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DISCRETE CHAOS:
SEQUENCES SATISFYING "STRANGE" RECURSIONS

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1. Historical Summary

Analogous to the Fibonacci Sequence, defined by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ $\forall\, n > 2$, D. Hofstadter (in [1]) defined a sequence $\{q_n\}$ by $q_1 = q_2 = 1$, $q_n = q_{n-q_{n-1}} + q_{n-q_{n-2}}$, which he called a "strange" recursion, in that the subscripts depend on terms in the sequence itself. He asserted that this sequence has no discernable regularities, and this remains very nearly true.

A somewhat better behaved sequence $\{c_n\}$, proposed by J.H. Conway (private communication) is defined by $c_1 = c_2 = 1$, $c_n = c_{n-c_{n-1}} + c_{c_{n-1}}$. Unlike $\{q_n\}$, the sequence $\{c_n\}$ is monotone non-decreasing, and in fact, $d_n = c_n - c_{n-1}$ is restricted to the values 0 and 1. Regularities include: $n \geq c_n \geq \frac{n}{2}$ for all n, with $c_n = \frac{n}{2}$ iff $n = 2^k$, $k \geq 1$. Even so, $\{d_n\}$ is a good "pseudo-random" binary sequence, and appears to approximate "G-randomness" as defined in [2].

Golomb proposed the recursion $a_n = a_{n-a_{n-1}}$, with a choice of initial conditions, as a very simple example of a "strange" recursion. U. Cheng showed [3] that even in this very simple case, some quite unusual and "strange" things can happen.

The design of virtually all modern digital computers makes it easy to perform arithmetic on indices, and hence to carry out the calculation of sequences which are well-defined by a "strange" recursion and appropriate initial conditions. Some of these sequences are likely to be useful in applications where "pseudo-random" sequences of integers are required.

The theory of "strange" recursions may be regarded as the discrete case of the theory of "strange attractors," which has become very fashionable in the last few years, and which is also referred to as the theory of "chaos" [8].

2. Classical Recursions

The archetype of classical recursions is the Fibonacci sequence, defined by

$$f_n = f_{n-1} + f_{n-2}, \quad f_1 = f_2 = 1. \tag{1}$$

The first forty terms of this sequence are given in Table 1, together with two different "closed-form" expressions for the n^{th} term of the sequence.

In the nineteenth century, Edouard Lucas in France described the analysis and properties of sequences satisfying any "linear recurrence" of any degree k over the real number field:

$$a_n = \sum_{j=1}^{k} c_j a_{n-j}. \tag{2}$$

The behavior of such linear recursions has also been studied as over finite fields [4], and over polynomial rings [5]. Classes of nonlinear recursions over the real numbers have been studied in [6] and [7], and over finite fields in [4].

$$f_n = f_{n-1}+f_{n-2}, \qquad f_1 = f_2 = 1$$

n	f_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
12	144
13	233
14	377
15	610
16	987
17	1,597
18	2,584
19	4,181
20	6,765
21	10,946
22	17,711
23	28,657
24	46,368
25	75,025
26	121,393
27	196,418
28	317,811
29	514,229
30	832,040
31	1,346,269
32	2,178,309
33	3,524,578
34	5,702,887
35	9,227,465
36	14,930,352
37	24,157,817
38	39,088,169
39	63,245,986
40	102,334,155

FIBONACCI'S SEQUENCE

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

$$f_n = \sum_{1\le j\le\left[\frac{n+1}{2}\right]} \binom{n-j}{j-1}.$$

$$I(f_n) = D(f_n)+D^2(f_n),$$

$$(D^2+D-I)(f_n) \equiv (0).$$

$$\lambda = \frac{-1\pm\sqrt{5}}{2}.$$

Other initial conditions will produce other sequences with the same characteristic equation, and with the "largest eigenvalue" ($|\lambda_1| = \left|\frac{-1-\sqrt{5}}{2}\right| = 1.618\ldots$) determining the asymptotic rate of growth.

TABLE 1

3. Hofstadter's Sequence $\{q_n\}$

The first 280 terms of Hofstadter's sequence $\{q_n\}$ are shown in Table 2. Hofstadter's sequence has the following properties:

1. Unless $1 \le q_n \le n$ for all $n \ge 1$, the sequence will not be well-defined. It is almost certainly true that $1 \le q_n \le n$ for all $n \ge 1$, but no proof of this has yet been given.

2. The sequence is *very* sensitive to its initial conditions, and the given sequence is the *only* one which neither "blows up" (by becoming undefined) nor degenerates into a rather deterministic pattern, for the given recursion. (The subscripts can all be translated a uniform amount without affecting the sequence itself.)

Has this been proved?

3. With $Q_1 = 3$, $Q_2 = 2$, $Q_3 = 1$, and Hofstadter's recursion, the resulting sequence is quasi-periodic with a quasi-period of 3:

n	Q_n
1	3
2	2
3	1
4	3
5	5
6	4
7	3
8	8
9	7
10	3
11	11
12	10

$$\boxed{\begin{aligned} Q_{3k+1} &= 3 \\ Q_{3k+2} &= 3k+2 \\ Q_{3k} &= 3k-2 \end{aligned}}$$

(It is easy to prove that this sequence satisfies the recursion, by induction.)

n	Q_n
13	3
14	14
15	13
16	3
17	17
18	16
19	3
20	20
21	19
22	3
23	23
24	22

4. *If* there is a limiting value l such that $\lim_{n\to\infty} \frac{q_n}{n} = l$, *then* $l = ½$.

Proof. Assume l exists, $0 \le l \le 1$. Then for very large n, the recursion is arbitrarily well approximated by

$$ln \approx l(n-l(n-1)) + l(n-l(n-2))$$

$$n \approx (n-l(n-1)) + \cancel{l}(n-l(n-2))$$

?

$$l(n-1) + l(n-2) \approx n, \quad 2nl - 3l \approx ln, \quad l \approx ½ + \frac{3\lambda}{2n},$$

$l \approx \frac{n}{2n-3}$

and since $n \to \infty$, $l = ½$.

5. *Apparent* regularities in Hofstadter's sequence include:

$q_4=3=2+1$	$q_7=5$	$q_3=2$	$:q_{192}=128$:	$q_5=3.$
$q_8=5=4+1$	$q_{15}=10$	$q_6=4$	$:q_{384}=256$:	$q_{10}=6.$
$q_{16}=9=8+1$	$q_{31}=20$	$q_{12}=8$	$:q_{768}=512$: BUT	$q_{20}=12.$
$q_{32}=17=16+1$	$q_{63}=40$	$q_{24}=16$	$:q_{1536}<1024$:	BUT
$q_{64}=33=32+1$	BUT	$q_{48}=32$	$:q_{3072}<2048$:	$q_{40}=22.$
BUT: $q_{128}=64.$	$q_{127}=68$	$q_{96}=64$	$:q_{6144}<4096$	

$q_{256} = 128-5$

However, none of these regularities persist!

6. *Statistical "regularities": $Pr\{|q_n - \frac{n}{2}| > \eta\} \to 0$* in the sense that, for fixed $\eta > 0$, the fraction of the set of integers $[10^k, 10^{k+1}]$ for which $|q_n - \frac{n}{2}| > \eta$ *appears* to go to 0 as $k \to \infty$. Statistics have been studied for $n \le 10^6$. Nothing has been *proved.*

7. Note the swings and oscillations around $n = 3 \cdot 2^k$. E.g. $q_{186} = q_{187} = \cdots = q_{191} = 96$, $q_{192} = 128$, $q_{193} = 72$.

8. The apparent regularities noted above, which do not persist, are strongly reminiscent of the "chaos" in the theory of "strange attractors" [8].

$$q_n = q_{n-q_{n-1}} + q_{n-q_{n-2}}, \quad n \geq 2; \quad q_1 = q_2 = 1.$$

n	q_n	n	q_n	n	q_n	n	q_n	n	q_n	n	q_n	n	q_n
1	1	41	23	81	44	121	72	161	82	201	106	241	132
2	1	42	23	82	43	122	58	162	85	202	124	242	113
3	2	43	24	83	43	123	61	163	84	203	82	243	133
4	3	44	24	84	46	124	78	164	84	204	101	244	123
5	3	45	24	85	44	125	57	165	88	205	111	245	118
6	4	46	24	86	45	126	71	166	83	206	108	246	125
7	5	47	24	87	47	127	68	167	87	207	118	247	121
8	5	48	32	88	47	128	64	168	88	208	104	248	129
9	6	49	24	89	46	129	63	169	87	209	108	249	122
10	6	50	25	90	48	130	73	170	86	210	106	250	136
11	6	51	30	91	48	131	63	171	90	211	114	251	129
12	8	52	28	92	48	132	71	172	88	212	104	252	116
13	8	53	26	93	48	133	72	173	87	213	114	253	149
14	8	54	30	94	48	134	72	174	92	214	109	254	137
15	10	55	30	95	48	135	80	175	90	215	100	255	120
16	9	56	28	96	64	136	61	176	91	216	109	256	123
17	10	57	32	97	41	137	71	177	92	217	120	257	143
18	11	58	30	98	52	138	77	178	92	218	112	258	146
19	11	59	32	99	54	139	65	179	94	219	108	259	107
20	12	60	32	100	56	140	80	180	92	220	118	260	139
21	12	61	32	101	48	141	71	181	93	221	106	261	138
22	12	62	32	102	54	142	69	182	94	222	105	262	139
23	12	63	40	103	54	143	77	183	94	223	130	263	135
24	16	64	33	104	50	144	75	184	96	224	110	264	120
25	14	65	31	105	60	145	73	185	94	225	114	265	146
26	14	66	38	106	52	146	77	186	96	226	115	266	135
27	16	67	35	107	54	147	79	187	96	227	112	267	143
28	16	68	33	108	58	148	76	188	96	228	107	268	129
29	16	69	39	109	60	149	80	189	96	229	120	269	151
30	16	70	40	110	53	150	79	190	96	230	114	270	133
31	20	71	37	111	60	151	75	191	96	231	122	271	135
32	17	72	38	112	60	152	82	192	128	232	121	272	136
33	17	73	40	113	52	153	77	193	72	233	120	273	148
34	20	74	39	114	62	154	80	194	96	234	114	274	148
35	21	75	40	115	66	155	80	195	115	235	138	275	136
36	19	76	39	116	55	156	78	196	100	236	110	276	144
37	20	77	42	117	62	157	83	197	84	237	122	277	143
38	22	78	40	118	68	158	83	198	114	238	119	278	152
39	21	79	41	119	62	159	78	199	110	239	120	279	129
40	22	80	43	120	58	160	85	200	93	240	130	280	139

HOFSTADTER'S SEQUENCE

TABLE 2

4. Conway's Sequence $\{c_n\}$

The first 160 terms of Conway's sequence $\{c_n\}$ are given in Table 3. This sequence has the following *provable* regularities:

Properties of Conway's Sequence $\{c_n\}$

1. $\{c_n\}$ is monotonic non-decreasing.

2. In fact, $c_{n+1} - c_n = 0$ or 1, for all n.

3. Fact 2 is proved by induction, using the further fact that: of the two summands which combine to form c_{n+1}, one of them is one of the two summands for c_n, and the other is the other summand of c_n with its argument advanced by 1. To illustrate:

$$c_{100} = c_{56}+c_{100-56} = c_{56}+c_{44} = 31+26 = 57,$$

$$\text{Hence,}\quad c_{101} = c_{57}+c_{101-57} = c_{57}+c_{44} = 31+26 = 57,$$

$$\text{and}\quad c_{102} = c_{57}+c_{102-57} = c_{57}+c_{45} = 31+26 = 57.$$

4. $\{d_n\} = \{c_{n+1}-c_n\}$ is a reasonable pseudo-random binary sequence, especially in view of Fact 5.

5. $c_n \geq \frac{n}{2}$ for all $n \geq 1$, with $c_n = \frac{n}{2}$ iff $n=2^k$, k=1,2,3,4,5,....

6. $\{d_n\}$ has arbitrarily long *runs* of 0's (*e.g.* terminating at the values $n=2^k$, $\{d_n\}$ has $\geq$ $k-1$ consecutive 0's; followed by $\geq k$ consecutive 1's) and of 1's.

7. $\lim_{n\to\infty} \frac{c_n}{n}$ *exists* and equals ½. (The second part follows from Fact 5.)

8. Formulas can be given for c_n by relating n to the "nearest" power of 2.

9. The sequence $\{d_n\}$ seems to closely approximate "G-randomness", as defined in [2], based on the "randomness properties" of Chapter 3 of [4].

10. In ~~the~~ terminology of "strange attractor" theory, Conway's sequence $\{c_n\}$ is "tamely chaotic", while Hofstadter's sequence $\{q_n\}$ is "wildly chaotic". See notes by Reiter for more on this Check

CONWAY'S SEQUENCE: $c_n = c_{n-c_{n-1}} + c_{c_{n-1}}$, $c_1 = c_2 = 1$.

n	c_n	n	c_n	n	c_n	n	c_n
1	1	41	24	81	46	121	63
2	1	42	24	82	47	122	64
3	2	43	25	83	47	123	64
4	2	44	26	84	48	124	64
5	3	45	26	85	48	125	64
6	4	46	27	86	48	126	64
7	4	47	27	87	49	127	64
8	4	48	27	88	50	128	64
9	5	49	28	89	51	129	65
10	6	50	29	90	51	130	66
11	7	51	29	91	52	131	67
12	7	52	30	92	53	132	68
13	8	53	30	93	53	133	69
14	8	54	30	94	54	134	70
15	8	55	31	95	54	135	71
16	8	56	31	96	54	136	71
17	9	57	31	97	55	137	72
18	10	58	31	98	56	138	73
19	11	59	32	99	56	139	74
20	12	60	32	100	57	140	75
21	12	61	32	101	57	141	76
22	13	62	32	102	57	142	76
23	14	63	32	103	58	143	77
24	14	64	32	104	58	144	78
25	15	65	33	105	58	145	79
26	15	66	34	106	58	146	80
27	15	67	35	107	59	147	80
28	16	68	36	108	60	148	81
29	16	69	37	109	60	149	82
30	16	70	38	110	61	150	83
31	16	71	38	111	61	151	83
32	16	72	39	112	61	152	84
33	17	73	40	113	62	153	85
34	18	74	41	114	62	154	85
35	19	75	42	115	62	155	86
36	20	76	42	116	62	156	86
37	21	77	43	117	63	157	86
38	21	78	44	118	63	158	87
39	22	79	45	119	63	159	88
40	23	80	45	120	63	160	89

TABLE 3

5. The Recursion $a_n = a_{n-a_{n-1}}$

This simple-looking one-term "strange" recursion hides a great wealth of behaviors, which depend on the initial conditions which are used. Three simple examples are shown below.

I.

a_1	2
a_2	5
a_3	2
a_4	5
a_5	a_0=?

II.

a_0	5
a_1	2
a_2	5
a_3	2
a_4	5
a_5	5
a_6	2
a_7	5
a_8	2
a_9	5
a_{10}	5
a_{11}	2

III.

a_1	2	←(transient)
a_2	3	
a_3	2	
a_4	3	
a_5	3	
a_6	2	
a_7	3	
a_8	3	
a_9	2	
a_{10}	3	
a_{11}	3	
a_{12}	2	

Observations:

1. Not all initial conditions lead to well-defined sequences.

2. If the sequence is well-defined from the initial conditions, it will be *ultimately periodic*; and it can never have any term which did not already appear as a value among the initial conditions.

3. *Proof* of the "ultimately periodic" property:

   Let $\{a_1, a_2, \ldots, a_{n_0}\}$ be a "proper initial condition", so that $\{a_n\}$ is defined for *all* $n \geq 1$. The possible values for *any* a_n are the distinct integers (say k of them) in the set

$\{a_1, a_2, \ldots, a_{n_0}\}$. Let m be the *largest* of these. Then a_n (for $n > n_0$) is uniquely determined by the m-tuple $(a_{n-m}, a_{n-m+1}, \ldots, a_{n-1})$. The total number of m-tuples of k distinct integers is $\cancel{m^k}$ k^m. Hence there are integers n_1 and p, $1 \le n_1 \le k^m + m$, and $1 \le p \le k^m$, such that $a_n = a_{n-p}$ for all $n > n_1$.

Constructing Cycles with Long Periods, for $a_n = a_{n-a_{n-1}}$

FIRST EXAMPLES, for $p = 9$.

First Step:	(9,9,9,9,9,9,9,9,9).	$p=1$, $m=9$, $k=1$.
Second Step:	(9,9,3,9,9,9,9,9,9).	$p=9$, $m=9$, $k=2$.
Third Step:	(6,9,3,6,9,9,9,9,9).	$p=9$, $m=9$, $k=3$.
Fourth Step:	(6,9,3,6,9,4,3,9,6).	$p=9$, $m=9$, $k=4$.

Question: Can the *period* be bigger than the biggest term in the sequence?

Answer: *Yes!* (due to Unjeng Cheng [3]).

SECOND EXAMPLES, $p \ge 9$.

#1. (6,9,3,6,3,3;6,9,6,6,3,6). $p=12$, $m=9$, $k=3$. Compare the "first half" and "second half" of the cycle!

#2. (8,12,8,12,8,4,8,4;8,12,2,12,8,4,2,4). $p=16$, $m=12$, $k=4$.

#3. (18,3,3;18,3,3;18,21,18;18,3,18;18,21,18,18,3,18;18,3,18;18,3,18;18,21,3;
18,3,3;18,21,18;18,3,18;18,3,18;18,3,18;18,21,81;18,3,18;18,21,3;18,3,3).
$p=54$, $m=21$, $k=3$.

In general, U. Cheng showed [3] that examples exist which make p/m arbitrarily large.

Suppose $u(u(n)) = u(n)$.
Let $R = \{n : u(n) = n\}$
(i) $R = \emptyset$: [illegible] $= S_n$ [illegible]
$u(S_n) = S_n$ so $S_n \in R = \emptyset$,
[illegible] Thus $R \neq \emptyset$
(ii) $R \neq \emptyset$: For any $n \notin R$, [illegible]
must be true that $u(u(n)) = u(n) \neq n$
so $u(n) \in R$ [illegible]
values in R.

6. "Strange" Recursions with Known Solutions

The simplest "strange" recursion is

$$u_{u_n} = u_n,$$

for which the most general solution is given as follows:

Let R be any non-empty subset of the positive integers. If $n \in R$, require $u_n = n$. For each $n \notin R$, arbitrarily pick a value r_n from the set R, and define $u_n = r_n$.

It is easy to prove that this is the general solution. As an example, let $R = \{2,7,10\}$, and use these values as the *range* of $u(n)$ described above. For example, we could set

n	u_n
1	7
2	2
3	10
4	10
5	2
6	7

n	u_n
7	7
8	7
9	2
10	10
11	2
12	2

We then verify:

$$n = 1,\ 7 = u_1 = u_{u_1} = u_7 = 7$$

$$n = 2,\ 2 = u_2 = u_{u_2} = u_2 = 2$$

$$n = 3,\ 10 = u_3 = u_{u_3} = u_{10} = 10$$

$$n = 4,\ 10 = u_4 = u_{u_4} = u_{10} = 10$$

etc.

From this starting point, progressively more complicated "strange" recursions can be

$b(\underbrace{b(n)+kn}_{m}) = 2b(n)+kn$

$b(m) = 2b(m-kn)+kn$ $\neq (m - b(n))$

investigated.

For example, for any positive integer k, there is a "strange" recursion

$$2b_n + kn = b_{b_n+kn}\,. \qquad (3)$$

arbitrary?

As initial condition, set $b_1 = 1$, and $b_2 = 3$ if $k=1$ but $b_2 = 2$ for all $k>1$. Then the sequence given by

e.g. $k=1$: $2b_3+3 = b_{b_3+3}$

The "+" sign is the problem doesn't permit unique identification of term based on preceding ones

$$b_n^{(k)} = [n\,\alpha^{(k)}] \qquad (4)$$

satisfies the recursion (3), where $\alpha^{(k)}$ is the positive root of $x^2+(k-2)x-k=0$, namely $\alpha^{(k)} = \frac{2-k+\sqrt{k^2+4}}{2}$, and $[y]$ denotes the greatest integer $\le y$.

Proof? False for k=1

Although (4) is a solution of the recursion (3), it is not the only solution. It appears to be the only monotonically increasing solution, however. Conway's sequence, which is monotone non-decreasing, is a close relative of these sequences, but is uniquely specified by its recursion and a simple (single two) initial condition. No finite (number of) initial conditions is sufficient to uniquely specify the solution sequence of (3) for any given k. A. Fraenkel (private communication) suggested the study of the sequences given by (4) in this context. fraenkel@wisdom.weizmann.ac.il.

A recursion which has a much simpler solution than one might expect from its "strange" appearance is

what if we used some other values?

$$g_n = g_{n-g_{n-1}}+1, \quad g_1 = 1. \qquad (5)$$

There is a uniquely determined solution sequence for (5), with $g_1 = 1$, $g_2 = g_3 = 2$, $g_4 = g_5 = g_6 = 3$, and in general each positive integer k occurring successively k times as the value of g_n. (See Table 4.) The transitions occur after each time that n is a "triangular number,"

n: 1 2 3 4 5 6 7 8 9 10
g_n: 1 2 2 3 3 3 4 4 4 4

$10 = 4+3+2+1$

$\frac{n(n+1)}{2}$: 1 3 6 ....

specifically where

$$n = \frac{g_n^2 + g_n}{2} . \quad = \quad \frac{g_n(g_n+1)}{2}$$

From this, we have the quadratic equation

$$g_n^2 + g_n - 2n = 0,$$

so that for all "triangular" values of n,

$$g_n = \frac{-1+\sqrt{8n+1}}{2} ,$$

and it is easy to show that for *general* n,

$$g_n = \left[\frac{[\sqrt{8n}\,]+1}{2} \right],$$

$n=2$ $g_2 =$ [illegible] 2

$g_4 =$ [illegible] 3

where $[y]$ denotes the greatest integer not exceeding y.

$g_6 =$ [illegible] $= 3$

This furnishes an important example of a recursion which looks as "strange" as several others we have considered, but where the resulting sequence is completely regular and predictable. It is a challenging unsolved problem to categorize those "strange" recursions which have well-behaved, closed-form solutions.

n	g(n)	n	g(n)	n	g(n)
1	1	16	6	31	8
2	2	17	6	32	8
3	2	18	6	33	8
4	3	19	6	34	8
5	3	20	6	35	8
6	3	21	6	36	8
7	4	22	7	37	9
8	4	23	7	38	9
9	4	24	7	39	9
10	4	25	7	40	9
11	5	26	7	41	9
12	5	27	7	42	9
13	5	28	7	43	9
14	5	29	8	44	9
15	5	30	8	45	9

TABLE 4

The Sequence $g_n = g_{n-g_{n-1}}+1, \quad g_1 = 1$

References

[1] D. Hofstadter, *Gödel, Escher, Bach, An Eternal Golden Braid*, Random House, New York, 1979.

[2] H. Beker and F. Piper, *Cipher Systems, the Protection of Communications*, John Wiley and Sons, 1982.

*[3] U. Cheng, "Properties of Sequences," Ph.D. Dissertation, USC Dept. of Electrical Engineering, 1981.

[4] S.W. Golomb, *Shift Register Sequences*, Holden-Day, Inc., 1967. Revised edition, Aegean Park Press, 1982.

[5] S.W. Golomb and A. Lempel, "Second Order Polynomial Recursions," *SIAM Journal on Applied Mathematics*, vol. 33, no. 4, December 1977, pp. 587-592.

[6] S.W. Golomb, "On Certain Nonlinear Recurring Sequences," *American Math. Monthly*, vol. 70, no. 4, April 1963. nothing of interest

[7] J.N. Franklin and S.W. Golomb, "A Function-Theoretic Approach to the Study of Nonlinear Recurring Sequences," *Pacific Journal of Mathematics*, February 1975.

[8] J. Gleick, *CHAOS*, Penguin-Viking, 1988.

Are there measures of "randomness" (like G-random, etc, pseudorandomness)?
If so, can we relate MF sequences, or even portions of them (like blocks) to these measures, so we can categorize how initial conditions affect their measure of randomness?