y \subseteq X$ is a smooth subvariety of codimension q then we have seen that complex orientability yields a fundamental class $[Y] \in H^{2q}(X)$ and a relative fundamental class $[Y]_{rel} \in H^{2q}(X, X-Y)$. This comes about by choosing a tubular neighborhood U of Y that is homeomorphic to the normal bundle, and using the isomorphisms

$$H^{2q}(X, X-Y) \cong H^{2q}(U, U-Y) \cong H^{2q}(N, N-0).$$

Assuming Y is connected it follows that $H^{2q}(X, X-Y) \cong \mathbb{Z}$ (by Thom isomorphism), and $[Y]_{rel}$ is defined to be the image of the Thom class $U_N \in H^{2q}(N, N-0)$. The fundamental class $[Y]$ is just the image of $[Y]_{rel}$ under $H^*(X, X-Y) \to H^*(X)$. Note that the argument shows that choice of U and homeomorphism $U \cong N$ to be irrelevant: there are only two generators in $H^{2q}(X, X-Y)$, and ???

We aim to show the existence of fundamental classes $[Y]$ even when the subvariety Y is not smooth.

Lemma 28.4. *Let X be a smooth algebraic variety and let $W \hookrightarrow Y \hookrightarrow X$ be subvarieties. Assume that Y has codimension q inside of X, and that W has codimension at least one inside of Y.*
(a) $H^*(X) \to H^*(X - Y)$ is an isomorphism for $* \leq 2q - 2$.
(b) $H^*(X, X - Y) \to H^*(X - W, X - Y)$ is an isomorphism for $* \leq 2q$.
(c) If Y is irreducible then $H^{2q}(X, X - Y) \cong \mathbb{Z}$.

Proof. First note that part (a) is true when Y is smooth, using the long exact sequence for the pair $(X, X-Y)$, the isomorphism $H^*(X, X-Y) \cong H^*(N, N-0)$, and the Thom isomorphism theorem. For the general case, we can filter Y by subvarieties

$$\emptyset \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_s = Y$$

where each Y_i is the singular set of Y_{i+1} (so that $Y_{i+1} - Y_i$ is smooth). Since Y_0 is smooth, we can assume by induction that $H^*(X) \to H^*(X - Y_i)$ is an isomorphism for $* \leq 2(\operatorname{codim} Y_i) - 2$. Now look at the composition

$$H^*(X) \to H^*(X - Y_i) \to H^*(X - Y_{i+1}) = H^*\big((X - Y_i) - (Y_{i+1} - Y_i)\big).$$

Since $Y_{i+1} - Y_i$ is smooth in $X - Y_i$, the second map is an isomorphism for $* \leq 2(\operatorname{codim} Y_{i+1}) - 2$. It follows that the composite is an isomorphism for $* \leq 2(\operatorname{codim} Y_{i+1}) - 2$ as well, and now the desired result follows by induction.

Part (b) follows from part (a) via the long exact sequences

$$\begin{array}{ccccccccc}
\cdots \to & H^{*-1}(X-Y) & \longrightarrow & H^*(X, X-Y) & \longrightarrow & H^*(X) & \longrightarrow & H^*(X-Y) & \to \cdots \\
& \uparrow \cong & & \uparrow & & \uparrow j^* & & \uparrow \cong & \\
\cdots \to & H^{*-1}(X-Y) & \to & H^*(X-W, X-Y) & \to & H^*(X-W) & \to & H^*(X-Y) & \to \cdots
\end{array}$$

By (a) the map labelled j^* is an isomorphism for $* \leq 2q$, and so the result follows by the five lemma.

Finally, for (c) we let Z be the singular set of Y. Then $H^{2q}(X, X - Y) \cong H^{2q}(X - Z, (X - Z) - (Y - Z))$ by (b). But $Y - Z$ is a smooth subvariety of $X - Z$ of codimension q, and it is connected because Y was irreducible. The remarks from the beginning of this section show that this cohomology group is isomorphic to Z, with generator $[Y - Z]_{rel}$. □

Now let $Y \hookrightarrow X$ be an algebraic subvariety of codimension q, and let Z be the singular set. Since $Y - Z$ is a smooth subvariety of $X - Z$, we have a fundamental class $[Y - Z] \in H^{2q}(X - Z)$. But by Lemma 28.4(a) the map $H^{2q}(X) \to H^{2q}(X - Z)$ is an isomorphism. We define $[Y] \in H^{2q}(X)$ to be the preimage of $[Y - Z]$ under this map.

Define $H^*_{alg}(X) \subseteq H^*(X)$ to be the subgroup generated by the fundamental classes of all the algebraic subvarieties of X.

In the next section it will help to be able to focus on classes $[Y]$ where Y is irreducible. To this end, the following is useful:

Lemma 28.5. *If $Y \hookrightarrow X$ is a codimension q subvariety with irreducible components $Y_1, \ldots, Y_k$ then $[Y] = \sum_i [Y_i]$????? The subgroup $H^*_{alg}(X)$ is spanned by classes*

Proof. ???? □

We need one more lemma before moving on:

Proposition 28.6. *Let M be a real manifold and let $N \hookrightarrow M$ be a codimension k real submanifold with a tubular neighborhood. Then $M - N \hookrightarrow M$ is $(k-1)$-connected.*

Sketch. ???? □

28.7. **Vanishing of differentials on algebraic classes.** Let X be a smooth algebraic variety, and let $Y \subseteq X$ be subvariety. Let $\mathcal{F}_\bullet$ be a resolution of $\mathcal{O}_Y$ by locally-free $\mathcal{O}_X$-modules, and write $F_\bullet$ for the associated chain complex of complex vector bundles on X. Then $[F_\bullet]$ defines a class in $K^0(X, X-Y)$ which we will denote $[Y]_{K,rel}$. Note that when Y is smooth this agrees with the relative fundamental class provided by the complex orientation of K-theory, by Theorem 18.8.

Proposition 28.8. *One has* $\mathrm{ch}([Y]_{K,rel}) = [Y]_{H,rel}$ + *higher order terms.*

Proof. We first prove this when Y is smooth. If N denotes the normal bundle for Y in X, then the result will follow once we know $\mathrm{ch}(U_N^K) = U_N^H$ +higher order terms—for both $[Y]_{K,rel}$ and $[Y]_{H,rel}$ are obtained from the Thom classes by applying natural maps. However, we have already seen in our discussion of Riemann-Roch that the complete formula is in fact

$$\mathrm{ch}(\mathcal{U}_N^K) = \mathcal{U}_N^H \cdot \mathrm{Td}(N)^{-1}$$

(see Proposition 25.3). Now just observe that $\mathrm{Td}(N)^{-1} = 1 +$ higher order terms.

Now let Y be arbitrary. Let Z be the singular locus, and the q denote the codimension of Y. Consider the diagram

$$\begin{array}{ccc} K^0(X, X-Y) & \xrightarrow{j^*} & K^0(X-Z, X-Y) \\ \downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\ H^*(X, X-Y) & \xrightarrow{j^*} & H^*(X-Z, X-Y). \end{array}$$

Both maps j^* are simply restriction to an open set, and so $j^*([Y]_{K,rel}) = [Y - Z]_{K,rel}$ and $j^*([Y]_{H,rel}) = [Y-Z]_{H,rel}$. By Lemma 28.4(b) the bottom map is an isomorphism for $* \leq 2q$. So the desired result for Y follows from the corresponding result for $Y - Z$, which has already been proven because $Y - Z$ is smooth in $X - Z$. □

Theorem 28.9. *Let X be a smooth algebraic variety, and let $Y \subseteq X$ be a subvariety. Then $[Y]_H$ survives the Atiyah-Hirzebruch spectral sequence; that is, all differentials vanish on this class.*

Proof. This is now easy. The class $[Y]_H$ is the image of $[Y]_{H,rel}$ under the natural map $H^*(X, X-Y) \to H^*(X)$. By naturality of the Atiyah-Hirzebruch spectral sequence, it suffices to show that all differentials on $[Y]_{H,rel}$ are zero. This follows from Proposition 28.8 and ????. □

Corollary 28.10. *Let X be a smooth algebraic variety, and let p be a fixed prime. If a p^e-torsion class $u \in H^{ev}(X)$ is algebraic then $\beta P^1(\overline{u}) = 0$.*

Proof. Immediate from Theorem 28.9 and Proposition 29.20(b). □

28.11. **Construction of varieties with non-algebraic cohomology classes.** At this point our job is to construct a smooth, projective algebraic variety X that has a class $u \in H^{ev}(X)$ for which $\beta P^1 x \neq 0$, for some prime p. Such a class cannot be algebraic by Corollary 28.10. It turns out that p can be any prime we like—that is, for any given p we can find an example of an X and a u. Moreover, u can be taken to lie in degree 4. The construction comes out of the following three results:

Theorem 28.12 (Serre)**.** *Let G be a finite group and let $n \geq 0$.*

(a) There exists a linear action of G on a projective space $\mathbb{C}P^N$ together with an n-dimensional, closed, smooth subvariety $X \hookrightarrow \mathbb{C}P^N$ which is a complete intersection, invariant under G, and has G acting freely.
(b) If X is a variety having the properties in (a), then its homotopy $(n-1)$-type is the same as that of $K(\mathbb{Z}, 2) \times BG$.

Corollary 28.13. *Let G be a finite group and let $n \geq 0$. There exists an n-dimensional, smooth, projective variety whose $(n-1)$-homotopy type is the same as that of $K(\mathbb{Z}, 2) \times BG$.*

Proposition 28.14. *Let p be a prime. Then there exists a finite group G and a class $u \in H^4(BG; \mathbb{Z})$ that is killed by p and is such that $\beta P^1(u) \neq 0$.*

We postpone the proofs for one moment so that we can observe the immediate consequence:

Corollary 28.15 (Atiyah-Hirzebruch)**.** *Fix a prime p. There exists a smooth, projective, complex algebraic variety X and a class $u \in H^{ev}(X)$ that is killed by p such that u is not algebraic.*

Proof. By Proposition 28.14 there exists a finite group G and a p-torsion class $u \in H^4(BG; \mathbb{Z})$ such that $\beta P^1(u) \neq 0$. By Theorem 28.12 there is a smooth, projective variety X whose $(2p+4)$-type is the same as $K(\mathbb{Z}, 2) \times BG$. But then $H^*(BG; \mathbb{Z})$ is a direct summand of $H^*(X)$ up through dimension $2p+4$. By Corollary 28.10 the class in $H^4(X)$ corresponding to u cannot be algebraic. □

Theorem 28.12 requires some algebraic geometry, but the proofs of both Corollary 28.13 and Proposition 28.14 are purely topological. We tackle these in reverse order:

Proof of Proposition 28.14. We start by considering $p = 2$ (the odd case turns out to be extremely similar). As a first attempt we might try to take $G = \mathbb{Z}/2$. Then $B\mathbb{Z}/2 = \mathbb{R}P^\infty$, $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$, and $H^*(\mathbb{R}P^\infty; \mathbb{Z}) = \mathbb{Z}/2[x^2]$. Unfortunately $\mathrm{Sq}^3 = \beta\,\mathrm{Sq}^2$ vanishes on all the integral classes, by an easy calculation.

Next look at $B(\mathbb{Z}/2 \times \mathbb{Z}/2) = \mathbb{R}P^\infty \times \mathbb{R}P^\infty$. The mod 2 cohomology is $\mathbb{Z}/2[x, y]$, and the integral cohomology is the subring consisting of all elements whose Bockstein vanishes. Such elements of course include all polynomials in x^2 and y^2, but it also includes $\theta = x^2y + xy^2 = \beta(xy)$. In fact all the elements of the integral cohomology look like $x^{2i}y^{2j} \cdot \theta$. Another easy calculation shows that Sq^3 applied to such an element is $x^{2i}y^{2j}\,\mathrm{Sq}^3(\theta)$, and $\mathrm{Sq}^3(\theta) = x^4y^2 + x^2y^4 \neq 0$. This gives us lots of classes that are not killed by Sq^3, however they are all in odd dimensions. So this still doesn't solve our problem.

In the preceding paragraph, the reason things didn't work ultimately came down to the fact that $\beta(xy)$ had odd dimension. This gets fixed once we move to

$B(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)$. The mod 2 cohomology is $\mathbb{Z}/2[x, y, z]$, and again the integral cohomology is the subring consisting of the elements where the Bockstein vanishes. One such element is $\theta = \beta(xyz) = x^2yz + xy^2z + xyz^2$. It is easy to calculate that $\mathrm{Sq}^3 \theta = x^4y^2z + x^4yz^2 + \cdots \neq 0$. So finally we have an even-dimensional integral cohomology class where Sq^3 vanishes.

The argument for odd primes works the same way. Recall that $H^*(B\mathbb{Z}/p; \mathbb{Z}/p) = \Lambda(u) \otimes \mathbb{F}_p[v]$, with $\beta(u) = v$ and $P^1(v) = v^p$. Take $G = \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$ and look at $\theta = \beta(u_1u_2u_3) \in H^4(BG; \mathbb{Z})$. A simple calculation shows that $\beta P^1(\theta) \neq 0$. $\square$

Next, the corollary to Serre's theorem:

Proof of Corollary 28.13. Let $X \hookrightarrow \mathbb{C}P^N$ be the subvariety provided by Theorem 28.12. Since $X \hookrightarrow \mathbb{C}P^N$ is a complete intersection, the strong form of the Weak Lefschetz Theorem ??? yields that $X \to \mathbb{C}P^N$ is an $(n-1)$-equivalence. Let $\eta \to \mathbb{C}P^N$ be the tautological line bundle $\mathbb{C}^{N+1} - 0 \to \mathbb{C}P^N$. Note that G acts on $\mathbb{C}^{N+1}$, and so G acts on the bundle η. Hence G acts on the pullback bundle $j^*\eta \to X$. Since the action of G on X is free we get a line bundle $(j * \eta)/G \to X/G$ which pulls back to $j^*\eta$ along the projection $X \to X/G$. Let the classifying map for this line bundle be $X/G \to \mathbb{C}P^\infty$. The diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{C}P^N \\ \downarrow & & \downarrow \\ X/G & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

necessarily commutes up to homotopy, as the two compositions classify the same bundle $j^*\eta$. Since $X \to \mathbb{C}P^N$ is an $(n-1)$-equivalence, so is the composite $X \to \mathbb{C}P^\infty$.

Consider the composite map $X \to X/G \to \mathbb{C}P^\infty$. This is a G-equivariant map, where the target is given the trivial G-action. Since the map is an $(n-1)$-equivalence, the induced map on homotopy orbit spaces $X_{hG} \to (\mathbb{C}P^\infty)_{hG}$ is also an $(n-1)$-equivalence. (Recall that $Z_{hG} = (Z \times EG)/G$. Crossing with EG preserves the $(n-1)$-equivalence, and since $Z \times EG \to Z_{hG}$ is a covering space it follows that quotienting by G also preserves the $(n-1)$-equivalence). But since the G-action on X is free one has $X_{hG} \simeq X/G$, and since the action on $\mathbb{C}P^\infty$ is trivial one has $(\mathbb{C}P^\infty)_{hG} \simeq \mathbb{C}P^\infty \times BG$. This completes the proof. $\square$

Finally, we prove Serre's theorem.

Proof of Theorem 28.12. We first give the proof for $G = \mathbb{Z}/2$. Even though the general case is basically the same, certain steps can be made more concrete by restricting to this case.

Let $\mathbb{C}^{2M}$ have coordinates $x_1, \ldots, x_M, y_1, \ldots, y_M$, and let G act trivially on the x's and by negation on the y's. Consider the induced action on $P = \mathbb{P}(\mathbb{C}^{2M}) = \mathbb{C}P^{2M-1}$. The homogeneous coordinate ring is $R = \mathbb{C}[x_1, \ldots, x_M, y_1, \ldots, y_M]$, and the ring of invariants $S = R^G$ is the subring generated by the x_i's and the y_iy_j's (including $i = j$). An easy argument shows that every homogeneous element of S having even degree is a polynomial in the elements x_ix_j and y_iy_j. That is, if $S(2) \subseteq S$ is the $\mathbb{C}$-linear span of all even-dimensional homogeneous elements then $S(2)$ is generated (as a subring) by elements of degree 2.

Let $\alpha_1, \dots, \alpha_s$ denote the degree 2 monomials in x, in some order. Similarly, let $\beta_1, \dots, \beta_s$ denote the degree 2 monomials of y, and let $b = 2s - 1$. Let $f\colon P \to \mathbb{C}P^b$ be the map

$$f([x_1 : \cdots : x_N : y_1 : \ldots : y_N]) \mapsto [\alpha_1 : \ldots : \alpha_s : \beta_1 : \ldots : \beta_s].$$

Clearly f induces a map $P/G \to \mathbb{C}P^b$. It is easy to see that this is a projective embedding (???). Its image Z is the closed subvariety whose homogeneous coordinate ring is $S(2)$, regarded as a quotient of the polynomial algebra $\mathbb{C}[\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_s]$.

Let $A \subseteq P$ be the set of fixed points for the $\mathbb{Z}/2$-action. One readily checks that A is the disjoint union $\mathbb{C}P^{M-1} \amalg \mathbb{C}P^{M-1}$, where the first piece corresponds to the vanishing of the x_i's and the second piece to the vanishing of the y_j's. The fibers of f are sets with at most two elements, and so $f(A) \subseteq Z$ is a subvariety having the same dimension as A. Note that $P - A \to Z - f(A)$ is a covering space, and so $Z - f(A)$ is nonsingular.

Let $H \subseteq \mathbb{C}P^b$ be a generic plane of dimension $b + n - (2M - 1)$. We can choose H so that it misses the singular set of Z and intersects Z transversally, and if $\dim f(A) + \dim H < b$ we can simultaneously require that H does not intersect $f(A)$. This dimension criterion is $M - 1 + b - (2M - 1) + n < b$, or just $n < M$. Our choice of H guarantees that $Z \cap H$ is a nonsingular variety of dimension n. Let $X = f^{-1}(Z \cap H)$. The criterion that $H \cap f(A) = \emptyset$ implies that $X \cap A = \emptyset$, and so G acts freely on X and the map $f|_X\colon X \to Z \cap H$ is a two-fold covering space. So X is also nonsingular of dimension n.

The plane H is defined by the vanishing of linear elements $g_1, \dots, g_t$ in the ring S, where $t = (2M - 1) - n$. Via the inclusion $S \subseteq R$ we can regard these as elements of R, where they are homogeneous of degree 2. The variety X is the vanishing set of these polynomials. Given that X is of codimension t, we find that X is a set-theoretic complete intersection: it is the intersection of the t hypersurfaces defined by each of the g_i's. It remains to show that X is actually a scheme-theoretic complete intersection: i.e., that the ideal of functions vanishing on X is generated by a regular sequence.

Since X has codimension t is follows that $\operatorname{ht}(g_1, \dots, g_t) = t$. By ??? this implies that $g_1, \dots, g_t$ is a regular sequence. If we let $I = (g_1, \dots, g_t)$ then the ideal of functions vanishing on X is $\operatorname{Rad}(g_1, \dots, g_t)$. We will show that $\operatorname{Rad}(I) = I$, as this proves that X is a scheme-theoretic complete intersection.

By Macaulay's Unmixedness Theorem [E, Corollary 18.14], all associated primes of I are minimal primes of I. So I has a primary decomposition $I = Q_1 \cap \cdots \cap Q_k$ where each Q_i is primary and $\operatorname{Rad}(Q_i) = P_i$ is a minimal prime of I. The $V(P_i)$'s are the irreducible components of X, and so for each i we can choose a closed point $m \in V(P_i)$ that does not belong to any other component. So m is a maximal ideal containing P_i, and $IR_m = Q_i R_m$.

Let $m' = f(m)$ and consider the diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X,m} & \longleftarrow & \mathcal{O}_{Z,m'} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}}_{X,m} & \xleftarrow{\cong} & \widehat{\mathcal{O}}_{Z,m'}. \end{array}$$

The bottom map is an isomorphism because $X \to Z$ is a two-fold covering space. The assumption that H meets Z transversally implies that $g_1, \dots, g_t$ is part of a regular system of parameters for $\mathcal{O}_{Z,m'}$; that is to say, their images in the Zariski cotangent space $(m')/(m')^2$ are independent. The same is therefore true in $\mathcal{O}_{X,m}$, because all maps in the above square induce isomorphisms on the Zariski cotangent space.

In particular, the fact that $g_1, \dots, g_t$ is part of a regular system of parameters in $\mathcal{O}_{X,m}$ implies that IR_m is prime. So $Q_iR_m = IR_m = \mathrm{Rad}(IR_m) = P_iR_m$. This can only happen if $Q_i = P_i$ (if Q is primary with radical P and $P \subseteq m$, then $QR_m = PR_m$ if and only if $Q = P$). We have thus proven that $Q_i = P_i$ for every i, and this implies $I = \mathrm{Rad}(I)$.

This proof is now completed for the case $G = \mathbb{Z}/2$.

For a general finite group G let V be the regular representation and let $P = \mathbb{P}(V^M)$. Let R be the homogeneous coordinate ring of P, and let $S = R^G$. If $S(d) \subseteq S$ is the subring spanned by homogeneous elements in degrees a multiple of d, one can prove that for some value of d the ring $S(d)$ is generated as an algebra by its elements of degree d. Choose a $\mathbb{C}$-basis $f_0, \dots, f_b$ for these generators and let $f\colon P \to \mathbb{C}P^b$ be the map $x \mapsto [f_0(x) : f_1(x) : \cdots : f_b(x)]$. This map induces a projective embedding $P/G \hookrightarrow \mathbb{C}P^b$; call the image Z. Let $A \subseteq P$ be the set of elements with nontrivial stabilizer under G. For $g \in G$ an easy argument shows that any eigenspace of g acting on V must have dimension equal to at most the number of right cosets of $\langle g \rangle$ in G. So the dimension of the eigenspace is at most $\#G/\#\langle g \rangle$, and therefore is bounded above by $\#G/2$. The eigenspaces of g acting on V^M thus have dimension at most $M \cdot \#G/2$, and from this one derives that $\dim A \leq (M \cdot \#G/2) - 1$. The rest of the argument proceeds almost identically to the $G = \mathbb{Z}/2$ case, the only change being that we take H to have dimension $b + n - (M \cdot \#G - 1)$ and that we only need to require $M > \frac{2n}{\#G}$ in order to choose H so that it avoids $f(A)$. □

28.16. **Thom's theory.** Although this part of the story doesn't use K-theory, the obstructions to algebraicity obtained from resolution of singularities are so simple that they are worth discussing here.

Theorem 28.17. *Let X be a smooth algebraic variety, and let p be a fixed prime. If $u \in H^{ev}(X;\mathbb{Z})$ is algebraic then all odd-degree cohomology operations vanish on the mod p reduction $\bar{u} \in H^{ev}(X;\mathbb{Z}/p)$. In particular, all the odd Steenrod squares vanish on the mod 2 reduction of u.*

The above theorem was probably folklore since the 1960s. It finally explicitly appeared in the beautiful paper [T].

The first key to the theorem is that there is a map of cohomology theories $MU^*(-) \to H^*(-)$, and that this map is compatible with the complex orientations. In particular, if $Y \hookrightarrow X$ is a smooth subvariety it sends $[Y]_{MU}$ to $[Y]_H$. We will not explain this claim in detail, but it is not hard.

The next part of the argument is best explained using the language of spectra. The above map of cohomology theories comes from a map of spectra $MU \to H\mathbb{Z}$. If $Y \hookrightarrow X$ is a smooth subvariety of codimension q then $[Y]_{MU} \in MU^{2q}(X)$ is represented by a map $X \to \Sigma^{2q}MU$, and likewise $[Y]_H \in H^{2q}(X)$ is represented by a map $X \to \Sigma^{2q}H\mathbb{Z}$. In the homotopy category of spectra we have the commutative diagram

$$\begin{array}{ccc} & & \Sigma^{2q}MU \\ & \overset{[Y]_{MU}}{\nearrow} & \downarrow \\ X & \xrightarrow[{[Y]_H}]{} & \Sigma^{2q}H\mathbb{Z}. \end{array}$$

Let θ be a cohomology operation of degree r on $H^*(-;\mathbb{Z}/p)$. This is a map of spectra $H\mathbb{Z}/p \to \Sigma^r H\mathbb{Z}/p$. The application of this operation to the mod p reduction of $[Y]_H$ enhances our diagram:

$$\begin{array}{ccccccc} & & \Sigma^{2q}MU & & \overset{f}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & & \searrow \\ & \overset{[Y]_{MU}}{\nearrow} & \downarrow & & & & \\ X & \xrightarrow[{[Y]_H}]{} & \Sigma^{2q}H\mathbb{Z} & \longrightarrow & \Sigma^{2q}H\mathbb{Z}/p & \xrightarrow[\theta]{} & \Sigma^{2q+r}H\mathbb{Z}/p. \end{array}$$

The map labelled f is just the evident composite. Note that f is an element of $H^r(MU;\mathbb{Z}/p)$. The "miracle" is that we can easily compute this group. The spaces making up the spectrum MU are just Thom spaces of the universal $\eta_n \to BU(n)$, and their integral cohomology is known by the Thom isomorphism. We leave the details to the reader, but the trivial conclusion here is that $H^*(MU;\mathbb{Z})$ is free abelian and concentrated in even degrees. It follows that $H^*(MU;\mathbb{Z}/p)$ vanishes in all odd degrees. In particular, the map f is null when r is odd! Thus, we have proven Thom's theorem:

Proposition 28.18 (Thom)**.** *Let X be a smooth algebraic variety and let $Y \hookrightarrow X$ be a smooth subvariety. Then all odd degree cohomology operations vanish on the class $[Y]_H \in H^*(X)$.*

Remark 28.19. It is interesting to note how easy the language of spectra makes the above argument. As a challenge, try to unwind the argument and rephrase it without using spectra—it is not so pleasant.

One can deduce Theorem 28.17 from Proposition 28.18 using resolution of singularities and a little work. We are not going to give complete details, but we give a rough sketch. Complete details (and much more) can be found in [T].

It suffices to prove Theorem 28.17 when u is the fundamental class of an irreducible subvariety $Y \hookrightarrow X$, say of codimension q. Such elements generate all algebraic cohomology classes. By Hironaka there is a resolution of singularities $\tilde{Y} \to Y$ obtained by successively blowing up Y at closed subschemes. Even more, one can successively blow up X at the same subschemes to produce a commutative square

$$\begin{array}{ccc} \tilde{Y} & \rightarrowtail & \tilde{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \rightarrowtail & X \end{array}$$

where the horizontal maps are closed inclusions and the vertical maps are compositions of successive blow-ups (in particular, they are proper). We of course have the class $[\tilde{Y}]_{MU} \in MU^{2q}(\tilde{X})$, and with a little work one can construct a pushforward map $\pi_!\colon MU^{2q}(\tilde{X}) \to MU^{2q}(X)$. We claim that $\pi_!([\tilde{Y}]_{MU})$ is a lift of the class $[Y]_H$. By ???? this can be checked by applying $j^*\colon H^*(X) \to H^*(X - Z)$

where Z is the singular set of Y and seeing that $j^*(\pi_!([\tilde{Y}]_{MU})) = [Y - Z]$. This can in turn be deduced from an appropriate push-pull formula and the fact that $\tilde{Y} - \pi^{-1}(Y) \to Y - Z$ is a homeomorphism. In any case, if you accept this last point then we now have the diagram

$$\begin{array}{ccc} & & \Sigma^{2q}MU \\ & \overset{\pi_!([\tilde{Y}]_{MU})}{\nearrow} & \downarrow \\ X & \xrightarrow[{[Y]_H}]{} & \Sigma^{2q}H\mathbb{Z} \end{array}$$

and at this point everything proceeds the same as before. This completes our sketch of a proof for Theorem 28.17.

Note that even with this approach, as opposed to the K-theory approach of Atiyah-Hirzebruch, one still needs to gives examples of algebraic varieties with nontrivial odd-degree operations on even-dimensional cohomology classes. So the hard work that went on in Section 28.11 is still necessary.

Part 5. Topological techniques and applications

In the next few sections we will mostly ignore the "geometric" perspective on K-theory that we have developed so far in these notes. Instead we will concentrate on the topological aspects of K-theory, in particular its use *as a cohomology theory* (forgetting about the complex-orientation). We will develop the basic topological tools for computing K-groups, use them to carry out some important computations, and then apply these computations to solve (or at least obtain information about) certain types of geometric, algebraic, and topological problems.

29. The Atiyah-Hirzebruch spectral sequence

Let $\mathcal{E}$ be a cohomology theory. Let X be a CW-complex with cellular filtration

$$\emptyset = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq X.$$

So each F_k/F_{k-1} is a wedge of k-spheres, and $X = \bigcup_k F_k$. Since one knows the cohomology groups $\mathcal{E}^*(F_k/F_{k-1})$, one can attempt to inductively determine the cohomology groups $\mathcal{E}^*(F_k)$ and thus to eventually determine $\mathcal{E}^*(X)$. A spectral sequence is a device for organizing all the information in such a calculation, and it has the surprising feature that one can determine $\mathcal{E}^*(X)$ without *explicitly* determining each of the steps $\mathcal{E}^*(F_k)$. It is somewhat magical that this can be done.

We will not try to teach the theory of spectral sequences from scratch here. For a thorough treatment the reader may refer to [Mc], for example. We will assume the reader has some familiarity with this theory, but at the same time we give a brief review.

29.1. **Generalities.** Each inclusion $F_{q-1} \hookrightarrow F_q$ yields a long exact sequence on cohomology, and these long exact sequences braid together to yield the following infinite diagram:

(29.2)

$$\begin{array}{ccccccccc}
 & & \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
\cdots & \longrightarrow & \boxed{\mathcal{E}^{p-1}(F_q)} & \xrightarrow{j} & \mathcal{E}^p(F_{q+1},F_q) & \xrightarrow{k} & \mathcal{E}^p(F_{q+1}) & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+2},F_{q+1}) \xrightarrow{k} \\
 & & \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
\cdots & \longrightarrow & \boxed{\mathcal{E}^{p-1}(F_{q-1})} & \xrightarrow{j} & \boxed{\mathcal{E}^p(F_q,F_{q-1})} & \xrightarrow{k} & \boxed{\mathcal{E}^p(F_q)} & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+1},F_q) \xrightarrow{k} \\
 & & \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
\cdots & \longrightarrow & \mathcal{E}^{p-1}(F_{q-2}) & \xrightarrow{j} & \mathcal{E}^p(F_{q-1},F_{q-2}) & \xrightarrow{k} & \boxed{\mathcal{E}^p(F_{q-1})} & \xrightarrow{j} & \boxed{\mathcal{E}^{p+1}(F_q,F_{q-1})} \xrightarrow{k} \\
 & & \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & &
\end{array}$$

The terms in bold-face constitute one long exact sequence: the one for the inclusion $F_{q-1} \hookrightarrow F_q$. Translating these terms vertically yields an infinite family of long exact sequences, each linked to the next via two of their three terms. A diagram such as this is called an *exact couple*. One obtains a spectral sequence of the form

$$E_1^{p,q} = \mathcal{E}^p(F_q, F_{q-1}) \Rightarrow \mathcal{E}^p(X).$$

Let us explain how this works, and in the course of doing so we will also explain what it means. By the way, notice that in our particular setup the columns of the diagram are eventually zero as one proceeds downward, because $F_i = \emptyset$ when i is negative. Notice as well that if $X = F_n$ for some value of n (so the filtration is finite) then the columns stabilize when moving upwards.

(0). Here is the basic idea for how the spectral sequence operates. Consider an element $x \in \mathcal{E}^p(F_q, F_{q-1})$, and proceed as follows:

(i) Let $x_0 = kx$.

(ii) If $j(x_0) = 0$ then $x_0 = i(x_1)$ for some $x_1 \in \mathcal{E}^p(F_{q+1})$. We may then look at jx_1.

(iii) If $jx_1 = 0$ then $x_1 = i(x_2)$ for some $x_2 \in \mathcal{E}^p(F_{q+2})$. We may then look at jx_2.

(iv) Continuing in this way, we get a sequence of "obstructions" jx_u, $u = 0, 1, 2, \ldots$. Each one only exists if the previous one vanishes. Note that at each stage the vanishing of jx_u doesn't depend on the choice of x_u; however, it *may* depend on the choice of x_v made at some previous stage $v < u$. In this sense the obstructions are not unique: different choices of lifts may lead to different obstructions later down the line.

(v) If we can make choices such that all of these obstructions vanish then we are able to lift kx arbitrarily far up in the diagram. If the filtration $F_\bullet$ was finite this means that we have produced an element of $\mathcal{E}^p(X)$; it turns out something like this also works for infinite filtrations, although the resulting element of $\mathcal{E}^p(X)$ is only uniquely determined in good cases. The spectral sequence is a device for keeping track of these obstructions and liftings, and what they ultimately produce.

We will now go through all of the machinery needed to define and work with the spectral sequence associated to our exact couple.

(1). For $1 \leq r \leq \infty$ write

$$Z_r^{p,q} = \{x \in \mathcal{E}^p(F_q, F_{q-1}) \,|\, kx \text{ may be lifted at least } r \text{ times under } i\}.$$

This is called the group of ***r*-cycles** in the spectral sequence. The phrasing is ambiguous when $r = \infty$, but we mean $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$. We also define $B_r^{p,q} \subseteq Z_r^{p,q}$ to be the subgroup generated by all "obstructions" that arise from at most $r-1$ layers lower down in the diagram. To be precise, $B_r^{p,q}$ is spanned by the sets

$$ji^{-s}k(\mathcal{E}^{p-1}(F_{q-s-1}, F_{q-s-2}))$$

for $0 \leq s \leq r-1$. It is best to immediately forget this precise description and just remember the idea.

Notice that everything in $B_r^{p,q}$ maps to zero under k, and hence is contained in every $Z_s^{p,q}$. That is, we have

$$0 = B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_2^{p,q} \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} = \mathcal{E}^p(F_q, F_{q-1}).$$

Note that $B_\infty^{p,q} = \bigcup_r B_r^{p,q}$ and $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$.

(2). It as an easy exercise to prove that $x \in B_r^{p,q}$ if and only if x can be written as $x = j(y_1 + y_2 + \cdots + y_r)$ for some y_u's such that $i^u(y_u) = 0$, for all u. As an immediate corollary, $B_\infty^{p,q}$ coincides with the image of j (or equivalently with the kernel of k).

Let $x_0 \in \mathcal{E}^p(F_q)$ and assume that we lifted x_0 a total of r times: that is, assume we have chosen elements $x_u \in \mathcal{E}^p(F_{q+u})$ for $1 \leq u \leq r$ such that $i(x_u) = x_{u-1}$. If $j(x_r) \in B_s$ for some s then (using the result of the previous paragraph) there exists a chain of elements $x'_u \in \mathcal{E}^p(F_{q+u})$ such that $i(x'_u) = x'_{u-1}$, $x'_u = x_u$ for $u \le r - s$, and $j(x'_r) = 0$. That is to say, we can alter our chain of x_u's in the top $s-1$ spots and end up with a chain that can be extended upwards one more level. Indeed, just define $x'_u = x_u - (y_{r-u+1} + \cdots + y_s)$ where $j(x_r) = j(y_1 + \cdots + y_s)$ and the y_i's are as in the preceding paragraph.

(3). Define $E_r^{p,q} = Z_{r-1}^{p,q}/B_{r-1}^{p,q}$. The process "apply k, lift $r-1$ times, then apply j" yields a well-defined map $d_r\colon E_r^{*,*} \to E_r^{*,*}$. This is our "obstruction" map, and it is now well-defined precisely because we are quotienting out by the subgroup $B_{r-1}^{*,*}$. Note that d_r shifts the bigrading, so that we have $d_r\colon E_r^{p,q} \to E_r^{p+1,q+r}$.

The map d_r satisfies $d_r^2 = 0$ because $kj = 0$. A little work shows that $E_{r+1}^{*,*}$ is precisely the homology of $E_r^{*,*}$ with respect to d_r. The sequence of chain complexes $E_1, E_2, E_3, \ldots$, each the homology of the previous one, is the **spectral sequence** associated to our exact couple.

(4). For any value of p and q we have "entering" and "exiting" differentials

$$E_r^{?,?} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{?,?}$$

for certain unimportant values of '?'. It is easy to check that

(i) The exiting map d_r is zero if and only if $Z_{r-1}^{p,q} = Z_r^{p,q}$, and
(ii) The entering map d_r is zero if and only if $B_{r-1}^{p,q} = B_r^{p,q}$.

(5). It remains to interpret the E_∞-term. Notice that, strictly speaking, this is not one of the stages of our spectral sequence—it is not obtained as the homology of a previous complex. Still, in many examples one finds that E_∞ agrees with some finite stage E_r, at least through a range of dimensions.

Write $\mathcal{E}^p(F_q)_\infty$ for the set of all $x \in \mathcal{E}^p(F_q)$ such that $i(x) = 0$ and x lifts arbitrarily far up in the diagram (and note that this is *not* the same as saying that x lifts to the inverse limit). The map k induces a surjection $Z_\infty^{p,q} \twoheadrightarrow \mathcal{E}^p(F_q)_\infty$, and $B_\infty^{p,q}$ is clearly the kernel; so we have an induced isomorphism

$$\Gamma\colon E_\infty^{p,q} \xrightarrow{\cong} \mathcal{E}^p(F_q)_\infty.$$

Consider the groups $\mathcal{E}^p(\mathcal{F}) = \lim_q \mathcal{E}^p(F_q)$. As for any inverse limit these come with a filtration where we define

$$\mathcal{E}^p(\mathcal{F})_{ZPq} = \{\alpha \in \mathcal{E}^p(\mathcal{F}) \,|\, \text{the image of } \alpha \text{ in } \mathcal{E}^p(F_{q-1}) \text{ is zero.}\}$$

The "ZPq" subscript is supposed to remind us "zero past F_q". We call this the "ZP-filtration":

$$\mathcal{E}^p(\mathcal{F}) = \mathcal{E}^p(\mathcal{F})_{ZP0} \supseteq \mathcal{E}^p(\mathcal{F})_{ZP1} \supseteq \mathcal{E}^p(\mathcal{F})_{ZP2} \supseteq \cdots$$

There is an evident map $\mathcal{E}^p(\mathcal{F})_{ZPq} \to \mathcal{E}^p(F_q)_\infty$ which induces an injection

$$\text{(29.3)}\quad \mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)} := \mathcal{E}^p(\mathcal{F})_{ZPq}/\mathcal{E}^p(\mathcal{F})_{ZP(q+1)} \hookrightarrow \mathcal{E}^p(F_q)_\infty = E_\infty^{p,q}.$$

Notice our notation for the associated graded of the ZP-filtration.

(6). We are ultimately trying to get information about $\mathcal{E}^*(X)$. Notice that we have a natural map $\mathcal{E}^p(X) \to \mathcal{E}^p(\mathcal{F})$. This is always surjective; while not obvious, it is a consequence of the fact that if Z is any topological space then $[X, Z] \to \lim_q [X_q, Z]$

is surjective, which in turn is a routine application of the homotopy extension property for cellular inclusions. Let

$$\mathcal{E}^p(X)_{ZPq} = \ker\big(\mathcal{E}^p(X) \to \mathcal{E}^p(F_{q-1})\big)$$

and note that we have a map of filtrations $\mathcal{E}^p(X)_{ZP_\bullet} \to \mathcal{E}^p(\mathcal{F})_{ZP_\bullet}$. The map on associated graded groups

$$\mathcal{E}^p(X)_{ZP(q/q+1)} \to \mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)}$$

is readily seen to be an isomorphism, for all q.

(7). If we are in "good" cases then the map in (29.3) will actually be an isomorphism. The question is whether an element of $\mathcal{E}^p(F_q)$ that can be lifted arbitrarily high in the diagram can also be lifted into the inverse limit—note that this is not automatic! It might be possible that higher and higher liftings exist but not "coherently"; that is, to get a higher lifting one needs to change arbitrarily many elements lower down in the chain.

Fix a p and consider the following condition:

(SSC_p) : There exists an N such that for all q the differentials entering and exiting the group $E_r^{p,q}$ are zero for all $r \geq N$.

If this "Spectral Sequence Convergence Condition" holds then by (4) we know $Z_{N-1}^{p,q} = Z_r^{p,q}$ and $B_{N-1}^{p,q} = B_r^{p,q}$ for all $r \geq N$. Therefore $Z_\infty^{p,q} = Z_{N-1}^{p,q}$, $B_\infty^{p,q} = B_{N-1}^{p,q}$, and consequently $E_\infty^{p,q} = E_N^{p,q}$. These statements hold for all values of q. So the $E_\infty^{p,*}$ groups coincide with the stable values of the $E_r^{p,*}$ groups.

The convergence condition (SSC_p) gives us one more important consequence, namely that the map of (29.3) is an isomorphism:

$$(SSC_p) \Rightarrow \Big[\ \mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)} \cong E_\infty^{p,q}, \ \text{ for all } q.\Big]$$

To prove this, let $x_0 \in \mathcal{E}^p(F_q)_\infty$. Then there exist elements $x_u \in \mathcal{E}^p(F_{q+u})$ for $1 \leq u \leq 2N$ uch that $i(x_u) = x_{u-1}$ for each u (nothing special about $2N$ is being used here, it is just a convenient large number). The element $j(x_{2N})$ lies in B_{2N+1}, which we have seen equals B_{N-1} by (SSC_p). Therefore by (2) we may modify the x_u's in dimensions $x_{2N}, x_{2N-1}, \ldots, x_{N+3}$ in such a way that the chain extends to an x_{2N+1}. It is important here that only the top $N-2$ elements are affected, for we can now continue by induction and build an element of $\mathcal{E}^p(\mathcal{F}) = \lim_s \mathcal{E}^p(F_s)$ that maps to x in $\mathcal{E}^p(F_q)$. This completes the proof.

To summarize, the statement that the spectral sequence "converges to $\mathcal{E}^p(X)$" is usually interpreted to mean that

- The groups $E_r^{p,*}$ stabilize, and equal $E_\infty^{p,*}$, at some value of r (usually one that is independent of the grading $*$); and,
- The maps in (29.3) are isomorphisms, for all values of q.

Under these conditions the stable values of the spectral sequence give the associated graded groups of the ZP-filtration on $\mathcal{E}^p(X)$. We have seen that (SSC_p) implies this kind of convergence.

(8). Sometimes it is convenient to have in mind a variation on the exact couple diagram (29.2). Fix an integer q and consider the following:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \\
\boxed{\mathcal{E}^p(F_{q+2},F_{q-1})} & \xrightarrow{k} & \mathcal{E}^p(F_{q+2}) & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+3},F_{q+2}) \\
\downarrow & & \downarrow & & \\
\boxed{\mathcal{E}^p(F_{q+1},F_{q-1})} & \xrightarrow{k} & \mathcal{E}^p(F_{q+1}) & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+2},F_{q+1}) \\
\downarrow & & \downarrow & & \\
\mathcal{E}^p(F_q,F_{q-1}) & \xrightarrow{k} & \mathcal{E}^p(F_q) & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+1},F_q)
\end{array}$$

The groups in the boxes are "new", in the sense that they are not part of the exact couple (and they are being drawn in places which previously were occupied by other groups from the exact couple). Each trio of groups $\mathcal{E}^p(F_{q+r},F_{q-1}) \to \mathcal{E}^p(F_{q+r-1},F_{q-1}) \xrightarrow{jk} \mathcal{E}^{p+1}(F_{q+r},F_{q+r-1})$ is part of the long exact sequence for a triple, and so is exact in the middle spot. From this one readily sees that a class $u \in \mathcal{E}^p(F_q,F_{q-1}) = E_1^{p,q}$ lies in Z_r if and only if u lifts to a class in $\mathcal{E}^p(F_{q+r},F_{q-1})$. So the differentials in the spectral sequence can be viewed as a sequence of obstructions for lifting u to a class in $\mathcal{E}^p(F_N,F_{q-1})$, for larger and larger N.

(9). (Summary of the general workings of spectral sequences). We have produced a sequence of bigraded chain complexes $E_1^{*,*}, E_2^{*,*}, \dots$ such that each equals the homology of the previous one. We also have a "limiting" collection of bigraded groups $E_\infty^{*,*}$, and we have seen that under certain convergence conditions these E_∞ groups really are the "stable values" in the sequence of E_r's, and moreover they give the associated graded groups for the ZP-filtration of $\mathcal{E}^*(X)$.

(10). Everything that we have said so far works for any increasing filtration $F_\bullet$ of X. Now we use the fact that a CW-filtration is very special. Notice that

$$E_1^{p,q} = \mathcal{E}^p(F_q,F_{q-1}) \cong \widetilde{\mathcal{E}}^p(F_q/F_{q-1}) \cong \widetilde{\mathcal{E}}^p\Big(\bigvee_\alpha S^q\Big) \cong \bigoplus_\alpha \widetilde{\mathcal{E}}^p(S^q) \cong \bigoplus_\alpha \mathcal{E}^{p-q}(pt)$$

where the wedges and direct sums are over the set of q-cells in X. We can identify this group with the cellular cochain group $C^q(X;E^{p-q}(pt))$. The differential $d_1\colon E_1^{p,q} \to E_1^{p+1,q+1}$ is a map $C^q(X;E^{p-q}(pt)) \to C^{q+1}(X;E^{p-q}(pt))$ and it is readily checked to coincide with the differential in the cellular cochain complex. We conclude that

$$E_2^{p,q} \cong H^q(X;E^{p-q}(pt)).$$

Notice that the E_2-term is a homotopy invariant of X, whereas the E_1-term was not.

(11). The bigraded groups forming each term of the spectral sequence can be reindexed in whatever way seems convenient, and topologists use various conventions in different settings. For the Atiyah-Hirzebruch spectral sequence the standard convention is to choose the grading $E_1^{p,q} = \mathcal{E}^{p+q}(F_p,F_{p-1})$ so that we get

$$E_2^{p,q} = H^p(X;\mathcal{E}^q(pt)).$$

Under this grading we have that the differential d_r is a map

$$d_r\colon E_2^{p,q} \to E_2^{p+r,q-r+1}.$$

The groups in the spectral sequence are drawn on a grid where p is the horizontal axis and q the vertical one, with $E_2^{p,q}$ drawn in the (p,q)-spot. Finally, in this grading the Γ-map relating the E_∞-term to the associated graded of $\mathcal{E}^*(X)$ has the form

$$\Gamma\colon \mathcal{E}^{p+q}(X)_{ZP(p/p+1)} \hookrightarrow E_\infty^{p,q}.$$

In terms of the charts, the "total degree" lines are the diagonals where $p+q$ is constant. The (SSC_t) condition, translated into this new indexing, says that on the diagonal $p+q=t$ all entering and exiting differentials vanish past some finite stage of the spectral sequence. When this condition holds we are guaranteed convergence for the groups along this diagonal.

Remark 29.4 (**Warning about indexing**.)**.** For the rest of these notes we will adopt the indexing conventions from (11) above, which are the standard ones for the Atiyah-Hirzebruch spectral sequence. This is **different** than the indexing we used in (0)–(10).

Remark 29.5 (Independence of filtration)**.** As we have defined things, the spectral sequence depends on the chosen CW-structure on X. However, this dependence actually goes away from the E_2-term onward. Let X_1 and X_2 denote the same space but with two different CW-structures. The identity map $X_1 \to X_2$ is homotopic to a cellular map $f\colon X_1 \to X_2$, and f gives us a map of spectral sequences by naturality. Homotopy invariance of cellular cohomology shows that f induces an isomorphism on the E_2-terms, and therefore on all the finite pages of the spectral sequence as well.

The ZP-filtration on X was defined in terms of the CW-structure, but we can define it in a different way that doesn't make use of that. We leave it as an exercise to check that

$$\mathcal{E}^p(X)_{ZPq} = \{\alpha \in \mathcal{E}^p(X) \mid u^*(\alpha) = 0 \text{ for any map } u\colon A \to X \text{ where } A \text{ is a } CW\text{-complex of dimension less than } q\}.$$

These remarks show us that the spectral sequence from E_2-onwards may be regarded as a natural homotopy invariant of X. In particular, any map $g\colon X \to Y$ gives a map of spectral sequences in the opposite direction (by replacing g with a cellular map).

29.6. **The Postnikov tower approach.** Let $\mathcal{E}$ be a spectrum representing the cohomology theory $\mathcal{E}^*$ and let $P_n\mathcal{E}$ denote the nth Postnikov section for $\mathcal{E}$. There is a tower of fibrations

$$\begin{array}{ccccccc}
\cdots \twoheadrightarrow & P_1\mathcal{E} & \twoheadrightarrow & P_0\mathcal{E} & \twoheadrightarrow & P_{-1}\mathcal{E} & \twoheadrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & \\
& \Sigma H(\mathcal{E}_1) & & H(\mathcal{E}_0) & & \Sigma^{-1}H(\mathcal{E}_{-1}) &
\end{array}$$

where $\mathcal{E}_i = \pi_i(\mathcal{E}) = \mathcal{E}^{-i}(pt)$ and HA denotes the Eilenberg-MacLane spectrum for the group A. If we apply function spectra $F(X,-)$ to all the spots in this diagram we get a new tower of fibrations

$$\begin{array}{ccccccccc} \cdots \twoheadrightarrow & F(X, P_1\mathcal{E}) & \twoheadrightarrow & F(X, P_0\mathcal{E}) & \twoheadrightarrow & F(X, P_{-1}\mathcal{E}) & \twoheadrightarrow \cdots \\ & \uparrow & & \uparrow & & \uparrow & \\ & F(X, \Sigma H(\mathcal{E}_1)) & & F(X, H(\mathcal{E}_0)) & & F(X, \Sigma^{-1} H(\mathcal{E}_{-1})) & \end{array}$$

Each fibration sequence $F(X, \Sigma^q H(\mathcal{E}_q)) \to F(X, P_q\mathcal{E}) \to F(X, P_{q-1}\mathcal{E})$ gives rise to a long exact sequence in homotopy groups, and these long exact sequences intertwine to form an exact couple. The associated spectral sequence has

$$E_1^{p,q} = \pi_{-p-q} F(X, \Sigma^q H(\mathcal{E}_q)) = H^{p+2q}(X; \mathcal{E}^{-q}),$$
$$d_r \colon E_r^{p,q} \longrightarrow E_r^{p-r+1,q+r},$$

and it is trying to converge to

$$\pi_{-p-q} F(X, \mathcal{E}) = \mathcal{E}^{p+q}(X).$$

It is not obvious, but with some trouble it can be seen that after re-indexing this is "the same" as the previously-constructed spectral sequence but with the E_1-term of this one corresponding to the E_2-term of the one constructed via CW-structures.

We won't really need this Postnikov version of the spectral sequence for anything, but it often provides a useful perspective. For example, note that this version of the spectral sequence is manifestly functorial in X and a homotopy invariant.

29.7. **Differentials.** The d_2-differential in the Atiyah-Hirzebruch spectral sequence is a map $H^p(X; \mathcal{E}^q) \to H^{p+2}(X; \mathcal{E}^{q-1})$. This is natural in X and it is also stable under the suspension isomorphism; so it is a stable cohomology operation. The d_3-differential is in some sense a secondary cohomology operation, and so on for all the differentials. This is often a useful perspective. For example, we can now prove the following general fact:

Proposition 29.8. *Suppose that the coefficient groups $\mathcal{E}^*(pt)$ are rational vector spaces. Then for any space X the differentials in the Atiyah-Hirzebruch spectral sequence all vanish, and so there are (non-canonical) isomorphisms*

$$\mathcal{E}^n(X) \cong \oplus_{p+q=n} H^p(X; \mathcal{E}^q(pt))$$

for every $n \in \mathbb{Z}$.

Proof. The point is that the only stable cohomology operation of nonzero degree on $H^*(-; \mathbb{Q})$ is the zero operation. This immediately yields that all d_2-differentials are zero. But then d_3 is a stable cohomology operation (not a secondary operation anymore) and therefore it also vanishes. Continue by induction. □

Remark 29.9. In the Postnikov approach to the Atiyah-Hirzebruch spectral sequence one sees the d_2-differentials very explicitly. The Postnikov tower has "k-invariants" of the form

$$\Sigma^q H(\mathcal{E}_q) \to \Sigma^{q+2} H(\mathcal{E}_{q+1})$$

which desuspend to give $H(\mathcal{E}_q) \to \Sigma^2 H(\mathcal{E}_{q+1})$. These are quite visibly stable cohomology operations $H^*(-; \mathcal{E}^{-q}) \to H^{*+2}(-; \mathcal{E}^{-q-1})$. The connection between higher differentials and higher cohomology operations has a similar realization, but the details are too cumbersome to be worth discussing here.

It is worth observing that from the E_2-term onward there are never any differentials emanating from the $p = 0$ line of the spectral sequence. For convenience assume X is connected and choose a cell structure on X where $F_0 = \{*\}$. By the remarks in (8) above, such differentials would be the obstructions for a class in $\mathcal{E}^q(F_0, \emptyset)$ to lift to $\mathcal{E}^q(F_r, \emptyset)$; but such a lifting necessarily exists, because $F_0 \hookrightarrow F_r$ is split. Note that when X is connected $\mathcal{E}^q(X)_{ZP1} = \widetilde{\mathcal{E}}^q(X)$, and $\mathcal{E}^q(X)_{ZP(1/2)} = \mathcal{E}^q(X)/\widetilde{\mathcal{E}}^q(X) = \mathcal{E}^q(pt)$, giving further confirmation that the E_∞-term coincides with the E_2-term on the $p = 0$ line.

In the original exact couple (29.2) we could replace all the $\mathcal{E}^*$ groups with $\widetilde{\mathcal{E}}^*$ groups and still have an exact couple, with the resulting spectral sequence having the form

$$E_2^{p,q} = \widetilde{H}^p(X; \mathcal{E}^q) \Rightarrow \widetilde{\mathcal{E}}^{p+q}(X).$$

This just amounts to removing the entire $p = 0$ line from the Atiyah-Hirzebruch spectral sequence. Sometimes it is convenient to consider this 'reduced' version of the spectral sequence.

29.10. **Multiplicativity.** Suppose that $\mathcal{E}$ is a multiplicative cohomology theory. Then for spaces X and Y we have the external product

$$\mathcal{E}^r(X) \otimes \mathcal{E}^s(Y) \to \mathcal{E}^{r+s}(X \times Y),$$

and this is readily checked to induce associated pairings on the ZP-filtration:

$$\mathcal{E}^r(X)_{ZPa} \otimes \mathcal{E}^s(Y)_{ZPb} \to \mathcal{E}^{r+s}(X \times Y)_{ZP(a+b)}$$

and

$$\mathcal{E}^r(Z)_{ZP(a/a+1)} \otimes \mathcal{E}^s(Y)_{ZP(b/b+1)} \to \mathcal{E}^{r+s}(X \times Y)_{ZP(a+b/a+b+1)}. \tag{29.11}$$

The pairings $\mathcal{E}^q(pt) \otimes \mathcal{E}^{q'}(pt) \to \mathcal{E}^{q+q'}(pt)$ also can be fed into the cup product machinery to give

$$H^p(X; \mathcal{E}^q(pt)) \otimes H^{p'}(Y; \mathcal{E}^{q'}(pt)) \longrightarrow H^{p+p'}(X \times Y; \mathcal{E}^{q+q'}(pt)). \tag{29.12}$$

Since the Atiyah-Hirzebruch spectral sequence starts with the groups $E_2(-) = H^*(-; \mathcal{E}^*)$ and then converges to the groups $\mathcal{E}^*(-)$, it is natural to ask if the pairings of (29.12) and (29.11) are connected via this convergence process. The machinery for making this connection is somewhat cumbersome to write out, although in practice not so cumbersome to use.

To say that there is a **pairing of spectral sequences** $E_*(X) \otimes E_*(Y) \to E_*(X \times Y)$ is to say that

(i) For each r there is a product $E_r^{p,q}(X) \otimes E_r^{p',q'}(Y) \to E_r^{p+p',q+q'}(X \times Y)$;
(ii) The differential d_r satisfies the Leibniz rule
$$d_r(a \cdot b) = d_r(a) \cdot b + (-1)^p a \cdot d_r b$$
for all $a \in E_r^{p,q}(X)$ and $b \in E_r^{p',q'}(Y)$, and therefore induces a product on $H_*(E_r) = E_{r+1}$;
(iii) The product on the E_{r+1}-term equals the one induced by the product on the E_r-term, for all r;
(iv) There is a product on the E_∞-term which agrees with the products on the E_r-terms where defined (???);

(v) The maps $\mathcal{E}^p(-)_{ZP(a/a+1)} \to E_\infty^{p,a}(-)$ are compatible with the products, in the evident sense.

To give a decent treatment of pairings between Atiyah-Hirzebruch spectral sequences it is best to work at the level of spectra, and to work with a category of spectra where there is a well-behaved smash product. This introduces several layers of foundational technicalities that we do not wish to dwell on, so let us just say that these things can all be worked out. In this setting the right notion of "multiplicative cohomology theory" consists of a spectrum $\mathcal{E}$ together with a map $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$ that is associative and unital. Both the complex and real K-theory spectra can be given this structure. One has the following general result:

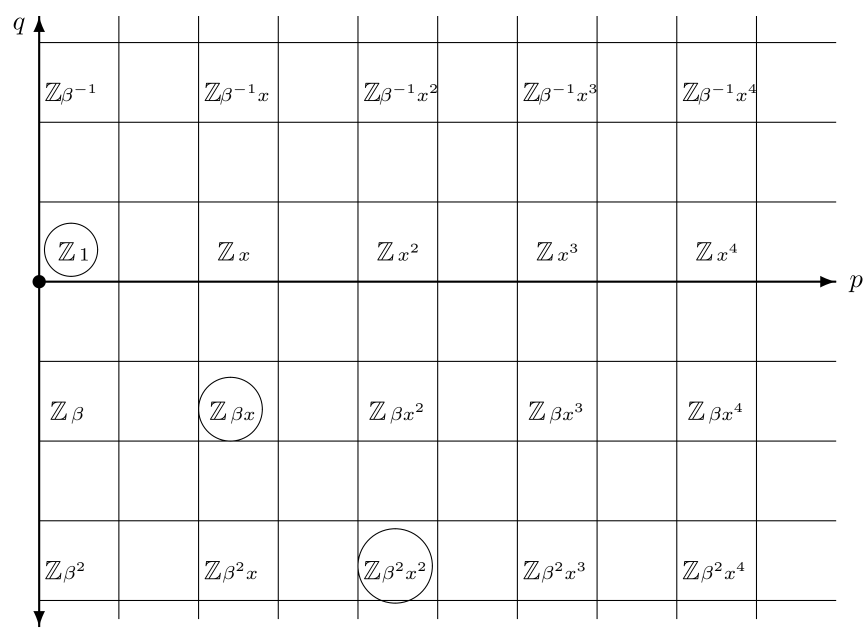
Theorem 29.13. *Let $\mathcal{E}$ be a spectrum with a product $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$. Then there is a pairing of Atiyah-Hirzebruch spectral sequences where the product on E_2-terms $H^p(X; \mathcal{E}^q) \otimes H^{p'}(Y; \mathcal{E}^{q'}) \to H^{p+p'}(X \times Y; \mathcal{E}^{q+q'})$ is equal to $(-1)^{p'q}$ times the cup product.*

We will not prove the above theorem here, as this would take us too far afield. For a proof, see [D2, Section 3]. What is more important is how to use the theorem; we will give some examples in the following section.

Remark 29.14. The signs in the above theorem cannot, in general, be neglected. See [D2, Section 2] for a complete discussion. However, notice that in the case of complex K-theory it is irrelevant because the groups $H^p(X; K^q)$ are only nonzero when q is even. This is a pleasant convenience. A similar convenience occurs for KO-theory: whereas the coefficients groups do have some nonzero terms in odd degrees, since these terms are all $\mathbb{Z}/2$'s one can once again neglect the signs.

29.15. **Some examples.** We now focus entirely on complex K-theory, considering two sample computations. Further examples, for both K and KO, are in Section 32.

Let us start by redoing the calculation $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$ where $X = L - 1$, now using the Atiyah-Hirzebruch spectral sequence. The following is the E_2-term:



Note that the E_2-term vanishes to the right of the line $p = 2n$, since $H^*(\mathbb{C}P^n)$ vanishes in this range. The circled groups (and others along the same diagonal) are the ones that contribute to $K^0(\mathbb{C}P^n)$. Note that there is no room for any differentials, because the nonzero groups only occur when both p and q are even. So the spectral sequence immediately collapses, and $E_2 = E_\infty$. It follows that the filtration quotients for the ZP-filtration on $K^0(\mathbb{C}P^n)$ are as follows:

$$K^0(\mathbb{C}P^n) \overset{\mathbb{Z}}{\longleftarrow} K^0(\mathbb{C}P^n)_{ZP1} \overset{0}{\longleftarrow} K^0(\mathbb{C}P^n)_{ZP2} \overset{\mathbb{Z}}{\longleftarrow} K^0(\mathbb{C}P^n)_{ZP3} \longleftarrow \cdots$$

with the $\mathbb{Z}$'s appearing exactly $n+1$ times. Since the quotients are free there are no extension problems and we conclude that $K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$ as abelian groups.

The spectral sequence also gives information about the ring structure on $K^0(\mathbb{C}P^n)$. Note that $K^0(\mathbb{C}P^n)_{ZP1} = \widetilde{K}^0(\mathbb{C}P^n)$, and that $K^0(\mathbb{C}P^n)_{ZP2} = K^0(\mathbb{C}P^n)_{ZP1}$ by the previous paragraph. Consider the canonical map

$$K^0(\mathbb{C}P^n)_{ZP(2/3)} \xrightarrow{\cong} E_\infty^{2,-2} = \mathbb{Z}\langle \beta x\rangle.$$

Let α denote a preimage for βx under this isomorphism. The multiplicativity of the spectral sequence tells us that α^k maps to $\beta^k x^k$ under the corresponding map $K^0(\mathbb{C}P^n)_{ZP(2k/2k+1)} \to E_\infty^{2k,-2k}$. In particular, α^k is nonzero for $k < n+1$. We know $\alpha^{n+1} = 0$, either by Lemma 20.2 or by the fact that $\alpha^{n+1} \in K^0(\mathbb{C}P^n)_{ZP(2n+2)}$ and this filtration group is zero by the spectral sequence.

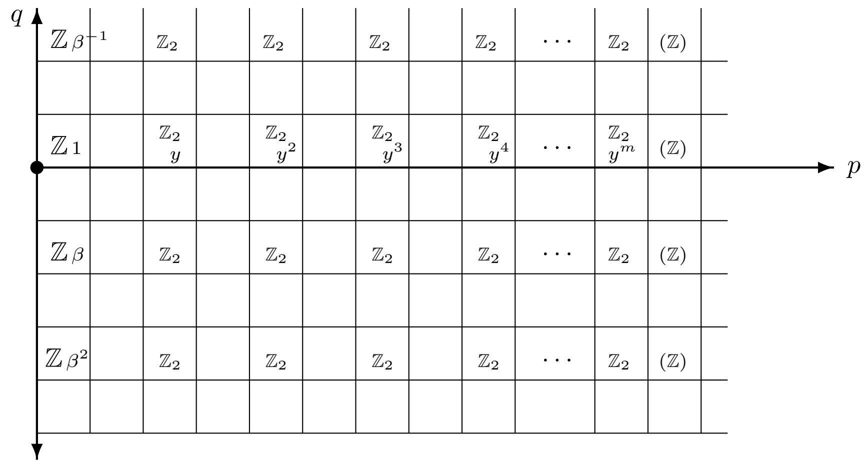
Now consider the map $\mathbb{Z}[\alpha]/(\alpha^{n+1}) \to K^0(\mathbb{C}P^n)$. This may be regarded as a map of filtered rings, where the domain is filtered by the powers of (α) and the target has the ZP-filtration. The spectral sequence tells us this is an isomorphism on the filtration quotients; but since there are only finitely many of these, it follows that the map is an isomorphism.

To complete our calculation is only remains to show that we may take $\alpha = \pm(1-[L])$. First verify this when $n = 1$, where it just comes down to the fact that $1-[L]$ is a generator for $\widetilde{K}^0(S^2)$. For general n we can now use the naturality of the spectral sequence, applied to the inclusion $j\colon \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$. The spectral sequences tell us that the natural map

$$K^0(\mathbb{C}P^n)_{ZP(2/3)} \to K^0(\mathbb{C}P^1)_{ZP(2/3)}$$

is an isomorphism. The element $1-[L]$ represents an element in the domain, which must be a generator precisely because it maps to a generator in the target. This tells us that one of $\pm(1-[L])$ maps to βx and is therefore a candidate for α, and this is enough to conclude $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$ where $X = 1-[L]$.

For our next example let us consider $K^0(\mathbb{R}P^n)$. The E_2-term is very similar to the one before, with the difference that most $\mathbb{Z}$'s are changed to $\mathbb{Z}/2$'s:



The $\mathbb{Z}$'s in parentheses lie in degree $p = n$ when n is odd, and are not present when n is even. As discussed in (29.7) above, there can be no differential emanating from the $p = 0$ column. So the only possible differentials allowed by the grading would occur when n is odd and would have a $\mathbb{Z}/2$ mapping into one of the $\mathbb{Z}$'s; but such a map must be zero. So all differentials vanish, and we again have $E_2 = E_\infty$. We conclude that the associated graded of the ZP-filtration on $\widetilde{K}^0(\mathbb{R}P^n)$ consists of $\lfloor \frac{n}{2} \rfloor$ copies of $\mathbb{Z}/2$, and so $\widetilde{K}^0(\mathbb{R}P^n)$ is an abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$. It remains to determine the group precisely. For this, use the map of spectral sequences induced by $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. The surjection on E_∞-terms shows that $\alpha = 1-[j^*L]$ generates the filtration quotient $K^0(\mathbb{R}P^n)_{ZP(2/3)}$, since $1 - [L]$ generates the corresponding quotient in the $\mathbb{C}P^n$ case. Note that $\alpha^2, \ldots, \alpha^{\lfloor \frac{n}{2} \rfloor}$ therefore generate the other filtration quotients, and so in particular are nonzero. But we can compute

$$\alpha^2 = (1 - [j^*L])^2 = 1 - 2[j^*L] + [j^*L]^2 = 1 - 2[j^*L] + 1 = 2(1 - [j^*L]) = 2\alpha,$$

where in the third equality we have used that the square of any real line bundle is trivial (Corollary 8.23). It follows that $\alpha^i = 2^{i-1}\alpha$. Since $\alpha^{\lfloor \frac{n}{2} \rfloor} \neq 0$ this gives $2^{(\lfloor \frac{n}{2} \rfloor - 1)}\alpha \neq 0$. The only abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$ that admits such an element is $\mathbb{Z}/(2^{\lfloor \frac{n}{2} \rfloor})$, and so $\widetilde{K}^0(\mathbb{R}P^n)$ is isomorphic to this cyclic group.

Remark 29.16. Note that we previously determined that $\widetilde{K}^0(\mathbb{R}P^2)$ was an abelian group of order 4, back in Section 13.10. Comparing the "brute force" approach used there to the spectral sequence machinery really shows the power of the latter: the argument is really the same, but the spectral sequence allows us to get at the conclusion much more quickly.

29.17. **More on differentials.** Since $K^{odd}(pt) = 0$ it follows for degree reasons that all differentials d_{2r} vanish in the Atiyah-Hirzebruch spectral sequence. So our first significant differential is d_3, which is a stable cohomology operation $H^*(-;\mathbb{Z}) \to H^{*+3}(-;\mathbb{Z})$. It is an easy matter to compute *all* such stable operations, as they are parameterized by the group $H^3(H\mathbb{Z})$ of stable homotopy classes $H\mathbb{Z} \to \Sigma^3 H\mathbb{Z}$; this can be computed as the cohomology group $H^{n+3}(K(\mathbb{Z}, n))$ for $n \geq 3$. A routine calculation (say, with the Serre spectral sequence) shows that this group is $\mathbb{Z}/2$. The nonzero element is an operation α that is an integral lift of Sq^3, in the sense that if $u \in H^*(X;\mathbb{Z})$ then

$$\overline{\alpha(u)} = \mathrm{Sq}^3(\overline{u})$$

where $\overline{x}$ denotes the mod 2 reduction of a class x.

The above paragraph shows that our differential d_3 either equals zero (for all spaces X) or coincides with the operation α. The latter option is the correct one, and to see this it suffices to produce a single space X where d_3 is nonzero. For this we take the space X from Example 24.7, constructed as the cofiber of a map $\Sigma^3\mathbb{R}P^2 \to S^3$ that gives a null-homotopy for 2η. This space has $H^3(X) = \mathbb{Z}$ and $H^6(X) = \mathbb{Z}/2$, so there is a potential d_3 in the spectral sequence. If this d_3 were zero then we would have $\widetilde{K}^0(X) = \mathbb{Z}/2$, but we calculated in Example 24.7 (using the Chern character) that $\widetilde{K}^0(X) = 0$. So d_3 is nonzero here. We have therefore proven:

Proposition 29.18. *The differential d_3 in the Atiyah-Hirzebruch spectral sequence is the unique nonzero cohomology operation $H^*(-;\mathbb{Z}) \to H^{*+3}(-;\mathbb{Z})$. It satisfies $2d_3(x) = 0$, for all x.*

Remark 29.19. We have now explained the motivation for the space X from Example 24.7. It is literally the smallest space for which there is a nonzero α-operation in its cohomology.

The fact that $2d_3(x) = 0$ shows that d_3 must vanish on any class $x \in H^*(X)$ whose order is prime to 2. Alternatively, if we tensor the Atiyah-Hirzebruch spectral sequence with $\mathbb{Z}[\frac{1}{2}]$ then all d_3 differentials vanish. In that case d_5 is a cohomology operation $H^*(-;\mathbb{Z}[\frac{1}{2}]) \to H^{*+5}(-;\mathbb{Z}[\frac{1}{2}])$, and such things are classfied by $H^5(H\mathbb{Z};\mathbb{Z}[\frac{1}{2}])$. This group is readily calculated to be $\mathbb{Z}/3$. If one then also inverts 3 this will kill d_5, but it turns out to also kill d_7 because $H^7(H\mathbb{Z};\mathbb{Z}[\frac{1}{6}]) = 0$. The d_9 differential becomes a cohomology operation in $H^9(H\mathbb{Z};\mathbb{Z}[\frac{1}{6}]) \cong \mathbb{Z}/5$, and so one can kill it by inverting 5. This process continues, and shows that inverting all primes smaller than p kills all differentials below d_{2p-1}. Note that this gives another proof of Proposition 29.8 (but with more precise information), saying that after tensoring with $\mathbb{Q}$ all Atiyah-Hirzebruch differentials vanish.

The following result summarizes and expands the discussion in the last paragraph. The two parts are closely related and almost equivalent, but it is useful to have them both stated explicitly.

Proposition 29.20. *Fix a prime p.*

(a) Inverting $(p-1)!$ in the Atiyah-Hirzebruch spectral sequence results in $d_r = 0$ for $r < 2p-1$, together with $d_{2p-1}(u) = (-1)^{p+1}\beta P^1(\bar{u})$ for all classes u, where $\bar{u}$ is reduction modulo p, P^1 is Steenrod's first reduced power operation (for the prime p), and β is the Bockstein for the sequence $0 \to \mathbb{Z}[\frac{1}{(p-1)!}] \xrightarrow{p} \mathbb{Z}[\frac{1}{(p-1)!}] \to \mathbb{Z}/p \to 0$.

(b) Let $u \in H^*(X)$ be p^e-torsion, where p is a prime. Then in the Atiyah-Hirzebruch spectral sequence $d_i(u) = 0$ for $i < 2p-1$, and $d_{2p-1}(u) = (-1)^{p+1}\beta P^1(\bar{u})$ where β is the Bockstein for $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$ and P^1 is as in (a).

Proof. Fixing a prime p, the following is known about $H^*(K(\mathbb{Z},n))$ assuming $n > 2p-1$:

(i) $H^i(K(\mathbb{Z},n)) = 0$ for $0 < i < n$ and $H^n(K(\mathbb{Z},n)) \cong \mathbb{Z}$;
(ii) $H^i(K(\mathbb{Z},n))$ is torsion, with all orders prime to p, for $n < i < n + 2p - 1$;
(iii) $H^{n+2p-1}(K(\mathbb{Z},n))$ is isomorphic to a direct sum $\mathbb{Z}/p \oplus A$ where A is a torsion group whose order only has prime factors smaller than p. The $\mathbb{Z}/p$ summand is generated by $\beta P^1(\overline{u})$, where $u \in H^n(K(\mathbb{Z},n))$ is the fundamental class.

Alternatively, the above results say that

$$H^*\Big(K(\mathbb{Z},n); \mathbb{Z}[\tfrac{1}{(p-1)!}]\Big) \cong \begin{cases} 0 & \text{if } i < n, \\ \mathbb{Z}[\frac{1}{(p-1)!}] & \text{if } i = n, \\ 0 & \text{if } n < i < n + 2p - 1, \\ \mathbb{Z}/p & \text{if } i = n + 2p - 1. \end{cases}$$

These facts are easy calculations with the Serre spectral sequence, using the methods of [MT, Chapter ???].

Consider the Atiyah-Hirzebruch spectral sequence for $K(\mathbb{Z},n)$, and in particular the differentials on the fundamental class u. By naturality of the spectral sequence this serves as a universal example for what happens on all spaces. Since inverting $(p-1)!$ kills all of the cohomology of $K(\mathbb{Z},n)$ in dimensions strictly between n and $n + 2p - 1$, this shows that it also kills the differentials $d_r(u)$ for $r < 2p - 1$. By universality, inverting $(p-1)!$ kills these differentials for any space X.

If $u \in H^*(X)$ is a p^e-torsion class then $d_r(u)$ for $r < 2p-1$ is killed by a power of $(p-1)!$ by the preceding paragraph, but it is also killed by p^e; since these integers are relatively prime it follows that $d_r(u) = 0$ for $r < 2p - 1$.

It remains to identify d_{2p-1} in both (a) and (b). We know by the calculation of $H^*(K(\mathbb{Z},n))$ for $n \gg 0$ that after localization at $(p-1)!$ one must have $d_{2p-1}(u) = \lambda \cdot \beta P^1(\bar{u})$, for some $\lambda \in \mathbb{Z}/p$. We need to determine λ, and for this we can examine a single well-chosen example space. The sample space we choose is a generalization of the one from Example 24.7. Consider the projection $\pi \colon \mathbb{C}P^p \to \mathbb{C}P^p/\mathbb{C}P^{p-1} \cong S^{2p}$. We claim that there is a stable map $f \colon S^{2p} \to \mathbb{C}P^p$ such that the composite $S^{2p} \to \mathbb{C}P^p \xrightarrow{\pi} S^{2p}$ has degree Mp for some integer M prime to p; for the proof, see Lemma 29.22 below. The "stable map" phrase is to indicate that f might only exist after suspensing some number of times, so really it is a map $f \colon \Sigma^r S^{2p} \to \Sigma^r \mathbb{C}P^p$. We can assume that r is even. Let X be the cofiber of f, and note that

$$\widetilde{H}^i(X) = \begin{cases} \mathbb{Z} & \text{if } r+2 \le i < r + 2p \text{ and } i \text{ is even}, \\ \mathbb{Z}/(Mp) & \text{if } i = r + 2p + 1, \\ 0 & \text{otherwise}. \end{cases}$$

Consider the cofiber sequence

$$\Sigma^r \mathbb{C}P^p \xrightarrow{j} X \xrightarrow{p} S^{r+2p+1}$$

Let $x \in H^2(\mathbb{C}P^p)$ be a chosen generator, and let $u \in H^{r+2}(X)$ be a class that j^* maps onto the suspension $\sigma^r(x)$. Let $v \in H^{r+2p+1}(X)$ be the image under p^* of the canonical generator from $H^{r+2p+1}(S^{r+2p+1})$. Note that j^* is an isomorphism on $H^*(-;\mathbb{Z}/p)$ for $* \le 2p$.

In $H^*(\mathbb{C}P^p; \mathbb{Z}/p)$ we have $P^1(\bar{x}) = \bar{x}^p$, as this is how P^1 behaves on classes of degree 2. So in $H^*(\Sigma^r \mathbb{C}P^p; \mathbb{Z}/p)$ we have $P^1(\sigma^r \bar{x}) = \sigma^r(\bar{x}^p)$, by stability of the P^1 operation. It follows that $\beta P^1(\bar{u}) = Mv$.

In the Atiyah-Hirzebruch spectral sequence for X, after inverting $(p-1)!$ there is only one possible nonvanishing differential, namely $d_{2p-1}\colon \mathbb{Z}[\frac{1}{(p-1)!}] \to \mathbb{Z}/p$. We know that

$$d_{2p-1}(u) = \lambda \cdot \beta P^1(\bar{u}) = \lambda M v. \tag{29.21}$$

But we can also compute $d_{2p-1}(u)$ directly. The class u is represented by an element in $K^0(X_{r+2}, X_{r+1})$ (a cellular $(r+2)$-cochain). The differential is represented by choosing a lift of $u \in K^0(X_{r+2}, X_{r+1})$ to $\xi \in K^0(X_{2p+r}, X_{r+1})$ and then applying the connecting homomorphism from the long exact sequence for the triple $(X_{2p+r+1}, X_{2p+r}, X_{r+1})$:

$$\begin{array}{ccc} \xi \in K^0(X_{2p+r+1}, X_{r+1}) & \xrightarrow{\delta} & K^1(X_{2p+r+1}, X_{2p+r}) \\ \downarrow & & \\ u \in K^0(X_{r+2}, X_{r+1}). & & \end{array}$$

Recall that $X_{r+1} = *$ and that u is represented by the class $\sigma^r(1-[L]) \in \widetilde{K}^0(\Sigma^r \mathbb{C}P^1)$. The element $1-[L] \in \widetilde{K}^0(\mathbb{C}P^1)$ lifts to the class with the same name in $\widetilde{K}^0(\mathbb{C}P^p)$, and so we may take $\xi = \sigma^r(1-[L]) \in K^0(\Sigma^r\mathbb{C}P^p, *)$. To compute $\delta(\xi)$ we can use the Chern character:

$$\begin{array}{ccc} K^0(X_{2p+r}, *) & \xrightarrow{\delta} & K^1(X_{2p+r+1}, X_{2p+1}) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(X_{2p+r}, *; \mathbb{Q}) & \xrightarrow{\delta} & H^{odd}(X_{2p+r+1}, X_{2p+1}; \mathbb{Q}). \end{array}$$

We know from Proposition 24.4 that the right vertical map is an injection and that its image is the integral subgroup $H^{odd}(X_{2p+r+1}, X_{2p+1}; \mathbb{Z})$. So we compute

$$\mathrm{ch}(\xi) = \mathrm{ch}\big(\sigma^r(1-[L])\big) = \sigma^r \,\mathrm{ch}(1-[L]) = \sigma^r(x - \tfrac{x^2}{2} + \tfrac{x^3}{6} - \cdots)$$

Applying δ to this expresstion kills everything except the class in degree $2p+r$, and we therefore get

$$\delta(\mathrm{ch}\,\xi) = (-1)^p \cdot \tfrac{1}{p!} \cdot Mp = (-1)^p \tfrac{M}{(p-1)!}$$

where the first two terms in the product come from $\mathrm{ch}(\xi)$ and the Mp comes from application of δ. Note that commutativity of the above square implies that $\frac{M}{(p-1)!}$ must be an integer, and that

$$\delta(\xi) = (-1)^p \tfrac{M}{(p-1)!} \cdot v$$

where v denotes the preferred generator of $K^1(X_{2p+r+1}, X_{2p+r}) \cong \mathbb{Z}$.

Putting everything together, we have just proven that in the Atiyah-Hirzebruch spectral sequence for X with $(p-1)!$ inverted one has

$$d_{2p-1}(u) = \Big[(-1)^p \tfrac{M}{(p-1)!}\Big]_p \cdot v$$

where $[-]_p$ denotes the residue modulo p. Wilson's Theorem says that $(p-1)! \equiv -1$ mod p, and so

$$d_{2p-1}(u) = \big[(-1)^{p+1}M\big]_p \cdot v.$$

Comparing to (29.21) gives $\lambda = (-1)^{p+1}$, and we are finished with (a).

Part (b) can be deduced from (a) using the commutative square

$$\begin{array}{ccc} H^*(X) & \xrightarrow{d_{2p-1}} & H^{*+2p-1}(X) \\ \downarrow & & \downarrow \\ H^*(X;\mathbb{Z}_f) & \xrightarrow{d_{2p-1}} & H^{*+2p-1}(X;\mathbb{Z}_f) \end{array}$$

where we are writing $\mathbb{Z}_f = \mathbb{Z}[\frac{1}{(p-1)!}]$. Let $B \subseteq H^{*+2p-1}(X)$ be the subgroup of elements killed by a power of p, and note that B injects into $H^{*+2p-1}(X;\mathbb{Z}_f)$. If $p^e u = 0$ then both $d_{2p-1}u$ and $(-1)^{p+1}\beta P^1(\bar{u})$ belong to B. The commutative square, together with part (a), show that the two classes map to the same element of $H^{*+2p-1}(X;\mathbb{Z}_f)$; hence, they are the same. □

Lemma 29.22. *Fix a prime p. Then for some $r > 0$ there exists a map $S^{2p+r} \to \Sigma^r\mathbb{C}P^p$ such that the composite*

$$S^{2p+r} \to \Sigma^r\mathbb{C}P^p \xrightarrow{\Sigma^r\pi} \Sigma^r S^{2p}$$

has degree equal to Mp for some M relatively prime to p. Here $\pi\colon \mathbb{C}P^p \to S^{2p}$ is the map that collapses $\mathbb{C}P^{p-1}$ to a point.

Proof. This is a computation with stable homotopy groups. Consider the homology theory $X \mapsto E_*(X) = \pi_*^s(X) \otimes \mathbb{Z}_{(p)}$. When $X = S^0$ the groups $E_*(X)$ are the p-components of the stable homotopy groups of spheres, and it is known that $E_i(X) = 0$ for $0 < i < 2p-3$ and $E_{2p-3}(X) \cong \mathbb{Z}/p$. An easy induction using the cofiber sequences $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n \to S^{2n}$ shows that $E_{2p-1}(\mathbb{C}P^n) \cong \mathbb{Z}/p$ for all $1 \le n \le p-1$. The long exact sequence for $\mathbb{C}P^{p-1} \hookrightarrow \mathbb{C}P^p \to S^{2p}$ then gives

$$\cdots \to \pi_{2p}^s(\mathbb{C}P^{p-1}) \to \pi_{2p}^s(\mathbb{C}P^p) \to \mathbb{Z} \to \pi_{2p-1}^s(\mathbb{C}P^{p-1}) \to \cdots$$

The element $p \in \mathbb{Z}$ necessarily maps to zero in $\pi_{2p-1}^s(\mathbb{C}P^{p-1}) \otimes \mathbb{Z}_{(p)}$, since the latter group is $\mathbb{Z}/p$. This means that there exists $M \in \mathbb{Z}$ prime to p such that Mp maps to zero in $\pi_{2p-1}^s(\mathbb{C}P^{p-1})$. But then Mp is the image of an element in $\pi_{2p}^s(\mathbb{C}P^p)$, and this element is what we were looking for. □

29.23. **Differentials and the Chern character.** In the proof of Proposition 29.20 there was a key step where we used the Chern character to help compute a differential in the Atiyah-Hirzebruch spectral sequence. We will next explain a generalization of this technique.

The E_1-term of the Atiyah-Hirzebruch spectral sequence breaks up into chain complexes that look like

$$\cdots \to K^{-1}(F_{q-1}, F_{q-2}) \to K^0(F_q, F_{q-1}) \to K^1(F_{q+1}, F_q) \to \cdots$$

If q is odd this is the zero complex, and if q is even we have seen that it is isomorphic to the cellular chain complex for X with $\mathbb{Z}$ coefficients. The latter is via isomorphisms $K^0(F_q, F_{q-1}) \cong \widetilde{K}^0(\vee_\alpha S^q) \cong \bigoplus_\alpha \widetilde{K}^0(S^q) \cong \bigoplus_\alpha \mathbb{Z}$. Notice that we can also use the Chern character to obtain such an isomorphism, as we know $\mathrm{ch}\colon K^0(F_q, F_{q-1}) \to H^*(F_q, F_{q-1};\mathbb{Q})$ to be injective with image equal to

$H^*(F_q, F_{q-1};\mathbb{Z})$ (Proposition 24.4). Since the Chern character is natural it actually gives us an isomorphism of chain complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K^{-1}(F_{q-1},F_{q-2}) & \longrightarrow & K^0(F_q,F_{q-1}) & \longrightarrow & K^1(F_{q+1},F_q) & \longrightarrow & \cdots \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ \cdots & \longrightarrow & H^*(F_{q-1},F_{q-2}) & \longrightarrow & H^*(F_q,F_{q-1}) & \longrightarrow & H^*(F_{q+1},F_q) & \longrightarrow & \cdots \end{array}$$

Taken on its own, this is just a matter of convenience: we already had a natural isomorphism between these complexes, so identifying it as the Chern character just gives it a nice name. But using the fact that the Chern character is defined more globally (i.e., on all pairs (Y,B)) allows us to push this a bit further and obtain a description of the Atiyah-Hirzebruch differentials. The following result is a combination of [AH2, Lemmas 1.2 and 7.3]

Proposition 29.24. *Let X be a CW-complex and let $u \in H^p(X)$. Then in the Atiyah-Hirzebruch spectral sequence one has $d_i u = 0$ for all $2 \leq i < r$ if and only if there exist $\tilde{u} \in H^p(F_{p+r-1}, F_{p-1})$ and $\xi \in K^*(F_{p+r-1}, F_{p-1})$ such that*

(i) $\tilde{u}$ is a lift for u under $H^p(F_{p+r-1},F_{p-1}) \to H^p(F_{p+r-1}) \xleftarrow{\cong} H^p(X)$, and
(ii) $\mathrm{ch}(\xi) = \tilde{u} +$ higher order terms.

Moreover, if in the above situation α is a cellular cochain representating $\mathrm{ch}(\xi)_{p+r-1}$ then $\delta\alpha$ is integral and represents the differential $d_r u$.

Proof. To symplify some typography we assume throughout the proof that p is even, although the odd case is identical (or else one could just replace X with its suspension).

Assume that $u \in H^p(X)$ satisfies $d_i u = 0$ for $2 \leq i < r$. Identifying u with an element in E_2, this condition says that u can be represented by a class $z \in E_1 = K^0(F_p, F_{p-1})$ with the property that $z \in Z_{r-1}$. The element associated to z by the isomorphism $K^0(F_p,F_{p-1}) \cong C^p_{cell}(X;\mathbb{Z})$ is a cellular p-cochain representative for u.

As remarked in (8) of Section 29.1, the condition $z \in Z_{r-1}$ is equivalent to saying that z lifts to a class $\tilde{z} \in K^0(F_{p+r-1}, F_{p-1})$. Now apply the Chern character to get the square

$$\begin{array}{ccc} K^0(F_{p+r-1},F_{p-1}) & \longrightarrow & K^0(F_p,F_{p-1}) \\ \mathrm{ch}\downarrow & & \downarrow\mathrm{ch} \\ H^*(F_{p+r-1},F_{p-1};\mathbb{Q}) & \xrightarrow{j^*} & H^*(F_p,F_{p-1};\mathbb{Q}), \end{array} \qquad \begin{array}{ccc} \tilde{z} & \longrightarrow & z \\ \downarrow & & \downarrow \\ \mathrm{ch}(\tilde{z}) & \longrightarrow & \mathrm{ch}(z). \end{array}$$

The element $\mathrm{ch}(z)$ is an (integral) cellular p-cochain that represents the class u. The groups $H^*(F_{p+r-1},F_{p-1})$ are zero in degrees $* < p$, so $\mathrm{ch}(\tilde{z})$ is of the form $\mathrm{ch}(\tilde{z})_p +$ higher order terms. The fact that $\mathrm{ch}(z)$ is a cellular cochain representing u says that any lift of $\mathrm{ch}(z)$ into $H^p(F_{p+1},F_{p-1})$ has the same image in $H^p(F_{p+1})$ as u:

$$H^p(X) \longrightarrow H^p(F_{p+1}) \longleftarrow H^p(F_{p+1},F_{p-1})$$

It follows readily that $\mathrm{ch}(\tilde{z})_p$ has the same image in $H^p(F_{p+r-1})$ as u. This completes the ($\Rightarrow$) direction of the first statement in the proposition. The ($\Leftarrow$) direction follows in the same way, as all the steps are reversible.

For the final statement of the proposition we continue to assume (for convenience) that p is even. Consider the diagram

$$\begin{array}{ccc} \xi \in K^0(F_{p+r-1}, F_{p-1}) & \xrightarrow{\delta} & K^1(F_{p+r}, F_{p+r-1}) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(F_{p+r-1}, F_{p-1}; \mathbb{Q}) & \xrightarrow{\delta} & H^{*+1}(F_{p+r}, F_{p+r-1}; \mathbb{Q}) \end{array}$$

where in both rows the map δ is the connecting homomorphism in the long exact sequence for the triple $(F_{p+r}, F_{p+r-1}, F_{p-1})$. Looking back on (8) of Section 29.1, the element $\delta(\xi)$ represents $d_r(u)$ in the E_r-term of the spectral sequence. But we know that the right vertical map is an injection whose image consists of the integral elements, and our isomorphism of the E_2-term with $H^*(X; \mathbb{Z})$ identifies $\delta(\xi)$ with $\mathrm{ch}(\delta(\xi))$. Commutativity of the square says this element is also $\delta((\mathrm{ch}\,\xi)_{q+r-1})$ (and also verifies that this class is integral). □

30. Operations on K-theory

Experience has shown that when studying a cohomology theory it is useful to look not just at the cohomology groups themselves but also the natural *operations* on the cohomology groups. In the case of singular cohomology this is the theory of Steenrod operations. In the present section we will construct some useful operations $K^0(X) \to K^0(X)$. We start with the λ-operations, which are easy to define but have the drawback that they are not group homomorphisms. Then we modify these to obtain the Adams operations ψ^k, which are more nicely behaved.

30.1. **The lambda operations.** Fix a topological space X. We start with the exterior power constructions $E \mapsto \Lambda^k E$ on vector bundles over X. These are, of course, not additive: $\Lambda^k(E \oplus F) \ncong \Lambda^k E \oplus \Lambda^k F$. So it is not immediately clear how these constructions induce maps on K-groups. The key lies in the formula

$$\Lambda^k(E \oplus F) \cong \bigoplus_{i+j=k} \Lambda^i E \otimes \Lambda^j F. \tag{30.2}$$

Construct a formal power series

$$\lambda_t(E) = \sum_{i=0}^{\infty} [\Lambda^i E] t^i = 1 + [E]t + [\Lambda^2 E]t^2 + \cdots \in K^0(X)[[t]].$$

Because the zero coefficient is 1, this power series is a unit in $K^0(X)[[t]]$. So λ_t is a function into the group of units inside $K^0(X)[[t]]$:

$$\begin{array}{ccc} \mathrm{Vect}(X) & \xrightarrow{\lambda_t} & \Big(K^0(X)[[t]]\Big)^* \\ \downarrow & \nearrow & \\ K^0(X) & & \end{array}$$

Formula (30.2) says that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$, and this implies the existence of the dotted-arrow group homomorphism in the above diagram. We will call this dotted arrow λ_t as well.

Finally, define $\lambda^k\colon K^0(X) \to K^0(X)$ by letting $\lambda^k(w)$ be the coefficient of t^k in $\lambda_t(w)$. Note that if E is a vector bundle over X then $\lambda^k([E]) = [\Lambda^k E]$. However, λ^k is not a group homomorphism; instead one has the formula

$$\lambda^k(u+w) = \sum_{i+j=k} \lambda^i(u)\lambda^j(v).$$

Example 30.3. To get a feeling for these operations let us compute $\lambda^k(-[E])$ for E a vector bundle over X. Note first that

$$\lambda_t(-[E]) = \tfrac{1}{\lambda_t([E])} = \tfrac{1}{1+[E]t+[\Lambda^2 E]t^2+\cdots}$$

For R a commutative ring and $a = 1 + a_1 t + a_2 t^2 + \cdots \in R[[t]]$, one has

$$\tfrac{1}{a} = 1 + P_1 t + P_2 t^2 + \cdots$$

where the P_i's are certain universal polynomials in the a_i's with coefficients in $\mathbb{Z}$. Equating coefficients in the identity $1 = (1 + a_1 t + a_2 t^2 + \cdots)(1 + P_1 t + P_2 t^2 + \cdots)$ gives

$$P_k + P_{k-1}a_1 + P_{k-2}a_2 + \cdots + P_1 a_{k-1} + a_k = 0,$$

which allows one to inductively determine each P_k. One finds that

$$P_1 = -a_1, \quad P_2 = a_1^2 - a_2, \quad P_3 = -a_1^3 - 2a_1a_2 + a_3$$

So we conclude that

$$\begin{aligned}
\lambda^1(-[E]) &= -[E],\\
\lambda^2(-[E]) &= [E]^2 - [\Lambda^2 E],\\
\lambda^3(-[E]) &= -[E]^3 - 2[E][\Lambda^2 E] + [\Lambda^3 E]
\end{aligned}$$

and so forth.

30.4. **Symmetric power operations.** One can repeat everything from the previous section using the symmetric product construction $E \mapsto \operatorname{Sym}^k E$ in place of the exterior product $\Lambda^k E$. One obtains a group homomorphism

$$\operatorname{sym}_t\colon K^0(X) \to \big(K^0(X)[[t]]\big)^*$$

and defines $\operatorname{sym}^k(w)$ to be the coefficient of t^k in $\operatorname{sym}_t(w)$. It turns out, however, that these operations do not give anything 'new'—they are related to the λ-operations by the formula

$$\operatorname{sym}^k(w) = \lambda^k(-w).$$

To explain this we need a brief detour on the deRham complex. The following material is taken from [FLS].

Let V be a vector space over a field F. Write $\operatorname{Sym}^*(V) = \oplus_k \operatorname{Sym}^k(V)$ and $\Lambda^*(V) = \oplus_k \Lambda^k V$. These each have a familiar algebra structure, and we have canonical isomorphisms

$$\operatorname{Sym}^*(V) \otimes \operatorname{Sym}^*(W) \xrightarrow{\cong} \operatorname{Sym}^*(V \oplus W), \qquad \Lambda^*(V) \otimes \Lambda^*(W) \xrightarrow{\cong} \Lambda^*(V \oplus W).$$

These isomorphisms allow us to equip both $\operatorname{Sym}^*(V)$ and $\Lambda^*(V)$ with coproducts making them into Hopf algebras. The coproducts are

$$\operatorname{Sym}^*(V) \longrightarrow \operatorname{Sym}^*(V \oplus V) \xleftarrow{\cong} \operatorname{Sym}^*(V) \otimes \operatorname{Sym}^*(V)$$

and

$$\Lambda^*(V) \longrightarrow \Lambda^*(V \oplus V) \xleftarrow{\cong} \Lambda^*(V) \otimes \Lambda^*(V)$$

where in each case the first map is the one induced by the diagonal $\Delta\colon V \to V \oplus V$. Write $e\colon \operatorname{Sym}^k(V) \to \operatorname{Sym}^{k-1} V \otimes \operatorname{Sym}^1(V)$ and $e'\colon \Lambda^k(V) \to \Lambda^1 V \otimes \Lambda^{k-1}(V)$ for the projections of the coproduct onto the indicated factors. Finally, write d and κ for the composites

$$\begin{array}{ccc}
\operatorname{Sym}^k(V) \otimes \Lambda^i(V) \xrightarrow{e\otimes \mathrm{id}} \operatorname{Sym}^{k-1}(V) \otimes \operatorname{Sym}^1(V) \otimes \Lambda^i(V) & & \\
\| & & \\
\operatorname{Sym}^{k-1}(V) \otimes \Lambda^1(V) \otimes \Lambda^i(V) & \longrightarrow & \operatorname{Sym}^{k-1}(V) \otimes \Lambda^{i+1}(V)
\end{array}$$

and

$$\begin{array}{ccc}
\operatorname{Sym}^k(V) \otimes \Lambda^i(V) \xrightarrow{\mathrm{id}\otimes e'} \operatorname{Sym}^{k}(V) \otimes \Lambda^1(V) \otimes \Lambda^{i-1}(V) & & \\
\| & & \\
\operatorname{Sym}^{k}(V) \otimes \operatorname{Sym}^1(V) \otimes \Lambda^{i-1}(V) & \longrightarrow & \operatorname{Sym}^{k+1}(V) \otimes \Lambda^{i-1}(V).
\end{array}$$

The maps d and κ are called the **de Rham** and **Koszul** differentials, respectively. The following diagram shows the maps d, and the maps κ go in the opposite direction:

(30.5) *The deRham and Koszul complexes:*

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
 & \overset{d}{\nearrow} & & \overset{d}{\nearrow} & & \overset{d}{\nearrow} & \\
\operatorname{Sym}^3 V \otimes \Lambda^2 V & & \operatorname{Sym}^2 V \otimes \Lambda^2 V & & \operatorname{Sym}^1 V \otimes \Lambda^2 V & & \operatorname{Sym}^0 V \otimes \Lambda^2 V \\
 & \overset{d}{\nearrow} & & \overset{d}{\nearrow} & & \overset{d}{\nearrow} & \\
\operatorname{Sym}^3 V \otimes \Lambda^1 V & & \operatorname{Sym}^2 V \otimes \Lambda^1 V & & \operatorname{Sym}^1 V \otimes \Lambda^1 V & & \operatorname{Sym}^0 V \otimes \Lambda^1 V \\
 & \overset{d}{\nearrow} & & \overset{d}{\nearrow} & & \overset{d}{\nearrow} & \\
\operatorname{Sym}^3 V \otimes \Lambda^0 V & & \operatorname{Sym}^2 V \otimes \Lambda^0 V & & \operatorname{Sym}^1 V \otimes \Lambda^0 V & & \operatorname{Sym}^0 V \otimes \Lambda^0 V
\end{array}$$

Let $e_1, \ldots, e_n$ be a basis for V. It will be convenient to use the notation de_j for the element of $\Lambda^1(V)$ corresponding to e_j under the canonical isomorphism $\Lambda^1(V) \cong V$. Let $m = e_{i_1} \otimes \cdots \otimes e_{i_r} \in \operatorname{Sym}^r(V)$ and $\omega = de_{j_1} \wedge \cdots \wedge de_{j_s} \in \Lambda^s(V)$. It is an exercise to verify that

$$e(m) = \sum_u \bigl(e_{i_1} \otimes \cdots \otimes \widehat{e_{i_u}} \otimes \cdots \otimes e_{i_r}\bigr) \otimes e_{i_u}$$

and

$$e'(\omega) = \sum_u (-1)^{u-1} de_{j_u} \otimes \bigl(de_{j_1} \wedge \cdots \wedge \widehat{de_{j_u}} \wedge \cdots \wedge de_{j_s}\bigr).$$

So

$$d(m \otimes \omega) = \sum_u \bigl(e_{i_1} \otimes \cdots \otimes \widehat{e_{i_u}} \otimes \cdots \otimes e_{i_r}\bigr) \otimes (de_{i_u} \wedge \omega)$$

and

$$\kappa(m \otimes \omega) = \sum_u (-1)^{u-1}(m \otimes e_{j_u}) \otimes \big(de_{j_1} \wedge \cdots \wedge \widehat{de_{j_u}} \wedge \cdots \wedge de_{j_s}\big).$$

From these descriptions one readily sees that d is the usual deRham differential and κ is the usual Koszul differential. Consequently, $d^2 = 0$ and $\kappa^2 = 0$. In the two-dimensional array (30.5), if we take direct sums inside of each row then we get

$$\operatorname{Sym}^*(V) \otimes \Lambda^0(V) \xrightarrow{d} \operatorname{Sym}^*(V) \otimes \Lambda^1(V) \xrightarrow{d} \cdots$$

and this is an algebraic version of the deRham complex. We also get

$$\cdots \longrightarrow \operatorname{Sym}^*(V) \otimes \Lambda^2(V) \xrightarrow{\kappa} \operatorname{Sym}^*(V) \otimes \Lambda^1(V) \xrightarrow{\kappa} \operatorname{Sym}^*(V) \otimes \Lambda^0(V)$$

which is a Koszul complex.

Proposition 30.6.

(a) $d\kappa + \kappa d\colon \operatorname{Sym}^r(V) \otimes \Lambda^s(V) \to \operatorname{Sym}^r(V) \otimes \Lambda^s(V)$ *is multiplication by the total degree* $r + s$.

(b) In (30.5) every diagonal deRham chain complex is exact in dimensions where the total degree is prime to the characteristic of F.

(c) In (30.5) every diagonal Koszul chain complex is exact, regardless of the characteristic of the ground field.

Proof. Part (a) is a computation that is tedious but not particularly hard. Part (b) follows from (a): the maps κ give a chain homotopy showing that multiplication by the total degree is homotopic to the zero map. If the total degree is invertible in the ground field, this implies that the homology must be zero in that dimension.

We do not actually need part (c) below, but we include it to complete the story. The maps d give a chain homotopy for the κ-complexes, much like in the proof of (b), but this gives exactness only for some spots in the complex. The proof of exactness at all spots is something we have already seen in a somewhat more general context, in Theorem 17.24(a). ∎

Now we apply the above results to K-theory. Since the deRham and Koszul complexes were canonical constructions, we can apply them to vector bundles. The deRham complex gives us exact sequences

$$0 \longrightarrow \operatorname{Sym}^k E \otimes \Lambda^0 E \longrightarrow \operatorname{Sym}^{k-1} E \otimes \Lambda^1 E \longrightarrow \cdots \longrightarrow \operatorname{Sym}^1 E \otimes \Lambda^{k-1} E \longrightarrow \operatorname{Sym}^0 E \otimes \Lambda^k E \longrightarrow 0.$$

These show that in $K^0(X)$ one has

$$\sum_{a+b=k} [\operatorname{Sym}^a E] \cdot [\Lambda^b E] = 0.$$

Consequently, $\operatorname{sym}_t([E]) \cdot \lambda_t([E]) = 1$. So

$$\operatorname{sym}_t([E]) = \tfrac{1}{\lambda_t([E])} = \lambda_t(-[E]) \qquad \text{and} \qquad \lambda_t([E]) = \tfrac{1}{\operatorname{sym}_t([E])} = \operatorname{sym}_t(-[E]).$$

Any class $w \in K^0(X)$ has the form $w = [E] - [F]$ for some vector bundles E and F, and therefore

$$\mathrm{sym}_t(w) = \mathrm{sym}_t([E]) \cdot \mathrm{sym}_t(-[F]) = \lambda_t(-[E]) \cdot \lambda_t([F]) = \lambda_t([F] - [E]) = \lambda_t(-w).$$

So the sym^k and λ^k operations on K-theory are essentially the same: $\mathrm{sym}^k(w) = \lambda^k(-w)$.

This has been a long discussion with somewhat of a negative conclusion: the sym^k operations can be completely ignored in favor of the λ^k's (or vice versa). We have learned some useful things along the way, however.

30.7. **The Newton polynomials.** The usefulness of the λ^k operations is limited by the fact that they are not group homomorphisms. There is a clever method, however, for combining the λ-operations in a way that does produce a collection of group homomorphisms. This is originally due to Frank Adams [Ad2]. Before describing this construction we take a brief detour to develop the algebraic combinatorics that we will need.

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30.8. **The Adams operations.** Recall that we have a map $\lambda_t \colon K^0(X) \to \big(K^0(X)[[t]]\big)^*$ and that this is a group homomorphism:

$$\lambda_t(x + y) = \lambda_t(x) \cdot \lambda_t(y).$$

If we want additive maps $K^0(X) \to K^0(X)$ a natural idea is to apply logarithms to the above formula. To be precise, start with the formal power series

$$\log(1 + z) = z - \tfrac{z^2}{2} + \tfrac{z^3}{3} - \cdots$$

Since $\lambda_t(x)$ has constant term equal to 1, we can use the above series to make sense of $\log(\lambda_t(x))$—but only provided that we add denominators into $K^0(X)$, say by tensoring with $\mathbb{Q}$. If we set $\mu_t(x) = \log(\lambda_t(x))$ then we would have

$$\mu_t(x + y) = \mu_t(x) + \mu_t(y).$$

The coefficients of powers of t in $\mu_t(x)$ then give additive operations, with the only difficulty being that they take values in $K^0(X) \otimes \mathbb{Q}$.

We can, however, eliminate the need for $\mathbb{Q}$-coefficients by applying the operator $\frac{d}{dt}$. Precisely, define

$$\nu_t(x) = \frac{d}{dt}\Big[\mu_t(x)\Big] = \tfrac{\lambda_t'(x)}{\lambda_t(x)} = (1 - z + z^2 - z^3 + \cdots)|_{z=\lambda_t(x)-1} \cdot \lambda_t'(x).$$

Clearly this eliminates the problem with denominators: $\nu_t(x) \in K^0(X)[[t]]$, yet we still have $\nu_t(x + y) = \nu_t(x) + \nu_t(y)$. Taking coefficients of $\nu_t(x)$ thereby yields additive operations $\nu^k \colon K^0(X) \to K^0(X)$.

We could stop here, but there is one more modification that makes things a bit simpler later on. Suppose that L is a line bundle over X, and take $x = [L]$. Then $\lambda_t(x) = 1 + [L]t = 1 + xt$, hence

$$\nu_t(x) = \tfrac{x}{1+xt} = x(1 - xt + x^2t^2 - x^3t^3 + \cdots) = x - x^2t + x^3t^2 - \cdots.$$

The ν-operations simply give powers of x, together with certain signs: $\nu^k(x) = (-1)^k x^{k-1}$. It is easy to adopt a convention that makes these signs disappear, and we might as well do this; and while we're at it, lets shift the indexing on the ν's so that the kth operation sends x to x^k, since that will be easier to remember.

Putting everything together, we have arrived at the following definition:

$$\psi_t(x) = t\frac{d}{dt}\Big[\log(\lambda_{-t}(x))\Big] = t \cdot \tfrac{\lambda'_{-t}(x)}{\lambda_{-t}(x)},$$

and $\psi^k(x)$ is the coefficient of t^k in $\psi_t(x)$. The operations ψ^k are called **Adams operations**. We have proven that

(1) Each ψ^k is a group homomorphism $K^0(X) \to K^0(X)$, natural in X;
(2) If $x = [L]$ for L a line bundle then $\psi^k(x) = x^k$.

Conditions (1) and (2) actually completely characterize the Adams operations, although we will not need this.

The following identities give a recursive formula for the Adams operations in terms of the λ-operations:

Proposition 30.9.
(a) $\psi^k = S_k(\lambda^1, \dots, \lambda^k)$*, where* S_k *is the* k*th Newton polynomial;*
(b) $\psi^k = \lambda^1\psi^{k-1} - \lambda^2\psi^{k-1} + \cdots + (-1)^k\lambda^{k-1}\psi^1 + (-1)^{k+1}k\lambda^k$.

Proof. Part (b) is, of course, an immediate consequence of (a) via Lemma B.1. Part (a) follows directly from Proposition B.3. □

We record the first few Adams operations:

$$\psi^1 = \lambda^1, \quad \psi^2 = (\lambda^1)^2 - 2\lambda^2, \quad \psi^3 = (\lambda^1)^3 - 3\lambda^1\lambda^2 + 3\lambda^3.$$

For more of these, just see Table 2.2.

30.10. **Properties of the Adams operations.**

Proposition 30.11. *Fix* $k, l \geq 1$ *and* $x, y \in K^0(X)$*. Then*
(a) $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$
(b) $\psi^k(xy) = \psi^k(x)\psi^k(y)$
(c) $\psi^k(\psi^l(x)) = \psi^{kl}(x)$
(d) If k *is prime then* $\psi^k(x) \equiv x^k$ *mod* k*.*

The proof will use the following terminology. A **line element** in $K^0(X)$ is any element $[L]$ where $L \to X$ is a line bundle. The span of the line elements consist of the classes of the form $[L_1] + \cdots + [L_a] - [L'_1] - \cdots - [L'_b]$ where the L_i's and L'_j's are all line bundles.

Proof of Proposition 30.11. Note that if x and y are line elements then all of the above results are obvious because $\psi^r([L]) = [L^r]$. More generally, the results follow easily if x and y are in the span of the line elements. The general result now follows from the splitting principle in Proposition 30.12 below. Specifically, choose a $p\colon X_1 \to X$ such that p^* is injective and $p^*(x)$ is in the span of line elements. Then choose a $q\colon X_2 \to X_1$ such that q^* is injective and $q^*(p^*(y))$ is in the span of line elements. The identities in (a)–(d) all hold for $x' = (pq)^*(x)$ and $y' = (pq)^*(y)$, and so the injectivity of $(pq)^*$ shows they hold for x and y as well. □

Proposition 30.12 (The Splitting Principle)**.** *Let* X *be any space, and let* $x \in K^0(X)$*. Then there exists a space* Y *and a map* $p\colon Y \to X$ *such that* $p^*\colon K^0(X) \to K^0(Y)$ *is injective and* $p^*(x)$ *is in the span of line elements.*

Proof. Write $x = [E] - [F]$ for vector bundles E and F. Consider the map $\pi\colon \mathbb{P}(E) \to X$. Then $\pi^*E \cong E' \oplus L$ where L is a line bundle, and $\pi^*\colon K^0(X) \to K^0(\mathbb{P}(E))$ is injective (????). Iterating this procedure we obtain a map $f\colon Y \to X$ such that f^*E is a sum of line bundles and f^* is injective. Now use the same process to obtain a map $g\colon Y' \to Y$ such that g^* is injective and $g^*(f^*F)$ is a sum of line bundles. The composite $Y' \to X$ has the properties from the statement of the proposition. □

Corollary 30.13 (Characterization of Adams operations)**.** *Fix $k \geq 1$. Suppose that $F\colon K^0(-) \to K^0(-)$ is a natural ring homomorphism such that $F([L]) = [L^k]$ for any line bundle $L \to X$. Then $F = \psi^k$.*

Proof. Fix a space X, and let $\alpha \in K^0(X)$. Then $\alpha = [E] - [F]$ for some vector bundles E and F on X. By the Splitting Principle there exists a map $p\colon Y \to X$ such that p^*E and p^*F are direct sums of line bundles, and such that $p^*\colon K^0(X) \to K^0(Y)$ is injective. Our assumption on F implies at once that $F(p^*\alpha) = \psi^k(p^*\alpha)$, or equivalently $p^*(F\alpha) = p^*(\psi^k\alpha)$. Injectivity of p^* now gives $F(\alpha) = \psi^k(\alpha)$. □

The fact that ψ^k is natural and preserves (internal) products immediately yields that it also preserves external products. Recall that if $x \in K^0(X)$ and $y \in K^0(Y)$ then the external product can be written as $x \times y = \pi_1^*(x) \cdot \pi_2^*(y) \in K^0(X \times Y)$. Clearly $\psi^k(x \times y) = \psi^k(x) \times \psi^k(y)$. Using this, we easily obtain the following:

Proposition 30.14. *For $k \geq 1$, ψ^k acts on $\widetilde{K}^0(S^{2n})$ as multiplication by k^n.*

Proof. Let $\beta = 1 - [L]$ be the Bott element in $\widetilde{K}^0(S^2)$, and recall that the interal square $\beta^2 = 0$. From this it follows readily that

$$\psi^k(\beta) = 1 - L^k = 1 - (1-\beta)^k = 1 - (1 - k\beta) = k\beta.$$

Now recall that the external power $\beta^{(n)} = \beta \times \beta \times \cdots \times \beta$ generates $\widetilde{K}^0(S^{2n})$. But

$$\psi^k(\beta^{(n)}) = (\psi^k\beta)^{(n)} = (k\beta)^{(n)} = k^n\beta^{(n)}.$$

□

Recall the canonical filtration of $K^0(X)$ discussed in ????. In particular, recall that $F^{2n-1}K^0(X) = F^{2n}K^0(X)$, for every n. The naturality of the Adams operations shows that they respect the filtration, and Proposition 30.14 shows that ψ^k acts as a scalar on the associated graded:

Proposition 30.15. *Let $k \geq 1$.*

(a) *If $x \in F^{2n}K^0(X)$ then $\psi^k(x) = k^n x$ + terms of higher filtration. That is, $\psi^k(x) - k^n x \in F^{2n+2}K^0(X)$.*

(b) *If the induced filtration on $K^0(X)_{\mathbb{Q}}$ is finite (e.g., if X is a finite-dimensional CW-complex) then the operations ψ^k are diagonalizable on $K^0(X)_{\mathbb{Q}}$, with eigenvalues of the form k^r for $r \geq 0$. The decomposition*

$$K^0(X)_{\mathbb{Q}} = \bigoplus_{r \geq 0} \mathrm{Eig}_{\psi^k}(k^r)$$

is independent of k, and it restricts to give

$$F^{2n}K^0(X)_{\mathbb{Q}} = \bigoplus_{r \geq n} \mathrm{Eig}_{\psi^k}(k^r).$$

Proof. For part (a) it suffices to replace X by a weakly equivalent CW-complex, so that $F^{2n}K^0(X) = \ker[K^0(X) \to K^0(X^{2n-1})]$. If $\alpha \in F^{2n}K^0(X)$ then let α_1 denote its image in $K^0(X^{2n})$. The cofiber sequence $X^{2n-1} \hookrightarrow X^{2n} \to X^{2n}/X^{2n-1}$ indcues a long exact sequence

$$\cdots \to \widetilde{K}^0(X^{2n}/X^{2n-1}) \to K^0(X^{2n}) \to K^0(X^{2n-1}) \to \cdots$$

The element $\alpha_1 \in K^0(X^{2n})$ maps to zero, and so it is the image of a class $\alpha_2 \in \widetilde{K}^0(X^{2n}/X^{2n-1})$. By Proposition 30.14 one knows $\psi^k(\alpha_2) = k^n\alpha_2$, and so $\psi^k\alpha_1 = k^n\alpha_1$. It follows that $\psi^k\alpha - k^n\alpha$ maps to zero in $K^0(X^{2n})$, and hence lies in $F^{2n+1}K^0(X^{2n})$.

For part (b), let $2n$ be the largest even integer such that $F^{2n}K^0(X)_{\mathbb{Q}} \neq 0$. It follows from (a) that ψ^k acts as multiplication by k^n on $F^{2n}K^0(X)_{\mathbb{Q}}$. We now prove by reverse induction that $F^{2i}K^0(X)_{\mathbb{Q}}$ is the sum of k^j-eigenspaces of ψ^k, for $j \geq i$. Assume this holds for a particular value of i, and let $\alpha \in F^{2i-2}K^0(X)$. We know $\psi^k\alpha = k^{i-1}\alpha + \gamma$, where γ is some element of $F^{2i}K^0(X)_{\mathbb{Q}}$. The induction hypothesis says that $\gamma = \sigma_i + \sigma_{i+1} + \cdots + \sigma_n$ where each σ_r is an eigenvector with eigenvalue k^r. A routine calculation now shows that $\alpha - \sum_{r\geq i} \frac{1}{k^r - k^{i-1}}\sigma_r$ is a k^{i-1}-eigenvector for ψ^k, and hence α belongs to the sum of eigenspaces for eigenvalues k^r, $r \geq i-1$. This completes the induction step. □

31. The Hopf invariant one problem

The Hopf invariant assigns an integer to every map $f\colon S^{2n-1} \to S^n$, giving a group homomorphism $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$. Elementary arguments show that 2 is always in the image, and the natural question is then whether 1 is also in the image. This is the Hopf invariant one problem—determine all values of n for which H is surjective (or said differently, all values of n for which there exists a map of Hopf invariant one).

It was known classically that H is surjective when $n \in \{1, 2, 4\}$, because the classical Hopf maps all have Hopf invariant equal to one. The question for other dimensions was first settled by Adams in [Ad1], who proved that no other Hopf invariant one maps exist. Adams's proof is not simple, even by modern standards, being based on secondary cohomology operations associated to the Steenrod squares. Several years after Adams gave his original proof, Adams and Atiyah used K-theory to give a much simpler solution to the Hopf invariant one problem. Their 'postcard proof' takes less than a page, in dramatic contrast to Adams's original method. This was seen—rightly so—as a huge demonstration of the power of K-theory.

Our goal in this section will be to present the Adams-Atiyah proof, although we will not quite do this in their style. Specifically, when Adams and Atiyah wrote their paper they clearly had an agenda: to write down the proof in as small a space as possible. If the goal is to accentuate how much the use of K-theory simplifies the solution, this makes perfect sense. But at the same time, writing the proof in this way results in a certain air of mystery: the proof involves a strange manipulation with the Adams operations ψ^2 and ψ^3 that comes out of nowhere—it seems like a magic trick.

In our presentation below we try to put this (ψ^2, ψ^3) trick into its proper context: it is part of a calculation of a certain Ext^1 group. The full calculation of this group is not hard, and quite interesting for other reasons—e.g., it connects deeply to the study of the J-homomorphism. Our presentation doesn't fit on a postcard, but by the time we are done we will have a good understanding of several neat and important things. Hopefully it won't seem like magic.

31.1. **Brief review of the problem.** Let $f\colon S^{2n-1} \to S^n$ and consider the mapping cone Cf. One readily computes that

$$H^i(Cf) \cong \begin{cases} \mathbb{Z} & \text{if } i \in \{0, n, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Fix an orientation on the two spheres, and let a and b be corresponding generators for $H^n(Cf)$ and $H^{2n}(Cf)$. The $b^2 = h \cdot a$ for a unique integer $h \in \mathbb{Z}$, and this integer is called the **Hopf invariant** of f: we write $h = H(f)$.

Note that if n is odd then $b^2 = -b^2$ and so $h = 0$. Therefore the Hopf invariant is only interesting when n is even.

Remark 31.2. We follow [Ha, Proposition 4B.1] to see that $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a group homomorphism. Given $f, g\colon S^{2n-1} \to S^n$ consider the diagram of mapping cones

$$\begin{array}{ccccc} S^{2n-1} & \xrightarrow{f+g} & S^n & \longrightarrow & C_{f+g} \\ \downarrow & & \| & & \downarrow \\ S^{2n-1} \vee S^{2n-1} & \xrightarrow{f\vee g} & S^n & \longrightarrow & X \end{array}$$

where the left vertical map is the equatorial collapse and $X = C_{f\vee g}$. Note that there are inclusions $Cf \hookrightarrow X$ and $Cg \hookrightarrow X$. The cohomology group $H^{2n}(X)$ has two generators a_1 and a_2, and naturality applied to those inclusions shows that $b^2 = H(f)a_1 + H(g)a_2$. But under the map $C_{f+g} \to X$ both a_1 and a_2 are sent to our usual generator a, and from this one gets that $H(f + g) = H(f) + H(g)$.

Remark 31.3. It is easy to see that 2 (and therefore any even integer) is always in the image of H. We again follow [Ha] here and let X be the pushout

$$\begin{array}{ccc} S^n \vee S^n & \rightarrowtail & S^n \times S^n \\ {\scriptstyle\nabla}\downarrow & & \vdots\,{\scriptstyle\alpha} \\ S^n & \cdots\cdots\rightarrow & X. \end{array}$$

One readily checks that the cohomology of X consists of two copies of $\mathbb{Z}$, in degrees n and $2n$. So X is the mapping cone of a certain map $f\colon S^{2n-1} \to S^n$, the attaching map of the top cell. If $x \in H^n(S^n)$ is a fixed generator, then there is a generator $b \in H^n(X)$ that maps to $x \otimes 1 + 1 \otimes x$ under α^*. It follows that b^2 maps to

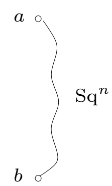
$$(x \otimes 1 + 1 \otimes x)^2 = 2(x \otimes x)$$

and therefore b^2 is twice a generator of $H^{2n}(X)$. It follows that the Hopf invariant of f is ± 2, depending on one's sign choices.

The problem arises of determining the precise image of $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$, when n is even. By Remarks 31.2 and 31.3 the image is a subgroup that contains $2\mathbb{Z}$, so there are only two possibilities: either the image equals $2\mathbb{Z}$, or else it equals all of $\mathbb{Z}$. The latter happens if and only if there exists an element in $\pi_{2n-1}(S^n)$ having Hopf invariant equal to one. Thus, this is the "Hopf invariant one" problem.

The following several paragraphs involve the Steenrod squares. The results will not be needed later in this section, but they constitute an interesting part of the overall story.

As soon as one is versed in the Steenrod squares it is easy to obtain a necessary condition for the existence of a Hopf invariant one map $f\colon S^{2n-1} \to S^n$. In the mod 2 cohomology of Cf we have $\mathrm{Sq}^n(b) = b^2 = h \cdot a$. So if f has odd Hopf invariant then $\mathrm{Sq}^n(b) = a$, and the mod 2 cohomology of Cf looks like this:



This picture just says that the cohomology has generators a and b together with a Sq^n connecting b to a. As an immediate consequence we obtain Adem's theorem:

Proposition 31.4 (Adem). *If $f\colon S^{2n-1} \to S^n$ has Hopf invariant one then n is a power of 2.*

Proof. The above picture represents a module over the Steenrod algebra only if Sq^n is indecomposable. But by our knowledge of the Steenrod algebra, the indecomposables all have degrees equal to a power of 2 (they are represented by the elements Sq^{2^i}). □

The reader might have noticed that there actually seem to be *two* problems here, that are interelated. There is the Hopf invariant one problem, and there is the question of whether there exists a map $S^{k+n-1} \to S^k$ whose cofiber has a nonzero Sq^n in mod 2 cohomology. The first problem is inherently *unstable* in nature because it deals with the cup product, whereas the second problem is clearly stable. It is useful to note that the two problems are actually equivalent:

Proposition 31.5. *Fix $n \geq 1$. The following two statements are equivalent:*
(a) There exists a map $S^{2n-1} \to S^n$ of Hopf invariant one;
(b) There exists a $k \geq 0$ and a map $S^{k+n-1} \to S^k$ whose mapping cone has a nonzero Sq^n operation.

Proof. In the discussion preceding Adem's theorem we saw that (a) implies (b) by taking $k = n$. Conversely, if (b) holds for a certain map g then by suspending if necessary we can assume $k \geq n$. The Freudenthal Suspension Theorem guarantees that $\pi_{2n-1}(S^n) \to \pi_{k+n-1}(S^k)$ is surjective, so choose map a map $f\colon S^{2n-1} \to S^n$ that is a preimage of g. The spaces Cf and Cg are homotopy equivalent after appropriate suspensions, so the mod 2 cohomology of Cf has a nonzero Sq^n. It immediately follows that f has odd Hopf invariant, and consequently there exists a map of Hopf invariant one. □

Adem's theorem is really an analysis of the stable problem, and it may be rephrased as follows. If $f\colon S^{n+k-1} \to S^k$ then there is a cofiber sequence $S^{n+k-1} \to S^k \to Cf$, and the long exact sequence on mod 2 cohomology breaks up into a family of short exact sequences

$$0 \leftarrow \tilde{H}^*(S^k;\mathbb{Z}/2) \leftarrow \tilde{H}^*(Cf;\mathbb{Z}/2) \leftarrow \tilde{H}^*(S^{n+k};\mathbb{Z}/2) \leftarrow 0.$$

These are maps of modules over the Steenrod algebra $\mathcal{A}$, and both the left and right terms are isomorphic to the trivial $\mathcal{A}$-module $\mathbb{F}_2$ (graded to lie in the appropriate dimension). So the above short exact sequence represents an element of $\operatorname{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. Standard homological algebra identifies this Ext^1 with the module of indecomposables I/I^2, where I is the augmentation ideal of $\mathcal{A}$. Adem's Theorem works because we know this module of indecomposables precisely, and therefore can identify the Ext^1 groups precisely.

31.6. **An** Ext **calculation.** Most of this section will be spent in pursuit of a purely algebraic question, somewhat related to what we just saw. Let $(\mathbb{N}, \cdot)$ be the monoid of natural numbers under multiplication, and let $\mathcal{B} = \mathbb{Z}[\mathbb{N}]$ be the corresponding monoid ring. Write ψ^k for the element of $\mathcal{B}$ corresponding to $k \in \mathbb{N}$. Then $\mathcal{B}$ is simply the polynomial ring

$$\mathcal{B} = \mathbb{Z}[\psi^2, \psi^3, \psi^5, \dots],$$

with one generator corresponding to each prime number. We think of $\mathcal{B}$ as the ring of formal Adams operations, and note that $K^0(X)$ is naturally a $\mathcal{B}$-module for any space X.

Let $\mathbb{Z}(r)$ denote the following module over $\mathcal{B}$: as an abelian group it is a copy of $\mathbb{Z}$, with chosen generator g, and the $\mathcal{B}$-module structure is $\psi^k.g = k^r g$. Our goal will be to compute the groups

$$\operatorname{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$$

for all values of r and s.

Before exploring this algebraic problem let us quickly indicate the application to topology. Let $f\colon S^{n+k} \to S^n$ be a map of spheres, and write Cf for the mapping cone. The Puppe sequence looks like

$$S^{n+k} \to S^n \to Cf \to S^{n+k+1} \to S^{n+1} \to \cdots$$

and applying $\widetilde{K}^0(-)$ to this yields

$$\widetilde{K}^0(S^{n+k}) \leftarrow \widetilde{K}^0(S^n) \leftarrow \widetilde{K}^0(Cf) \leftarrow \widetilde{K}^0(S^{n+k+1}) \leftarrow \widetilde{K}^0(S^{n+1}) \leftarrow \cdots$$

Any $\widetilde{K}^0(-)$ group is naturally a $\mathcal{B}$-module, via the Adams operations; and all the maps in the above sequence are maps of $\mathcal{B}$-modules. Under the hypotheses that n is even and k is odd, the groups on the two ends vanish and we get a short exact sequence

$$0 \leftarrow \widetilde{K}^0(S^n) \leftarrow \widetilde{K}^0(Cf) \leftarrow \widetilde{K}^0(S^{n+k+1}) \leftarrow 0.$$

Proposition 30.14 says that as a $\mathcal{B}$-module $\widetilde{K}^0(S^{2r})$ is isomorphic to $\mathbb{Z}(r)$, and hence the above sequence yields an element

$$A(f) \in \operatorname{Ext}^1_{\mathcal{B}}\big(\mathbb{Z}(\tfrac{n+k+1}{2}), \mathbb{Z}(\tfrac{n}{2})\big).$$

That is to say, we have obtained a topological invariant of f taking values in this Ext group.

Now we begin our computation. Let X be a $\mathcal{B}$-module that sits in a short exact sequence

$$0 \to \mathbb{Z}(s) \to X \to \mathbb{Z}(r) \to 0. \tag{31.7}$$

Write a for a chosen generator of $\mathbb{Z}(s)$ (as well as its image in X) and $\tilde{b}$ for a chosen generator of $\mathbb{Z}(r)$. Write b for a preimage of $\tilde{b}$ in X. Then we have

$$\psi^k b = k^r b + P_k a$$

for a unique $P_k \in \mathbb{Z}$. The $\mathcal{B}$-module structure on X is completely determined by the ∞-tuple of integers $\mathbf{P} = (P_2, P_3, P_5, \ldots)$.

Does any choice of $\mathbf{P}$ correspond to a $\mathcal{B}$-module? To be a $\mathcal{B}$-module one must have $\psi^k\psi^l = \psi^l\psi^k$ on $\mathcal{B}$. But we can compute

$$\begin{aligned}\psi^k(\psi^l b) = \psi^k(l^r b + P_l a) &= l^r \cdot \psi^k(b) + P_l \cdot \psi^k a \\ &= l^r \cdot (k^r b + P_k a) + P_l k^s a\end{aligned}$$

and likewise

$$\begin{aligned}\psi^l(\psi^k b) = \psi^l(k^r b + P_k a) &= k^r \cdot \psi^l(b) + P_k \cdot \psi^l a \\ &= k^r \cdot (l^r b + P_l a) + P_k l^s a.\end{aligned}$$

Equating these expressions we find that $(l^r - l^s)P_k a = (k^r - k^s)P_l a$. Since a is infinite order in X it must be that

$$(l^r - l^s)P_k = (k^r - k^s)P_l,$$

and this holds for every two primes k and l. If $r = s$ this gives no condition and it is indeed true that any choice of $\mathbf{P}$ corresponds to a module X. But in the case $r \neq s$ we can write

$$\tfrac{P_l}{P_k} = \tfrac{l^r - l^s}{k^r - k^s}.$$

So once we fix a prime k, all other P_l's are determined by P_k. For convenience we take k to be the smallest prime, and obtain

$$P_l = P_2 \cdot \left(\tfrac{l^r - l^s}{2^r - 2^s}\right)$$

for every prime l. This shows that the module X depends on the single parameter P_2; however, it is still not true that all possible integral choices for P_2 correspond to $\mathcal{B}$-modules. Indeed, we will only get a $\mathcal{B}$-module if the above formula for P_l yields an *integer* for every choice of l. To this end define

$$Z_{r,s} = \left\{ P \in \mathbb{Z} \,\middle|\, P \cdot \left(\tfrac{l^r - l^s}{2^r - 2^s}\right) \in \mathbb{Z}, \text{ for all primes } l \right\}.$$

For $P \in Z_{r,s}$ let X_P denote the corresponding $\mathcal{B}$-module for which $P_2 = P$.

Note that $Z_{r,s} \subseteq \mathbb{Z}$ is an ideal, and nonzero because it contains $2^r - 2^s$. In a moment we will compute this ideal in some examples. For now simply note that we have a map (in fact a surjection) $Z_{r,s} \to \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$ sending P to the extension (31.7) in which $X = X_P$. It is an exercise to check that this is indeed a map of abelian groups.

We next need to understand when X_P and X_Q are isomorphic as elements of $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$. This is when there is a map of $\mathcal{B}$-modules $X_P \to X_Q$ yielding a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(s) & \longrightarrow & X_P & \longrightarrow & \mathbb{Z}(r) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}(s) & \longrightarrow & X_Q & \longrightarrow & \mathbb{Z}(r) & \longrightarrow & 0 \end{array}$$

Such an f must satisfy $f(a) = a$, and $f(b) = b + Ja$ for some $J \in \mathbb{Z}$ (note that the symbols a and b are being used to simultaneously represent different elements of X_P and X_Q). The condition that f be a map of $\mathcal{B}$-modules is that $\psi^k(f(b)) = f(\psi^k b)$ for all primes k. For $k = 2$ the left-hand-side is

$$\psi^2(b + Ja) = 2^r b + Qa + J \cdot 2^s a$$

and the right-hand-side is

$$f(\psi^2 b) = f(2^r b + Pa) = 2^r(b + Ja) + Pa.$$

We obtain the condition

$$Q - P = (2^r - 2^s)J.$$

The reader may check as an exercise that the condition for the other ψ^k's follows as a consequence of this one.

The conclusion is that X_P and X_Q are isomorphic as elements of $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$ precisely when $P - Q$ is a multiple of $2^r - 2^s$. The map $Z_{r,s} \to \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$ therefore descends to an isomorphism

$$Z_{r,s}/(2^r - 2^s) \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s)).$$

Finally, it remains to determine the group $Z_{r,s}/(2^r - 2^s)$. This is a cyclic group (since $Z_{r,s} \cong \mathbb{Z}$), and we need to find its order. To this end, note that the condition $P(\frac{l^r - l^s}{2^r - 2^s}) \in \mathbb{Z}$ is equivalent to

$$\tfrac{2^r - 2^s}{\gcd(2^r - 2^s, l^r - l^s)} \Big| P.$$

This is true for all l if and only if P is a multiple of

$$\mathrm{lcm}\Big\{ \tfrac{2^r - 2^s}{\gcd(2^r - 2^s, l^r - l^s)} \;\Big|\; l \text{ prime} \Big\} = \tfrac{2^r - 2^s}{\gcd(\{l^r - l^s \mid l \text{ prime}\})}.$$

So define

$$N_{r,s} = \gcd(2^r - 2^s, 3^r - 3^s, 5^r - 5^s, 7^r - 7^s, \ldots).$$

Then we have just determined that $Z_{r,s} = ((2^r - 2^s)/N_{r,s})$, and hence

$$\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s)) \cong (\tfrac{2^r - 2^s}{N_{r,s}})/(2^r - 2^s) \cong \mathbb{Z}/N_{r,s}.$$

We will explore the numbers $N_{r,s}$ in a moment, but we have already done enough to be able to solve the Hopf invariant one problem. So let us pause and tackle that first.

31.8. **Solution to Hopf invariant one.** We are ready to give the Adams-Atiyah [AA] solution to the Hopf invariant one problem:

Theorem 31.9. *If $f \colon S^{2n-1} \to S^n$ has Hopf invariant one then $n \in \{1, 2, 4, 8\}$.*

Proof. We assume $n > 1$ and prove that $n \in \{2, 4, 8\}$. We of course know that n is even, since otherwise the Hopf invariant is necessarily zero. Write $n = 2r$, and let X be the mapping cone of f. We have an exact sequence of $\mathcal{B}$-modules

$$0 \leftarrow \widetilde{K}^0(S^n) \leftarrow \widetilde{K}^0(X) \leftarrow \widetilde{K}^0(S^{2n}) \leftarrow 0$$

which has the form

$$0 \leftarrow \mathbb{Z}(r) \leftarrow \widetilde{K}^0(X) \leftarrow \mathbb{Z}(2r) \leftarrow 0.$$

Let $a \in \widetilde{K}^0(X)$ be the image of a chosen generator for $\widetilde{K}^0(S^{2n})$ and let $b \in \widetilde{K}^0(X)$ be an element that maps to a chosen generator of $\widetilde{K}^0(S^n)$. Then b^2 maps to 0 in

$\widetilde{K}^0(S^n)$, so we have $b^2 = h \cdot a$ for a unique $h \in \mathbb{Z}$. A little thought shows that, up to sign, h is the Hopf invariant of the map f.

The key to the argument is the equivalence $\psi^2(b) \equiv b^2 \mod 2$ (Proposition 30.11(d)). Using our assumption that h is odd, this gives $\psi^2(b) \equiv a \mod 2$. However, recall our classification of extensions in $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(2r))$. Such extensions are determined by an integer $P_2 \in \mathbb{Z}$ satisfying

$$P_2 \cdot \left(\frac{l^{2r}-l^r}{2^{2r}-2^r}\right) \in \mathbb{Z} \qquad (31.10)$$

for all primes l, where P_2 is defined by the equation $\psi^2 b = 2^r b + P_2 a$. Our assumption about the Hopf invariant of f now gives that P_2 is odd. But equation (31.10) says that

$$P_2 \cdot \frac{l^r}{2^r} \cdot \frac{l^r - 1}{2^r - 1} \in \mathbb{Z},$$

and if P_2 is odd and l is odd then this implies that $2^r | l^r - 1$.

Let us pause here and summarize. From the topology we have extracted a number-theoretic condition: if $n = 2r$ and $S^{2n-1} \to S^n$ has Hopf invariant one, then $2^r | l^r - 1$ for all odd primes l.

This number-theoretic condition is very restrictive, and it turns out just looking at $l = 3$ is enough to give us what we want. The lemma below shows that r lies in $\{1, 2, 4\}$, impying that our original n belongs to $\{2, 4, 8\}$ as desired. □

Lemma 31.11. *If* $2^r | 3^r - 1$ *then* $r \in \{0, 1, 2, 4\}$.

Proof. Let $\nu(n)$ be the 2-adic valuation of an integer n: that is, $n = 2^{\nu(n)} \cdot (\text{odd})$. Here is a table showing the numbers $\nu(3^r - 1)$ for small values of r:

Table 31.12.

r	1	2	3	4	5	6	7	8	9	10	11	12
$\nu(3^r - 1)$	1	3	1	4	1	3	1	5	1	3	1	4

The reader will reach the natural guess that $\nu(3^r - 1) = 1$ when r is odd, and this is easy to prove by working modulo 4. In $\mathbb{Z}/4$ we have $3 = -1$, and so $3^r = (-1)^r = -1$ when r is odd. Thus $3^r - 1 = 2$ in $\mathbb{Z}/4$, which confirms that $\nu(3^r - 1) < 2$.

When r is even the reader will note from the table that $\nu(3^r - 1)$ seems to grow quite slowly as a function of r. Again, this is easy enough to prove as soon as one has the idea to do so. If $r = 2u$ then

$$3^r - 1 = 3^{2u} - 1 = (3^u - 1)(3^u + 1).$$

Modulo 8 the powers of 3 are just 1 and 3, so the possible values for $3^u + 1$ are only 2 and 4. In particular, 8 does not divide $3^u + 1$: that is, $\nu(3^u + 1) < 3$ for all values of u. We therefore have $\nu(3^r - 1) \leq \nu(3^u - 1) + 2$. If u is odd we stop here, otherwise we again divide by 2 and apply the same formula; a simple induction along these lines yields

$$\nu(3^r - 1) \leq 1 + 2\nu(r).$$

The bound $1 + 2\nu(r)$ is generally substantially smaller than r. An easy exercise verifies that $r \leq 1 + 2\nu(r)$ only when $r \in \{1, 2, 4\}$. So to summarize, we have shown that if $r \notin \{1, 2, 4\}$ then $1 + 2\nu(r) < r$; hence $\nu(3^r - 1) < r$, and so $2^r \nmid 3^r - 1$. □

31.13. **Completion of the** Ext **calculation.** At this point we have finished with the solution to the Hopf invariant one problem. But there is another interesting problem that is still on the table, namely the exact computation of the groups

$$\operatorname{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s)) \cong \mathbb{Z}/N_{r,s}.$$

We need to determine the numbers $N_{r,s}$.

This calculation, of course, is intriguing from a purely algebraic perspective—when an answer comes down to finding one specific number, it would be difficult not to take the extra step and determine just what that number is. But the answer is also interesting for topological reasons. We have seen that if $f\colon S^{n+k-1} \to S^n$ where n and k are both even, then we get an extension $A(f) \in \operatorname{Ext}^1_{\mathcal{B}}(\mathbb{Z}(\frac{n}{2}), \mathbb{Z}(\frac{n+k}{2}))$. A little work shows that this actually gives a group homomorphism

$$A\colon \pi_{n+k-1}(S^n) \to \operatorname{Ext}^1_{\mathcal{B}}\big(\mathbb{Z}(\tfrac{n}{2}), \mathbb{Z}(\tfrac{n+k}{2})\big) \cong \mathbb{Z}/N_{\frac{n}{2},\frac{n+k}{2}}.$$

It is important to determine how large the target group is, and how close A is to being an isomorphism. This was investigated by Adams [A3].

We begin our investigation of the numbers $N_{r,s}$ by looking at $N_{r,r-1}$. Since $l^r - l^{r-1} = l^{r-1}(l-1)$ this is

$$N_{r,r-1} = \gcd(2^{r-1},\ 3^{r-1}\cdot 2,\ 5^{r-1}\cdot 4,\ 7^{r-1}\cdot 6,\ \ldots).$$

The 2^{r-1} in the first entry tells us that the gcd will be a power of 2, and the $3^{r-1}\cdot 2$ tells us that it will be at most 2^1. A moment's thought reveals that the gcd is precisely 2^1, as long as $r \geq 2$. When $r = 1$ the gcd is just 1:

$$N_{r,r-1} = \begin{cases} 1 & \text{if } r = 1, \\ 2 & \text{if } r \geq 2. \end{cases}$$

Next consider the numbers $N_{r,r-2}$, requiring us to look at $l^r - l^{r-2} = l^{r-2}(l^2-1)$. We have

$$\begin{aligned} N_{r,r-2} &= \gcd(2^{r-2}(2^2-1), 3^{r-2}(3^2-1), 5^{r-2}(5^2-1), 7^{r-2}(7^2-1), \ldots) \\ &= \gcd(2^{r-2}\cdot 3,\ 3^{r-2}\cdot 8,\ 5^{r-2}\cdot 24,\ 7^{r-2}\cdot 48,\ \ldots). \end{aligned}$$

From the first entry we see that the gcd will only have twos and threes in its factorization, with at most one 3. Later entries show that the gcd has at most 3 twos, and a brief inspection leads to the guess that the gcd is 24 as long as $r \geq 5$. To prove this we need to verify that $24 | l^2 - 1$ for primes $l > 3$. This is easy, though. Consider the numbers $l-1$, l, and $l+1$. At least one is a multiple of 3, and our hypotheses on l say that it isn't l. So 3 divides $(l-1)(l+1) = l^2 - 1$. Likwise, both $l-1$ and $l+1$ are even and at least one must be a multiple of 4: so $8 | l^2 - 1$ as well. The reader will now find it easy to check the following numbers:

$$N_{r,r-2} = \begin{cases} 1 & \text{if } r = 2, \\ 6 & \text{if } r = 3, \\ 12 & \text{if } r = 4, \\ 24 & \text{if } r \geq 5. \end{cases}$$

Remark 31.14. The two cases we have analyzed so far yield an evident conjecture: that $N_{r,r-t}$ is independent of r for $r \gg 0$. We will see below that this is indeed the case.

Let us work out two more cases before discussing the general pattern.

$$\begin{aligned} N_{r,r-3} &= \gcd(2^{r-3}(2^3-1),\ 3^{r-3}(3^3-1),\ 5^{r-3}(5^3-1),\ \dots) \\ &= \gcd(2^{r-3}\cdot 7,\ 3^{r-3}\cdot 26,\ 5^{r-2}\cdot 124,\ \dots). \end{aligned}$$

A very quick investigation shows that

$$N_{r,r-3} = \begin{cases} 1 & \text{if } r=3, \\ 2 & \text{if } r\geq 4. \end{cases}$$

Moving to $N_{r,r-4}$ we have

$$\begin{aligned} N_{r,r-4} &= \gcd(2^{r-4}(2^4-1),\ 3^{r-4}(3^4-1),\ 5^{r-4}(5^4-1),\ \dots) \\ &= \gcd(2^{r-4}\cdot 15,\ 3^{r-4}\cdot 80,\ 5^{r-4}\cdot 624,\ 7^{4-r}\cdot 2400,\ \dots). \end{aligned}$$

The numbers are getting larger now, and it is harder to see the patterns. The relevant fact is that l^4-1 is a multiple of $2^4\cdot 3\cdot 5$ for all primes $l>5$; and for $l=2$ it is a multiple of $3\cdot 5$, for $l=3$ it is a multiple of $2^4\cdot 5$, and for $l=5$ it is a multiple of $2^4\cdot 3$. We leave it as an exercise for the reader to prove this, using the factorization $l^4-1=(l-1)(l+1)(l^2+1)$ and some easy number theory. The conclusion is that

$$N_{r,r-4} = \begin{cases} 1 & \text{if } r=4, \\ 30 & \text{if } r=5, \\ 60 & \text{if } r=6, \\ 120 & \text{if } r=7, \\ 240 & \text{if } r\geq 8. \end{cases}$$

By now it should be clear what the general pattern is, if not the specifics. To understand $N_{r,r-t}$ we consider the numbers

$$2^t-1,\ 3^t-1,\ 5^t-1,\ 7^t-1,\ \dots$$

Excluding some finite set of primes at the beginning, there will be an "interesting" gcd to this set of numbers. When r is large the bad primes at the beginning become irrelevant to the computation, and so $N_{r,r-t}$ is equal to the aforementioned "interesting" gcd when r is large. We encourage the reader to do some investigation on his or her own at this point. The "large r" values of $N_{r,r-t}$ are listed in the following table, together with their prime factorizations:

t	1	2	3	4	5	6	7	8	9	10	11	12
$N_{r,r-t}$	2	24	2	240	2	504	2	480	2	264	2	65520
p.f.	2	$2^3\cdot 3$	2	$2^4\cdot 5$	2	$2^3\cdot 3^2\cdot 7$	2	$2^5\cdot 3\cdot 5$	2	$2^3\cdot 3\cdot 11$	2	$2^4\cdot 3^2\cdot 5\cdot 7\cdot 13$

If you have been around the stable homotopy groups of spheres you will see some familiar numbers in this table, which might make you sit up and take notice. For example: $\pi_3^s\cong\mathbb{Z}/24$, $\pi_7^s\cong\mathbb{Z}/240$, $\pi_{11}^s\cong\mathbb{Z}/504$, and $\pi_{15}^s\cong\mathbb{Z}/960$ (note that the last one does not quite match). It is remarkable to have these numbers coming up in a purely algebraic computation! It turns out that what we are seeing here is the so-called "image of J". We will say more about this at a later time.

Fix $t \in \mathbb{Z}_+$. It turns out that there is a simple formula for the "stable" values of $N_{r,r-t}$, as a function of t. These stable values are also closely connected to the denominators of Bernoulli numbers. We close this section by explaining this.

Our examples have led to the hypothesis that the sequence of numbers

$$2^N(2^t-1),\ 3^N(3^t-1),\ 5^N(5^t-1),\ 7^N(7^t-1),\ldots$$

has a greatest common divisor that is independent of N when $N \gg 0$. Our aim is to prove this, and to investigate this gcd. To this end, let $m_N(t)$ be this gcd:

$$m_N(t) = \gcd\{l^N(l^t-1)\,|\,l \text{ is prime}\}.$$

Also define

$$m'_N(t) = \gcd\{k^N(k^t-1)\,|\,k \in \mathbb{Z}_+\}.$$

Clearly $m'_N(t)$ divides $m_N(t)$, but in fact the two are equal:

Lemma 31.15. *For all t and N, $m'_N(t) = m_N(t)$.*

Proof. It will suffice to show that $m_N(t)$ divides $m'_N(t)$, or equivalently that every prime-power factor of the former is also a factor of the latter. So let p be a prime and suppose $p^e|m_N(t)$. Then $p^e|p^N(p^t-1)$, so $e \le N$. For any l such that $(l,p)=1$ we have $p^e|l^t-1$, so $l^t = 1$ in $\mathbb{Z}/p^e$.

Now let $k \in \mathbb{Z}$ with $k \ge 2$. If $p \mid k$ then $p^e \mid k^N(k^t-1)$ since $e \le N$. If $p \nmid k$ then write $k = l_1 l_2 \ldots l_r$ where each l_r is a prime different from p. We know that $l_i^t = 1$ in $\mathbb{Z}/p^e$ for each i, and so $k^t = l_1^t l_2^t \ldots l_r^t = 1$ in $\mathbb{Z}/p^e$ as well. That is, $p^e|k^t-1$. We have therefore shown that $p^e|m'_N(t)$, which is what we wanted. □

The next proposition proves that $m_N(t)$ stabilizes for $N \gg 0$, and it also determines an explicit formula for the stable value in terms of the prime factorization of t. Let $\nu_p(t)$ denote the exponent of the prime p in the prime factorization of t.

Proposition 31.16. *Let L be the supremum of all exponents in the prime factorization of t. Then $m_N(t)$ is independent of N for $N \ge L+2$. If we call this stable value $m(t)$ then*

(a) $m(t) = 2$ when t is odd;

(b) When t is even $m(t) = 2^{2+\nu_2(t)} \cdot \prod_{p \text{ odd},(p-1)|t} p^{1+\nu_p(t)}$.

(c) More generally,

$$m_N(t) = 2^{\min\{2+\nu_2(t),N\}} \cdot \prod_{p \text{ odd},(p-1)|t} p^{\min\{1+\nu_p(t),N\}}.$$

Remark 31.17. The notation $m(t)$ comes from Adams [A3].

Before proving the proposition let us look at a couple of examples. To compute $m(50)$ we write $50 = 2 \cdot 5^2$. Next we make a list of all odd primes p such that $p-1$ divides 50; these are 3 and 11. So

$$m(50) = 2^3 \cdot 3 \cdot 11 = 264.$$

For a harder example let us compute $m(12)$. We write $12 = 2^2 \cdot 3$, and our list of p such that $p-1$ divides 12 is 3, 5, 7, and 13. So

$$m(12) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 65520.$$

Further, we have $m(12) = m_4(12)$ and

$$\begin{aligned} m_3(12) &= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \\ m_2(12) &= 2^2 \cdot 3^2 \cdot 5 \cdot 7 \\ m_1(12) &= 2^1 \cdot 3^1 \cdot 5 \cdot 7 \\ m_0(12) &= 1. \end{aligned}$$

Here are several more values of m for the reader's curiosity (the numbers are, of course, better understood in terms of their prime factorizations):

t	2	4	6	8	10	12	14	16	18
$m(t)$	24	240	504	480	264	65520	24	16320	28728

To prove Proposition 31.16 we need a lemma from algebra. Most basic algebra courses prove that the group of units in $\mathbb{Z}/p$ is a cyclic group, necessarily isomorphic to $\mathbb{Z}/(p-1)$. One can also completely describe the group of units inside the ring $\mathbb{Z}/p^e$, for any e. Recall that this group simply consists of all residue classes of integers k such that $(p,k) = 1$. Here is the result:

Lemma 31.18. *Fix a prime p and consider the group of units $(\mathbb{Z}/p^e)^*$ inside the ring $\mathbb{Z}/p^e$.*

(a) If p is odd then $(\mathbb{Z}/p^e)^* \cong \mathbb{Z}/((p-1)p^{e-1}) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p^{e-1})$.
(b) If $e \geq 2$ then $(\mathbb{Z}/2^e)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/(2^{e-2})$. Here the $\mathbb{Z}/2$ is the subgroup $\{1,-1\}$ and the $\mathbb{Z}/(2^{e-2})$ is the subgroup of all numbers congruent to 1 mod 4.
(c) $(\mathbb{Z}/2)^* = \{1\}$.

Proof. We first recall the proof that $(\mathbb{Z}/p)^*$ is cyclic. If a finite abelian group is noncyclic, then it contains a subgroup isomorphic to $\mathbb{Z}/k \times \mathbb{Z}/k$, for some prime k (this follows readily from the structure theorem for finite abelian groups). But if this were true for $(\mathbb{Z}/p)^*$ then the field $\mathbb{Z}/p$ would have k^2 solutions to the polynomial $x^k - 1$, and this is a contradiction.

Assume that p is odd. Reduction modulo p gives a surjective map $(\mathbb{Z}/p^e)^* \to (\mathbb{Z}/p)^*$. Let K denote the kernel. Note that $(\mathbb{Z}/p^e)^*$ coincides with the set

$$\{1, 2, \ldots, p^e - 1\} - \{p, 2p, 3p, \ldots, (p^{e-1} - 1)p\}$$

and so has order $p^e - p^{e-1}$. Thus, $|K| = p^{e-1}$. It remains to show that K is cyclic, and for this it suffices to verify that K has exactly $p-1$ elements of order p. Let $a \in K - \{1\}$, and let the base p representation of a be

$$a = 1 + a_f p^f + a_{f+1} p^{f+1} + \cdots + a_{e-1} p^{e-1},$$

for $0 \leq a_i < p - 1$ and $a_f \neq 0$ (note that $a_0 = 1$ by the definition of K). Write $b = a_f + a_{f+1} p + a_{f+2} p^2 + \cdots$, so that

$$a^p = (1 + p^f b)^p = 1 + p \cdot p^f b + \tbinom{p}{2} p^{2f} b^2 + \cdots$$

The terms after $p^{1+f} b$ all contain at least $f+2$ factors of p; so modulo p^{f+2} one has $a^p \equiv 1 + p^{f+1} b \equiv 1 + a_f p^{f+1}$. So we can have $a^p = 1$ in $\mathbb{Z}/p^e$ only if $f = e - 1$. Thus, we have shown that the elements of K that are pth roots of unity are precisely the elements $1 + ap^{e-1}$, for $0 \leq a < p$. In particular, we have only $p - 1$ of these (excluding the identity element). This completes the proof.

The proof for $p = 2$ is similar. Of course $(\mathbb{Z}/4)^* \cong \mathbb{Z}/2$. For $e \geq 3$ consider the sequence $0 \to K \to (\mathbb{Z}/2^e)^* \to (\mathbb{Z}/4)^* \to 0$, where the right map is reduction modulo 4. This reduction map is split-surjective, with the splitting sending the generator of $(\mathbb{Z}/4)^*$ to -1. The proof that K is cyclic proceeds exactly as in the odd primary case. □

Proof of Proposition 31.16. Let p be an odd prime. Then one has

$$\begin{aligned} p^e \mid m_N(t) &\iff p^e \mid p^N(p^t - 1) \text{ and } p^e \mid l^N(l^t - 1) \text{ for all primes } l \neq p \\ &\iff e \leq N \text{ and } p^e \mid l^t - 1 \text{ for all primes } l \neq p \\ &\iff e \leq N \text{ and } p^e | k^t - 1 \text{ for all } k \in \mathbb{Z}_+ \text{ such that } p \nmid k \\ &\iff e \leq N \text{ and all units in } \mathbb{Z}/p^e \text{ are } t\text{th roots of } 1 \\ &\iff e \leq N \text{ and } (p-1)p^{e-1} | t. \end{aligned}$$

The second equivalence is by Lemma 31.15, and in the last equivalence we have used that $(\mathbb{Z}/p^e)^* \cong \mathbb{Z}/((p-1)p^{e-1})$. This last line shows why N is redundant when it is large enough: the condition $p^{e-1}|t$ already forces $e \leq \nu_p(t) + 1$, and so $e \leq N$ is redundant if $N \geq \nu_p(t) + 1$.

Assuming now that N is large enough so that we are in the stable case, the above equivalences show that $p|m(t)$ only when $p - 1|t$; and also that in this case $\nu_p(m(t)) = 1 + \nu_p(t)$.

The analysis of $p = 2$ is very similar. One finds

$$2^e \mid m_N(t) \iff e \leq N \text{ and all units in } \mathbb{Z}/2^e \text{ are } t\text{th roots of } 1.$$

When $e = 1$ the latter condition is just $e \leq N$. When $e > 1$ the latter condition is equivalent to $e \leq N$ and $2^{e-2}|t$, using Lemma 31.18(b). This readily yields the desired result. □

Our final task is to make the connection between the numbers $m(t)$ and the Bernoulli numbers. For a review of the Bernoulli numbers and their basic properties, see Appendix A. The result we are after is the following:

Theorem 31.19. *When t is even, $m(t)$ is the denominator of the fraction $\frac{B_t}{2t}$ when expressed in lowest terms.*

The following table demonstrates this result in the first few cases:

t	2	4	6	8	10	12
$\lvert B_t \rvert$	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$
$\frac{\lvert B_t \rvert}{2t}$	$\frac{1}{24}$	$\frac{1}{240}$	$\frac{1}{504}$	$\frac{1}{480}$	$\frac{1}{264}$	$\frac{691}{65520}$
$m(t)$	24	240	504	480	264	65520

It should be remarked, perhaps, that this connection between $m(t)$ and Bernoulli numbers is worth more in effect than it is in practical value. The explicit formula for $m(t)$ from Proposition 31.16 is much more useful than its description as the denominator of $B_t/(2t)$. Moreover, the denominators of Bernoulli numbers are very simple: for B_n it is simply the product of all primes p such that $p - 1$ divides n. So in the end the result of Theorem 31.19 is neither deep nor particularly useful. Still, it provides a nice-sounding connection between topology and number theory.

Proof of Proposition 31.19. By the theorem of von Staudt and Claussen (Theorem A.5) we know that the denominator of B_t (in lowest terms) is the product of all primes p such that $p-1$ divides t. Note that one such prime is $p=2$, so the denominator is even (in fact, congruent to 2 modulo 4) and the numerator is odd. Note also that these are the same primes appearing in the factorization of $m(t)$, by Proposition 31.16(b).

Consider now $\alpha = B_t/(2t)$. Since the numerator of B_t is odd, the number of twos in the denominator of α is $1+1+\nu_2(t)$. This is the same as the number of twos in the prime factorization of $m(t)$.

For every prime p such that $p-1|t$ we have one p appearing in the denominator of B_t and $\nu_p(t)$ of them appearing in t, so the total number of p's in the denominator of α is $1+\nu_p(t)$. At this point we have thereby shown that $m(t)|\operatorname{den}(\alpha)$.

It remains to check that if $p^e \mid t$ and $p-1 \nmid t$ then p^e divides the numerator of B_t and therefore disappears from the denominator of α. This is the nontrivial part of the proof.

Let p be an odd prime in the denominator of α, appearing with multiplicity e. Then p^e is also in the denominator of B_t/t. By Proposition A.6 we know that

$$\tfrac{k^t(k^t-1)B_t}{t} \in \mathbb{Z}$$

for all $k \in \mathbb{Z}$. Consequently we have $p^e|k^t(k^t-1)$ for all $k\in\mathbb{Z}$. But this exactly says that p^e divides the gcd $m(t)$. This verifies that the 'odd part' of $\operatorname{den}(\alpha)$ divides $m(t)$, and we have already checked the factors of 2 in a previous paragraph. So $\operatorname{den}(\alpha) \mid m(t)$, and therefore the two are equal. □

32. Calculation of KO for stunted projective spaces

The goal in this section is to determine $KO^0(\mathbb{R}P^n)$ and $KO^0(\mathbb{R}P^n/\mathbb{R}P^a)$ for all values of n and a, together with the Adams operations on these groups. These computations are the key to solving the vector field problem, which we do in the next section. As intermediate steps we also compute $K^*(\mathbb{R}P^n)$ and $K^*(\mathbb{R}P^n/\mathbb{R}P^a)$. The original written source for this material is Adams [Ad2] (although he acknowledges unpublished work of Atiyah-Todd and Bott-Shapiro for portions of the calculation). We follow Adams's approach very closely.

Some words of warning about this material are in order. The complete calculation of KO for stunted projective spaces is fairly involved. Several things end up going on at once, so that there is a bunch of stuff to keep track of. And calculations are just never very fun to read in the first place. We have attempted to structure our presentation to try to help with this, but of course it only goes so far. After some preliminary material we give a section which has just the statements of the results, with a minimal amount of discussion in between (and no proofs). The intent is to give the reader the general picture, and also a convenient reference section. All of the proofs are then given in a subsequent section.

Some readers might want to skip the proofs the first time through, and this is not a problem. Later applications in the text only need the results, not details from the proofs. However, I **highly recommend** that algebraic topologists go through the proofs carefully at an early stage in their career. I cannot stress this enough. Going through the proofs will teach you something important about this subject that I do not have words for, and it will open up doors for you down the road. Trust me that this is an important thing to do.

32.1. **Initial material.** The results in this section will make heavy use of the interplay between $\mathbb{R}P^n$ and $\mathbb{C}P^n$ in the homotopy category of spaces. We begin by reviewing the basics of what we will need.

Let η be the tautological complex line bundle on $\mathbb{C}P^n$, and let L be the tautological real line bundle on $\mathbb{R}P^n$. Let $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ be the inclusion.

Lemma 32.2. *The complexification of L is the pullback of η: that is, $cL \cong j^*\eta$.*

Proof. Complex line bundles on a space X are classified by homotopy classes in $[X, \mathbb{C}P^\infty]$. The real line bundle L is classified by the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$, so the complexificaton of L is classified by the composition $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty \hookrightarrow \mathbb{C}P^\infty$. The complex line bundle η is classified by $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$, so $j^*\eta$ is classified by the composition $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$. The result follows from the commutative diagram

$$\begin{array}{ccc} \mathbb{R}P^n & \longrightarrow & \mathbb{C}P^n \\ \downarrow & & \downarrow \\ \mathbb{R}P^\infty & \longrightarrow & \mathbb{C}P^\infty. \end{array}$$

□

Note that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, so complex line bundles on X are classified by $[X, \mathbb{C}P^\infty] = H^2(X)$. If $n > 1$ then $H^2(\mathbb{R}P^n) = \mathbb{Z}/2$, so there are only two isomorphism classes of complex line bundles: the trivial bundle and the nontrivial bundle. The bundle $cL = j^*\eta$ is nontrivial, since its classification map $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty \hookrightarrow \mathbb{C}P^\infty$ represents the generator of $H^2(\mathbb{R}P^n)$.

Remark 32.3. The powers η^k are all distinct line bundles on $\mathbb{C}P^n$ (e.g., the first Chern classes are $c_1(\eta^k) = kc_1(\eta) = kx$ where x is the generator of $H^2(\mathbb{C}P^n)$). The situation upon pulling back to $\mathbb{R}P^n$ is very different, however. We have

$$[j^*(\eta)]^2 = (cL)^2 = c(L^2) = c(1_{\mathbb{R}}) = 1_{\mathbb{C}}.$$

So the even powers of $j^*(\eta)$ are all trivial, and the odd powers are just $j^*(\eta)$. We will see that this accounts for the main difference between $K^0(\mathbb{C}P^n)$ and $K^0(\mathbb{R}P^n)$.

In addition to the inclusion $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ there is another interesting map from real to complex projective space. Every real line in $\mathbb{C}^{n+1}$ determines a complex line by taking the $\mathbb{C}$-linear span, and therefore we get a map $\mathbb{P}_{\mathbb{R}}(\mathbb{C}^{n+1}) \to \mathbb{P}_{\mathbb{C}}(\mathbb{C}^{n+1}) = \mathbb{C}P^n$. It is easy to see that this is a fiber bundle with fiber S^1 (the space of real lines in $\mathbb{C}$). Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ shows that the domain is homeomorphic to $\mathbb{R}P^{2n+1}$, giving us a fiber bundle

$$S^1 \longrightarrow \mathbb{R}P^{2n+1} \xrightarrow{q} \mathbb{C}P^n.$$

In terms of homogeneous coordinates, q sends the point $[x_0 : x_1 : \cdots : x_{2n} : x_{2n+1}]$ to $[x_0 + ix_1 : x_2 + ix_3 : \cdots : x_{2n} + ix_{2n+1}]$.

Lemma 32.4. *The diagram*

$$\begin{array}{ccc} \mathbb{R}P^n & \xrightarrow{i} & \mathbb{R}P^{2n+1} \\ & \searrow^{j} & \downarrow q \\ & & \mathbb{C}P^n \end{array}$$

commutes up to homotopy, where i is the standard inclusion. Consequently, q^* sends $x \in H^2(\mathbb{C}P^n)$ to the nonzero element of $H^2(\mathbb{R}P^{2n+1})$. (The latter statement can also be proven via the Serre spectral sequence for q).

Proof. The diagram commutes on the nose if i is replaced by the inclusion sending $[x_0 : x_1 : \cdots : x_n]$ to $[x_0 : 0 : x_1 : 0 : \cdots : x_n : 0]$. But all linear inclusions from one projective space to another are homotopic. □

Corollary 32.5. *There is an isomorphism of bundles $q^*\eta \cong cL$.*

Proof. As we have remarked before, there are only two isomorphism classes of complex bundles on $\mathbb{R}P^{2n+1}$. Since $j^*\eta = i^*(q^*\eta)$ is not trivial, the bundle $q^*\eta$ cannot be trivial. So the only possibility is $q^*\eta \cong cL$. □

Note that for $a \leq n$ one has the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}P^{2a+1} & \xrightarrow{i} & \mathbb{R}P^{2n+1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{C}P^a & \xrightarrow{i} & \mathbb{C}P^n, \end{array}$$

and therefore q induces a map $\mathbb{R}P^{2n+1}/\mathbb{R}P^{2a+1} \to \mathbb{C}P^n/\mathbb{C}P^a$. We will also use q to denote this induced map on quotients, as well as various small modifications. Note that these induced maps fit together to give a big commutative diagram

$$\begin{array}{cccccc} (32.6) \quad S^{2a+2} & = & \mathbb{R}P^{2a+2}/\mathbb{R}P^{2a+1} & & & \\ & & \downarrow & & & \\ & & \mathbb{R}P^{2a+3}/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^{a+1}/\mathbb{C}P^a & = S^{2a+2} \\ & & \downarrow & & \downarrow & \\ & & \vdots & & \vdots & \\ & & \downarrow & & \downarrow & \\ & & \mathbb{R}P^{2n+1}/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^n/\mathbb{C}P^a & \\ & & \downarrow & & \Big\downarrow & \\ & & \mathbb{R}P^{2n+2}/\mathbb{R}P^{2a+1} & & & \\ & & \downarrow & & & \\ & & \mathbb{R}P^{2n+3}/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^{n+1}/\mathbb{C}P^a & \\ & & \downarrow & & \downarrow & \\ & & \vdots & & \vdots & \\ & & \downarrow & & \downarrow & \\ & & \mathbb{R}P^{\infty}/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^{\infty}/\mathbb{C}P^a. & \end{array}$$

We will tend to use 'q' as a name for any composition of maps from this diagram that involves one horizontal step.

Notice that at the very top of the diagram we have a map from S^{2a+2} to itself. The next result identifies this map:

Lemma 32.7. *For $n \geq 1$ the composite*

$$S^{2n} \cong \mathbb{R}P^{2n}/\mathbb{R}P^{2n-1} \hookrightarrow \mathbb{R}P^{2n+1}/\mathbb{R}P^{2n-1} \xrightarrow{q} \mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}.$$

is a homeomorphism.

Proof. The space $\mathbb{R}P^{2n}/\mathbb{R}P^{2n-1}$ consists of the basepoint and the affine space $\mathbb{R}^{2n}$ made up of points $[x_0 : x_1 : \cdots : x_{2n-2} : x_{2n-1} : 1]$. Likewise, the space $\mathbb{C}P^n/\mathbb{C}P^{n-1}$ consists of the basepoint and the affine space $\mathbb{C}^n$ made up of the points $[z_0 : z_1 : \cdots : z_{n-1} : 1]$. One readily uses the formula for q to see that it gives a bijective correspondence, and is therefore a homeomorphism. □

32.8. **The main results.** Here we state the main theorems about the K-theory of real and complex projective spaces. The proofs will be deferred until the next section.

Theorem 32.9 (Complex K-theory of complex projective spaces and stunted projective spaces)**.** *Let η be the tautological line bundle on $\mathbb{C}P^n$, and write $\mu = [\eta]-1 \in \widetilde{K}^0(\mathbb{C}P^n)$.*

(a) $K^0(\mathbb{C}P^n) = \mathbb{Z}[\mu]/(\mu^{n+1})$ and $K^1(\mathbb{C}P^n) = 0$.
(b) The Adams operations on $K^0(\mathbb{C}P^n)$ are given by

$$\psi^k(\mu^s) = \left[(1+\mu)^k - 1\right]^s = k^s\mu^s + s\tbinom{k}{2}\mu^{s+1} + \text{(higher order terms)}.$$

(c) The sequence $0 \to \widetilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^a) \to \widetilde{K}^0(\mathbb{C}P^n) \to \widetilde{K}^0(\mathbb{C}P^a) \to 0$ is exact, and identifies $\widetilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^a)$ as the free abelian group $\mathbb{Z}\langle\mu^{a+1}, \mu^{a+2}, \ldots, \mu^n\rangle \subseteq K^0(\mathbb{C}P^n)$. The ring structure and action of the Adams operations are determined by the corresponding structures on $K^0(\mathbb{C}P^n)$. Also, $K^1(\mathbb{C}P^n/\mathbb{C}P^a) = 0$.

Remark 32.10. Following Adams [Ad2] we write $\mu^{(i)}$ for the element of $K^0(\mathbb{C}P^n/\mathbb{C}P^a)$ that maps to μ^i in $K^0(\mathbb{C}P^n)$. The extra parentheses in the exponent remind us that this class is not a true ith power in the ring $K^0(\mathbb{C}P^n/\mathbb{C}P^a)$.

Theorem 32.11 (Complex K-theory of real projective spaces)**.** *Let ν be the element $[j^*\eta] - 1 \in \widetilde{K}^0(\mathbb{R}P^n)$.*

(a) $\widetilde{K}^0(\mathbb{R}P^n) \cong \mathbb{Z}/\big(2^{\lfloor \frac{n}{2} \rfloor}\big)$ with generator ν. The ring structure has $\nu^2 = -2\nu$ and $\nu^{f+1} = 0$, where $f = \lfloor \frac{n}{2} \rfloor$.
(b) $K^1(\mathbb{R}P^n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$
(c) The Adams operations on $K^0(\mathbb{R}P^n)$ are given by

$$\psi^k(\nu^e) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \nu^e & \text{if } k \text{ is odd.} \end{cases}$$

Recall that we have calculated $\widetilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^{a-1})$ to be the free abelian group $\mathbb{Z}\langle\mu^{(a)}, \mu^{(a+1)}, \ldots, \mu^{(n)}\rangle$. For $k \le 2n+1$ we may pull back these classes along the map $q\colon \mathbb{R}P^k/\mathbb{R}P^{2a-1} \to \mathbb{C}P^n/\mathbb{C}P^{a-1}$ to get elements of $\widetilde{K}^0(\mathbb{R}P^k/\mathbb{R}P^{2a-1})$. We again follow Adams [Ad2] and set

$$\bar{\nu}^{(t)} = q^*(\mu^{(t)}) \in \widetilde{K}^0(\mathbb{R}P^k/\mathbb{R}P^{2a-1})$$

for $a \le t \le n$. Note that these elements correspond nicely as k and n vary, due to the commutative diagram (32.6). We may as well take $n \mapsto \infty$ so that we have classes $\bar{\nu}^{(t)}$ for all $t \ge a$.

We claim that upon pulling back along the projection $\pi\colon \mathbb{R}P^k \to \mathbb{R}P^k/\mathbb{R}P^{2a-1}$ we have $\pi^*(\bar{\nu}^{(t)}) = \nu^t$; this explains our choice of notation. To prove this claim we can deal with the cases $k = 2u$ and $k = 2u+1$ simultaneously. Consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{R}P^{2u}/\mathbb{R}P^{2a-1} & \rightarrowtail & \mathbb{R}P^{2u+1}/\mathbb{R}P^{2a-1} & \xrightarrow{q} & \mathbb{C}P^u/\mathbb{C}P^{a-1} \\ {\scriptstyle\pi}\uparrow & & {\scriptstyle\pi}\uparrow & & {\scriptstyle\pi_{\mathbb{C}}}\uparrow \\ \mathbb{R}P^{2u} & \rightarrowtail & \mathbb{R}P^{2u+1} & \xrightarrow{q} & \mathbb{C}P^u. \end{array}$$

We have $\pi^*(q^*(\mu^{(t)})) = q^*(\pi^*_{\mathbb{C}}(\mu^{(t)})) = q^*(\mu^t) = (q^*\mu)^t = \nu^t$, where the last equality is by Corollary 32.5.

Observe that $\widetilde{K}^0(\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1}) = \widetilde{K}^0(S^{2a}) \cong \mathbb{Z}$ and the class $\bar{\nu}^{(a)}$ is a generator. This follows because $\mu^{(a)}$ generates $\widetilde{K}^0(\mathbb{C}P^a/\mathbb{C}P^{a-1}) \cong \mathbb{Z}$ and $\bar{\nu}^{(a)}$ is the pullback of $\mu^{(a)}$ along the map $\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1} \to \mathbb{C}P^a/\mathbb{C}P^{a-1}$, which by Lemma 32.7 is a homotopy equivalence.

The inclusion $i\colon \mathbb{R}P^{2a}/\mathbb{R}P^{2a-1} \hookrightarrow \mathbb{R}P^n/\mathbb{R}P^{2a-1}$ induces a map

$$\mathbb{Z} = \widetilde{K}^0(S^{2a}) = \widetilde{K}^0(\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1}) \xleftarrow{i^*} \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2a-1}).$$

This map is surjective, because the class $\bar{\nu}^{(a)}$ in the domain maps to the class $\bar{\nu}^{(a)}$ in the target, and the latter is a generator. In particular, not only do we know that the above map i^* is surjective but the class $\bar{\nu}^{(a)}$ in the domain gives a choice of splitting. This is their main use to us.

Theorem 32.12 (Complex K-theory of real stunted projective spaces)**.**

(a) If $a = 2t$ then the sequence

$$0 \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{K}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^a) \to 0$$

is exact. It identifies $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ with the subgroup of $\widetilde{K}^0(\mathbb{R}P^n)$ generated by $\nu^{t+1} = (-2)^t\nu$. As a group, $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z}/(2^g)$ where $g = \lfloor \frac{n-a}{2} \rfloor$. The ring structure and Adams operations are determined by the structures in $K^0(\mathbb{R}P^n)$. In particular, ψ^k acts as zero when k is even and as the identity when k is odd.

(b) Let $a = 2t-1$. The cofiber sequence $\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1} \to \mathbb{R}P^n/\mathbb{R}P^{2t-1} \to \mathbb{R}P^n/\mathbb{R}P^{2t}$ induces a sequence

$$0 \leftarrow \widetilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) \leftarrow \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \leftarrow \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t}) \leftarrow 0$$

that is short exact. Consequently, $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \cong \mathbb{Z} \oplus \mathbb{Z}/(2^f)$ where $f = \lfloor \frac{n}{2} \rfloor - t$; the former summand is generated by $\bar{\nu}^{(t)}$ and the latter summand is generated by $\nu^{(t+1)}$.

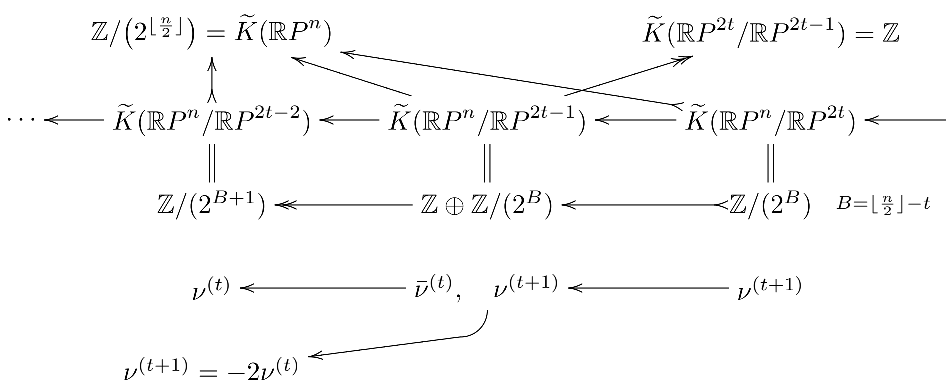
(c) The action of ψ^k on $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1})$ is given by

$$\psi^k(\nu^{(t+1)}) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \nu^{(t+1)} & \text{if } k \text{ is odd,} \end{cases}$$

and

$$\psi^k(\bar{\nu}^{(t)}) = k^t\bar{\nu}^{(t)} + \begin{cases} \frac{1}{2}k^t\nu^{(t+1)} & \text{if } k \text{ is even} \\ \frac{1}{2}(k^t-1)\nu^{(t+1)} & \text{if } k \text{ is odd.} \end{cases}$$

Most of the content to the above theorem is represented in the following convoluted but useful diagram:



The indicated maps are injections/surjections, and our chosen generators of the groups are written in the bottom two lines. The generators $\nu^{(t+1)}$ and $\bar{\nu}^{(t)}$ map to the elements ν^{t+1} and ν^t in $\widetilde{K}(\mathbb{R}P^n)$, and $\bar{\nu}^{(t)}$ maps to a generator of $\widetilde{K}(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) = \widetilde{K}(S^{2t})$. The action of the Adams operations on $\nu^{(t+1)}$ is completely determined by what happens in $\widetilde{K}(\mathbb{R}P^n)$. Likewise, the action on $\bar{\nu}^{(t)}$ is completely determined by using the surjection onto $\widetilde{K}(S^{2t})$ together with the map into $\widetilde{K}(\mathbb{R}P^n)$. These are instructive exercises; but if necessary see the proofs in Section 32.22 for details.

Remark 32.13. The action of ψ^k on the element $\bar{\nu}^{(t)}$ is of crucial importance to the solution of the vector field on spheres problem.

We now move from the realm of K-theory to KO-theory. Recall that $L \to \mathbb{R}P^n$ always denotes the tautological line bundle.

Theorem 32.14 (Real K-theory of real projective spaces)**.**
$\widetilde{KO}^0(\mathbb{R}P^n) \cong \mathbb{Z}/(2^f)$ where $f = \#\{s \,|\, 0 < s \leq n,\ s \equiv 0, 1, 2,$ or 4 mod $8\}$. The group is generated by $\lambda = [L] - 1$, which satisfies $\lambda^2 = -2\lambda$ and $\lambda^{f+1} = 0$. The Adams operations are given by

$$\psi^k(\lambda) = \begin{cases} 0 & k \text{ even}, \\ \lambda & k \text{ odd}. \end{cases}$$

Remark 32.15. Because this number comes up so often, write

$$\varphi(n) = \#\{s \,|\, 0 < s \leq n,\ s \equiv 0, 1, 2, \text{ or } 4 \text{ mod } 8\}.$$

The following chart shows the groups $\widetilde{KO}^0(\mathbb{R}P^n)$ and $\widetilde{K}^0(\mathbb{R}P^n)$ as functions of n. To save space we write $\mathbb{Z}_n$ instead of $\mathbb{Z}/n$; but all the groups are cyclic, and so really one only needs to keep track of the order.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
KO	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{128}$	$\mathbb{Z}_{128}$	$\mathbb{Z}_{128}$
K	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{128}$

Observe that the $\widetilde{KO}^0(\mathbb{R}P^n)$ groups follow the by-now-familiar 8-fold pattern from Bott periodicity and Clifford algebras: the orders of the groups jump according to the pattern "jump-jump-nothing-jump-nothing-nothing-nothing-jump", where the pattern starts in multiples of 8 (the first few are not on the chart because the associated projective spaces are exceptions in some way). In particular, every eight steps a total of four jumps have occurred, resulting in the orders being multiplied by 16. This is the quasi-periodicity of the first line. The second line has the simpler quasi-periodocity of length 2, where every two steps the order of the group gets doubled. Note that the groups on the two lines coincide in dimensions congruent to 6, 7, and 8 modulo 8; in other dimensions there is a difference of a factor of 2.

For reference purposes we also include a table showing the numbers $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$. Even though this is really the same information as in the previous table, it is very useful to have around.

TABLE 32.15. Comparison of $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\varphi(n)$	0	1	2	2	3	3	3	3	4	5	6	6	7	7	7	7	8
$\lfloor \frac{n}{2} \rfloor$	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8

The table provides some useful information about the comparison between $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$. We record it in a proposition:

Proposition 32.16. *For every $n \geq 2$, the number $\varphi(n)$ equals either $\lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1$. The former occurs precisely when n is congruent to 6, 7, or 8 modulo 8.*

Recall the complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^n)$. Both groups are cyclic, and by Lemma 32.2 the map sends the generator $\lambda = L - 1$ of the domain to the generator $\nu = j^*\eta - 1$ of the target. Hence, c is surjective. Our observations about the orders now proves part (a) of the following result. Part (b) follows at once from $c(\lambda) = \nu$ and the fact that $r_{\mathbb{R}}c = 2$.

Theorem 32.17. *Let $n \geq 2$.*

(a) The complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^n)$ is always surjective. It is an isomorphism if n is congruent to 6, 7, or 8 modulo 8, and it has kernel $\mathbb{Z}/2$ otherwise.

(b) The map $r_{\mathbb{R}}\colon \widetilde{K}^0(\mathbb{R}P^n) \to \widetilde{KO}^0(\mathbb{R}P^n)$ sends ν to 2λ.

Finally, we turn our attention to KO-theory of the spaces $\mathbb{R}P^n/\mathbb{R}P^a$. It is almost true that the Atiyah-Hirzebruch spectral sequence for $\mathbb{R}P^n/\mathbb{R}P^a$ is a truncation of the one for $\mathbb{R}P^n$. The mod 2 cohomology groups $H^*(\mathbb{R}P^n/\mathbb{R}P^a;\mathbb{Z}/2)$ are indeed a truncation of $H^*(\mathbb{R}P^n;\mathbb{Z}/2)$, and the integral cohomology groups are a similar truncation when a is even. But when a is odd there is a $\mathbb{Z}$ in $H^{a+1}(\mathbb{R}P^n/\mathbb{R}P^a)$ that does not appear in $H^{a+1}(\mathbb{R}P^n)$. This new $\mathbb{Z}$ will contribute to $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ only if it shows up along the main diagonal in the Atiyah-Hirzebruch spectral sequence, which will happen precisely when $a+1$ is a multiple of 4. This explains the two cases in the following result:

Theorem 32.18 (Real K-theory of real, stunted projective spaces; part 1)**.**

(a) Suppose $a \not\equiv -1 \bmod 4$. Then the map $\pi^*\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{KO}^0(\mathbb{R}P^n)$ is an injection whose image is the subgroup generated by $\lambda^{\varphi(a)+1}$. So

$\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z}/(2^g)$ *where* $g = \varphi(n) - \varphi(a) = \#\{s \,|\, a < s \le n, s \equiv 0, 1, 2,$ *or* 4 *mod* $8\}$. *Let* $\lambda^{(\varphi(a)+1)}$ *be the preimage for* $\lambda^{\varphi(a)+1}$ *under* π^*, *which generates the group. Then*

$$\psi^k(\lambda^{(u)}) = \begin{cases} 0 & k \text{ even}, \\ \lambda^{(u)} & k \text{ odd}. \end{cases}$$

(b) Assume that $a \equiv -1$ *mod* 4. *The sequence*

$$S^{a+1} = \mathbb{R}P^{a+1}/\mathbb{R}P^a \to \mathbb{R}P^n/\mathbb{R}P^a \to \mathbb{R}P^n/\mathbb{R}P^{a+1}$$

induces a split-exact sequence in KO-theory:

$$0 \leftarrow \widetilde{KO}^0(S^{a+1}) \leftarrow \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \leftarrow \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{a+1}) \leftarrow 0.$$

In particular,

$$\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z} \oplus \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{a+1}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^h$$

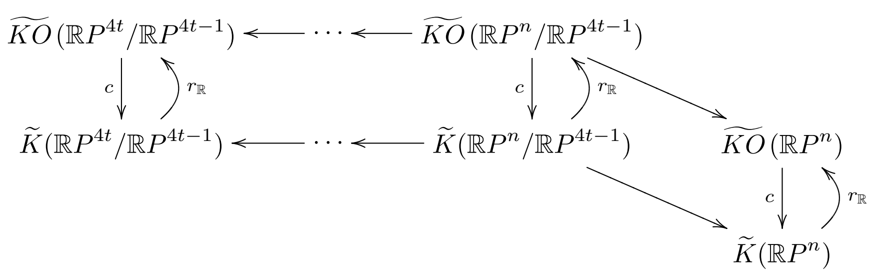
where $h = \varphi(n) - \varphi(a+1) = \#\{s \,|\, a+1 < s \le n,\ s \equiv 0, 1, 2,$ *or* 4 *mod* $8\}$.

Notice that the above result does not give the action of the Adams operations in part (b). To do this we need to choose a specific generator for the $\mathbb{Z}$ summand, and this requires some explanation. It turns out (and this is not obvious) that the generator can always be chosen so that it maps to $\lambda^{\varphi(a+1)}$ in $\widetilde{KO}^0(\mathbb{R}P^n)$. This property is all that we will really need, but it is not so easy to prove; in fact there are always *two* such generators, and proving the desired existence seems to be best accomplished by having a method for systematically choosing a preferred generator out of the two possibilities. This is what we do next; in the chain

$$\widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \leftarrow \widetilde{KO}(\mathbb{R}P^{4t+1}/\mathbb{R}P^{4t-1}) \leftarrow \cdots \leftarrow \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \leftarrow \cdots$$

we wish to choose elements $\overline{\lambda}^{(\varphi(4t))}$ in each group with the property that they all map onto each other, they all map to a generator of the left-most group, and upon pulling back along the projection $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{4t-1}$ the element $\overline{\lambda}^{(\varphi(4t))}$ maps to $\lambda^{\varphi(4t)}$ (for any choice of n).

These elements will be produced by starting with the elements $\bar{\nu}^{(2t)}$ that we have already constructed, living in the bottom groups of the following diagram:



First note that it suffices to construct the $\overline{\lambda}^{(\varphi(4t))}$ classes for n sufficiently large, as we can then construct the classes for smaller n by naturality. In particular, we may assume that n is congruent to 6 (or 7 or 8) modulo 8. This forces the right-most vertical c map to be an isomorphism, by Theorem 32.17(a).

The rest of the argument breaks into two cases, depending on whether t is even or odd. When t is even, $\varphi(4t) = 2t - 1$ (see Table 32.15) and the vertical maps c in the above diagram are all isomorphisms; this will be proven carefully in Theorem 32.20(b) below, but for now we just accept it. Define

$$\overline{\lambda}^{(\varphi(4t))} = c^{-1}(\bar{\nu}^{(2t)}).$$

The desired properties of $\overline{\lambda}^{(\varphi(4t))}$ are immediate by the naturality of c and the known properties of $\bar{\nu}^{(2t)}$.

When t is odd one has $\varphi(4t) = 2t+1$. The vertical maps c are no longer isomorphisms (except the rightmost one), but we can use the map $r_{\mathbb{R}}$ instead. The idea for this comes from the fact that when t is odd the map $c\colon \widetilde{KO}(S^{4t}) \to \widetilde{K}(S^{4t})$ is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, and $r_{\mathbb{R}}$ sends a generator to a generator. So $r_{\mathbb{R}}(\bar{\nu}^{(2t)})$ will gives us an element of $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ that maps to a generator in $\widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$.

However, note that $r_{\mathbb{R}}(\bar{\nu}^{2t})$ maps to $-\lambda^{(2t+1)}$ in $\widetilde{KO}(\mathbb{R}P^n)$. This follows at once from a simple calculation:

$$r_{\mathbb{R}}(\nu^{2t}) = r_{\mathbb{R}}((-2)^{2t-1}\cdot\nu) = (-2)^{2t-1}\cdot r_{\mathbb{R}}(\nu) = (-2)^{2t-1}\cdot 2\lambda = -(-2)^{2t}\cdot\lambda = -\lambda^{2t+1}.$$

The extra minus sign leads us to make the definition

$$\overline{\lambda}^{(\varphi(4t))} = -r_{\mathbb{R}}(\bar{\nu}^{(2t)})$$

in the case when t is odd.

We have now constructed the desired generators $\overline{\lambda}^{(\varphi(4t))}$. The group $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ is generated by the two elements $\lambda^{\varphi(4t+1)}$ (which is torsion) and $\overline{\lambda}^{(\varphi(4t))}$ (which is non-torsion). We next use these generators to describe the action of the Adams operations:

Theorem 32.19 (Real K-theory of real, stunted projective spaces; part 2)**.** *Let $t \geq 1$ and let $f = \varphi(4t)$. The Adams operations on $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ are given by the formulas*

$$\psi^k\left(\lambda^{(f+1)}\right) = \begin{cases} 0 & k \text{ even,} \\ \lambda^{(f+1)} & k \text{ odd;} \end{cases}$$

$$\psi^k\left(\overline{\lambda}^{(f)}\right) = k^{2t}\overline{\lambda}^{(f)} + \begin{cases} \frac{1}{2}k^{2t}\lambda^{(f+1)} & k \text{ even,} \\ \frac{1}{2}(k^{2t}-1)\lambda^{(f+1)} & k \text{ odd.} \end{cases}$$

Just as we saw for $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^a)$, much of the information about KO-theory of stunted projective spaces is represented in the following useful diagram. The above formulas for the Adams operations are obtained easily by chasing information around the diagram.

$$\begin{array}{ccccccc}
\mathbb{Z}/(2^{\varphi(n)}) = \widetilde{K}(\mathbb{R}P^n) & & & & \widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) = \mathbb{Z} & & \\
\uparrow & & & & & & \\
\longleftarrow \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-2}) & \longleftarrow & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & \longleftarrow & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \longleftarrow & \\
\| & & \| & & \| & & \\
\mathbb{Z}/(2^{B+1}) & \twoheadleftarrow & \mathbb{Z} \oplus \mathbb{Z}/(2^B) & \leftarrowtail & \mathbb{Z}/(2^B) & {\scriptstyle B=\varphi(n)-\varphi(4t)} &
\end{array}$$

$$\begin{array}{ccccc}
\lambda^{(f)} & \longleftarrow & \overline{\lambda}^{(f)}, \quad \lambda^{(f+1)} & \longleftarrow & \lambda^{(f+1)} \qquad {\scriptstyle f=\varphi(4t)} \\
\lambda^{(f+1)} = -2\lambda^{(f)} & \longleftarrow & & &
\end{array}$$

Due to lack of space the diagram does not show the groups $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-3})$, but these are similar to the $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-2})$ and $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t})$ cases in that these groups all inject into $\widetilde{KO}(\mathbb{R}P^n)$ and have the "expected" image.

Because we have needed this already in the process of defining the classes $\overline{\lambda}^{(\varphi(4t))}$, we also include some more information on the complexification map for stunted projective spaces. We want to investigate $c\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$, but results are awkward to state in this generality: one runs into a multitude of cases depending on the congruences classes of n and a modulo 8. We start with the observation that it is essentially enough to solve the problem for n large enough. If $N \geq n$ then we have the diagram

$$\begin{array}{ccc}
\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \twoheadleftarrow & \widetilde{KO}^0(\mathbb{R}P^N/\mathbb{R}P^a) \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\
\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \twoheadleftarrow & \widetilde{K}^0(\mathbb{R}P^N/\mathbb{R}P^a).
\end{array}$$

So if we know the right vertical map then we can also figure out the left vertical map, using the horizontal surjections.

Notice that by choosing N so that it is congruent to 6 (or 7 or 8) modulo 8, we can get ourselves in the situation where $c\colon \widetilde{KO}^0(\mathbb{R}P^N) \to \widetilde{K}^0(\mathbb{R}P^N)$ is an isomorphism (see Theorem 32.17)—clearly this will simplify some matters in our analysis. This explains why we focus on this special case in the following result.

Here is a notational simplification that is very useful. At this point we have specified particular generators for the groups $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a)$, for all values of n and a. These are the elements $\lambda^{(i)}$ and $\overline{\lambda}^{(j)}$ for certain values of i and j that depend on a. To actually name i and j precisely requires separating various cases for a, and it is convenient to not always have to do this. We will write λ° and $\overline{\lambda}^{\circ}$ as abbreviations for our generators, but where we have not bothered to write the exact number in the exponent (it is uniquely specified, anyway). We also write ν° and $\bar{\nu}^{\circ}$ for our generators in $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$.

Theorem 32.20. *Consider the map* $c\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ *and the map* $r_{\mathbb{R}}$ *going in the opposite direction. Assume that* n *is congruent to* 6, 7, *or* 8 *modulo* 8.

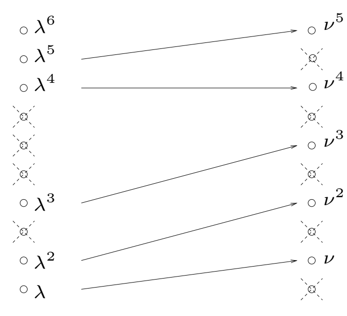
(a) Suppose a *is even, so that both the groups are torsion.*
  (i) If $a \equiv 6, 8 \bmod 8$ *then* c *is an isomorphism,* $c(\lambda^\circ) = \nu^\circ$, *and* $r_{\mathbb{R}}(\nu^\circ) = 2\lambda^\circ$.
  (ii) If $a \equiv 2, 4 \bmod 8$ *then* c *is an injection with cokernel* $\mathbb{Z}/2$. *One has* $c(\lambda^\circ) = -2\nu^\circ$ *and* $r_{\mathbb{R}}(\nu^\circ) = -\lambda^\circ$.

(b) Suppose that $a = 4t - 1$. *Here both the domain and target of* c *have copies of* $\mathbb{Z}$ *inside them.*
  (i) If $a \equiv 7 \bmod 8$ *(i.e.,* t *is even) then* c *is an isomorphism.*
  (ii) If $a \equiv 3 \bmod 8$ *(i.e.,* t *is odd) then* c *is a monomorphism, and the cokernel is* $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

(c) Suppose that $a = 4t + 1$. *In this case* c *maps its domain isomorphically onto the torsion subgroup of the target of* c. *One has* $c(\lambda^\circ) = \nu^\circ$ *and* $r_{\mathbb{R}}(\nu^\circ) = 2\lambda^\circ$.

32.21. **An extended example.** Let us demonstrate much of what we have learned by looking at a specific example. The Atiyah-Hirzebruch spectral sequence for computing $\widetilde{KO}(\mathbb{R}P^{10})$ gives one $\mathbb{Z}/2$ for every dimension from 1 through 10 that is congruent to 0, 1, 2, or 4 modulo 8. These are the dimensions 1, 2, 4, 8, 9, and 10—so we have six $\mathbb{Z}/2$'s, and $\widetilde{KO}(\mathbb{R}P^{10}) \cong \mathbb{Z}/(2^6)$. In comparison, $\widetilde{K}(\mathbb{R}P^{10})$ is just $\mathbb{Z}/(2^5)$ (as $5 = \frac{10}{2}$).

There is a visual way of representing this information that is useful, especially when it comes to the stunted projective spaces. Draw a cell diagram for $\mathbb{R}P^{10}$, leaving out the 0-cell. For $\widetilde{KO}(\mathbb{R}P^{10})$ discard all cells except the ones in dimensions congruent to 0, 1, 2, or 4 modulo 8; then label the remaining cells with ascending powers of λ. For $\widetilde{K}(\mathbb{R}P^{10})$ discard all the odd-dimensional cells and label the remaining ones with ascending powers of ν. Always remembering that $\lambda^2 = -2\lambda$ (and $\nu^2 = -2\nu$), the cells now represent the associated graded of $\widetilde{KO}(\mathbb{R}P^{10})$ (or $\widetilde{K}(\mathbb{R}P^{10})$) with respect to the 2-adic filtration. The picture below also shows the complexification map $c\colon \widetilde{KO}(\mathbb{R}P^{10}) \to \widetilde{K}(\mathbb{R}P^{10})$. Recall that this is a ring map and sends λ to ν:



$\widetilde{KO}(\mathbb{R}P^{10}) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10})$

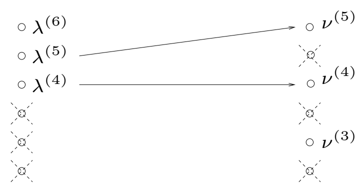
We see in this case that $c\colon \widetilde{KO}(\mathbb{R}P^{10}) \to \widetilde{K}(\mathbb{R}P^{10})$ is surjective with kernel $\mathbb{Z}/2$ (generated by $\lambda^6 = -32\lambda$). One has $r_{\mathbb{R}}(\nu) = r_{\mathbb{R}}(c(\lambda)) = 2\lambda$, and more generally

$$r_{\mathbb{R}}(\nu^k) = r_{\mathbb{R}}((-2)^{k-1}\nu) = (-2)^{k-1} r_{\mathbb{R}}(\nu) = (-2)^{k-1} \cdot 2\lambda = -(-2)^k\lambda = -\lambda^{k+1}.$$

Next let us consider the K-groups of $\mathbb{R}P^{10}/\mathbb{R}P^4$, referring to the diagram

$$\begin{array}{ccc} \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) & \rightarrowtail & \widetilde{KO}(\mathbb{R}P^{10}) \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) & \rightarrowtail & \widetilde{K}(\mathbb{R}P^{10}). \end{array}$$

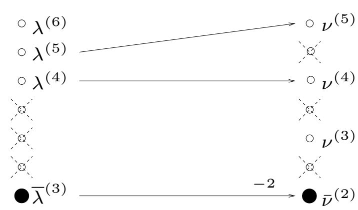
In relation to our cell-diagrams, the K-groups of $\mathbb{R}P^{10}/\mathbb{R}P^4$ are obtained by throwing away the bottom four cells. We obtain the picture



$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4)$

This picture tells us that $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \cong \mathbb{Z}/(2^3)$, generated by $\lambda^{(4)}$, and also $\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) \cong \mathbb{Z}/(2^3)$ with generator $\nu^{(3)}$. These each embed into the respective K-group of $\mathbb{R}P^{10}$. The complexification map therefore sends $\lambda^{(4)}$ to $\nu^{(4)}$, and we find that this map has both kernel and cokernel isomorphic to $\mathbb{Z}/2$.

The situation is a little different if we consider the K-groups of $\mathbb{R}P^{10}/\mathbb{R}P^3$. Here the bottom cell of $\mathbb{R}P^{10}/\mathbb{R}P^3$ gives rise to a $\mathbb{Z}$ in singular cohomology, and a corresponding $\mathbb{Z}$ in the K-groups. The picture becomes as follows:



$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3)$

Here the black dots represent copies of $\mathbb{Z}$, so that $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ with the two summands generated by $\overline{\lambda}^{(3)}$ and $\lambda^{(4)}$, respectively. Likewise, $\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ with the two summands generated by $\bar{\nu}^{(2)}$ and $\nu^{(3)}$. The various maps

$$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \longrightarrow \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \longrightarrow \widetilde{KO}(\mathbb{R}P^{10}), \text{ and}$$
$$\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) \longrightarrow \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3) \longrightarrow \widetilde{K}(\mathbb{R}P^{10})$$

are the evident ones suggested by the diagrams. The only subtlety lies in determining the complexification map c. Of course $c(\lambda^{(i)}) = \nu^{(i)}$ for $i = 4, 5$, as this is forced by that happens on the subgroup $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \subseteq \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3)$. To compute $c(\overline{\lambda}^{(3)})$ we must remember that $\overline{\lambda}^{(3)}$ is defined by the equation

$$\overline{\lambda}^{(3)} = -r_{\mathbb{R}}(\nu^{(2)})$$

(note that $3 = 4t - 1$ where $t = 1$, and so we are in the case where t is odd; in the case where t is even the definition of the $\overline{\lambda}$ classes is different). So we obtain

$$c\big(\overline{\lambda}^{(3)}\big) = -c(r_{\mathbb{R}}(\nu^{(2)})) = -(1+\psi^{-1})(\bar{\nu}^{(2)}) = -[\bar{\nu}^{(2)} + \bar{\nu}^{(2)}] = -2\bar{\nu}^{(2)}.$$

Here we have used Theorem 32.12(c) for evaluating $\psi^{-1}(\bar{\nu}^{(2)})$. Note that the pull-back map induced by $\mathbb{R}P^{10} \to \mathbb{R}P^{10}/\mathbb{R}P^3$ sends $\overline{\lambda}^{(3)}$ to λ^3 and sends $-2\bar{\nu}^{(2)}$ to $-2\nu^2 = \nu^3$; hence the above formula is consistent with our previous computation of $c\colon \widetilde{KO}(\mathbb{R}P^{10}) \to \widetilde{K}(\mathbb{R}P^{10})$.

32.22. **The proofs.** We now give proofs for all of the results previously stated in this section.

Proof of Theorem 32.9. This is straightforward, and left to the reader. □

Proof of Theorem 32.11. There is no room for differentials in the Atiyah-Hirzebruch spectral sequence for $K^*(\mathbb{R}P^n)$, so it collapses at E_2. Part (b) follows immediately. It is also a direct consequence that $\widetilde{K}^0(\mathbb{R}P^n)$ is an abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$. It remains to solve the extension problems to determine precisely which group it is.

Observe that $L^2 = 1$, hence $(cL)^2 = c(L^2) = c(1) = 1$. So

$$\nu^2 = (cL-1)^2 = (cL)^2 - 2(cL) + 1 = 2(1 - c(L)) = -2\nu.$$

Note that an immediate consequence is $\nu^t = (-2)^{t-1}\nu$.

Let $F^i = \ker\big(\widetilde{K}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^{i-1})\big)$. So

$$\widetilde{K}^0(\mathbb{R}P^n) = F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots \supseteq F^{n+1} = 0.$$

The quotients F^i/F^{i+1} are the groups in the E_∞ term of the spectral sequence, and so are

$$F^i/F^{i+1} = \begin{cases} \mathbb{Z}/2 & 0 < i \le 2\lfloor \frac{n}{2} \rfloor \text{ and } i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

So $F^0 = F^1 = F^2$ and $F^2/F^3 \cong \mathbb{Z}/2$. The element ν generates F^2/F^3: we know this by naturality of the spectral sequence, applied to the map $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. The element μ generates F^2/F^3 for $\widetilde{K}^0(\mathbb{C}P^n)$, and $H^2(\mathbb{C}P^n) \to H^2(\mathbb{R}P^n)$ is the projection $\mathbb{Z} \to \mathbb{Z}/2$: the image of a generator is another generator. So $j^*\mu$ generates F^2/F^3 for $\widetilde{K}^0(\mathbb{R}P^n)$, and of course $\nu = j^*\mu$.

The multiplicativity of the spectral sequence then gives us that ν^2 generates F^4/F^5, and in general ν^j generates F^{2j}/F^{2j+1} for $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. In particular, ν^j is not equal to zero for j in this range. But $\nu^j = (-2)^{j-1}\nu$, so $2^{\lfloor \frac{n}{2} \rfloor - 1}\nu \neq 0$. This proves that the only possibility for $\widetilde{K}^0(\mathbb{R}P^n)$ is $\mathbb{Z}/2^{\lfloor \frac{n}{2} \rfloor}$, and that ν is a generator.

Note that we than have $0 = (-2)^{\lfloor \frac{n}{2} \rfloor}\nu = \nu^{\lfloor \frac{n}{2} \rfloor + 1}$.

For part (c) we just observe that $\psi^k(cL) = (cL)^k = c(L^k)$ and this equals 1 if k is even, and cL if k is odd. □

Proof of Theorem 32.12. Parts (a) and (b) are trivial. In each case one writes down the evident long exact sequence and quickly sees that the given sequence is short exact. The only slight subtlety is seeing in the case $a = 2t - 1$ that $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \to \widetilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1})$ is surjective, but this was explained when we constructed the element $\bar{\nu}^{(t)}$ (which maps to a generator in the target group).

For part (c), the action of ψ^k on $\nu^{(t+1)}$ is determined by the corresponding action in $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t})$; so there is nothing to prove here. The action on $\bar{\nu}^{(t)}$ is more interesting. We can, of course, write

$$\psi^k(\bar{\nu}^{(t)}) = A\bar{\nu}^{(t)} + B\nu^{(t+1)} \tag{32.23}$$

where A is a unique integer and B is unique modulo 2^f. Applying the map $i^*\colon \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \to \widetilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1})$ kills $\nu^{(t+1)}$ and sends $\bar{\nu}^{(t)}$ to a generator g, so this equation becomes $\psi^k(g) = Ag$. But we already know that ψ^k acts on such a generator by k^t, so $A = k^t$.

Next we apply the map $\pi^*\colon \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t}) \to \widetilde{K}^0(\mathbb{R}P^n)$ to equation (32.23). The map π^* sends $\bar{\nu}^{(t)}$ to ν^t and $\nu^{(t+1)}$ to ν^{t+1}, so using $A = k^t$ we obtain

$$\psi^k(\nu^t) = k^t\nu^t + B\nu^{t+1}$$

in $\widetilde{K}^0(\mathbb{R}P^n)$. Now use that ψ^k is a ring homomorphism, together with $\nu^{t+1} = -2\nu^t$. We get

$$(k^t - 2B)\nu^t = \left[\psi^k(\nu)\right]^t = \begin{cases} 0 & \text{if } k \text{ is even} \\ \nu^t & \text{if } k \text{ is odd.} \end{cases}$$

The group $\widetilde{K}^0(\mathbb{R}P^n)$ is $\mathbb{Z}/(2^g)$ with generator ν, and $\nu^t = (-2)^{t-1}\nu$. So the additive order of ν^t is 2^{g-t+1}, or equivalently 2^{f+1}. In the case that k is even it follows that $k^t - 2B$ is a multiple of 2^{f+1}, so that $\frac{k^t}{2} \equiv B \bmod 2^f$ (recall that B is only well-defined modulo 2^f in the first place).

In the remaining case where k is odd we get $k^t - 2B \equiv 1$ modulo 2^{f+1}. So $\frac{k^t-1}{2} \equiv B$ modulo 2^f, which is what we wanted. □

Proof of Theorem 32.14. In the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}(\mathbb{R}P^n)$, the diagonal of the E_2-term that is relevant to $\widetilde{KO}^0(\mathbb{R}P^n)$ consists of $\varphi(n)$ copies of $\mathbb{Z}/2$. The first concern is to determine if there are any differentials causing some of these copies to disappear by E_∞, and the second concern is the problem of extensions.

Observe that the complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^n)$ is surjective, because ν generates the target and $\nu = c(L-1)$. So it follows from Theorem 32.11 that at least $\lfloor \frac{n}{2} \rfloor$ among our $\varphi(n)$ copies of $\mathbb{Z}/2$ must survive the spectral sequence.

The trick now is to not consider one n at a time, but rather to consider them all at once. When n is congruent to 6, 7, or 8 modulo 8 then we know $\varphi(n) = \lfloor \frac{n}{2} \rfloor$, and so here it must be that all the $\mathbb{Z}/2$'s along the main diagonal survive. That is, all differentials entering or exiting the main diagonal are zero. But then by naturality of the spectral sequence this is true for all n. We conclude that the order of $\widetilde{KO}^0(\mathbb{R}P^n)$ is $2^{\varphi(n)}$, no matter what n is.

When n is congruent to 6, 7, or 8 modulo 8 we now know that the orders of $\widetilde{KO}^0(\mathbb{R}P^n)$ and $\widetilde{K}^0(\mathbb{R}P^n)$ are the same. Since the complexification map is surjective, it is therefore an isomorphism. So $\widetilde{KO}^0(\mathbb{R}P^n)$ is cyclic. Since $c(\lambda) = \nu$ it follows that λ is a generator. In particular, λ is a generator for the quotient F^1/F^2.

But then by naturality of the spectral sequence (and with it, naturality of the filtration F^i) it follows that λ generates F^1/F^2 for *every* value of n. Since $L^2 = 1$ we of course have $\lambda^2 = -2\lambda$. At this point the argument follows the one in the proof of Theorem 32.11 to show that $\widetilde{KO}^0(\mathbb{R}P^n)$ is cyclic, for all values of n.

The computation of the Adams operations again follows from $\psi^k(L) = L^k$, which equals 1 if k is even and L if k is odd. □

Proof of Theorem 32.17. This was given just prior to the statement of the theorem. □

Proof of Theorem 32.18. For part (a) one examines the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}^*(\mathbb{R}P^n/\mathbb{R}P^a)$. Note that the quotient $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^a$ induces a map of spectral sequences in the other direction. The diagonal groups in the E_2-term for $\widetilde{KO}^*(\mathbb{R}P^n/\mathbb{R}P^a)$ are a truncation of the diagonal groups appearing in the E_2-term for $\widetilde{KO}^*(\mathbb{R}P^n)$. Since there are no entering or exiting differentials (along the diagonal) in the latter case, naturality of the spectral sequence guarantees there are no entering or exiting differentials for $\mathbb{R}P^n/\mathbb{R}P^a$. Passing to E_∞-terms now, we see that the associated graded groups for $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a)$ are a truncation of the associated graded groups for $\widetilde{KO}(\mathbb{R}P^n)$. Examining the map of filtered groups

$$
\begin{array}{ccccccc}
\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a) & \leftarrowtail & F_1 & \leftarrowtail & F_2 & \leftarrowtail & \cdots \\
\downarrow{\scriptstyle \pi^*} & & \downarrow & & \downarrow & & \\
\widetilde{KO}(\mathbb{R}P^n) & \leftarrowtail & F_1' & \leftarrowtail & F_2' & \leftarrowtail & \cdots
\end{array}
$$

we now find that $F_k \to F_k'$ is an isomorphism for $k \geq a+1$ and $F_k/F_{k+1} = 0$ for $k \leq a$. It follows that $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{KO}(\mathbb{R}P^n)$ is an injection, with image equal to F_{a+1}'. In our analysis of $KO(\mathbb{R}P^n)$ we have already seen that $F_{a+1}' \subseteq \widetilde{KO}(\mathbb{R}P^n)$ is the subgroup generated by λ^{a+1}. Everything else in part (a) is then immediate.

For (b) we only need to prove that $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \to \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$ is surjective, since the latter group is isomorphic to $\widetilde{KO}^0(S^{4t}) \cong \mathbb{Z}$. Everything else in part (b) is routine. To do this, consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & \mathbb{Z}/2 & & & & \\
 & & & & \| & & & & \\
\cdots \leftarrow & \widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \leftarrow & \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2}) & \overset{j_1^*}{\twoheadleftarrow} & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2}) & \leftarrow \cdots \\
 & \uparrow & & {\scriptstyle i^*}\twoheaduparrow & & \twoheaduparrow & & \\
\cdots \leftarrow & \widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \leftarrow & \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) & \overset{j_2^*}{\leftarrow} & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & \leftarrow \cdots \\
 & & & \| & & & & \\
 & & & \mathbb{Z} & & & &
\end{array}
$$

The two indicated vertical maps are surjections because they sit inside long exact sequences where the third term is $\widetilde{KO}^0(\mathbb{R}P^{4t-1}/\mathbb{R}P^{4t-2}) = \widetilde{KO}^0(S^{4t-1}) = 0$. The indicated group is $\mathbb{Z}/2$ by part (a) of the theorem, which also yields the diagram

$$
\begin{array}{ccc}
\widetilde{KO}^0(\mathbb{R}P^{4t}) & \twoheadleftarrow & \widetilde{KO}^0(\mathbb{R}P^n) \\
\uparrow & & \uparrow \\
\widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2}) & \overset{j_1^*}{\leftarrowtail} & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2}).
\end{array}
$$

The group $\widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2})$ is the subgroup of $\widetilde{KO}^0(\mathbb{R}P^{4t})$ generated by $\nu^{1+\varphi(4t-2)}$, and $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2})$ is the similarly-described subgroup of $\widetilde{KO}^0(\mathbb{R}P^n)$. It follows at once that j_1^* is surjective.

Returning to the earlier diagram, the image of j_2^* is an ideal (r) inside of $\mathbb{Z}$. The fact that $i^*j_2^*$ is surjective (readily observed from the diagram) proves that r must be odd. But the quotient $\mathbb{Z}/r$ will inject into $\widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t})$, by the long exact sequence. The Atiyah-Hirzebruch spectral sequence shows that this latter group has no odd torsion, because it has a filtration where the quotients are only $\mathbb{Z}$'s and $\mathbb{Z}/2$'s. So the conclusion is that $r = 1$, hence j_2^* is surjective. □

Proof of Theorem 32.19. The evaluation of $\psi^k\left(\lambda^{(f+1)}\right)$ is immediate using naturality and Theorem 32.14. For the evaluation of $\psi^k\left(\overline{\lambda}^{(f)}\right)$ one can repeat the proof of Theorem 32.12(c) almost verbatim. Alternatively, one can use the result of Theorem 32.12(c) together with the complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$, which is a monomorphism for n congruent to 6, 7, or 8 modulo 8; the result for other values of n can then be deduced by naturality. □

Proof of Theorem 32.20. For part (a) we consider the two short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \longrightarrow & \widetilde{KO}^0(\mathbb{R}P^n) & \longrightarrow & \widetilde{KO}^0(\mathbb{R}P^a) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c}\,\cong & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \longrightarrow & \widetilde{K}^0(\mathbb{R}P^n) & \longrightarrow & \widetilde{K}^0(\mathbb{R}P^a) & \longrightarrow & 0.
\end{array}$$

If a is congruent to 6 or 8 modulo 8 then the right vertical map is an isomorphism, which means the left vertical map is as well. The desired results are immediate.

If a is congruent to 2 or 4 modulo 8 then the right vertical map is a surjection with kernel $\mathbb{Z}/2$. It follows from the zig-zag lemma that the left vertical map is an injection with cokernel $\mathbb{Z}/2$. The generator for the domain is $\lambda^{(\varphi(a)+1)}$, and we are in the case where $\varphi(a) = \lfloor \frac{a}{2} \rfloor + 1$. So c maps this generator to $\nu^{(\lfloor \frac{a}{2} \rfloor+2)} = -2\nu^{(\lfloor \frac{a}{2} \rfloor+1)}$. The statement $r(\nu^\circ) = -\lambda^\circ$ then follows using that $rc = 2$.

For (b) we look at the short exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \to & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & \to & \widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) & \to & 0 \\
 & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) & \longrightarrow & 0.
\end{array}$$

When t is even the right vertical map is an isomorphism by Bott's calculation, and the left vertical map is an isomorphism by (a). So the middle vertical map is also an isomorphism.

When t is odd the right vertical map is an injection with cokernel $\mathbb{Z}/2$, by Bott. The left vertical map is an injection with cokernel $\mathbb{Z}/2$ by part (a). So by the Snake Lemma the middle vertical map is also an injection, and its cokernel is either $(\mathbb{Z}/2)^2$ or $\mathbb{Z}/4$. The element $\bar{\nu}^{(2t)}$ maps to a generator for the right bottom group $\widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$. If we verify that $2\bar{\nu}^{(2t)} = 0$ in the cokernel of c then we will have proven that this cokernel is $(\mathbb{Z}/2)^2$, not $\mathbb{Z}/4$. But note that

$$cr_{\mathbb{R}}(\bar{\nu}^{(2t)}) = (1 + \psi^{-1})(\bar{\nu}^{(2t)}) = 2\bar{\nu}^{(2t)}$$

where in the last equality we have used the formula for $\psi^{-1}(\bar{\nu}^{(2t)})$ from Theorem 32.12(c).

For (c) we consider the following:

$$\begin{array}{ccccccccc}
0 \to & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+2}) & \xrightarrow{p^*} & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \to & \widetilde{KO}(\mathbb{R}P^{4t+2}/\mathbb{R}P^{4t+1}) & \to 0 \\
& \downarrow c & & \downarrow c & & \downarrow c & \\
0 \to & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+2}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^{4t+2}/\mathbb{R}P^{4t+1}) & \to 0.
\end{array}$$

We have not yet discussed exactness of the top row. On the left end this follows because $\widetilde{KO}^{-1}(S^{4t+2}) = 0$. By our computations, the cokernel of p^* is a group of order $2^{\varphi(4t+2)-\varphi(4t+1)}$; but by inspection this number is equal to 1 when t is odd and 2 when t is even. As this is the same as the order of the group $\widetilde{KO}(S^{4t+2})$, this justifies exactness on the right.

If t is odd then the left vertical map is an isomorphism by (a), and the horizontal map p^* is an isomorphism; the desired claims follow at once. When t is even we must argue more carefully. Note that the image of $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+2})$ inside $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is precisely the torsion subgroup; let us call this image T. Since the group $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is torsion, its image under c is also torsion; so this image is a subgroup of T. Moreover, $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is generated by $\lambda^{(\varphi(4t+1)+1)}$, and one readily computes that $\varphi(4t+1) = 2t+1$. Consider the square

$$\begin{array}{ccc}
\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \rightarrowtail & \widetilde{KO}(\mathbb{R}P^n) \\
\downarrow c & & \downarrow c\ \cong \\
\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^n).
\end{array}$$

The left vertical map clearly must be injective. Compute that $\pi^*(c(\lambda^{(2t+2)})) = c(\lambda^{2t+2}) = \nu^{2t+2}$. There is only one element of the torsion subgroup of $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ that pulls back to ν^{2t+2}, namely $\nu^{(2t+2)}$. It follows that $c(\lambda^{(2t+2)}) = \nu^{(2t+2)}$. But $\nu^{(2t+2)}$ generates T, so c maps $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ isomorphically onto T. □

33. Solution to the vector field problem

In this section we conclude our story of the vector field problem, following the original paper by Adams [Ad2]. Let us first recall the Hurwitz-Radon function $\rho(n)$: if $n = 2^{4b+a} \cdot (\text{odd})$ then $\rho(n) = 2^a + 8b - 1$. We have seen in Theorem 14.5 that one can construct $\rho(n)$ independent vector fields on S^{n-1}. The vector field problem will be settled once we prove the following:

Theorem 33.1 (Adams)**.** *There do not exist $\rho(n)+1$ independent vector fields on S^{n-1}.*

Remark 33.2. There will inevitably come a time when the reader wishes to remember the formula for $\rho(n)$ but cannot immediately look it up. The key facts about the formula are:

(i) $\rho(n)$ only depends on the power of 2 in the prime factorization of n;
(ii) For $a \leq 3$ one has $\rho(2^a) = 2^a - 1$;
(iii) $\rho(16n) = \rho(n) + 8$.

These facts of course uniquely determine $\rho(n)$. Personally, I find the exact form of (iii) hard to remember when I haven't been working with this stuff for a while. What *is* able to stick in my head is that there are zero vector fields on S^0, one on S^1, three on S^3, seven on S^7—and then I have to remember that there are only eight on S^{15}. The jump from zero on S^0 to eight on S^{15} is the quasi-periodicity; so there are nine on S^{31}, eleven on S^{63}, fifteen on S^{127}, and so forth. From this it is easy to recover the formula $\rho(16n) = \rho(n) + 8$, and onward to the general formula for ρ.

The proof of Theorem 33.1 is quite involved—it requires a surprising amount of algebraic topology. KO-theory is usually regarded as the key tool in the proof, but one also needs Steenrod operations, James periodicity, and Atiyah duality in the stable homotopy category. This adds up to a sizable amount of material. We will take a modular approach to things; we start by giving an outline of the proof, and then we will fill in the details one by one.

33.3. **Outline of the proof.**

Step 1: We have the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv):

(i) There exist $k-1$ vector fields on S^{n-1}.
(ii) There exist $k-1$ vector fields on S^{un-1} for every $u \geq 1$.
(iii) The map $\pi_1 \colon V_k(\mathbb{R}^{un}) \to S^{un-1}$ has a section, for every $u \geq 1$.
(iv) The map $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1} \to S^{un-1}$ (projection onto the top cell) has a section in the homotopy category, for every u such that $un+2 > 2k$. That is to say, $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1}$ splits off the top cell in the stable homotopy category.

We have seen these implications back in Section 14, but let us briefly recall why they hold. For (i)$\Rightarrow$(ii) it is a direct construction: given $k-1$ orthogonal vector fields made from vectors with n coordinates, one can repeat those patterns in successive groups of coordinates to make $k-1$ orthogonal vector fields in un coordinates, for any u. The step (ii)$\Rightarrow$(iii) is a triviality, essentially just a restatement of the problem. Then for (iii)$\Rightarrow$(iv) it is because for $a+2 > 2b$ the space $V_b(\mathbb{R}^a)$ has a cell structure where the a-skeleton is $\mathbb{R}P^{a-1}/\mathbb{R}P^{a-b-1}$ (Proposition 14.23).

Step 2: Steenrod operations allow one to prove that if $a+1 = 2^r \cdot (\text{odd})$ then $\mathbb{R}P^a/\mathbb{R}P^{a-b}$ does not split off the top cell for $b > 2^r$.

The proof will be described in detail below, but here is a short summary. The hypothesis says that $a = 2^r(2t+1) - 1 = 2^{r+1}t + 2^r - 1$, and this guarantees that there is a Sq^{2^r} operation in $H^*(\mathbb{R}P^\infty)$ connecting the class in degree $2^{r+1}t-1$ to the class in degree a. If $b > 2^r$ then that Sq^{2^r} operation is still present in $\mathbb{R}P^a/\mathbb{R}P^{a-b}$, and this obstructs the splitting off of the top cell.

Step 3: Putting steps 1 and 2 together, we have that if $n = 2^m \cdot (\text{odd})$ then there do not exist 2^m vector fields on S^{n-1}.

For $m \leq 3$ this solves the vector field problem, because in this case $\rho(m) = 2^m - 1$.

Step 4: There are periodicities to the spaces $\mathbb{R}P^a/\mathbb{R}P^{a-b}$. If $L-1$ has finite order r_b in $\widetilde{KO}(\mathbb{R}P^{b-1})$, then

$$\mathbb{R}P^a/\mathbb{R}P^{a-b} \simeq \Sigma^{-sr_b}\Big(\mathbb{R}P^{a+sr_b}/\mathbb{R}P^{a+sr_b-b}\Big)$$

for every $s \geq 1$, where the homotopy equivalence is in the stable homotopy category. This is called *James periodicity*. The proof was given in Proposition 15.16.

Step 5: Now things get a bit more sophisticated. Atiyah proved that if M is any compact manifold and $E \to M$ is a real vector bundle, then $\mathrm{Th}(E \to M)$ is Spanier-Whitehead dual to $\mathrm{Th}(-E - T_M \to M)$, where T_M is the tangent bundle to M (for Thom spaces of negative bundles, see Section 15.12).

Recall from Example 15.9 that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \cong \mathrm{Th}((n-k)L \to \mathbb{R}P^{k-1})$. Also, for $M = \mathbb{R}P^{k-1}$ one has $T_M = kL - 1$ in $KO(M)$ (Example 23.10). So Atiyah Duality gives that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ is Spanier-Whitehead dual to

$$\mathrm{Th}\begin{pmatrix} -(n-k)L-T \\ \downarrow \\ \mathbb{R}P^{k-1}\end{pmatrix} = \mathrm{Th}\begin{pmatrix} -(n-k)L-kL+1 \\ \downarrow \\ \mathbb{R}P^{k-1}\end{pmatrix} = \mathrm{Th}\begin{pmatrix} -nL+1 \\ \downarrow \\ \mathbb{R}P^{k-1}\end{pmatrix} = \Sigma\,\mathrm{Th}\begin{pmatrix} -nL \\ \downarrow \\ \mathbb{R}P^{k-1}\end{pmatrix}.$$

If r_k is the order of $L-1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$ then the last spectrum may be interpreted as

$$\begin{aligned}\Sigma\,\mathrm{Th}\begin{pmatrix} -nL \\ \downarrow \\ \mathbb{R}P^{k-1}\end{pmatrix} &\simeq \Sigma\Sigma^{-sr_k}\,\mathrm{Th}\begin{pmatrix} sr_kL-nL \\ \downarrow \\ \mathbb{R}P^{k-1}\end{pmatrix} \\ &= \Sigma^{1-sr_k}\,\mathrm{Th}\begin{pmatrix} (sr_k-n)L \\ \downarrow \\ \mathbb{R}P^{k-1}\end{pmatrix} \\ &= \Sigma^{1-sr_k}\Big[\mathbb{R}P^{sr_k-n+k-1}/\mathbb{R}P^{sr_k-n-1}\Big],\end{aligned}$$

where s is any integer sufficiently large so that $sr_k - n - 1 \geq 0$. We have therefore proven:

The Spanier-Whitehead dual of $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ is (up to suspension) $\mathbb{R}P^{sr_k-n+k-1}/\mathbb{R}P^{sr_k-n-1}$, where s is any integer such that $sr_k - n - 1 \geq 0$.

Step 6: A direct consequence of the previous statement is that

$\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ splits off its top cell (stably)

if and only if

$\mathbb{R}P^{sr_k-n+k-1}/\mathbb{R}P^{sr_k-n-1}$ splits off its bottom cell (stably),

where $s \gg 0$ as above.

Step 7: Using step 6 we can add a condition onto the list of implications from step 1. Namely, we have (iv)⇒(v) where the latter is

(v) $\mathbb{R}P^{sr_k-un+k-1}/\mathbb{R}P^{sr_k-un-1}$ splits off its bottom cell (stably), for any $u \gg 0$ and any $s \gg 0$.

Step 8: Adams calculated $\widetilde{KO}(\mathbb{R}P^a)$ for all a, together with the Adams operations on these groups. He used this knowledge, together with Step 2 above, to prove the following:

For any $m \geq 0$, $\mathbb{R}P^{m+\rho(m)+1}/\mathbb{R}P^{m-1}$ does not split off its bottom cell in the stable homotopy category.

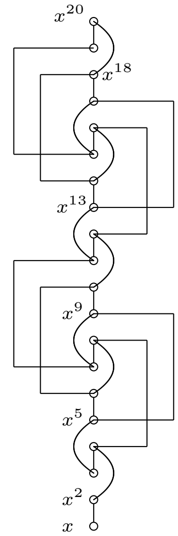
Step 9: Completion of the proof.

Proof of Theorem 33.1. Suppose there are $k-1$ vector fields on S^{n-1}. Then by Step 7 the space $\mathbb{R}P^{sr_k-un+k-1}/\mathbb{R}P^{sr_k-un-1}$ stably splits off its bottom cell for any u and s such that $un+2 > 2k$ and $sr_k - un - 1 \geq 0$. Choose u to be odd, and choose s to be a multiple of $2n$. Set $m = sr_k - un$, and note that m is an odd multiple of n; consequently, we have $\rho(m) = \rho(n)$.

We have that $\mathbb{R}P^{m+k-1}/\mathbb{R}P^{m-1}$ splits off its bottom cell. By Step 8 this implies that $k-1 \leq \rho(m) = \rho(n)$. So there are at most $\rho(n)$ vector fields on S^{n-1}. □

The missing pieces from our outline are: **Step 2, Step 5, and Step 8**. We now fill in the details for these steps, one by one—but not quite in the above order. We save Atiyah duality for last, only because the other two pieces belong more to the same theme.

33.4. **Steenrod operations and stunted projective spaces (steps 2 and 3).** Let $x \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ be the nonzero element. The Steenrod operations on $\mathbb{R}P^\infty$ are easily computed from the facts $\mathrm{Sq}^1(x) = x^2$, $\mathrm{Sq}^i(x) = 0$ for $i > 1$, and the Cartan formula. We leave this as an exercise for the reader. In the following picture we show the Sq^1, Sq^2, and Sq^4 operations on $H^*(\mathbb{R}P^{20}; \mathbb{Z}/2)$:



The Sq^1 operations are depicted as vertical lines, the Sq^2 operations as curved lines, and the Sq^4s as "offseted vertical lines". For example, one can read off of the diagram that $\mathrm{Sq}^1(x^5) = x^6$, $\mathrm{Sq}^2(x^{10}) = x^{12}$, and $\mathrm{Sq}^4(x^{10}) = 0$ (in the latter case because the diagram does not have a Sq^4 emanating from the x^{10} class).

The pattern of Sq^{2^r} operations in $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$ is very simple. The first Sq^{2^r} operation occurs on x^{2^r} and thereafter they follow the pattern of "2^r on/2^r off". This is captured by the formula

$$\mathrm{Sq}^{2^r}(x^a) = \begin{cases} x^{a+2^r} & \text{if } a \geq 2^r \text{ and } a \equiv 2^r, 2^r+1, \ldots, 2^{r+1}-1 \bmod 2^{r+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Of course for $H^*(\mathbb{R}P^n;\mathbb{Z}/2)$ the formulas must be truncated to account for the fact that classes above dimension n are not present.

To see how Steenrod operations give obstructions to stable splittings, consider $\mathbb{R}P^9/\mathbb{R}P^6$. Its cohomology has a Sq^2 connecting the class in degree 7 to the class in degree 9. Suppose the projection onto the top cell $p\colon \mathbb{R}P^9/\mathbb{R}P^6 \to S^9$ has a splitting χ in the stable homotopy category. Then the composite

$$H^*(S^9) \xrightarrow{p^*} H^*(\mathbb{R}P^9/\mathbb{R}P^6) \xrightarrow{\chi^*} H^*(S^9)$$

is an isomorphism. Write x_i for the generator in $H^i(\mathbb{R}P^9/\mathbb{R}P^6)$, so that in this notation we have $\mathrm{Sq}^2(x_7) = x_9$. Necessarily we must have $\chi^*(x_7) = 0$, therefore $0 = \mathrm{Sq}^2(\chi^* x_7) = \chi^*(\mathrm{Sq}^2 x_7) = \chi^*(x_9)$. But x_9 is in the image of p^*, so this is a contradiction.

Clearly this kind of argument will work for any $\mathbb{R}P^n/\mathbb{R}P^{n-b}$ where we have a nontrivial cohomology operation hitting the top class. Based on this, it is now easy to prove the following:

Proposition 33.5. *Write $n+1 = 2^s \cdot odd$. If $\mathbb{R}P^n/\mathbb{R}P^{n-b}$ splits off its top cell stably then $b \leq 2^s$.*

Proof. If n is even then in $H^*(\mathbb{R}P^\infty)$ there is a Sq^1 hitting the class in degree n, and this operation will be present in $H^*(\mathbb{R}P^n/\mathbb{R}P^{n-b})$ as long as $b > 1$. So the top cell can not split off in this case. In other words, if $n+1 = 2^0 \cdot (\text{odd})$ then splitting of the top cell can only happen if $b \leq 2^0$.

Similarly, if $n = 4e+1$ then in $H^*(\mathbb{R}P^\infty)$ there is a Sq^2 hitting the class in degree n. This Sq^2 will be present in $H^*(\mathbb{R}P^n/\mathbb{R}P^{n-b})$ as long as $b > 2$, and again we find that under this criterion the top cell can not split off. So $n+1 = 2^1 \cdot (\text{odd})$ implies splitting of the top cell can only happen if $b \leq 2^1$.

The same style of argument continues. If $n = 2^r e + (2^{r-1}-1)$ then there is a $\mathrm{Sq}^{2^{r-1}}$ hitting our class in degree n, and this obstructs the splitting of the top cell as long as $b > 2^{r-1}$. Rephrased, this says that if $n+1 = 2^{r-1}(2e+1)$ then the splitting does not exist when $b > 2^{r-1}$. Replacing $r-1$ with s, we have the desired result. □

Corollary 33.6. *If $n = 2^s \cdot odd$ then there are at most $2^s - 1$ independent vector fields on S^{n-1}.*

Proof. If there are $k-1$ vector fields on S^{n-1} then $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1}$ splits off its top cell for all $u \gg 0$. Choose a u that is odd, so that $un = 2^s \cdot \text{odd}$. By Proposition 33.5 we conclude that $k \leq 2^s$. □

The upper bounds provided by Corollary 33.6 agree with the Hurwitz-Radon lower bounds when $s \leq 3$. Note that these few cases cover quite a bit more than one first might think. For example, for spheres of dimension less than 50 it follows that the Hurwitz-Radon number of vector fields is the maximum possible for all but three cases, namely the spheres S^{15}, S^{31}, and S^{47} (multiples of 16 minus one). Corollary 33.6 yields that there exist at most 15 vector fields on S^{15}, 31 on S^{31}, and 15 on S^{47}, whereas the Hurwitz-Radon construction only gives 8 vector fields on S^{15} and S^{47}, and 9 vector fields on S^{31}. This demonstrates that our bounds for the spheres S^{16e-1} are still far away from our goal.

33.7. **Adams's Theorem and KO-theory (Step 8).** Next we move to the piece that finally cracked the proof, namely the following theorem of Adams [Ad2]:

Theorem 33.8. *Let $m \geq 1$. Then the space $\mathbb{R}P^{m+\rho(m)+1}/\mathbb{R}P^{m-1}$ does not split off its bottom cell in the stable homotopy category.*

Proof. Write $m = 2^u \cdot \text{odd}$, and consider $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ for $N \geq m$. Note that there is a Sq^{2^u} operation on $H^*(\mathbb{R}P^N/\mathbb{R}P^{m-1};\mathbb{Z}/2)$ connecting the generator in degree m to the generator in degree $m+2^u$. This proves that the bottom cell does not split off when $N \geq m+2^u$. This settles the theorem in the case $u \leq 3$, as here $\rho(m) = 2^u - 1$ and so $m + 2^u = m + \rho(m) + 1$.

We will next do a similar argument—but using K-theory—in the case $u \geq 4$. Actually, the K-theory argument only uses $u \geq 3$ so for the sake of pedagogy let us just make this weaker assumption.

Since $m \equiv 0 \bmod 4$ we know by Theorem 32.18 that there is a short exact sequence

(33.9)
$$\begin{array}{ccccccc} 0 \leftarrow \widetilde{KO}(\mathbb{R}P^m/\mathbb{R}P^{m-1}) & \leftarrow & \widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^{m-1}) & \leftarrow & \widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^m) \leftarrow 0 \\ \downarrow \cong & & & & \downarrow \cong \\ \mathbb{Z} & & & & \mathbb{Z}/(2^f) \end{array}$$

where f is a certain integer we will recall in a moment. Each of the groups has Adams operations on it, and the maps are compatible with these operations. If we let $\mathcal{B} = \mathbb{Z}[\psi^2, \psi^3, \psi^5, \ldots]$, then (33.9) is an exact sequence of $\mathcal{B}$-modules (note that $\mathcal{B}$ is just the monoid ring $\mathbb{Z}[\mathbb{N}]$ from Section 31). If $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell then this extension is split; so we will attempt to algebraically analyze when such a splitting exists.

As a $\mathcal{B}$-module the group $\widetilde{KO}(\mathbb{R}P^m/\mathbb{R}P^{m-1})$ is $\mathbb{Z}(\frac{m}{2})$, meaning that each ψ^k acts as multiplication by $k^{\frac{m}{2}}$. It will be convenient to set $r = \frac{m}{2}$. Also, we know by Theorem 32.18(a) that on $\widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^m)$ the operation ψ^k acts as zero when k is even, and the identity when k is odd. We also have determined the action of the ψ^k's on the middle group, but let us ignore this for the moment and consider the situation in a bit more generality.

Let A be any abelian group, and let $A[2]$ be the $\mathcal{B}$-module whose underlying abelian group is A and where ψ^k acts as zero when k is even, and the identity when k is odd. Consider an extension of $\mathcal{B}$-modules

$$0 \leftarrow \mathbb{Z}(r) \leftarrow E \leftarrow A[2] \leftarrow 0.$$

Let g be an element of E that maps onto a generator for $\mathbb{Z}(r)$. Then we can write $\psi^k g = k^r g + \alpha_k$ for unique elements $\alpha_k \in A$, and the relations $\psi^k\psi^l = \psi^l\psi^k$ show that we must have

$$(33.10)\qquad \begin{cases} k^r\alpha_l = (l^r-1)\alpha_k & \text{whenever } k \text{ is even and } l \text{ is odd,}\\ k^r\alpha_l = l^r\alpha_k & \text{whenever } k \text{ and } l \text{ are both even,}\\ (k^r-1)\alpha_l = (l^r-1)\alpha_k & \text{whenever } k \text{ and } l \text{ are both odd.}\end{cases}$$

So the extension E is determined by the elements $\alpha_k \in A$, for $k \geq 2$, satisfying the above equations.

To analyze when the sequence is split, let $\tilde{g}$ denote the image of g in $\mathbb{Z}(r)$. A splitting would send $\tilde{g}$ to an element $g + a$, for some $a \in A$. Since $\psi^k(\tilde{g}) = k^r\tilde{g}$ we find that

$$k^r(g+a) = \psi^k(g+a) = k^r g + \alpha_k + \begin{cases} 0 & \text{if } k \text{ is even,}\\ a & \text{if } k \text{ is odd.}\end{cases}$$

Rearranging to solve for α_k, we obtain

$$\alpha_k = \begin{cases} k^r a & \text{if } k \text{ is even,}\\ (k^r-1)a & \text{if } k \text{ is odd.}\end{cases}$$

So these are the properties of an (α_k) sequence that are equivalent to the extension $0 \leftarrow \mathbb{Z}(r) \leftarrow E \leftarrow A[2] \leftarrow 0$ being split. Notice that such sequences are in some sense the "trivial" solutions of the relations (33.10).

We make one more general comment before returning to our specific situation. Let $a \in A$ and consider the sequence defined by

$$(33.11)\qquad \alpha_k = \begin{cases} \frac{1}{2}k^r.a & k \text{ even}\\ \frac{1}{2}(k^r-1).a & k \text{ odd.}\end{cases}$$

Note that the fractions multiplying a are in fact integers. So this sequence defines a valid extension E, and if a is not a multiple of 2 in A then the extension doesn't "look" split. Precisely, if A is torsion-free and $a \notin 2A$, then the extension is clearly nonsplit. We will see that the case where A is torsion is a bit more subtle.

Now let us return to the extension in (33.9). Here $A = \mathbb{Z}/(2^f)$, where $f = \varphi(N) - \varphi(m)$. Since m is a multiple of 8 (and this is the first place where we use this assumption), f also equals $\varphi(N-m)$.

In Theorem 32.19 we previously computed the action of the Adams operations on $\widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^{m-1})$. The corresponding α-sequence is precisely the one given by (33.11), where a is a generator for $\mathbb{Z}/(2^f)$. So the extension is split if and only if there exists a $B \in \mathbb{Z}$ such that

$$\tfrac{1}{2}k^r.a = k^r.(Ba) \quad (k \text{ even}), \qquad \tfrac{1}{2}(k^r-1).a = (k^r-1).(Ba) \quad (k \text{ odd})$$

for all k. Phrased differently, these say that

$$2^f \text{ divides } \quad \begin{cases} \frac{1}{2}k^r(1-2B) & \text{if } k \text{ is even,}\\ \frac{1}{2}(k^r-1)(1-2B) & \text{if } k \text{ is odd.}\end{cases}$$

Since $1-2B$ is always odd, this is equivalent to

$$2^{f+1} \text{ divides } \quad \begin{cases} k^r & k \text{ even,}\\ k^r - 1 & k \text{ odd.}\end{cases}$$

Let $\nu(x)$ denote the 2-adic valuation of the integer x, and let us restate what we have now shown: If $8|m$ and $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell, then

$$\varphi(N-m)+1 \leq \min\bigl\{r,\ \nu(3^r-1),\ \nu(5^r-1),\ \nu(7^r-1),\ \nu(9^r-1),\ \dots\bigr\}$$

where $r = m/2$. Here we have used $\nu(2^r) = r$ and have also left out $\nu(k^r)$ for even integers $k > 2$, as these numbers are at least r and hence irrelevant for the minimum.

Our next task is to consider the numbers $\nu(3^r - 1)$, and for these we refer to Lemma 33.12 below. Since in our case r is even one has $\nu(3^r - 1) = \nu(r) + 2$. Note that as $r \geq 4$ this term is no larger than r, and so the first term in the above minimum is irrelevant.

We could proceed to analyze the terms $\nu(k^r - 1)$ for odd $k > 3$, which is not hard, but in fact we have done enough to conclude the proof already. We have shown that if $8|m$ and $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell, then

$$\varphi(N-m)+1 \leq \nu(r)+2 = \nu(m)+1,$$

or simply $\varphi(N-m) \leq \nu(m)$. So our task is to find the largest x for which $\varphi(x) = \nu(m)$. To do this, write $\nu(m) = 4b + a$ where $0 \leq a \leq 3$. Let "φ-count" stand for counting the integers that are congruent to 0, 1, 2, or 4 modulo 8. Every cycle of 8 consecutive integers contribules 4 to the φ-count, and so for $\varphi(x) \geq 4b$ we would need $x \geq 8b$. The cases $a = 0, 1, 2, 3$ can now be analyzed by hand: for $a = 0$ we have $x = 8b$; for $a = 1$ we have $x = 8b + 1$; for $a = 2$ we have $x = 8b + 3$; and for $a = 3$ we have $x = 8b + 7$. So in general the largest x such that $\varphi(x) = 4b + a$ is $x = 8b + 2^a - 1$.

Putting everything together, if $\varphi(N - m) \leq \nu(m) = 4b + a$ then $N - m \leq 8b + 2^a - 1 = \rho(m)$. So if $N \geq m + \rho(m) + 1$ then $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ cannot split off its bottom cell. □

Lemma 33.12. *If r is even then $\nu(3^r-1) = \nu(r)+2$. If r is odd then $\nu(3^r-1) = 1$.*

Proof. If r is odd then modulo 4 we have $3^r = (-1)^r = -1$, so $3^r - 1 \equiv 2 \bmod 4$. This proves that $\nu(3^r - 1) = 1$.

If $r = 2^f \cdot u$ where u is odd, we prove by induction on f that $\nu(3^r - 1) = f + 2$. The base case is $f = 1$, and here we use $3^r - 1 = 3^{2u} - 1 = (3^u - 1)(3^u + 1)$. We know $\nu(3^u - 1) = 1$ by the preceding paragraph. Modulo 4 one has $3^u + 1 = (-1)^u + 1 = 0$, but modulo 8 one has $3^u + 1 = 4$. So $\nu(3^u + 1) = 2$, which confirms that $\nu(3^r - 1) = 3$.

For the inductive step, if $r = 2^{f+1}u$ where u is odd then write

$$3^r - 1 = (3^{2^f u} - 1)(3^{2^f u} + 1).$$

By induction we know $\nu(3^{2^f u} - 1) = f + 2$. Modulo 4 we have $3^{2^f u} + 1 = \bigl((-1)^{2^f}\bigr)^u + 1 = 1 + 1 = 2$. So $\nu(3^{2^f u} + 1) = 1$, hence $\nu(3^r - 1) = f + 3$. □

Remark 33.13. It is intriguing that a number-theoretic analysis of $\nu(3^r - 1)$ was the ultimate step in both the Hopf invariant one problem and the vector fields on spheres problem. To my knowledge, there is no reason to suspect any connection between these two problems.

Exercise 33.14. If k is any odd number, prove that $\nu(k^r - 1) \geq \nu(r) + 2$ when r is even. This confirms that the terms for $k > 3$ were irrelevant for the minimum considered in the above proof.

Example 33.15. To demonstrate the proof of Adams's Theorem, consider $m = 576 = 2^6 \cdot 9$. Starting strictly above 576, we mark off numbers until we have exceeded a φ-count of $\nu(m) = 6$. In the following sequence, the numbers contributing to the φ-count have boxes around them:

$$\mathbf{576} \mid \boxed{577}, \boxed{578}, 579, \boxed{580}, 581, 582, 583, \boxed{585}, \boxed{585}, \boxed{586}, 587, \boxed{588}$$

Adams's argument shows that $\mathbb{R}P^{588}/\mathbb{R}P^{575}$ does not split off its bottom cell. Note, of course, that $\rho(576) = 11$ and $588 = 576 + 11 + 1$. The point, however, is that one does not need to remember the awkward formula for $\rho(m)$; the procedure is simply to count past m until the φ-count exceeds $\nu(m)$.

33.16. **Atiyah duality (Step 5).** This is the final piece. The material in this section will complete our proof of Theorem 33.1.

Consider the space $\mathbb{R}P^9/\mathbb{R}P^4$. Its cohomology is shown below in the diagram on the left:

We obtained the picture on the right by simply turning the left diagram upside down; is this also the cohomology of a space? It is easy to see that the answer is yes: the right diagram is $H^*(\mathbb{R}P^{10}/\mathbb{R}P^5)$. This turns out to be a general phenomenon, first discovered by Atiyah. And the kind of 'duality' we are seeing actually takes place at a deeper level than just that of cohomology. It is essentially a geometric duality, taking place inside of the stable homotopy category.

The stable homotopy category is symmetric monoidal: the monoidal product is the smash $E, F \mapsto E \wedge F$, and the unit is the sphere spectrum S. It is also *closed* symmetric monoidal, meaning that there exist function objects $E, F \mapsto \mathcal{F}(E, F)$ and a natural adjunction

$$\mathrm{Hom}_{\mathrm{Ho}\,(Sp)}(E, \mathcal{F}(X, Y)) \cong \mathrm{Hom}_{\mathrm{Ho}\,(Sp)}(E \wedge X, Y).$$

Spanier-Whitehead duality has to do with the functor $X \mapsto DX = \mathcal{F}(X, S)$. This functor preserves cofiber sequences and it sends the n-sphere $S^n = \Sigma^\infty S^n$ to the $(-n)$-sphere S^{-n}. So if a certain spectrum X is built from cells in dimensions 0 through n, the spectrum DX is built from cells in dimensions $-n$ through 0.

For nice enough spectra X one has the property that $D(D(X)) \simeq X$; such spectra are called **dualizable**. All finite cell complexes are dualizable. One can prove that $H^*(DX)$ agrees with 'taking $H^*(X)$ and turning it upside down'.

The essentials of Spanier-Whitehead duality were known long before the details of the stable homotopy category had all been worked out (particularly the details behind the smash product). Here is the main result for finite complexes:

Proposition 33.17. *Let X be a finite cell complex that is embedded in S^n as a subcomplex (of some chosen cell structure on S^n). Then*

$$\Sigma^{n-1} D(\Sigma^\infty X) \simeq \Sigma^\infty (S^n - X).$$

Example 33.18. Let us check the above proposition in some very easy examples. We use the form $D(X) \simeq \Sigma^{-n+1}(S^n - X)$.

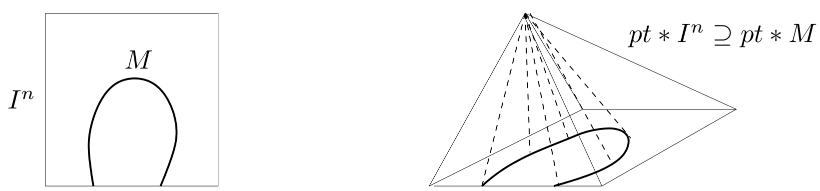
(a) $X = S^0$. Then $S^n - S^0 \simeq S^{n-1}$, and so we find $D(S^0) \simeq \Sigma^{-n+1}S^{n-1} \simeq S^0$. The 0-sphere is self-dual.
(b) Let $X = S^{n-1}$, embedded as the equator in S^n. The complement $S^n - X$ is S^0 (up to homotopy), so we have $D(S^{n-1}) \simeq \Sigma^{-n+1}S^0 = S^{-(n-1)}$. Again, this is as expected.

For some classical references on Spanier-Whitehead duality, see [A4, Chapter III.5] and [Swz, Chapter 14].

We can now state Atiyah's main theorem. The proof is taken directly from [At2]. See Section 15.12 for a discussion of Thom spaces of virtual bundles.

Theorem 33.19 (Atiyah Duality). *Let M be a compact, smooth manifold.*
(a) If M has boundary then $D(M/\partial M) \simeq \mathrm{Th}(-T_M)$ where $T_M \to M$ is the tangent bundle.
(b) Now assume that M is closed, and let $E \to M$ be a real vector bundle. Then
$$D(\mathrm{Th}\,E) \simeq \mathrm{Th}(-E - T_M).$$

Proof. For (a), embed M into I^n (nicely) in such a way that ∂M maps into $I^{n-1} \times \{0\}$. (See [At2] for details). Consider the join $pt * I^n$, which is a pyramid; its boundary is homeomorphic to S^n. Refer to the following picture for an example:



Consider the subcomplex
$$X = M \cup (pt * \partial M) \subseteq \partial(pt * I^n) \cong S^n.$$
We have $M/\partial M \simeq M \cup (pt * \partial M)$, and so
$$D(M/\partial M) \simeq D(X) \simeq \Sigma^{-n+1}(S^n - X).$$
Projection away from pt gives a deformation retraction $S^n - X \xrightarrow{\sim} I^n - M$. Next observe $I^n - M \simeq I^n - U$, where U is a tubular neighborhood of M in I^n. Finally, notice that since I^n is contractible we have that $I^n/(I^n - U)$ is a model for $\Sigma(I^n - U)$ (up to homotopy). Putting everything together, we have
$$\begin{aligned} D(M/\partial M) \simeq \Sigma^{-n+1}(I^n - U) \simeq \Sigma^{-n}(I^n/(I^n - U)) &= \Sigma^{-n}\,\mathrm{Th}(N_{I^n/M}) \\ &\simeq \mathrm{Th}(N_{I^n/M} - \underline{n}). \end{aligned}$$
Now use that $T_M \oplus N_{I^n/M} \cong \underline{n}$.

For (b) use that $\mathrm{Th}(E) \cong Di(E)/S(E)$, where $Di(E)$ is the disk bundle and $S(E)$ is the sphere bundle of E. The disk bundle is a compact manifold with boundary $S(E)$, so by (a) one has
$$D(\mathrm{Th}\,E) = D\big(Di(E)/S(E)\big) \simeq \mathrm{Th}(-T_{Di(E)}).$$

If $\pi\colon Di(E)\to M$ is the bundle map then it is easy to see that $T_{Di(E)}\cong \pi^*(E\oplus T_M)$. Since π is a homotopy equivalence,

$$\mathrm{Th}(-T_{Di(E)})=\mathrm{Th}(-\pi^*(E\oplus T_M))\simeq \mathrm{Th}(-(E\oplus T_M)).$$

This finishes the proof. □

We next apply what we just learned to stunted projective spaces. Recall from Example 15.9 that all stunted projective spaces are Thom spaces:

$$\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}\cong \mathrm{Th}\begin{pmatrix}(n-k)L\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}.$$

Recall as well that the tangent bundle to $\mathbb{R}P^{k-1}$ satisfies $T\oplus 1\cong kL$ (Example 23.10). Using these two facts, Atiyah Duality now gives that

$$D(\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1})\simeq \mathrm{Th}\begin{pmatrix}-(n-k)L-T\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}=\mathrm{Th}\begin{pmatrix}-nL+1\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}\simeq \Sigma\,\mathrm{Th}\begin{pmatrix}-nL\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}.$$

Let r_{k-1} be the additive order of $[L]-1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$. Then $r_{k-1}L\cong \underline{r_{k-1}}$ (stably), and hence for any $s\in\mathbb{Z}$ we have

$$\begin{aligned}\mathrm{Th}\begin{pmatrix}-nL\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}&\simeq \Sigma^{-sr_{k-1}}\,\mathrm{Th}\begin{pmatrix}-nL+sr_{k-1}\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}\simeq \Sigma^{-sr_{k-1}}\,\mathrm{Th}\begin{pmatrix}-nL+sr_{k-1}L\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}\\ &\simeq \Sigma^{-sr_{k-1}}\,\mathrm{Th}\begin{pmatrix}(sr_{k-1}-n)L\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}\\ &\simeq \Sigma^{-sr_{k-1}}\Big[\mathbb{R}P^{sr_{k-1}-n+k-1}/\mathbb{R}P^{sr_{k-1}-n-1}\Big].\end{aligned}$$

In the last line we imagine s chosen to be large enough so that $sr_{k-1}-n-1\geq 0$. Putting everything together, we have proven the following:

Corollary 33.20. *Let r_{k-1} be the additive order of $[L]-1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$. Then there is a stable homotopy equivalence*

$$D(\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1})\simeq \Sigma^{1-sr_{k-1}}\Big[\mathbb{R}P^{sr_{k-1}-n+k-1}/\mathbb{R}P^{sr_{k-1}-n-1}\Big]$$

where s is any integer such that $sr_{k-1}-n-1\geq 0$.

Example 33.21. Let us consider the Spanier-Whitehead dual of $\mathbb{R}P^9/\mathbb{R}P^4$, as in the beginning of this section. Relative to our above discussion, $n=10$ and $k=5$. By Theorem 32.14 we know $\widetilde{KO}(\mathbb{R}P^4)\cong\mathbb{Z}/8$, so the order of $[L]-1$ is 8. The above corollary gives

$$D(\mathbb{R}P^9/\mathbb{R}P^4)\simeq \Sigma^{1-8s}\Big[\mathbb{R}P^{8s-6}/\mathbb{R}P^{8s-11}\Big]$$

for any s where the right-hand-side makes sense. The smallest choice is $s=2$, giving $D(\mathbb{R}P^9/\mathbb{R}P^4)\simeq\Sigma^{-15}(\mathbb{R}P^{10}/\mathbb{R}P^5)$.

34. The immersion problem for $\mathbb{R}P^n$

Let M be a compact, n-dimensional, real manifold. It is a classical theorem of Whitney from the 1940s that M can be immersed in $\mathbb{R}^{2n-1}$ and embedded into $\mathbb{R}^{2n}$ [Wh1, Wh2]. A much more difficult result, proved by Cohen [C] in 1985, says that M can be immersed in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the binary expansion of n. As a general result this is known to be the best possible, but for specific choices of M one could conceivably do better. Define the **immersion dimension** (resp. the **embedding dimension**) of M to be the smallest k such that M immerses (resp., embeds) into $\mathbb{R}^k$.

In general, determining the immersion and embedding dimensions of a given manifold seem to be difficult problems. Over the last 60+ years they have been extensively studied, particularly for the manifold $\mathbb{R}P^n$. The problem tends to involve two distinct components. As one aspect, clever geometric constructions are used to produce immersions (or embeddings) and therefore upper bounds on the immersion dimension. For lower bounds one must prove *non*-immersion results, and this is usually done by making use of some sort of homotopical invariants. Over the years the problem for $\mathbb{R}P^n$ has been used as a sort of testing ground for every new homotopical technique to come along.

Our intent here is not to give a complete survey of this problem, as this would take far too long. We will be content to give a small taste, entirely concentrating our focus on some easily-obtained lower bounds in the case of $\mathbb{R}P^n$. The methods involve Stiefel-Whitney classes (in singular cohomology) and some related characteristic classes in KO-theory.

34.1. **A short survey.** Before jumping into our analysis, let us give some sense of what is known about the problem. The following table shows the current knowledge (as of January 2013) about the immersion and embedding dimensions for $\mathbb{R}P^n$ when $n \leq 24$:

Table 34.2. Immersion and embedding dimensions for $\mathbb{R}P^n$

$\mathbb{R}P^n$	2	3	4	5	6	7	8	9	10	11	12	13
imm. dim.	3	4	7	7	7	8	15	15	16	16	18	22
emb. dim.	4	5	8	9	$[9,11]$	$[9,12]$	16	17	17	18	$[18,21]$	$[22,23]$

$\mathbb{R}P^n$	14	15	16	17	18	19	20	21	22	23	24
imm.	22	22	31	31	32	32	34	38	38	38	$[38,39]$
emb.	$[22,23]$	$[23,24]$	32	33	33	$[33,34]$	$[34,37]$	39	39	39	$[39,42]$

In the table, entries in brackets are given when the exact answer is not known. For example, the embedding dimension of $\mathbb{R}P^6$ is only known to lie in the interval $[9,11]$. $\mathbb{R}P^6$ definitely embeds into $\mathbb{R}^{11}$ and does not embed into $\mathbb{R}^8$—but it is not known if $\mathbb{R}P^6$ embeds into $\mathbb{R}^9$ or $\mathbb{R}^{10}$. In comparison, we know much more about the immersion problem; the smallest unknown case is $\mathbb{R}P^{24}$.

The above data on the immersion and embedding dimensions was taken from a table compiled by Don Davis [Da]. Davis's table contains substantially more data, covering slightly past $\mathbb{R}P^{100}$.

One of the earliest results is due to Milnor: if $n = 2^r$ then the immersion dimension of $\mathbb{R}P^n$ equals $2n-1$ (showing that the Whitney upper bound is sharp in this case). Peterson proved that if $n = 2^r$ then the embedding dimension equals $2n$, again showing that the Whitney bound is sharp here. In general, if $n = 2^r + d$ for $0 \leq d < 2^r$ and d is relatively small, then one can expect the immersion and embedding dimensions to be $2^{r+1} + x$ where x is a known quantity or one that is tightly constrained. The following theorem encompasses most of what is known for $d \leq 10$:

Theorem 34.3. *Write $n = 2^i + d$ where $0 \leq d < 2^i$. Then the immersion dimension of $\mathbb{R}P^n$ equals $2^{i+1} + e$ and the embedding dimension equals $2^{i+1} + f$ where the following is known:*

d	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
i	≥ 1	≥ 2	≥ 3	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4
e	-1	-1	0	0	2	6	6	6	$[6, 7]$	14	14
f	0	1	1	$[1, 2]$	$[2, 5]$	7	7	7	$[7, 10]$	$[14, 15]$	$[14, 16]$

The above theorem is not credited because it represents the combined work over many years of a dozen authors. Much credit should be given to Davis, who has brought all the results together and given complete references. The above theorem is just the first few lines of the table [Da].

34.4. **Stiefel-Whitney techniques.** Suppose that M is a compact manifold of dimension n, and that M is immersed in $\mathbb{R}^{n+k}$. The immersion has a normal bundle ν, and there is an isomorphism of bundles $T_M \oplus \nu \cong \underline{n+k}$. Taking total Stiefel-Whitney classes of both sides gives

$$w(T_M) \cdot w(\nu) = w(T_M \oplus \nu) = w(\underline{n+k}) = 1.$$

Recall that the total Stiefel-Whitney class of a bundle E is $w(E) = 1 + w_1(E) + w_2(E) + \cdots$. Because the zero-coefficient is 1 we can formally invert this expression, and because $H^*(M)$ is zero in sufficiently large degrees this formal inverse actually makes sense as an element of $H^*(M)$. So we can feel free to write $w(E)^{-1}$, and we obtain

$$w(T_M)^{-1} = w(\nu).$$

We don't know anything about ν except its rank, which is equal to k. This guarantees that $w(\nu)$ does not have any terms of degree larger than k, and so we obtain the following simple result [MS, material preceding Theorem 4.8]:

Proposition 34.5. *Let M be a compact manifold of dimension n. If M immerses in $\mathbb{R}^{n+k}$ then $w(T_M)^{-1}$ vanishes in degrees larger than k.*

Let us apply this proposition to $\mathbb{R}P^n$. Here we have the identity $T_{\mathbb{R}P^n} \oplus 1 = (n+1)L$ (Example 23.10) and so

$$w(T_{\mathbb{R}P^n}) = w(T_{\mathbb{R}P^n} \oplus 1) = w((n+1)L) = w(L)^{n+1} = (1+x)^{n+1},$$

where x denotes the generator for $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. Taking inverses gives

$$w(T_{\mathbb{R}P^n})^{-1} = (1+x)^{-(n+1)} = \sum_{i=0}^{\infty} \binom{-(n+1)}{i} x^i.$$

We can rewrite the coefficient of x^i, since

$$\binom{-(n+1)}{i} = (-1)^i \frac{(n+1)\cdot(n+2)\cdots(n+i)}{i!} = (-1)^i \binom{n+i}{i}.$$

Putting everything together we obtain the following:

Corollary 34.6. *If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then $\binom{n+i}{i}$ is even for $k < i \le n$.*

The above corollary yields very concrete non-immersion results, but to obtain these we need to be good at checking when binomial coefficients are even. Historically, topologists got pretty good at this because of the presence of binomial coefficients in the Adem relations. The important result is the following:

Lemma 34.7. *Let $n_j n_{j-1} \dots n_0$ be the base 2 representation for n; that is, each $n_j \in \{0,1\}$ and $n = \sum n_j 2^j$. Similarly, let $k_j k_{j-1} \dots k_0$ be the base 2 representation for k. Then*

$$\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \mod 2.$$

Proof. The result follows easily from three points: (i) $\binom{2n}{i}$ is even if i is odd; (ii) $\binom{2n}{2k} \equiv \binom{n}{k}$ (mod 2); and (iii) $\binom{2n+1}{2k} \equiv \binom{2n}{2k}$ (mod 2). For (i) and (ii), imagine a column of the numbers 1 through n and a second "mirror" column containing the same entries. If i is odd, the i-element subsets of the two columns together may be partitioned into two classes: those which contain more elements from column A than column B, and those which contain less elements from column A. The operation of "switch entries between the two columns" gives a bijection between these two classes, thereby showing that $\binom{2n}{i}$ is even.

For (ii), note that the i-element subsets can be partitioned into groups determined by the number of elements from column A. This gives rise to the formula

$$\binom{2n}{2k} = \binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n}{k-1} + \cdots + \binom{n}{k-1}\binom{n}{1} + \binom{n}{k}\binom{n}{0}.$$

The terms on the right-hand-side are symmetric and so can be grouped in pairs, except for the middle term which is $\binom{n}{k}^2$. So working mod 2 we have

$$\binom{2n}{2k} \equiv \binom{n}{k}^2 \equiv \binom{n}{k}.$$

For (iii) just use Pascal's identity $\binom{2n+1}{2k} = \binom{2n}{2k} + \binom{2n}{2k-1}$ together with (i). □

Example 34.8. To determine if $\binom{20}{9}$ is even then we note that 20 is 10100 in base 2, and 9 is 1001. Using the above lemma we compute

$$\binom{20}{9} = \binom{1\,0\,1\,0\,0}{0\,1\,0\,0\,1} \equiv \binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{0}{1} = 1\cdot 0\cdot 1\cdot 1\cdot 0 = 0.$$

So $\binom{20}{9}$ is even.

Notice that $\binom{1}{1} = \binom{1}{0} = \binom{0}{0} = 1$, whereas $\binom{0}{1} = 0$. So in computations like the one above, the fnal answer is even if and only if $\binom{0}{1}$ appears at least once in the product—that is, if n has a certain bit turned "off" and the corresponding bit of k is "on". In particular, the following three statements are now obvious:

(i) If $n = 2^r$ then $\binom{n}{i}$ is even for all i in the range $0 < i < n$.
(ii) If $n = 2^r - 1$ then $\binom{n}{i}$ is odd for all $i \le n$.
(iii) $\binom{2n}{n}$ is always even.

The point for the first two is that 2^r has all of its bits turned off except for the rth, whereas 2^{r-1} has all of its bits turned on. For statement (iii) just consider the smallest bit of n that is turned on, and note that the corresponding bit is off in $2n$.

Corollary 34.6 is most often used in the form below. The proof is immediate from Corollary 34.6.

Corollary 34.9. *Fix $n \geq 2$, and let k be the largest integer such that $k \leq n$ and $\binom{n+k}{k}$ is odd. Then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{n+k-1}$.*

Note that k will be strictly less than n, as $\binom{2n}{n}$ is always even; this conforms with the Whitney immersion result. As an application of the above corollary, suppose we want to immerse $\mathbb{R}P^{10}$. We start with

$$\binom{20}{10} = \frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12\cdot 11}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

We have 2^{10} dividing the numerator and 2^8 dividing the denominator. Start removing factors from the left, one by one from the numerator and denominator simultaneously, watching what happens to the number of 2's in each. The fraction does not become odd until we are looking at $\binom{15}{5}$. So the conclusion is that $\mathbb{R}P^{10}$ does not immerse into $\mathbb{R}P^{14}$.

The above process is cumbersome, and with a little investigation it is not hard to produce a shortcut.

Proposition 34.10. *Write $n = 2^i + d$ where $0 \leq d < 2^i$. Then the largest k in the range $0 \leq k \leq n$ such that $\binom{n+k}{k}$ is odd is $k = 2^i - d - 1$.*

Proof. Note that if $j = 2^i - d - 1$ then $n + j = 2^{i+1} - 1$ and so $\binom{n+j}{j}$ is certainly odd. We must show that $\binom{n+j}{j}$ is even for j in the range $2^i - d \leq j \leq n$. This is the kind of analysis that is perhaps best left to the reader, but we will give a sketch. Suppose to the contrary that j is in this range and $\binom{n+j}{j}$ is odd. Let $e = j - (2^i - d - 1)$, so that $n + j = (2^i + d) + e + (2^i - d - 1) = e + (2^{i+1} - 1)$. Let the smallest bit of e that is turned on be the rth bit; this is also the smallest bit of $n + j$ that is turned off. Since $\binom{n+j}{n}$ is odd, this bit must be also off in n. If we write $e = e' + 2^r$, then $n + j = e' + (2^r + 2^{i+1} - 1)$. The term in parentheses has all bits off from the $(r + 1)$st through the ith, and so the bits of $n + j$ agree with the bits of e' (and of e) in this range. Since $\binom{n+j}{n}$ is odd, every bit of n that is turned on in this range must also be turned on in $n + j$—and therefore also in e. We have thus shown that

- The rth bit is off in n but on in e, and
- All bits greater than the rth that are on in n are also on in e.

These two facts show that $e > n$, which is not allowed since $e \leq j \leq n$. □

Corollary 34.11. *If $n = 2^i + d$ where $0 \leq d \leq 2^i - 1$ then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{2^{i+1}-2}$. In particular, if $n = 2^i$ then the immersion dimension of $\mathbb{R}P^n$ equals $2n - 1$.*

Proof. The first line is immediate from Corollary 34.9 and Proposition 34.10. The second statement follows from the first together with the Whitney theorem saying that $\mathbb{R}P^n$ immerses into $\mathbb{R}^{2n-1}$. □

Let us now change gears just a bit and consider embeddings. We can also use characteristic classes to give obstructions in this setting. The key result is the following:

Proposition 34.12. *Suppose M is a compact n-manifold that is embedded in $\mathbb{R}^{n+k}$. Then $w_k(\nu) = 0$ where ν is the normal bundle.*

Proof. Choose a metric on ν and let $S(\nu)$ be the sphere bundle. If $p\colon S(\nu) \to M$ denotes the projection map, then clearly $p^*\nu$ splits off a trivial bundle: $p^*\nu = \underline{1} \oplus E$ for some rank $k-1$ bundle E on $S(\nu)$. From this it immediately follows that $0 = w_k(p^*\nu) = p^*(w_k(\nu))$.

The proof will be completed by showing that $p^*\colon H^*(M;\mathbb{Z}/2) \to H^*(S(\nu);\mathbb{Z}/2)$ is injective. Let U be a tubular neighborhood of M in $\mathbb{R}^{n+k}$, arranged so that its closure $\overline{U}$ is homeomorphic to the disk bundle of ν. Write $\partial\overline{U}$ for the boundary, which is isomorphic to $S(\nu)$. We have the long exact sequence

$$\cdots \to H^i(\overline{U}, \partial\overline{U}) \to H^i(\overline{U}) \to H^i(\partial\overline{U}) \to \cdots$$

where all cohomology groups have $\mathbb{Z}/2$-coefficients. The projection $\overline{U} \to M$ is a homotopy equivalence, so our map p^* is isomorphic to $H^i(\overline{U}) \to H^i(\partial\overline{U})$. We can verify that this is injective by checking that the previous map in the long exact sequence is zero. We look only at $i > 0$, as the $i = 0$ case is trivial.

To this end, consider the diagram below:

$$\begin{array}{ccc} H^i(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} - M) & \longrightarrow & H^i(\mathbb{R}^{n+k}) \\ \cong \big\downarrow & & \big\downarrow \\ H^i(\overline{U}, \partial\overline{U}) & \longrightarrow & H^i(\overline{U}) \end{array}$$

The left vertical map is an isomorphism by excision, and the group in the upper right corner is zero. So the bottom horizontal map is zero, as we desired. □

Compare the next result to Proposition 34.5:

Corollary 34.13. *Let M be a compact n-manifold. If M embeds into $\mathbb{R}^{n+k}$ then $w(T_M)^{-1}$ vanishes in degrees k and larger.*

Proof. We saw in the proof of Proposition 34.5 that $w(T_M)^{-1} = w(\nu)$. Since ν has rank k, this forced the Stiefel-Whitney classes to vanish in degrees larger than k. But now Proposition 34.12 also gives us the vanishing in degree k. □

Corollary 34.14. *If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $\binom{n+j}{j}$ is even for $k \le j \le n$.*

Proof. Same as for Corollary 34.6. □

Corollary 34.15. *Let $n = 2^i + d$ where $0 \le d \le 2^i - 1$. Then $\mathbb{R}P^n$ does not embed into $\mathbb{R}^{2^{i+1}-1}$. In particular, if $n = 2^i$ then the embedding dimension of $\mathbb{R}P^n$ equals $2n$.*

Proof. Let k be the largest integer in the range $0 \le k \le n$ such that $\binom{n+k}{k}$ is odd. Then Corollary 34.14 shows that $\mathbb{R}P^n$ does not embed into $\mathbb{R}^{n+k}$. But Proposition 34.10 identifies $k = 2^i - d - 1$, and so $n + k = 2^{i+1} - 1$. This proves the first statement. The second statement is then a consequence of the first together with the Whitney theorem that $\mathbb{R}P^n$ embeds into $\mathbb{R}^{2n}$. □

To gauge the relative strength of Corollaries 34.11 and 34.15, see the table below and compare to Table 34.2. One gets a clear sense of how far algebraic topology has progressed since the early days of Stiefel-Whitney classes!

Table 34.16. Stiefel-Whitney lower bounds for the immersion and embedding dimensions of $\mathbb{R}P^n$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
imm$\geq$	3	3	7	7	7	7	15	15	15	15	15	15	15	15	15	15	15
emb$\geq$	4	4	8	8	8	8	16	16	16	16	16	16	16	16	16	16	16

34.17. **K-theoretic techniques.** One can readily imagine taking the basic approach from the last section and replacing the Stiefel-Whitney classes with characteristic classes taking values in some other cohomology theory. Atiyah [At3] pursued this idea using KO-theory, and certain constructions of Grothendieck provided the appropriate theory of characteristic classes. In this way he obtained some new non-immersion and non-embedding theorems. We describe this work next.

Earlier in these notes (????) we described the construction of the γ-operations in complex K-theory. The same formulas work just as well for KO-theory. Explicitly, for an element $x \in \widetilde{KO}(X)$ define

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = 1 + \tfrac{t}{1-t}[\lambda^1 x] + \left(\tfrac{t}{1-t}\right)^2[\lambda^2 x] + \cdots$$

Note that $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$. Define $\gamma^i(x)$ to be the coefficient of t^i in $\gamma_t(x)$. So we have

$$\gamma_k(x+y) = \sum_{i+j=k} \gamma^i(x)\gamma^j(y).$$

Note also that $\gamma_t(1) = 1 + \frac{t}{1-t} = \frac{1}{1-t} = 1 + t + t^2 + \cdots$

For a vector bundle E over X define

$$\tilde{\gamma}_t(E) = \gamma_t(E - \underline{\operatorname{rank} E}) = \frac{\gamma_t(E)}{\gamma_t(\underline{\operatorname{rank} E})} = \frac{\gamma_t(E)}{\gamma_t(1)^{\operatorname{rank} E}} = \gamma_t(E)\cdot(1-t)^{\operatorname{rank} E}.$$

One should think of this as just being a renormalization of the γ_t construction; note that $\tilde{\gamma}_t(E) = 1$ if E is a trivial bundle. Observe that we still have the analog of the Whitney formula:

$$\begin{aligned}\tilde{\gamma}_t(E \oplus F) = \gamma_t(E \oplus F - \underline{\operatorname{rank}(E+F)}) &= \gamma_t\big((E - \underline{\operatorname{rank} E}) + (F - \underline{\operatorname{rank} F})\big)\\ &= \gamma_t(E - \underline{\operatorname{rank} E})\gamma_t(F - \underline{\operatorname{rank} F})\\ &= \tilde{\gamma}_t(E)\tilde{\gamma}_t(F).\end{aligned}$$

If L is a line bundle then

$$\tilde{\gamma}_t(L) = \gamma_t(L)\cdot(1-t) = \big[1 + \tfrac{t}{1-t}[L]\big]\cdot(1-t) = 1 - t + t[L] = 1 + t([L]-1).$$

So $\tilde{\gamma}_1(L) = [L] - 1$ and $\tilde{\gamma}_i(L) = 0$ for $i > 1$.

Finally, we observe that if E is a rank k bundle then $\tilde{\gamma}_i(E) = 0$ for all $i > k$. This is because

$$\tilde{\gamma}_t(E) = \lambda_{\frac{t}{1-t}}(E)(1-t)^k = \sum_{i=0}^{\infty} (\tfrac{t}{1-t})^i[\Lambda^i E]\cdot(1-t)^k = \sum_{i=0}^{k} t^i(1-t)^{k-i}[\Lambda^i E].$$

Clearly the final expression is a polynomial in t of degree at most k.

Compare the following result to Proposition 34.5 and Corollary 34.13.

Proposition 34.18. *Let M be a compact n-manifold. If M immerses into $\mathbb{R}^{n+k}$ then the power series $\tilde{\gamma}_t(T_M)^{-1}$ vanishes in degrees larger than k. If M embeds into $\mathbb{R}^{n+k}$ then $\tilde{\gamma}_t(T_M)^{-1}$ vanishes in degrees k and larger.*

Proof. The proofs are the same as before. If M immerses into $\mathbb{R}^{n+k}$ then $T_M \oplus \nu \cong \underline{n+k}$ where ν is the normal bundle. So $\tilde{\gamma}_t(T_M)\tilde{\gamma}_t(\nu) = \tilde{\gamma}_t(T_M \oplus \nu) = \tilde{\gamma}_t(\underline{n+k}) = 1$, and so $\tilde{\gamma}_t(T_M)^{-1} = \tilde{\gamma}_t(\nu)$. But since ν has rank k we have $\tilde{\gamma}_i(\nu) = 0$ for $i > k$.

For the second part of the proposition we need to prove that if M embeds into $\mathbb{R}^{n+k}$ then $\tilde{\gamma}_k(\nu) = 0$. The proof is exactly the same as for Proposition 34.12. □

For the following proposition, recall that $\varphi(n)$ denotes the number of integers s such that $0 < s \leq n$ and s is congruent to 0, 1, 2, or 4 modulo 8.

Corollary 34.19 (Atiyah)**.**
(a) If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then $2^{\varphi(n)-j+1}$ divides $\binom{n+j}{j}$ for $k < j \leq \varphi(n)$.
(b) If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $2^{\varphi(n)-j+1}$ divides $\binom{n+j}{j}$ for $k \leq j \leq \varphi(n)$.

Proof. Recall that $1 \oplus T_{\mathbb{R}P^n} \cong (n+1)L$, as in Example 23.10. We get

$$\tilde{\gamma}_t(T_{\mathbb{R}P^n}) = \tilde{\gamma}_t(T_{\mathbb{R}P^n} \oplus 1) = \tilde{\gamma}_t((n+1)L) = \tilde{\gamma}_t(L)^{n+1} = \big(1 + t\lambda\big)^{n+1}$$

where $\lambda = [L] - 1 \in \widetilde{KO}^0(\mathbb{R}P^n)$. So

$$\tilde{\gamma}_t(T_{\mathbb{R}P^n})^{-1} = (1+t\lambda)^{-(n+1)} = \sum_{j=0}^{\infty} \binom{-(n+1)}{j} \lambda^j \cdot t^j = \sum_{j=0}^{\infty} (-1)^j \binom{n+j}{j} \lambda^j \cdot t^j.$$

If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then by Proposition 34.18 $\binom{n+j}{j}\lambda^j = 0$ in $\widetilde{KO}^0(\mathbb{R}P^n)$ for all $k < j$. If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $\binom{n+j}{j}\lambda^j = 0$ for all $k \leq j$.

Now we recall from Theorem 32.14 that $\widetilde{KO}^0(\mathbb{R}P^n) \cong \mathbb{Z}/(2^{\varphi(n)})$ and that λ is a generator. Also recall that $\lambda^2 = -2\lambda$, or $\lambda^j = (-2)^{j-1}\lambda$. The desired conclusions follow immediately. □

Corollary 34.19 is best used in the following form. Let $\sigma(n)$ denote the largest value of j in the range $1 \leq j \leq \varphi(n)$ for which $\binom{n+j}{j}$ is not divisible by $2^{\varphi(n)+1-j}$; if no such j exists then set $\sigma(n) = 0$ by default. Then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{n+\sigma(n)-1}$ and does not embed into $\mathbb{R}^{n+\sigma(n)}$.

For some values of n the result of Corollary 34.19 is stronger than what we obtained from Corollaries 34.6 and 34.15, and for some values of n it is weaker. We demonstrate some examples:

Example 34.20. For the question of immersions of $\mathbb{R}P^8$, we have $\varphi(8) = 4$ and $\sigma(8) = 4$. So Atiyah's result (34.19) gives that $\mathbb{R}P^8$ does not immerse into $\mathbb{R}^{11}$. The Stiefel-Whitney classes, however, told us that $\mathbb{R}P^8$ does not immerse into $\mathbb{R}^{14}$.

In contrast, for $\mathbb{R}P^{15}$ we have $\varphi(15) = 7$ and $\sigma(15) = 4$. So Atiyah's result tells us that $\mathbb{R}P^{15}$ does not immerse into $\mathbb{R}^{18}$. The method of Stiefel-Whitney classes (34.6) gives no information in this case.

The table below shows the lower bounds for the immersion dimension of $\mathbb{R}P^n$ obtained from Stiefel-Whitney techniques versus the KO-theoretic techniques. The reader will notice that the Stiefel-Whitney bounds are significantly better when $n = 2^i + d$ and d is small, whereas the KO-theoretic bounds are better for $n = 2^i + d$ when d is close to (but not exceeding) $2^i - 1$.

Table 34.21. Lower bounds for the immersion dimension of $\mathbb{R}P^n$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
S-W	3	3	7	7	7	7	15	15	15	15	15	15	15	15	31	31	31
KO	3	3	7	7	7	7	12	13	15	15	17	17	19	19	24	25	27

n	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
S-W	31	31	31	31	31	31	31	31	31	31	31	31	31	63	63
KO	27	31	31	31	31	34	35	38	39	40	41	42	43	48	49

As an example when n is much larger, the Stiefel-Whitney classes give no information on the immersion dimension of $\mathbb{R}P^{255}$. By contrast, the Atiyah result gives that the immersion dimension is at least 355. For $n = 2^i - 1$ the improvement of the Atiyah bound over the Hopf bound is on the order of $\frac{n}{2}$.

34.22. **Immersions and geometric dimension.** So far we have used various characteristic classes to give lower bounds on the immersion/embedding dimensions for $\mathbb{R}P^n$. To close this section we will show how to produce upper bounds for the *immersion* dimension, via a geometric result of Hirsch that translates this into a bundle-theoretic problem. The central tool again ends up being KO-theory. Using these methods we will completely determine the immersion dimension of $\mathbb{R}P^n$ for $n \leq 9$.

Let $E \to X$ be a vector bundle. Define the **virtual dimension** of E by the formula

$$\mathrm{v.}\dim E = \min\{k \,|\, E \cong F \oplus \underline{(\operatorname{rank} E) - k} \text{ for some bundle } F \text{ of rank } k\}.$$

We might say that the virtual dimension is the smallest k such that E is isomorphic to a stablized rank k bundle. If X is compact then we also have

$$\mathrm{v.}\dim E = \operatorname{rank} E - \max\{j \,|\, E \text{ has } j \text{ independent sections}\}.$$

Note that the virtual dimensions of E and $E \oplus \underline{1}$ might be different; the latter might be smaller than the former. With this in mind we can also introduce the **stable virtual dimension**:

$$\begin{aligned}\mathrm{sv.}\dim E &= \min\{\mathrm{v.}\dim(E \oplus \underline{r}) \,|\, r \geq 0\} \\ &= \min\{\operatorname{rank} F \,|\, F \text{ is a bundle that is stably equivalent to } E\}.\end{aligned}$$

Finally, we introduce the following related concept. For $\alpha \in \widetilde{KO}^0(X)$, define the **geometric dimension** of α to be

$$\mathrm{g.}\dim \alpha = \min\{k \,\big|\, \alpha + \underline{k} = [F] \text{ for some vector bundle } F \text{ on } X\}.$$

The above three concepts are related as follows:

Proposition 34.23. *Let $E \to X$ be a vector bundle, where X is compact.*
(a) $\mathrm{g.}\dim(E - \underline{\operatorname{rank} E}) = \mathrm{sv.}\dim(E) \leq \mathrm{v.}\dim(E)$.
(b) If X is compact and $\operatorname{rank} E > \dim X$ *then* $\mathrm{sv.}\dim(E) = \mathrm{v.}\dim(E)$.

Proof. For (a) only the first equality requires proof. This equality is almost a tautology: for any integer $d \geq 0$ we have

$$\begin{aligned} \operatorname{sv.dim}(E) \leq d &\iff \text{there exists an } F \text{ of rank } d \text{ such that } E \cong_{st} F \\ &\iff \text{there exists an } F \text{ of rank } d \text{ such that } [E] - \underline{\operatorname{rank} E} = [F] - d \\ &\iff \text{there exists a bundle } F \text{ such that } [E] - \underline{\operatorname{rank} E} + d = [F] \\ &\iff \operatorname{g.dim}(E - \underline{\operatorname{rank} E}) \leq d. \end{aligned}$$

The desired equality follows immediately.

For (b), let $r = \operatorname{sv.dim}(E)$, $k = \operatorname{rank}(E)$, and note that $r \leq k$. Then there exists a rank r bundle F such that E and F are stably isomorphic: $E \oplus \underline{N} \cong F \oplus \underline{(N + k - r)}$ for some $N > 0$. Since $\operatorname{rank}(E) > \dim X$ we can cancel the $\underline{N}$ factors to get $E \cong F \oplus \underline{(k - r)}$, by Proposition 11.10. Hence $\operatorname{v.dim}(E) \leq \operatorname{rank}(F) = r = \operatorname{sv.dim}(E)$. □

The following result of Hirsch [Hi], and its corollary, translate the immersion problem into a purely homotopy-theoretic question. This is the key to why immersions are better understood than embeddings.

Theorem 34.24 (Hirsch)**.** *Let M be a compact manifold of dimension n. For $k \geq 1$ the following statements are equivalent:*

(a) M can be immersed in $\mathbb{R}^{n+k}$
(b) There exists a bundle F of rank k such that $T_M \oplus F$ is trivial.
(c) There exists an O_n-equivariant map $\operatorname{Fr}(T_M) \to V_n(\mathbb{R}^{n+k})$, where $\operatorname{Fr}(T_M)$ is the bundle of n-frames in T_M.

Observe that (a) implies (b) by taking F to be the normal bundle of the immersion. Also, if $\phi\colon T_M \oplus F \to \underline{n+k}$ is an isomorphism then any n-frame in T_M yields an n-frame in $\mathbb{R}^{n+k}$ by applying ϕ; thus, one gets an equivariant map $\operatorname{Fr}(T_M) \to V_n(\mathbb{R}^{n+k})$. This shows (b) implies (c). So the content of the above theorem is really in (c)⇒(a); this is what was proven by Hirsch, via geometric arguments [Hi, Theorem 6.1 (taking $r = 0$ there)]. He actually showed much more, essentially proving that homotopy classes of immersions from M to $\mathbb{R}^{n+k}$ are in bijective correspondence with equivariant homotopy classes of maps $\operatorname{Fr}(T_M) \to V_n(\mathbb{R}^{n+k})$. We will not give Hirsch's proof here, but we will use the following corollary of his result. This corollary first appeared in [At3, Proposition 3.2] and in [Sa1, Theorem 2.1].

Corollary 34.25. *Let $k \geq 1$, and let M be a compact manifold of dimension n. Then M immerses in $\mathbb{R}^{n+k}$ if and only if* $\operatorname{g.dim}(n - T_M) \leq k$.

Proof. We have already seen the 'only if' direction when we obtained obstructions to immersions: if an immersion exists then $\underline{n+k} \cong T_M \oplus \nu$ where ν is the normal bundle, therefore $n - T_M = \nu - k$ and hence $\operatorname{g.dim}(n - T_M) = \operatorname{g.dim}(\nu - k) \leq k$.

For the other direction, assume $\operatorname{g.dim}(n - T_M) \leq k$. So there exists a rank k vector bundle F such that $n - T_M + k = F$ in $KO(M)$. This implies $n + k = T_M + F$ in $KO(M)$, which in turn yields that $\underline{n+k+N} \cong T_M \oplus F \oplus \underline{N}$ for some $N \geq 0$. Since the rank of $T_M \oplus F$ is larger than $\dim M$, it follows by Proposition 11.10 that we can cancel the $\underline{N}$ on both sides to get $\underline{n+k} \cong T_M \oplus F$. Then by Theorem 34.24 we know that M immerses into $\mathbb{R}^{n+k}$. □

Remark 34.26. Note in particular that if M is parallelizable then M immerses into $\mathbb{R}^{n+1}$ (taking $k = 1$ in Corollary 34.25, since $k = 0$ is not allowed).

We now specialize again to the case of $M = \mathbb{R}P^n$. Here we have

$$n - T_{\mathbb{R}P^n} = (n+1) - (1 \oplus T_{\mathbb{R}P^n}) = (n+1) - (n+1)L = (n+1)(1-L).$$

Proposition 34.27. *For $n \leq 8$ the immersion dimension of $\mathbb{R}P^n$ is as given in Table 34.2.*

Proof. Given the lower bounds given by Stiefel-Whitney classes (see Table 34.21), we only have to demonstrate the required immersions. The fact that $\mathbb{R}P^8$ immerses in $\mathbb{R}^{15}$ is a special case of Whitney's classical theorem. Both $\mathbb{R}P^3$ and $\mathbb{R}P^7$ have trivial tangent bundles, and so by Remark 34.26 they immerse into $\mathbb{R}^4$ and $\mathbb{R}^8$, respectively.

For $\mathbb{R}P^6$ we must calculate the geometric dimension of $7(1-L) = -7(L-1)$. But $\varphi(6) = 3$, and so $\widetilde{KO}(\mathbb{R}P^6) \cong \mathbb{Z}/8$. Hence $8(L-1) = 0$, and so $-7(L-1) = L-1$. The geometric dimension of $L-1$ is clearly at most 1, and so by Corollary 34.25 $\mathbb{R}P^6$ immerses into $\mathbb{R}^7$.

A similar argument works to show that $\mathbb{R}P^2$ immerses into $\mathbb{R}^3$ (or one can just construct the immersion geometrically). □

The reader should note why the above result stopped with $\mathbb{R}P^8$. For $\mathbb{R}P^9$ one finds that the immersion problem boils down to determining the geometric dimension of $-10(L-1) = 22(L-1)$ (here we used that $\varphi(9) = 5$ and so $\widetilde{KO}^0(\mathbb{R}P^9) = \mathbb{Z}/32$). The precise value of this geometric dimension is far from clear. We will close this section by analyzing it completely, following Sanderson [Sa1]. However, we take a short detour to illustrate some general principles.

Recall that for general n we have $n - T_{\mathbb{R}P^n} = (n+1)(1-L)$. We would like to interpret the geometric dimension of this class as being a stable virtual dimension, but for this we would need to be looking at a positive multiple of $L-1$ rather than $1-L$. There are two ways to get ourselves into this position. The first, which we have already seen, proceeds by recalling that $L-1$ has order $2^{\varphi(n)}$ in $\widetilde{KO}(\mathbb{R}P^n)$. So we can write

$$n - T_{\mathbb{R}P^n} = -(n+1)(L-1) = \big(2^{\varphi(n)} - (n+1)\big)(L-1)$$

and hence

$$\begin{aligned} \operatorname{g.dim}(n - T_{\mathbb{R}P^n}) &= \operatorname{g.dim}\Big(\big[2^{\varphi(n)} - (n+1)\big]L - \big[2^{\varphi(n)} - (n+1)\big]\Big) \\ &= \operatorname{sv.dim}\Big(\big[2^{\varphi(n)} - (n+1)\big]L\Big). \end{aligned}$$

The second approach is from Sanderson [Sa2, Lemma 2.2]:

Proposition 34.28. *For the bundle $L \to \mathbb{R}P^n$, the statement* $\operatorname{g.dim}(a(L-1)) \leq b$ *is equivalent to* $\operatorname{g.dim}((b-a)(L-1)) \leq b$*, for any $a, b \in \mathbb{Z}$ with $b \geq 0$.*

Proof. It suffices to prove the implication in one direction, by symmetry. So suppose $\operatorname{g.dim}(a(L-1)) \leq b$. This implies that $a(L-1) + b = E$ in $KO^0(\mathbb{R}P^n)$, for some rank b bundle E. Multiply by L to get $a(1-L) + bL = E \otimes L$, and then rearrange to find $(b-a)(L-1) + b = E \otimes L$. This yields that $\operatorname{g.dim}((b-a)(L-1)) \leq b$. □

Corollary 34.29. *Let $L \to \mathbb{R}P^n$ be the tautological bundle. For $k > 0$ the following statements are equivalent:*

(1) $\mathbb{R}P^n$ *immerses into* $\mathbb{R}^{n+k}$,

(2) $\operatorname{g.dim}(-(n+1)(L-1)) \leq k$,

(3) $\operatorname{g.dim}((n+k+1)(L-1)) \le k$,
(4) $\operatorname{g.dim}\big([2^{\varphi(n)}-(n+1)]\,(L-1)\big) \le k$.

Proof. The equivalence (1) $\iff$ (2) comes from Corollary 34.25, and (2) $\iff$ (3) is by Proposition 34.28. Finally, (2) $\iff$ (4) is true because $2^{\varphi(n)}(L-1)=0$ in $\widetilde{KO}^0(\mathbb{R}P^n)$. $\square$

Part (3) of the above result, which is the part that comes from Proposition 34.28, will not be needed in the remainder of this section. But we record it here for later use.

We close this section by settling the immersion problem for $\mathbb{R}P^9$:

Proposition 34.30 (Sanderson)**.** *The immersion dimension of* $\mathbb{R}P^9$ *equals* 15.

This result is from [Sa1, Theorem 5.3]. Sanderson proves much more than this, for example that $\mathbb{R}P^n$ immerses into $\mathbb{R}^{2n-3}$ whenever n is odd. He also proved that $\mathbb{R}P^{11}$ immerses into $\mathbb{R}^{16}$, which ends up being the best result for both $\mathbb{R}P^{11}$ and $\mathbb{R}P^{10}$.

Proof. The lower bound of 15 is given by Stiefel-Whitney classes, as in Table 34.21. So we only need to prove that $\mathbb{R}P^9$ immerses into $\mathbb{R}^{15}$. By Corollary 34.29 this is equivalent to $\operatorname{g.dim}(22(L-1)) \le 6$, and is also equivalent to $\operatorname{g.dim}(16(L-1)) \le 16$. The proof below works for both statements, but for specificity we just prove the former. Note that what we must prove is equivalent to $\operatorname{sv.dim}(22L) \le 6$, by Proposition 34.23. We will outline the steps for this, and then give more details afterwards.
Step 1: There exists a rank 4 complex bundle E on $\mathbb{R}P^9$ such that $r_{\mathbb{R}}E$ is stably equivalent to $22L$ (recall that $r_{\mathbb{R}}E$ denotes the real bundle obtained from E by forgetting the complex structure).
Step 2: The bundle $E|_{\mathbb{R}P^8}$ has a nonzero section s.
Step 3: The bundle $r_{\mathbb{R}}E|_{\mathbb{R}P^8}$ has a field of (real) 2-frames.
Step 4: For $N \gg 0$ the field of real $(2+N)$-frames of $(r_{\mathbb{R}}E \oplus N)|_{\mathbb{R}P^8}$ may be extended over $\mathbb{R}P^9$.
Step 5: $\operatorname{sv.dim}(22L) = \operatorname{sv.dim}(r_{\mathbb{R}}E) \le 6$, hence $\mathbb{R}P^9$ can be immersed into $\mathbb{R}^{15}$.

We now justify each of these steps. For step 1 use that $r_{\mathbb{R}}\colon \widetilde{K}(\mathbb{R}P^n) \to \widetilde{KO}(\mathbb{R}P^n)$ has image equal to $\langle 2(L-1)\rangle$, by Theorems 32.14 and 32.17. Since 22 is even, there is a complex bundle E on $\mathbb{R}P^9$ such that $r_{\mathbb{R}}E$ is stably equivalent to $22L$. The bundle E is represented by a map $\mathbb{R}P^9 \to BU$, and such a map necessarily factors up to homotopy through $BU(4)$: for this, use obstruction theory and the homotopy fiber sequences $S^{2n-1} \to BU(n-1) \to BU(n)$. For a map $\mathbb{R}P^9 \to BU(n)$ to lift (up to homotopy) into $BU(n-1)$, one has obstructions in the groups $H^i(\mathbb{R}P^9;\pi_{i-1}S^{2n-1})$ for $0 \le i \le 9$. But as long as $5 \le n$ the homotopy groups $\pi_{i-1}S^{2n-1}$ vanish in this range, so all the obstruction groups are zero.

For step 2 we again proceed by obstruction theory. We have the sphere bundle $S(E|_{\mathbb{R}P^8}) \to \mathbb{R}P^8$ with fiber S^7, and all the obstruction groups vanish except for the last one: $H^8(\mathbb{R}P^8;\pi_7S^7)$. The coefficents are untwisted because the complex structure gives a canonical orientation to each fiber. This final obstruction class is the same as the Euler class of $r_{\mathbb{R}}E$, or equivalently the top Chern class of E. Since $H^8(\mathbb{R}P^8;\mathbb{Z}) = \mathbb{Z}/2$ it will be sufficient to compute the mod 2 reduction of this class, which is the top Stiefel-Whitney class $w_8(r_{\mathbb{R}}E)$. Now we use that $r_{\mathbb{R}}E$ is stably isomorphic to $22L$, so the total Stiefel-Whitney class is $w(r_{\mathbb{R}}E) = w(22L) =$

$w(L)^{22} = (1+x)^{22}$. Hence $w_8(r_{\mathbb{R}}E) = \binom{22}{8}x^8$. Since $\binom{22}{8}$ is even, the obstruction class vanishes and we indeed have a nonzero section.

Step 3 is trivial: the sections s and is give the field of real 2-frames.

Step 4 is obstruction theory yet again. If $F = r_{\mathbb{R}}E \oplus \underline{N}$ then we have a (partial) section of $V_{N+2}(F) \to \mathbb{R}P^9$ defined over $\mathbb{R}P^8$, and we must extend this to all of $\mathbb{R}P^9$. The obstruction lies in $H^9(\mathbb{R}P^9; \pi_8(V_{N+2}(\mathbb{R}^{N+8})))$. But the homotopy group in the coefficients is known to be zero for large enough N, by ????.

Step 5 is now immediate: $r_{\mathbb{R}}E \oplus N$ has rank $8+N$ and has $2+N$ independent sections, hence $\operatorname{v.dim}(r_{\mathbb{R}}E \oplus N) \leq 6$. So $\operatorname{sv.dim}(r_{\mathbb{R}}E) \leq 6$ as well. Since $r_{\mathbb{R}}E$ is stably equivalent to $22L$, $\operatorname{sv.dim}(22L) = \operatorname{sv.dim}(r_{\mathbb{R}}E) \leq 6$. $\square$

The arguments in this section naturally suggest the following problem, which is open:

Problem: Given n and k, compute the geometric dimension of $k([L]-1) \in \widetilde{KO}(\mathbb{R}P^n)$.

Over the years this problem has been extensively studied by Adams, Davis, Gitler, Lam, Mahowald, Randall, and many others. See Section 35.14 for a bit more discussion.

34.31. **Summary.** In this section we obtained two sets of non-immersion/non-embedding results, one using Stiefel-Whitney classes in mod 2 singular cohomology and the other using the $\tilde{\gamma}$ classes in KO-theory. We also translated the immersion problem into a question about geometric dimension of reduced bundles, and for $\mathbb{R}P^n$ we completely solved this question for $n \leq 9$. As we have said before, this is far from the whole story—in fact it is just the very tip of a large and interesting iceberg. We refer the reader to the references cited in [Da] for other pieces of the story.

35. The sums-of-squares problem and beyond

This section is in some ways an epilogue to the previous one. In the last section we started with a geometric problem, that of immersing $\mathbb{R}P^n$ into Euclidean space. We then used cohomology theories and characteristic classes to obtain necessary conditions for such an immersion to exist: we obtained two sets of conditions, one from mod 2 singular cohomology and one from KO-theory. In the present section we start with an *algebraic* problem, one that at first glance seems completely unrelated to immersions. It is the problem of finding sums-of-squares formulas in various dimensions, which we encountered already back in Section 14 (we will review the problem below). Once again we will use cohomology theories to obtain necessary conditions for the existence of such formulas. The surprise is that these conditions are basically the same as the ones that arose in the immersion problem! This is because both problems lead to the same homotopy-theoretic situation involving bundles over real projective space.

In theory the present section could be read completely independently of the last one. But because the underlying homotopy-theoretic problem is the same, we refer to the previous section for many details of its analysis.

35.1. **Review of the basic problem.** Recall that a **sums-of-squares formula of type $[r, s, n]$** (over $\mathbb{R}$) is a bilinear map $\phi\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ with the property that

$$|\phi(x,y)|^2 = |\phi(x)|^2 \cdot |\phi(y)|^2 \tag{35.2}$$

for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. If we write $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_s)$ then $\phi(x,y) = (z_1, \ldots, z_n)$ where each z_i is a bilinear expression in the x's and y's. Formula (35.2) becomes

$$(x_1^2 + \cdots + x_r^2) \cdot (y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2.$$

We will sometimes refer to an "$[r, s, n]$-formula" for short. Note that if an $[r, s, n]$-formula exists then one trivially has $[i, j, k]$-formulas for any $i \leq r$, $j \leq s$, and $k \geq n$.

For what values of r, s, and n does an $[r, s, n]$-formula exist? This is the **sums-of-squares** problem. Said differently, given a specific r and s what is the smallest value of n for which an $[r, s, n]$-formula exists? Call this number $r * s$. As with the immersion problem, there are two aspects here. One is the problem of constructing sums-of-squares formulas, thereby giving upper bounds for $r * s$; the other is the problem of finding necessary conditions for their existence, thereby giving lower bounds. The latter is the part that involves topology.

The sums-of-squares formulas that everyone knows are the ones coming from the multiplications on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. These haves types $[1, 1, 1]$, $[2, 2, 2]$, $[4, 4, 4]$, and $[8, 8, 8]$. Hurwitz proved than an $[n, n, n]$ formula only exists when $n \in \{1, 2, 4, 8\}$, and the Hurwitz-Radon theorem generalizes this:

Theorem 35.3 (Hurwitz-Radon)**.** *A sums-of-squares formula of type $[r, n, n]$ exists if and only if $r \leq \rho(n) + 1$.*

Proof. Write $n = (\text{odd}) \cdot 2^{a+4b}$ with $0 \leq a \leq 3$. Recall from Theorem 14.12 that an $[r, n, n]$-formula exists if and only if there exists a Cl_{r-1}-module structure on $\mathbb{R}^n$. We saw in Section 14 that representations of Cl_{r-1} only exist on vector spaces whose dimension is a multiple of $2^{\varphi(r-1)}$. Thus, we have the chain of equivalences

$$\begin{aligned}
\text{an } [r,n,n]\text{-formula exists} &\iff \text{there exists a } \mathrm{Cl}_{r-1}\text{-module structure on } \mathbb{R}^n \\
&\iff 2^{\varphi(r-1)} \big| n \\
&\iff \varphi(r-1) \leq a + 4b \\
&\iff r - 1 \leq 2^a + 8b - 1 = \rho(n).
\end{aligned}$$

For the last equivalence note that $\varphi(2^a + 8b - 1) = \varphi(2^a - 1) + 4b = a + 4b$, where the first equality is the 8-fold periodicity of φ and the second is just a calculation for $0 \leq a \leq 3$. Moreover, $2^a + 8b - 1$ is the *largest* number whose φ-value is $a + 4b$; by periodicity this can again be checked just for $b = 0$ and $0 \leq a \leq 3$. So in general we have $\varphi(s) \leq a + 4b = \varphi(2^a + 8b - 1)$ if and only if $s \leq 2^a + 8b - 1$; this is what is used in the final equivalence. □

Remark 35.4. Note that if there exist formulas of type $[r, s_1, n_1]$ and $[r, s_2, n_2]$ then there is also a formula of type $[r, s_1 + s_2, n_1 + n_2]$ (by distribuitivity). This says that

$$r * (s_1 + s_2) \leq r * s_1 + r * s_2.$$

Also notice that $r * s \leq (r+a) * (s+b)$ whenever $a, b \geq 0$, because a formula of type $[r+a, s+b, n]$ automatically yields one of type $[r, s, n]$ by plugging in zeros for a of the x's and b of the y's.

The classical identities show that $2 * 2 = 2$, $4 * 4 = 4$, and $8 * 8 = 8$, and it is trival that $n * 1 = n$. Using these together with the observations of the previous paragraph, one can obtain upper bounds on $r * s$. For example, $3 * 10 \leq 12$ because

$$3 * 10 \leq 3 * 8 + 3 * 2 \leq 8 * 8 + 4 * 4 = 8 + 4 = 12.$$

The following table shows what is known about $r * s$ for small values of r and s. For $r \leq 8$ the values completely agree with the upper bounds obtained by the above methods.

Table 35.4. Values of $r * s$

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	2	4	4	6	6	8	8	10	10	12	12	14	14	16	16	18
3		4	4	7	8	8	8	11	12	12	12	15	16	16	16	19
4			4	8	8	8	8	12	12	12	12	16	16	16	16	20
5				8	8	8	8	13	14	15	16	16	16	16	16	21
6					8	8	8	14	14	16	16	16	16	16	16	22
7						8	8	15	16	16	16	16	16	16	16	23
8							8	16	16	16	16	16	16	16	16	24
9								16	16	[16,17]	??					
10									16	[16,17]						
11										17						

To justify the numbers in the above table we have to produce lower bounds for $r * s$. For example, we have to explain why there do not exist formulas of type $[5, 10, 13]$. Almost all the known lower bounds come from topological methods; we will describe some of these next.

35.5. **Lower bounds via topology.** Here is the key result that shows how a sums-of-squares formula gives rise to something homotopy-theoretic:

Proposition 35.6. *If an $[r, s, n]$-formula exists then there exists a rank $n-r$ bundle E on $\mathbb{R}P^{s-1}$ such that $rL \oplus E \cong \underline{n}$ (here rL is the direct sum of r copies of L).*

Proof. Let $\phi\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ be the map giving the sums-of-squares formula. If $u \in \mathbb{R}^s$ is a unit vector, check that $\phi(e_1, u), \ldots, \phi(e_r, u)$ is an orthonormal frame in $\mathbb{R}^n$. This is an easy consequence of the sums-of-squares identity; it is an exercise, but see the proof of Corollary 14.9 if you get stuck. In this way we obtain a map $f\colon S^{s-1} \to V_r(\mathbb{R}^n)$. Compose with the projection $V_r(\mathbb{R}^n) \to \mathrm{Gr}_r(\mathbb{R}^n)$ and then note that the map factors to give $F\colon \mathbb{R}P^{s-1} \to \mathrm{Gr}_r(\mathbb{R}^n)$. Precisely, given a line in $\mathbb{R}^s$ spanned by a vector u its image under F is the r-plane spanned by $\phi(e_1, u), \ldots, \phi(e_r, u)$.

Let η be the tautological r-plane bundle on $\mathrm{Gr}_r(\mathbb{R}^n)$. We claim that $F^*\eta = rL$. This follows from the commutative diagram

$$\begin{array}{ccc} rL & \xrightarrow{\tilde{F}} & \eta \\ \downarrow & & \downarrow \\ \mathbb{R}P^{s-1} & \xrightarrow{F} & \mathrm{Gr}_r(\mathbb{R}^n) \end{array}$$

where the top map is described as follows. Given r points on the same line $\langle u\rangle$ we write them as $\lambda_1 u, \dots, \lambda_r u$ and then send them to the element $\lambda_1\phi(e_1, u) + \cdots + \lambda_r\phi(e_r, u)$ on the r-plane $F(\langle u\rangle)$. One readily checks that this does not depend on the choice of u; in fact, we could just say that points $z_1, \dots, z_r$ on a common line ℓ are sent to the element $\phi(e_1, z_1) + \cdots + \phi(e_r, z_r)$ on $F(\ell)$. The fact that $\phi(e_1, u), \dots, \phi(e_r, u)$ are orthonormal (hence independent) shows that $\tilde{F}$ is injective on fibers, hence an isomorphism on fibers. This shows that $F^*\eta \cong rL$.

We have now done all the hard work. To finish, just recall that η sits inside a short exact sequence $0 \to \eta \to \underline{n} \to Q \to 0$ where Q is the standard quotient bundle. This sequence is split because $\mathrm{Gr}_r(\mathbb{R}^n)$ is compact. Pulling back along F now gives $rL \oplus F^*Q \cong \underline{n}$, as desired. □

The following result was originally proven independently by Hopf [Ho] and Stiefel [St]; Stiefel's method is the one we follow here.

Corollary 35.7 (Hopf-Stiefel)**.** *If an $[r, s, n]$-formula exists then the following two equivalent conditions hold:*
(1) $\binom{r+i-1}{i}$ is even for $n - r < i < s$;
(2) $\binom{n}{i}$ is even for $n - r < i < s$.

Proof. By Proposition 35.6 we know that $rL \oplus E \cong \underline{n}$ for some rank $n - r$ bundle on $\mathbb{R}P^{s-1}$. Applying total Stiefel-Whitney classes gives $w(rL)w(E) = w(rL \oplus E) = w(\underline{n}) = 1$, or $w(E) = w(rL)^{-1} = w(L)^{-r}$. So $w(L)^{-r}$ vanishes in degrees larger than $n-r$. But $w(L) = 1+x$ where x is the generator for $H^1(\mathbb{R}P^{s-1}; \mathbb{Z}/2)$, and the coefficient of x^i in $(1 + x)^{-r}$ is $\binom{-r}{i} = \binom{r+i-1}{i}$ (recall that we are working modulo 2). So $\binom{r+i-1}{i}$ is even for $n - r < i < s$.

The equivalence of the conditions in (1) and (2) follows at once from the lemma below, taking $k = n - r + 1$ and $i = r + s - n - 2$ (note that conditions (1) and (2) are both vacuous unless $i \geq 0$). □

Lemma 35.8. *For any non-negative integers n, k, and i, the following $\mathbb{Z}$-linear spans are the same inside of $\mathbb{Q}$:*

$$\mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \dots, \tbinom{n}{k+i} \Big\rangle = \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n+1}{k+1}, \dots, \tbinom{n+i}{k+i} \Big\rangle.$$

Consequently, an integer is a common divisor of the first set of binomial coefficients if and only if it is a common divisor of the second set.

Proof. Taking first differences and using Pascal's identity shows (via multiple iterations) that

$$\begin{aligned}\mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n+1}{k+1}, \dots, \tbinom{n+i}{k+i} \Big\rangle &= \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \tbinom{n+1}{k+2}, \tbinom{n+2}{k+3}, \dots, \tbinom{n+i-1}{k+i} \Big\rangle \\ &= \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \tbinom{n}{k+2}, \tbinom{n+1}{k+3}, \dots, \tbinom{n+i-2}{k+i} \Big\rangle \\ &= \cdots \\ &= \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \dots, \tbinom{n}{k+i} \Big\rangle.\end{aligned}$$

□

Example 35.9. Does a formula of type $[10, 10, 15]$ exist? If it did, statement (2) of Corollary 35.7 would imply that $\binom{15}{i}$ is even for $5 < i < 10$. But $\binom{15}{6}$ is odd.

The full power of the numerical conditions in Corollary 35.7 is subtle, and one really needs a computer to thoroughly investigate them. But the following consequence represents much of the information buried in those conditions:

Corollary 35.10. *If $r + s > 2^k$ then $r * s \geq 2^k$.*

Proof. We must show that if $r + s > 2^k$ then formulas of type $[r, s, 2^k - 1]$ do not exist. If they did, the Hopf-Stiefel conditions would imply that $\binom{2^k-1}{i}$ is even for i in the range $2^k - 1 - r < i < s$. But $\binom{2^k-1}{i}$ is odd no matter what i is, so the conditions are only consistent if the range is empty—or equivalently, if $2^k - 1 - r \geq s - 1$. The hypothesis $r + s > 2^k$ guarantees that this is not the case. □

For example, sums-of-squares formulas of type $[16, 17, n]$ must all have $n \geq 32$.

For $r \leq 8$ the Hopf-Stiefel lower bounds for $r * s$ turn out to exactly match the upper bounds obtained via the constructive methods of Remark 35.4. So this justifies the numbers in Table 35.4 for the range $r \leq 8$.

35.11. ***K*-theoretic techniques.** We can also analyze the implications of Proposition 35.6 using KO-theory. This was first done by Yuzvinsky [Y]. Note that one could also use complex K-theory here, but KO-theory gives stronger results: the point is that $\widetilde{K}^0(\mathbb{R}P^m)$ and $\widetilde{KO}^0(\mathbb{R}P^m)$ are almost the same, but for certain values of m the latter group is slightly bigger (by a factor of 2).

Corollary 35.12 (Yuzvinsky)**.** *If an $[r, s, n]$-formula exists then the following two equivalent conditions hold:*
(1) $2^{\varphi(s-1)-i+1}$ divides $\binom{r+i-1}{i}$ for $n - r < i \leq \varphi(s-1)$;
(2) $2^{\varphi(s-1)-i+1}$ divides $\binom{n}{i}$ for $n - r < i \leq \varphi(s-1)$.

Proof. Proposition 35.6 gives that $rL \oplus E \cong \underline{n}$ for some bundle E on $\mathbb{R}P^{s-1}$. We again use characteristic classes, but this time the $\tilde{\gamma}$ classes in KO-theory. We find that $\tilde{\gamma}_t(E) = \tilde{\gamma}_t(L)^{-r}$, and so $\tilde{\gamma}_t(L)^{-r}$ must vanish in degrees larger than $n - r$. Recall that $\tilde{\gamma}_t(L) = 1 - t\lambda$ where $\lambda = 1 - [L]$, and so the coefficient of t^i in $\tilde{\gamma}_t(L)^{-r}$ is $\pm\binom{-r}{i}\lambda^i = \pm\binom{r+i-1}{i}2^{i-1}\lambda$. Here we have used $\lambda^2 = -2\lambda$.

Recalling that $\widetilde{KO}^0(\mathbb{R}P^{s-1}) \cong \mathbb{Z}/(2^{\varphi(s-1)})$, we find that $2^{\varphi(s-1)}$ divides $2^{i-1}\binom{r+i-1}{i}$ for all $i > n - r$. This statement only has content for $i \leq \varphi(s-1)$, and thus we obtain the condition in (1).

The equivalence of (1) and (2) is an instance of the following general observation: the sequence of conditions

$$2^N \big| \tbinom{n}{k},\ 2^{N-1} \big| \tbinom{n}{k+1},\ 2^{N-2} \big| \tbinom{n}{k+2}, \ldots,\ 2^{N-j} \big| \tbinom{n}{k+j}$$

is equivalent to the sequence of conditions

$$2^N \big| \tbinom{n}{k},\ 2^{N-1} \big| \tbinom{n+1}{k+1},\ 2^{N-2} \big| \tbinom{n+2}{k+2}, \ldots,\ 2^{N-j} \big| \tbinom{n+j}{k+j}.$$

This follows at once by applying Lemma 35.8 multiple times, with $i = 1$, $i = 2$, ..., $i = j$. $\square$

The Hopf-Stiefel conditions are symmetric in r and s, but this is not true for the KO-theoretic conditions in the above proposition. For example, applying the conditions yields no information on $[3, 6, n]$-formulas except $n \geq 6$, whereas applying the conditions to $[6, 3, n]$-formulas yields $n \geq 8$. One must therefore apply the conditions to both $[r, s, n]$ and $[s, r, n]$ to get the best information.

Example 35.13. Like we saw for the immersion problem, in some dimensions the KO-theoretic conditions are stronger than the Hopf-Stiefel conditions—and in some dimensions they are weaker. Neither result is strictly stronger than the other.

For example, the Hopf-Stiefel conditions show that $4 * 5 \geq 8$ whereas the KO-conditions only show $4 * 5 \geq 7$. The smallest dimension for which the KO-theoretic conditions are stronger is when $r = 10$ and $s = 15$. The Hopf-Stiefel conditions rule out the existence of $[10, 15, 15]$-formulas, but not $[10, 15, 16]$. The KO-conditions rule out $[15, 10, 16]$, however, and therefore also $[10, 15, 16]$ by symmetry.

To pick a larger example, the Hopf-Stiefel conditions show that $127 * 127 \geq 128$ but the KO-conditions show that $127 * 127 \geq 183$. The KO-conditions seem to give their greatest power when r and s are slightly less than a power of 2.

35.14. **Other problems.** A careful look at Proposition 35.6 shows that one can make the argument work with something much weaker than a sums-of-squares formula. Specifically, all we needed was a bilinear map $f\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ such that $f(x \otimes y) = 0$ only when $x = 0$ or $y = 0$. Such a bilinear map is usually called **nonsingular**. Given such a map and a nonzero $u \in \mathbb{R}^r$, the elements $\phi(u, e_1), \ldots, \phi(u, e_r)$ are necessarily linearly independent—and this is really all that was needed in the proof of Proposition 35.6.

We can replace our sums-of-squares problem with the following: given r and s, for what values of n does there exist a nonsingular bilinear map of type $[r, s, n]$? The topological obstructions we found for sums-of-squares formula are of course still valid in this new context.

The existence of nonsingular bilinear maps turns out to be related to the immersion problem for real projective spaces. More than this, both problems are connected to a number of similar questions that have been intently studied by algebraic topologists since the 1940s. Many of these problems were originally raised by Hopf [Ho]. This material takes us somewhat away from our main theme of K-theory, but it seems worthwhile to tell a bit of this story since we have encountered it.

To start with, let us introduce the following classes of statements:

SS$[r,s,n]$:	there exists a sums-of-squares formula of type $[r,s,n]$
NS$[r,s,n]$:	there exists a nonsingular bilinear map $\mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$
RR$[r,s,n]$:	there exist $n \times s$ matrices $A_1, \dots, A_r$ with the property that every nonzero linear combination of them has rank s
T$[r,s,n]$:	the tangent bundle $T_{\mathbb{R}P^n}$ has s independent sections when restricted to $\mathbb{R}P^r$
IS$[r,s,n]$:	the bundle $nL \to \mathbb{R}P^r$ has s independent sections
GD$[r,s,n]$:	over $\mathbb{R}P^r$ one has $\mathrm{g.\,dim}(n(L-1)) \le n-s$
ES$[r,s,n]$:	the 'first-vector' map $p_1\colon V_s(\mathbb{R}^{n+1}) \to S^n$ has a $\mathbb{Z}/2$-equivariant section over the subspace $S^r \subseteq S^n$. Here $\mathbb{Z}/2$ acts antipodally on $\mathbb{R}^{n+1}$, and both $V_s(\mathbb{R}^{n+1})$ and S^n get the induced action.
AX$[r,s,n]$:	there exists an "axial map" $\mathbb{R}P^r \times \mathbb{R}P^s \to \mathbb{R}P^n$; this is a map with the property that the restrictions $\mathbb{R}P^r \times \{*\} \to \mathbb{R}P^n$ and $\{*\} \times \mathbb{R}P^s \to \mathbb{R}P^n$ are homotopic to linear embeddings, for some choice of basepoints in $\mathbb{R}P^r$ and $\mathbb{R}P^s$
IM$[r,n]$:	$\mathbb{R}P^r$ immerses into $\mathbb{R}P^n$
VF$[k,n]$:	there exist k independent vector fields on S^n.

The acronyms are mostly self-evident, except for a few: RR stands for "rigid rank", IS for "independent sections", and ES for "equivariant sections".

The above statements are closely interrelated, as the next result demonstrates. We should point out that very little from this result will be needed in our subsequent discussion. We are including it because most of the claims are easy to prove, and because the various statements get used almost interchangeably (often without much explanation) in the literature on the immersion problem.

Proposition 35.15.

(a) $\mathrm{GD}[r,s,n] \iff \mathrm{GD}[r,-n,-s]$

(b) $\mathrm{IM}[r,n] \iff \mathrm{GD}[r,-(n+1),-(r+1)] \iff \mathrm{GD}[r,r+1,n+1]$

(c) $T[n,k,n] \Rightarrow \mathrm{VF}[k,n]$

(d) One has the following implications:

$$
\begin{array}{ccccccc}
 & \mathrm{SS}[r,s,n] \overset{1}{\Longrightarrow} & \mathrm{NS}[r,s,n] & \overset{2}{\Longleftrightarrow} & \mathrm{RR}[r,s,n] & & \mathrm{ES}[r-1,s,n-1] \\
 & & & & \Big\Downarrow 3 & \nearrow\!\!\swarrow & \Big\Downarrow 6 \\
\mathrm{GD}[r-1,s,n] & \overset{5}{\Longleftarrow} & \mathrm{IS}[r-1,s,n] & \overset{4}{\Longleftarrow} & \mathrm{T}[r-1,s-1,n-1] & & \mathrm{AX}[r-1,s-1,n-1]
\end{array}
$$

(e) If $r < n$ then implication 4 is reversible, and if $r \le n$ then 5 is reversible.

(f) If $r < n$ and $r \le 2(n-s)$ then implication 6 in (d) is also reversible.

Proof. Part (a) is Proposition 34.28, and part (b) is Corollary 34.29; we have seen these already. Part (c) follows from the fact that $T_{S^n} = p^* T_{\mathbb{R}P^n}$ where $p\colon S^n \to \mathbb{R}P^n$ is the projection.

For part (d), the first implication is obvious. The others we treat one by one.

$\mathbf{NS}[\boldsymbol{r,s,n}] \iff \mathbf{RR}[\boldsymbol{r,s,n}]$: The equivalence follows from adjointness, as bilinear maps $f\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ correspond bijectively to linear maps $F\colon \mathbb{R}^r \to \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$. The linear map F is specified by the $n \times s$ matrices $F(e_1), \dots, F(e_r)$. It is easy to verify that f is nonsingular if and only if all nontrivial linear combinations of these matrices have rank s.

$\mathbf{RR}[r,s,n] \Rightarrow \mathbf{T}[r-1,s-1,n-1]$: First note that the tangent bundle of $\mathbb{R}P^n$ is the collection of pairs (x,v) such that $x,v \in \mathbb{R}^n$, $|x|=1$, and $x \cdot v = 0$, modulo the identifications $(x,v) \sim (-x,-v)$. Secondly, note that since $\mathrm{NS}[r,s,n]$ is symmetric in r and s the same is true for $\mathrm{RR}[r,s,n]$. The condition $\mathrm{RR}[s,r,n]$ says that we have $n \times r$ matrices $A_1,\dots,A_s$ such that every nontrivial linear combination has rank r. Multiplying these matrices on the left by a fixed element of $GL_n(\mathbb{R})$, we can assume that the columns of A_1 are the standard basis $e_1,\dots,e_r$.

For every $x \in S^{r-1} \subseteq \mathbb{R}^r \subseteq \mathbb{R}^n$ consider the independent vectors $A_1x, A_2x, A_3x, \dots, A_sx$. Note that $A_1x = x$. Let $u_i(x) = A_ix - (A_ix \cdot x)x$. Then $u_2(x),\dots,u_r(x)$ are independent, and orthogonal to x. Since $u_i(-x) = -u_i(x)$ for each i, these give us $s-1$ independent sections of $T_{\mathbb{R}P^{n-1}}$ defined over $\mathbb{R}P^{r-1}$.

$\mathbf{T}[r-1,s-1,n-1] \Rightarrow \mathbf{IS}[r-1,s,n]$: This follows from the bundle isomorphism $T_{\mathbb{R}P^{n-1}} \oplus 1 \cong nL$.

$\mathbf{IS}[r-1,s,n] \Rightarrow \mathbf{GD}[r-1,s,n]$: Trivial.

$\mathbf{T}[r-1,s-1,n-1] \iff \mathbf{ES}[r-1,s,n-1]$: First note that the frame bundle $V_{s-1}(T_{\mathbb{R}P^{n-1}})$ is homeomorphic to $V_s(\mathbb{R}^n)/\pm 1$ in an evident way. Under this homeomorphism the projection $V_{s-1}(T_{\mathbb{R}P^{n-1}}) \to \mathbb{R}P^{n-1}$ corresponds to the first-vector map $V_s(\mathbb{R}^n)/\pm 1 \to S^{n-1}/\pm 1$. So $T[r-1,s-1,n-1]$ is equivalent to the latter bundle having a section over $S^{r-1}/\pm 1$. But then consider the diagram

$$\begin{array}{ccc} V_s(\mathbb{R}^n) & \longrightarrow & V_s(\mathbb{R}^n)/\pm 1 \\ \downarrow & & \downarrow \\ S^{n-1} & \longrightarrow & S^{n-1}/\pm 1 \end{array}$$

where the two horizontal maps are 2-fold covering spaces. This is a pullback square. It is easy to see that the right vertical map has a section defined over $S^{r-1}/\pm 1$ if and only if the left vertical map has a $\mathbb{Z}/2$-equivariant section defined over S^{r-1}.

$\mathbf{ES}[r-1,s,n-1] \Rightarrow \mathbf{AX}[r-1,s-1,n-1]$: Note that there is an evident map $V_s(\mathbb{R}^n) \to \mathcal{T}op(S^{s-1},S^{n-1})$ that sends a frame $v_1,\dots,v_s$ to the map $(a_1,\dots,a_s) \mapsto a_1v_1 + \cdots + a_sv_s$. So a section $\chi\colon S^{r-1} \to V_s(\mathbb{R}^n)$ gives by composition a map $S^{r-1} \to \mathcal{T}op(S^{s-1},S^{n-1})$, and then by adjointness a map $g\colon S^{r-1} \times S^{s-1} \to S^{n-1}$. The fact that χ was a section of the first-vector map shows that $g(x,e_1) = x$ for all $x \in S^{r-1}$. Also, it is clear that $g(e_1,-)$ is a linear inclusion $S^{s-1} \hookrightarrow S^{n-1}$ (there is nothing special about e_1 here). The $\mathbb{Z}/2$-equivariance of χ shows that $g(-x,y) = -g(x,y)$, and the similar identity $g(x,-y) = -g(x,y)$ is trivial. So g descends to give a map $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}$, and this map is axial.

For (e), the reversibility of both implications is governed by stability theory of vector bundles. For example, assume $\mathrm{IS}[r-1,s,n]$. Then $nL \cong s \oplus E$ for some bundle E of rank $n-s$, where we are working over $\mathbb{R}P^{r-1}$. Recall that $j^*T_{\mathbb{R}P^{n-1}} \oplus 1 \cong nL$, where $j\colon \mathbb{R}P^{r-1} \hookrightarrow \mathbb{R}P^{n-1}$ is the inclusion. So $j^*T_{\mathbb{R}P^{n-1}} \oplus 1 \cong s \oplus E \cong 1 \oplus ((s-1) \oplus E)$. Since we are working over $\mathbb{R}P^{r-1}$ and $r-1 < n-1$, we can cancel the 1 on both sides to get $j^*T_{\mathbb{R}P^{n-1}} \cong (s-1) \oplus E$ (see Proposition 11.10). This says that $\mathrm{T}[r-1,s-1,n-1]$ holds. The reversibility of implication 5 is very similar, and is left to the reader.

Part (f), on the reversibility of implication 6, is the only part of the proposition that is not elementary. Let $\mathcal{T}op_{\mathbb{Z}/2}(S^{s-1},S^{n-1})$ denote the space of $\mathbb{Z}/2$-equivariant maps, where the spheres have the antipodal action. This is a subspace of the usual function space $\mathcal{T}op(S^{s-1}, S^{n-1})$. Note that the space of equivariant maps has a $\mathbb{Z}/2$-action, given by composing (or equivalently, precomposing) a given map with the antipodal map. James [J1] shows that the evident map $\mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \to \mathcal{T}op(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1})$ is a principal $\mathbb{Z}/2$-bundle with respect to the above action.

We will also have need of the evaluation map $\mathrm{ev}\colon \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \to S^{n-1}$ sending $h \mapsto h(e_1)$. James [J1] shows that this is also a fibration.

Note that the Stiefel manifold $V_s(\mathbb{R}^n)$ is a subspace of $\mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ in the evident way, and that the following diagram commutes:

$$\begin{array}{ccc} V_s(\mathbb{R}^n) & \longrightarrow & \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle \mathrm{ev}} & \\ & S^{n-1}. & \end{array}$$

James [J1, Theorem 6.5] proved that $V_s(\mathbb{R}^n) \hookrightarrow \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ is $[2(n-s)-1]$-connected, and this is the crucial point of the whole argument.

Suppose $f\colon \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}$ is an axial map. By the homotopy extension property we can assume that $f(*,-)$ and $f(-,*)$ are both equal to the canonical embeddings, where $*$ refers to some chosen basepoints in $\mathbb{R}P^{r-1}$ and $\mathbb{R}P^{s-1}$. The axial map f gives a map $F\colon \mathbb{R}P^{r-1} \to \mathcal{T}op(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1})$ by adjointness. Regard the target as pointed by the canonical embedding j, and note that F is a pointed map.

Covering space theory gives that there is a unique map $\tilde{F}\colon S^{r-1} \to \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ such that $\tilde{F}(*)$ is the canonical embedding $S^{s-1} \hookrightarrow S^{n-1}$ and such that the diagram

$$\begin{array}{ccc} S^{r-1} & \xrightarrow{\tilde{F}} & \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \\ \downarrow & & \downarrow \\ \mathbb{R}P^{r-1} & \xrightarrow{F} & \mathcal{T}op(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1}) \end{array}$$

commutes. We claim that $\tilde{F}$ is $\mathbb{Z}/2$-equivariant, and that the composition $\mathrm{ev} \circ \tilde{F}$ is the standard inclusion $S^{r-1} \hookrightarrow S^{n-1}$. Both of these are easy exercises in covering space theory. The latter statement depends on the so-far unused portion of the axial map condition on f.

If $r-1 \leq 2(n-s)-1$ then by James's connectivity result the map $\tilde{F}$ can be factored up to homotopy through $V_s(\mathbb{R}^n)$. Moreover, this can be done in the category of $\mathbb{Z}/2$-equivariant pointed spaces over S^{n-1} (the only hard part here is the $\mathbb{Z}/2$-equivariance, and for this one uses that S^{r-1} has an equivariant cell decomposition made from free $\mathbb{Z}/2$-cells). In this way one produces the relevant equivariant section of the map $V_s(\mathbb{R}^n) \to S^{n-1}$. □

Remark 35.16. We have included the above proposition because it is very useful as a reference. However, it should be pointed out that there is something slightly deceptive about part (d). Some of the implications are more obvious than the long chains would suggest. For example, $\mathrm{NS}[r,s,n] \Rightarrow \mathrm{AX}[r-1,s-1,n-1]$ is a very easy argument of one or two lines. Likewise, $\mathrm{NS}[r,s,n] \Rightarrow \mathrm{GD}[r-1,s,n]$ is just the argument in Proposition 35.6. The picture in (d) is useful in showing *all* the relations at once, but it makes some of the statements seem more distant than they really are.

The reader will have noticed that Proposition 35.15 encodes several things that we have seen before. Some are transparently familiar, like parts (a) and (b). A less transparent example is

$$\mathrm{SS}[r,n,n] \iff \mathrm{SS}[n,r,n] \Rightarrow \mathrm{T}[n-1,r-1,n-1] \Rightarrow \mathrm{VF}[r-1,n-1].$$

This was the content of Corollary 14.9. In constrast, here is a similarly-obtained implication that we have *not* seen yet:

$$\mathrm{NS}[r,r,n] \Rightarrow \mathrm{GD}[r-1,r,n] \iff \mathrm{IM}[r-1,n-1].$$

From this we learn that immersion results for real projective space can be obtained by demonstrating the existence of nonsingular bilinear maps. This approach was successfully used by K.Y. Lam in [L1]. We briefly sketch his method simply to give the basic idea; for details the reader may consult [L1] and similar papers.

Recall that $\mathbb{O}$ denotes the octonions. Consider the map $f\colon \mathbb{O}^2 \times \mathbb{O}^2 \to \mathbb{O}^3$ given by

$$f\big((u_1,u_2),(x_1,x_2)\big) = (u_1x_1 - \overline{x}_2u_2, x_2u_1 + u_2\overline{x}_1, u_2x_2 - x_2u_2).$$

With a little work one can prove that this is nonsingular. Also, it is a general fact about the octonions that for any $a, b \in \mathbb{O}$ the commutator $[a,b] = ab - ba$ is imaginary. So the image of f actually lies in the 23-dimensional subspace of $\mathbb{O}^3$ where the real part of the third coordinate vanishes. So f gives a nonsingular bilinear map of type $[16,16,23]$. This shows that $\mathbb{R}P^{15}$ immerses into $\mathbb{R}^{22}$. By restricting f to appropriate subspaces Lam also obtained nonsingular bilinear maps of types $[11,11,17]$, $[13,13,19]$, and $[10,10,16]$, thereby proving that $\mathbb{R}P^{10}$ immerses into $\mathbb{R}^{16}$, $\mathbb{R}P^{12}$ immerses into $\mathbb{R}^{18}$, and $\mathbb{R}P^9$ immerses into $\mathbb{R}^{15}$.

35.17. **Summary.** In this section we examined the sums-of-squares problem, and saw how characteristic classes can be used to obtain lower bounds for the numbers $r*s$. Use of Stiefel-Whitney classes in singular cohomology yielded the Hopf-Stiefel lower bounds on $r*s$, whereas the use of the γ-classes in KO-theory gave the Yuzvinsky lower bounds. This story is very similar to the one for immersions of $\mathbb{R}P^n$ discussed in Section 34, and in fact the sums-of-squares problem is closely connected to this immersion problem. We closed the section by exploring the relations between these and a host of similar problems.

Part 6. Homological intersection theory

36. The theorem of Gillet-Soulé

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Part 7. Appendices

Appendix A. Bernoulli numbers

There are different conventions for naming the Bernoulli numbers, especially when one enters the topology literature. We adopt what seems to be the most common definition, which is the following:

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \cdot \frac{x^k}{k!}.$$

Expanding the power series yields

$$\begin{aligned}\frac{x}{e^x-1} &= 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots \\ &= 1 - \Big(\frac{1}{2}\Big)x + \Big(\frac{1}{6}\Big)\cdot\frac{x^2}{2} - \Big(\frac{1}{30}\Big)\cdot\frac{x^4}{4!} + \Big(\frac{1}{42}\Big)\cdot\frac{x^6}{6!} - \Big(\frac{1}{30}\Big)\cdot\frac{x^8}{8!}\cdots\end{aligned}$$

So we have

k	0	1	2	3	4	5	6	7	8	9
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0

From the table one guesses that $B_{2n+1} = 0$ for $n > 0$. This is easy to prove: if we set $f(x) = \frac{x}{e^x-1}$ then we can isolate the odd powers of x by examining $f(x)-f(-x)$. But algebra yields

$$\Big(\frac{x}{e^x-1}\Big) - \Big(\frac{-x}{e^{-x}-1}\Big) = -x.$$

Computing the coefficients of $\frac{x}{e^x-1}$ is not the most efficient way of computing Bernoulli numbers, as one can deduce from the large denominators in the above formula. A better method is via a certain recursive formula, and this is best remembered by a "mnemonic":

$$(B+1)^n = B^n. \tag{A.1}$$

Do not take this formula literally! It is shorthand for the following procedure. First expand the left-hand-side via the Binomial Formula, treating B as a formal variable. Then rewrite the formula by "lowering all indices", meaning changing every B^i to a B_i. This gives the desired recursive formula.

For example: $(B+1)^2 = B^2$ yields $B^2 + 2B + 1 = B^2$, which in turn gives $B_2 + 2B_1 + 1 = B_2$. Cancelling the B_2's we obtain $2B_1 + 1 = 0$, or $B_1 = -\frac{1}{2}$. Likewise, $(B+1)^3 = B^3$ yields $B_3 + 3B_2 + 3B_1 + 1 = B_3$, thereby giving

$$B_2 = -\tfrac{1}{3}(1+3B_1) = -\tfrac{1}{3}\cdot -\tfrac{1}{2} = \tfrac{1}{6}.$$

And so on. For the record here are a few more of the Bernoulli numbers, computed in this way:

k	0	1	2	4	6	8	10	12	14	16	18
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$

Note that we have not yet justified the recursive formula (A.1). We will do this after a short interlude.

A.2. **Sums of powers.** The Bernoulli numbers first arose in work of Jakob Bernoulli on computing formulas for the power sums

$$1^t + 2^t + 3^t + \cdots + n^t.$$

Most modern students have seen the formulas

$$1 + 2 + \cdots + n = \tfrac{n(n+1)}{2} \quad \text{and} \quad 1^2 + 2^2 + \cdots + n^2 = \tfrac{n(n+1)(2n+1)}{6}.$$

The Bernoulli formulas generalize these to give

$$1^t + 2^t + \cdots + n^t = P_t(n)$$

where P_t is a degree $t+1$ polynomial in n with rational coefficients. It is somewhat surprising that the formulas for the P_t's can be given using a single set of coefficients, the Bernoulli numbers.

The Bernoulli formulas are most succinctly written using our mnemonic device of lowering indices. We write

$$1^t + 2^t + \cdots + n^t = \tfrac{1}{t+1}\Big[\big(B + (n+1)\big)^{t+1} - B^{t+1}\Big]. \tag{A.3}$$

Let us work through the first few examples of this. For $t = 1$ we have

$$\begin{aligned} 1 + 2 + \cdots + n = \tfrac{1}{2}\Big[(B + (n+1))^2 - B^2\Big] &= \tfrac{1}{2}\Big[B_2 + 2B_1(n+1) + (n+1)^2 - B_2\Big] \\ &= \tfrac{1}{2}\Big[(n+1)^2 - (n+1)] \\ &= \tfrac{1}{2}(n+1)n. \end{aligned}$$

For $t = 2$ we have

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 &= \tfrac{1}{3}\Big[(B + (n+1))^3 - B^3\Big] \\ &= \tfrac{1}{3}\Big[3B_2(n+1) + 3B_1(n+1)^2 + (n+1)^3\Big] \\ &= \tfrac{1}{3}(n+1)\Big[\tfrac{1}{2} - \tfrac{3}{2}(n+1) + (n+1)^2\Big] \\ &= \tfrac{1}{3}(n+1)\Big[\tfrac{1}{2}n + n^2\Big] \\ &= \tfrac{1}{6}(n+1)n(2n+1). \end{aligned}$$

We leave it to the reader to derive the $t = 3$ formula:

$$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Proof of the Bernoulli formula (A.3). Start with the identity of power series

$$1 + e^x + e^{2x} + \cdots + e^{nx} = \frac{e^{(n+1)x} - 1}{e^x - 1} = \Big(\frac{x}{e^x - 1}\Big) \cdot \Big(\frac{e^{(n+1)x} - 1}{x}\Big).$$

The coefficient of x^t on the left-hand-side is

$$\frac{1}{t!}(1^t + 2^t + \cdots + n^t).$$

The coefficient of x^t on the right-hand-side is

$$\sum_{k=0}^{t} \frac{B_k}{k!} \cdot \frac{(n+1)^{t+1-k}}{(t+1-k)!}.$$

Equating coefficients and rearranging yields the Bernoulli formula immediately. □

Now let us return to our recursive formula (A.1) for computing the Bernoulli numbers. Note that it is an immediate consequence of (A.3) by taking $n = 0$ to get $0 = (B+1)^{t+1} - B^{t+1}$.

A.4. **Miscellaneous facts.**

Theorem A.5 (Claussen/von Staudt)**.**

(a) $(-1)^n B_{2n} \equiv \sum_p \frac{1}{p}$ *mod* $\mathbb{Z}$*, where the sum is taken over all primes* p *such that* $p-1$ *divides* $2n$*.*

(b) When expressed as a fraction in lowest terms, B_{2n} *has square-free denominator consisting of the product of all primes* p *such that* $p-1$ *divides* $2n$*.*

For example, we can now immediately predict that the denominator of B_{20} will be $2 \cdot 3 \cdot 5 \cdot 11 = 330$. Note that the primes 2 and 3 will always appear in the denominators of Bernoulli numbers.

The following strange fact is relevant to the appearance of Bernoulli numbers in topology:

Proposition A.6. *For any even* n *and any* $k \in \mathbb{Z}$*,* $\frac{k^n(k^n-1)B_n}{n} \in \mathbb{Z}$*.*

Proof. We follow Milnor and Stasheff here [MS]. Write

$$f(x) = 1 + e^x + e^{2x} + \cdots + e^{(k-1)x} = \frac{e^{kx}-1}{e^x - 1}.$$

Note that

$$f^{(r)}(0) = 1^r + 2^r + \cdots + (k-1)^r.$$

In particular, the derivatives of f evaluated at 0 are all integers.

Next consider the logarithmic derivative

$$\begin{aligned}\frac{f'(x)}{f(x)} = D(\log(f(x))) &= \frac{ke^{kx}}{e^{kx}-1} - \frac{e^x}{e^x-1} \\ &= k\Big[\frac{1}{1-e^{-kx}}\Big] - \Big[\frac{1}{1-e^{-x}}\Big] \\ &= \frac{1}{x}\Big[\frac{-kx}{e^{-kx}-1} - \frac{-x}{e^{-x}-1}\Big] \\ &= \frac{1}{x}\Big[\sum \tfrac{B_i}{i!}(-kx)^i - \sum \tfrac{B_i}{i!}(-x)^i\Big] \\ &= \sum_i (-1)^i \tfrac{B_i}{i!}(k^i - 1)x^{i-1} \\ &= \tfrac{k-1}{2} + \tfrac{B_2}{2!}(k^2-1)x + \tfrac{B_4}{4!}(k^4-1)x^3 + \cdots\end{aligned}$$

The $(2t-1)$st derivative of this expression, evaluated at 0, is $\frac{B_{2t}}{2t}(k^{2t}-1)$.

However, iterated use of the quotient rule shows that the $(2t-1)$st derivative of $f'(x)/f(x)$, evaluated at 0, can be written as an integral linear combination of

$f(0), f'(0), f''(0), \dots$ divided by $f(0)^{2t}$. Since $f(0) = k$ and all the deriviates of f have integral values at 0, this gives

$$\frac{B_{2t}}{2t}(k^{2t} - 1) \cdot k^{2t} \in \mathbb{Z}.$$

□

Appendix B. The algebra of symmetric functions

Let $S = \mathbb{Z}[x_1, \dots, x_n]$ be equipped with the evident Σ_n-action that permutes the indices. It is a well-known theorem that the ring of invariants is a polynomial ring on the elementary symmetric functions:

$$S^{\Sigma_n} \cong \mathbb{Z}[\sigma_1, \dots, \sigma_n].$$

Let $s_k = x_1^k + x_2^k + \cdots + x_n^k$, the ***k*th power sum** of the variables x_i. Since s_k is a Σ_n-invariant we have

$$s_k = S_k(\sigma_1, \dots, \sigma_n)$$

for a unique polynomial S_k in n variables (with integer coefficients). The polynomial S_k is called the kth **Newton polynomial**.

Let us calculate the simplest examples of the Newton polynomials. Clearly $s_1 = \sigma_1$, and so $S_1(\sigma_1, \dots, \sigma_n) = \sigma_1$. For s_2 we compute that

$$s_2 = x_1^2 + \cdots + x_n^2 = (x_1 + \cdots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n) = \sigma_1^2 - 2\sigma_2.$$

These calculations get more difficult as the exponents get larger.

It is useful to adopt the following notation when working with the ring of invariants. If m is a monomial in the x_i's then $[m]$ denotes the sum of all elements in the Σ_n-orbit of m. For example,

$$[x_1x_2] = \sigma_2, \quad [x_1^k] = s_k, \quad \text{and} \quad [x_1^2x_2] = \sum_{i \neq j} x_i^2 x_j.$$

If $H \leq \Sigma_n$ is the stabilizer of m then we can also write

$$[m] = \sum_{g \in \Sigma_n/H} gm.$$

Let us use the above notation to help work out the third Newton polynomial. Elementary algebra easily yields the equation

$$s_3 = [x_1^3] = [x_1]^3 - 3[x_1^2x_2] - 6[x_1x_2x_3].$$

Here one considers the product $(x_1 + \cdots + x_n)^3$ and reasons that a term like $x_1^2x_2$ appears three times in the expansion, and terms like $x_1x_2x_3$ appear six times. Via a similar process we work out that

$$[x_1^2x_2] = [x_1] \cdot [x_1x_2] - 3[x_1x_2x_3].$$

Putting everything together, we have found that

$$s_3 = \sigma_1^3 - 3(\sigma_1\sigma_2 - 3\sigma_3) - 6\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

This final expression is the third Newton polynomial S_3.

Lemma B.1 (The Newton identities)**.** *For $k \geq 2$ one has the identity*

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \cdots + (-1)^k \sigma_{k-1} s_1 + (-1)^{k+1} k\sigma_k.$$

Consequently, there is analogous inductive formula for the Newton polynomials:

$$S_k = \sigma_1 S_{k-1} - \sigma_2 S_{k-2} + \cdots + (-1)^k \sigma_{k-1} S_1 + (-1)^{k+1} k\sigma_k.$$

Proof. The key is the formula

$$[x_1 x_2 \cdots x_{j-1} x_j^k] = [x_1 \cdots x_j] \cdot [x_1^{k-1}] - [x_1 \cdots x_j x_{j+1}^{k-1}]$$

which holds for $k > 2$, whereas when $k = 2$ we have

$$[x_1 x_2 \cdots x_{j-1} x_j^2] = [x_1 \cdots x_j] \cdot [x_1] - (j+1)[x_1 \cdots x_j x_{j+1}].$$

In the latter case the point is that a term $x_1 \ldots x_{j+1}$ appears $j + 1$ times in the product $[x_1 \ldots x_j] \cdot [x_1]$.

When $k = 2$ the identity from the statement of the lemma has already been verified by direct computation. For $k > 2$ start with the simple formula

$$s_k = [x_1^k] = [x_1] \cdot [x_1^{k-1}] - [x_1 x_2^{k-1}] = \sigma_1 s_{k-1} - [x_1 x_2^{k-1}].$$

Next observe that

$$[x_1 x_2^{k-1}] = \begin{cases} [x_1 x_2] \cdot [x_1^{k-2}] - [x_1 x_2 x_3^{k-2}] & \text{if } k > 3, \\ [x_1 x_2] \cdot [x_1^{k-2}] - 3[x_1 x_2 x_3] & \text{if } k = 3. \end{cases}$$

If $k = 3$ we are now done, otherwise repeat the above induction step. The details are left to the reader. □

As an application of Lemma B.1 observe that we have

$$S_3 = \sigma_1 S_2 - \sigma_2 S_1 + 3\sigma_3 = \sigma_1(\sigma_1^2 - 2\sigma_2) - \sigma_2\sigma_1 + 3\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3,$$

agreeing with our earlier calculation. Here is a table showing the first few Newton polynomials:

Table 2.2. Newton polynomials

k	S_k
1	σ_1
2	$\sigma_1^2 - 2\sigma_2$
3	$\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$
4	$\sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4$
5	$\sigma_1^5 - 5\sigma_1^3\sigma_2 + 4\sigma_1^2\sigma_3 + 5\sigma_1\sigma_2^2 - 3\sigma_2\sigma_3 - 5\sigma_1\sigma_4 + 5\sigma_5$

The Newton polynomials also show up in the following:

Proposition B.3. *Let $\alpha = \alpha_1 t + \alpha_2 t^2 + \cdots \in R[[t]]$, where R is a commutative ring. Then*

$$\tfrac{d}{dt}\big(\log(1+\alpha)\big) = \tfrac{\alpha'}{1+\alpha} = \mu_1 + \mu_2 t + \mu_3 t^2 + \cdots$$

where $\mu_k = (-1)^k S_k(\alpha_1, \ldots, \alpha_k)$.

Proof. Equate coefficients in the identity

$$\alpha_1 + 2\alpha_2 t + 3\alpha_3 t^3 + \cdots \; = (1 + \alpha_1 t + \alpha_2 t^2 + \cdots) \cdot (1 + \mu_1 t + \mu_2 t^2 + \cdots).$$

This gives a series of identities for each μ_k that parallel the Newton identities. The result then follows by an easy induction. □

Appendix C. Homotopically compact pairs

By a **pair of topological spaces** we mean an ordered pair (X, A) where A is a subspace of A. A map of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subseteq B$, and such a map is said to be a weak equivalence if both $f\colon X \to Y$ and $f|A\colon A \to B$ are weak equivalences. Two maps $f, g\colon (X, A) \to (Y, B)$ are said to be homotopic if there is a map $H\colon (X \times I, A \times I) \to (Y, B)$ such that $H|_{X\times 0} = f$ and $H|_{X\times 1} = g$.

Define a topological space X to be **homotopically compact** if it is weakly equivalent to a finite CW-complex. Likewise, define a pair of topological spaces (X, A) to be homotopically compact if there exists a finite CW-pair (X', A') and a weak equivalence $(X', A') \to (X, A)$. In this case we call (X', A') a **finite model** for (X, A).

Proposition C.1. *A pair (X, A) is homotopically compact if and only if both X and A are homotopically compact.*

Proof. The "only if" direction is trivial, and the other direction is an immediate consequence of the slightly more general lemma below. □

Lemma C.2. *Let $f\colon A \to X$ be a map, where both A and X are homotopically compact. Let $\widetilde{A} \to A$ be any finite model for A. Then there exists a finite CW-complex $\widetilde{X}$, containing $\widetilde{A}$ as a subcomplex, together with a weak equivalence $\widetilde{X} \to X$ such that the square*

$$\begin{array}{ccc} \widetilde{A} & \rightarrowtail & \widetilde{X} \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & X \end{array}$$

commutes.

Proof. Let $\gamma_X : \widetilde{X} \to X$ be any finite model X. Since $[\widetilde{A}, \widetilde{X}] \to [\widetilde{A}, X]$ is a bijection, there is a map $\tilde{f}\colon \widetilde{A} \to \widetilde{X}$ such that $\gamma_X \tilde{f} \simeq f\gamma_A$. By cellular approximation, we may assume that $\tilde{f}$ is cellular. Choose such a homotopy. Let C_X denote the mapping cylinder of $\tilde{f}$, and let $\gamma_C : C_X \to X$ be the evident map. Note that γ_C gives a finite model for X, and that the diagram

$$\begin{array}{ccc} \widetilde{A} & \rightarrowtail & C_X \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ A & \xrightarrow{f} & X \end{array}$$

commutes. Here $\widetilde{A} \hookrightarrow C_X$ is the canonical inclusion into the top of the mapping cylinder. Since $(C_X, \tilde{A})$ is a finite CW-pair, the lemma is proven. □

Proposition C.3. *Let (X, A) be homotopically compact. If $f_0\colon (X_0, A_0) \to (X, A)$ and $f_1\colon (X_1, A_1) \to (X, A)$ are finite models for (X, A), then there exists a map $(X_0, A_0) \to (X_1, A_1)$ such that the triangle*

$$\begin{array}{ccc} (X_0, A_0) & \xrightarrow{\simeq} & (X, A) \\ {\simeq}\downarrow & \nearrow{\simeq} & \\ (X_1, A_1) & & \end{array}$$

commutes up to homotopy. Additionally, there exists a zig-zag of finite models

$$\begin{array}{ccccccccc} (Y_0,B_0) & \longrightarrow & (Y_1,B_1) & \longleftarrow & (Y_2,B_2) & \longrightarrow & \cdots & \longleftarrow & (Y_r,B_r) \\ & \searrow^{\simeq} & \downarrow^{\simeq} & & \downarrow^{\simeq} & & \swarrow^{\simeq} & & \\ & & & & (X,A) & & & & \end{array}$$

such that $(Y_0, B_0) \to (X, A)$ *equals* $(X_0, A_0) \to (X, A)$ *and* $(Y_r, B_r) \to (X, A)$ *equals* $(X_1, A_1) \to (X, A)$. *That is, the category of finite models of* (X, A) *is connected.*

Proof. First use that $[A_0, A_1] \to [A_0, A]$ is a bijection to produce a map $g \colon A_0 \to A_1$ whose image under the bijection is $f_0|_A$. We may assume that g is cellular. Choose a homotopy from $f_0|_A$ to $f_1|_A \circ g$, and using the Homotopy Extension Property extend this to a homotopy $H \colon X_0 \times I \to X$ such that $H_0 = f_0$. Let $f' = H_1$. We now have a commutative diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \simeq \\ X_0 & \xrightarrow{f'} & X \end{array}$$

and so by the Relative Homotopy Lifting Property (???) there exists a map $h \colon X_0 \to X_1$ such that the upper triangle commutes and the lower triangle commutes up to a homotopy relative to A_0. And again, we map assume that h is cellular. Putting our two homotopies together, we get the required homotopy-commutative triangle.

For the final statement of the proposition we can use a four-step zig-zag as follows:

$$\begin{array}{ccccccc} (X_0,A_0) & \xrightarrow{i_0} & (X_0 \times I, A_0 \times I) & \xleftarrow{i_1} & (X_0,A_0) & \xrightarrow{h} & (X_1,A_1) \\ & \searrow^{\simeq}_{f_0} & J \downarrow \simeq & \swarrow^{\simeq}_{f_1 h} & & \swarrow^{\simeq}_{f_1} & \\ & & (X,A). & & & & \end{array}$$

The map labelled J is a homotopy for the triangle in the first part of the proposition. We leave the details for the reader. □

For us what is very useful about the class of homotopically compact spaces is that it includes all algebraic varieties:

Theorem C.4. *If* X *is an algebraic variety over* $\mathbb{C}$ *then* X *is homotopically compact.*

Proof. When X is a subvariety of some $\mathbb{C}^n$ this is a consequence of [Hir, Theorem on page 170 and Remark 1.10]. For the general case we do an induction on the size of an affine cover for X. Suppose that $\{U_1, \ldots, U_n\}$ is an affine cover, and let $A = U_2 \cup \cdots \cup U_n$. Then we have the pushout diagram

$$\begin{array}{ccc} U_1 \cap A & \longrightarrow & A \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & X, \end{array}$$

which is also a homotopy pushout by [DI, Corollary 1.6]. By induction we know that A is homotopically compact. Moreover, since U_1 is affine it is a subvariety of some $\mathbb{C}^n$, and therefore the same is true of $U_1 \cap A$. So both U_1 and $U_1 \cap A$ are homotopically compact by the base case. The result then follows by Lemma C.5 below. □

Lemma C.5. *Let A, X, and Y be homotopically compact spaces. Then the homotopy pushout of any diagram $X \leftarrow A \rightarrow Y$ is also homotopically compact.*

Proof. Let $f\colon A \to X$ and $g\colon A \to Y$ denote the maps, and choose a finite model $\widetilde{A} \to A$. By Lemma C.2 (applied twice) there exists a diagram

$$\begin{array}{ccccc} \widetilde{X} & \longleftarrow & \widetilde{A} & \longrightarrow & \widetilde{Y} \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow\simeq \\ X & \longleftarrow & A & \longrightarrow & Y. \end{array}$$

where $(\widetilde{X}, \widetilde{A})$ and $(\widetilde{Y}, \widetilde{A})$ are finite CW-pairs. The homotopy pushout of $X \leftarrow A \rightarrow Y$ is therefore weakly equivalent to that of $\widetilde{X} \leftarrow \widetilde{A} \rightarrow \widetilde{Y}$, and the latter clearly has the homotopy type of a finite CW-complex (in fact, in the latter case the pushout is itself a model for the homotopy pushout). □

Corollary C.6. *If (X, A) is a pair of algebraic varieties over $\mathbb{C}$ then (X, A) is homotopically compact.*

Proof. This follows from Theorem C.4 and Proposition C.1. □

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