$$b + c = i \qquad \left(\leq \left[\frac{1}{2} n \right] \right),$$

gives

nd $\quad b + c \leq n$

$\leq a$, where

$+ b + c = n$

itions (b, c)

$+ c \leq a$, fa

nce betwee

o 2 parts o

the $p_3(n$

in

$p_2(j)$.

partitions, we see tha

for $\quad j = 2k + 1,$ the partitions are $(1, 2k)$
$\ldots, (k, k + 1)$;

for $\quad j = 2k,$ the partitions are $(1, 2k -$

Scan AS044
+2 more
Honsberger
Math Gems III
But just
a few pages
(4)

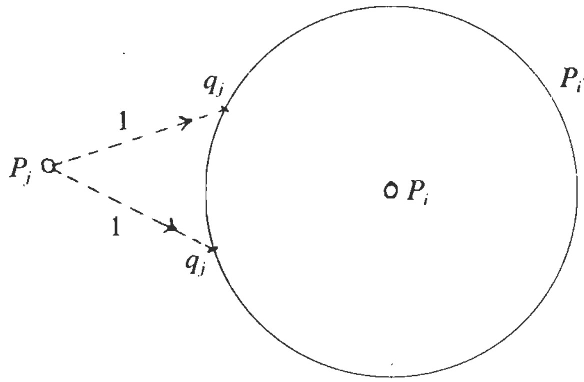
Ross Honsberger, Math Gems III

A 5044

A69905

A1399

tions for each point p_j. Taking into account all the pairs of points (p_i, p_j), and the inadvisable directions $p_i p_j$, noted earlier, a great many directions might be ruled out. However, for any n, the number of unsafe directions will be finite, leaving any number of satisfactory choices for d, and the conclusion follows by induction.



Directions to avoid

2. A Problem about Triangles

Our second problem was very kindly brought to my attention by the outstanding number theorist George Andrews (Pennsylvania State University).

> How many different triangles are there which have integral sides and perimeter n?

A complete specification of this number $T(n)$ is given in a paper by J. H. Jordan, Ray Walch, and R. J. Wisner [5]. In a brief note [6] George Andrews gives a beautiful solution which exploits the rather natural connection between $T(n)$ and $p_3(n)$ and $p_2(n)$, the number of ways of partitioning an integer n into 3 and 2 parts, respectively.

Andrews' Solution. A partition of n into 3 parts, $a + b + c = n$, generally defines a triangle counted by $T(n)$. The only way a partition will not do so is by failing to satisfy one of the triangle inequalities

$$a + b > c, \qquad b + c > a, \qquad c + a > b.$$

[5] is Jordan et al, "Triangles with integer sides." Amer. Math. Monthly, 86 (1979) 686-689.

But this can happen only when the sum of the two smaller parts, say $b + c$, fails to exceed the largest part a: i.e., when $b + c \le a$. In this case, b and c would add up to some integer $j \le (1/2)n$, which, because only integers are involved, is equivalent to

$$b + c = j \le \left[\frac{1}{2}n\right],$$

where $[x]$ is used to denote the greatest integer $\le x$.

Conversely, suppose j is a positive integer $\le [(1/2)n]$. Then each of the $p_2(j)$ partitions of j into 2 parts,

$$b + c = j \qquad \left(\le \left[\frac{1}{2}n\right]\right),$$

gives

$$b + c + j \le n, \qquad \text{and} \qquad b + c \le n - j.$$

Letting $n - j = a$, we have $b + c \le a$, where

$$a + b + c = n - j + b + c = n.$$

That is to say, each of the $p_2(j)$ partitions (b, c) corresponds to a partition (a, b, c) of n that, in view of $b + c \le a$, fails to generate a triangle. Thus there is a 1–1 correspondence between the failing partitions (a, b, c) of n and the partitions into 2 parts of the integers j in the range 1 to $[(1/2)n]$. Subtracting from the $p_3(n)$ possible partitions the total number of failures, we obtain

$$T(n) = p_3(n) - \sum_{1 \le j \le [(1/2)n]} p_2(j).$$

Now, by simply listing all the partitions, we see that $p_2(j)$ is always just $[(1/2)j]$:

for $j = 2k + 1$, the partitions are $(1, 2k)$, $(2, 2k - 1)$, $\ldots$, $(k, k + 1)$;

for $j = 2k$, the partitions are $(1, 2k - 1)$, $(2, 2k - 2)$, $\ldots$, (k, k);

thus $p_2(j) = k = [(1/2)j]$.

Using this result, it is an exercise in mathematical induction to establish that the sum in question is given by $[n/4][(n+2)/4]$, yielding

$$T(n) = p_3(n) - \left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right].$$

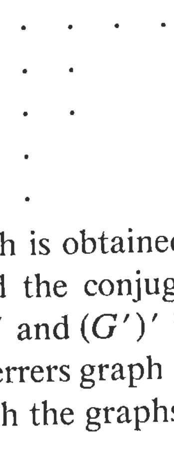
A1399 ? A69905

But, as we shall see, $p_3(n)$ is given by the formula $p_3(n) = \{n^2/12\}$, where $\{x\}$ denotes the integer *nearest* x (since no square is ever halfway between two multiples of 12, this is never ambiguous). Finally, then, we have the pretty result

$$T(n) = \left\{\frac{n^2}{12}\right\} - \left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right].$$

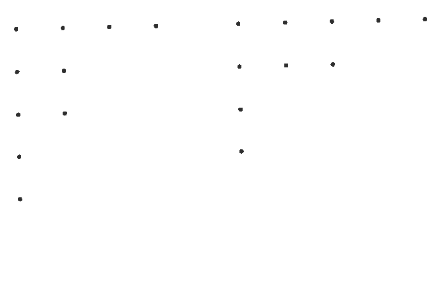
A5044

The Formula for $p_3(n)$. Let us prove this formula for $p_3(n)$. In order to do this, we introduce a simple geometric representation of a partition which is known as a Ferrers graph. In a Ferrers graph G, each part r in the partition is represented by a row of r equally spaced dots. In preparing a partition for representation by a Ferrers graph, it needs to be written in nonincreasing order. Accordingly, the rows in a Ferrers graph, which are lined up one under the other from the left so that the dots fall into columns, also occur in nonincreasing order. For example, $10 = 4 + 2 + 2 + 1 + 1$ yields the Ferrers graph:



The Ferrers graph G', which is obtained from G by interchanging its rows and columns, is called the conjugate of G. Clearly a graph G has exactly one conjugate G' and $(G')'$ is simply G, itself. Thus, if the conjugate is taken of each Ferrers graph in a set S, the set of conjugates is in 1–1 correspondence with the graphs of S.

G: G':



Let us consider, then, the collection of Ferrers graphs of the partitions that are counted by $p_3(n)$. Each of these graphs G will have exactly 3 rows, which means that each conjugate G' will have exactly 3 dots in its first row and no more than 3 dots in any other row. As such, each G' represents a partition of n in which every part

is either a 1, 2, or 3, and at least one part must be a 3. Since $p_3(n)$ counts *all* partitions containing exactly 3 parts, it is not difficult to see that the set of conjugates $\{G'\}$ must represent *all* partitions of this kind (i.e., with parts that are 1's, 2's, or 3's, with at least one 3). In fact, the whole purpose of these Ferrers graphs and their conjugates is simply to provide us with a nice way of seeing that there exists a 1–1 correspondence between the partitions counted by $p_3(n)$ and the partitions of n into 1's, 2's, and 3's, where at least one 3 must occur. We have, then, that the *number* $p_3(n)$ is the same as the number of partitions in this latter class.

Now, if this obligatory 3 is removed from a conjugate partition, we obtain a partition of $n - 3$ into 1's, 2's, and 3's which has no additional qualifying condition on its composition (it may or may not have any 3's left). If this is done to each conjugate partition of n, then, a set of partitions of $n - 3$ is obtained which is clearly in 1–1 correspondence with the set of conjugate partitions of n, and, therefore, is also in a 1–1 correspondence with the partitions counted by $p_3(n)$. Letting $p(\text{“}A\text{”}, m)$ denote the number of partitions of m which have parts from the set of numbers A, we have that

$$p_3(n) = p(\text{“}\{1, 2, 3\}\text{”}, n - 3).$$

At last we have arrived at an expression for $p_3(n)$ that we are able to handle.

In order to calculate this quantity we turn to one of the premier tools of the combinatorialist—generating functions [7]. Consider the product

$$f(x) = (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)$$
$$(1 + x^3 + x^6 + x^9 + \cdots).$$

In multiplying these series together, one of the terms in x^{16}, for example, is obtained by taking

x^3 from the first factor,

x^4 from the second, and

x^9 from the third.

This displays the exponent 16 in the form

$$16 = 3 + 4 + 9,$$

which we may construe to be

$$\begin{aligned} 16 &= 3(1) + 2(2) + 3(3) \\ &= 1 + 1 + 1 + 2 + 2 + 3 + 3 + 3, \end{aligned}$$

corresponding to a partition of 16 in which only 1's, 2's, and 3's occur (the number of 1's comes from the first factor in $f(x)$, and so on). Conversely, every such partition of 16 can be used as a prescription for selecting terms from the three factors of $f(x)$, based on the number of 1's, 2's, and 3's that are called for, that will generate a term in x^{16}. Consequently, the total coefficient of x^{16} in $f(x)$ is just $p(``\{1, 2, 3\}", 16)$, and in general, the desired $p(``\{1, 2, 3\}", n - 3)$ is the coefficient of x^{n-3} in $f(x)$. Fortunately we can determine this coefficient by elementary methods.

First of all, observe that the binomial theorem gives

$$(1 - x^k)^{-1} = 1 + x^k + x^{2k} + x^{3k} + \cdots,$$

making

$$\begin{aligned} f(x) &= (1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \\ &= \frac{1}{(1 - x)(1 - x^2)(1 - x^3)}. \end{aligned}$$

Now, by resolving $f(x)$ into its partial fractions, we obtain

$$f(x) = \frac{1/6}{(1-x)^3} + \frac{1/4}{(1-x)^2} + \frac{1/4}{1-x^2} + \frac{1/3}{1-x^3}$$

$$= \frac{1}{6}(1-x)^{-3} + \frac{1}{4}(1-x)^{-2} + \frac{1}{4}(1-x^2)^{-1} + \frac{1}{3}(1-x^3)^{-1}.$$

Therefore, the desired coefficient of x^{n-3} is the sum of the coefficients of x^{n-3} that are obtained from these four parts. But these may be extracted by straightforward applications of the binomial theorem. From the first part we get

$$\frac{1}{6}\cdot\frac{(-3)(-4)\cdots[-3-(n-3)+1]}{(n-3)!}(-1)^{n-3}$$
$$= \frac{1}{6}\cdot\frac{3\cdot 4\cdots(n-1)}{(n-3)!} = \frac{1}{6}\cdot\frac{(n-2)(n-1)}{2},$$

and from the second part we similarly obtain $(1/4)(n-2)$.
In the third part, only even powers of x occur, and we obtain the coefficient 0 if $n-3$ is odd, and $(1/4)(1) = 1/4$ if $n-3$ is even. We may express this by saying that the coefficient is $(1/4)k$, where k is either 0 or 1. Similarly, in the final part, the coefficient is $(1/3)t$, where t is either 0 or 1. In these terms, then, we have

$$p_3(n) = \frac{1}{6}\cdot\frac{(n-1)(n-2)}{2} + \frac{1}{4}(n-2) + \frac{1}{4}k + \frac{1}{3}t$$
$$= \frac{n^2 - 4 + 3k + 4t}{12}.$$

Now the most that $3k + 4t$ can be is 7 and the least is 0. Therefore, we have

$$\frac{n^2-4}{12} \le p_3(n) \le \frac{n^2+3}{12},$$

that is,

$$\frac{n^2}{12} - \frac{1}{3} \le p_3(n) \le \frac{n^2}{12} + \frac{1}{4}.$$

Thus the integer $p_3(n)$ does not differ from $n^2/12$ by more than 1/3, making it the integer nearest $n^2/12$, as claimed.

Another Approach to $T(n)$. Finally, let us close with a most elegant solution of this problem, which is based on the fact that

$$T(2n) = p_3(n), \tag{1}$$

an insight that was made independently by N. J. Fine and P. Pacitti of Pennsylvania State University. Combined with the property

$$T(2n-3) = T(2n), \tag{2}$$

the formula for $p_3(n)$ gives another complete solution. Again, I am indebted to George Andrews for this approach.

The key result $T(2n) = p_3(n)$ is established directly by displaying a 1-1 correspondence between the triangles counted by $T(2n)$ and the partitions of $p_3(n)$. Suppose that (a, b, c) is a triangle counted by $T(2n)$. In this case, $a + b + c = 2n$, and because each side of a triangle is less than one-half the perimeter, we have each of $a, b, c < n$. Consequently, each of the integers $n - a, n - b, n - c$ is positive, and $(n - a, n - b, n - c)$ is a partition counted by $p_3(n)$:

$$n - a + n - b + n - c = 3n - (a + b + c) = 3n - 2n = n.$$

Conversely, if (p, q, r) is a partition counted by $p_3(n)$, we have $p + q + r = n$, and that each of p, q, r is less than n. Then the 3 positive integers $n - p, n - q, n - r$ add to $2n$ and the sum of any two exceeds the third, for example, $n - p + n - q = 2n - (p + q) > 2n - n = n > n - r$. Therefore, $(n - p, n - q, n - r)$ is a triangle counted by $T(2n)$, and we have

$$T(2n) = p_3(n).$$

Now let us verify property (2). If (a, b, c) is a triangle counted by $T(2n - 3)$, it is easy to see that $(a + 1, b + 1, c + 1)$ is a triangle counted by $T(2n)$: from $a + b + c = 2n - 3$, we have $a + 1 + b + 1 + c + 1 = 2n$, and, from the known $b + c > a$, etc., we have $b + 1 + c + 1 > a + 2 > a + 1$, etc., satisfying the triangle inequalities. Hence $T(2n) \geq T(2n - 3)$.

We shall obtain the desired $T(2n) = T(2n - 3)$ by showing that $T(2n - 3) \geq T(2n)$. To this end, suppose (a, b, c) is a triangle counted by $T(2n)$. First we will show that none of a, b, c can be unity. Suppose to the contrary, for example, that $c = 1$. By the triangle inequality, we would then have that the integer

$$|a - b| < c = 1,$$

which would leave no option but $a = b$. In this case, however, the total perimeter would be

$$a + b + c = 2a + 1,$$

an odd number, not the $2n$ it is supposed to be. Thus each of a, b, c must exceed unity.

Finally, we complete the argument by showing that the triple of positive integers $(a - 1, b - 1, c - 1)$ determines a triangle that is counted by $T(2n - 3)$. Because (a, b, c) is counted by $T(2n)$, it is clear that $a - 1 + b - 1 + c - 1 = 2n - 3$; furthermore, the known triangle inequality

$$a + b > c,$$

for example, yields

$$a - 1 + b - 1 > c - 1 - 1,$$

or

$$a - 1 + b - 1 \geq c - 1;$$

now, if equality were to hold here, then the total perimeter of the triangle $(a - 1, b - 1, c - 1)$ would be $2(c - 1)$, an even number instead of the odd $2n - 3$. Thus the triangle inequalities are satisfied by $(a - 1, b - 1, c - 1)$, and we have the desired $T(2n - 3) \geq T(2n)$.

Consequently, we have

$$T(2n - 3) = T(2n) = p_3(n) = \left\{\frac{n^2}{12}\right\}.$$

This gives

$$T(2n) = \left\{\frac{n^2}{12}\right\} = \left\{\frac{(2n)^2}{48}\right\}$$

and

$$T(2n - 3) = \left\{ \frac{[(2n - 3) + 3]^2}{48} \right\}.$$

That is to say,

$$\text{if } n \text{ is even,} \quad \text{then} \quad T(n) = \left\{ \frac{n^2}{48} \right\},$$

$$\text{if } n \text{ is odd,} \quad \text{then} \quad T(n) = \left\{ \frac{(n + 3)^2}{48} \right\}.$$

Exercise

If a, b, c are positive integers such that $a^2 + b^2 = c^2$ (that is, if (a, b, c) is a Pythagorean triple), prove that

$$p_3(a) + p_3(b) = p_3(c).$$

(Proposed by Jack Garfunkel, Queen's College, New York, in *Pi Mu Epsilon Journal*, 1981, page 31).

References

1. H. Hadwiger, H. Debrunner, and V. Klee, Combinatorial Geometry in the Plane, Holt, Rinehart and Winston, New York, 1964.
2. I. M. Yaglom and V. G. Boltyanski, Convex Figures, Holt, Rinehart and Winston, New York, 1961.
3. R. Honsberger, Mathematical Gems II, vol. 2, Dolciani Mathematical Expositions, MAA, 1973.
4. Mathematical Spectrum, 6 (2) (1973/74); also the sequel in 7 (2) (1974/75) 69–70.
5. J. H. Jordan, R. Walch, and R. J. Wisner, Triangles with integer sides, Amer. Math. Monthly, 86 (1979) 686–689.
6. G. Andrews, A note on partitions and triangles with integer sides, Amer. Math. Monthly, 86 (1979) 477.
7. A. Tucker, Applied Combinatorics, John Wiley & Sons, New York, 1980.

Ross Honsberger was born in Toronto, Canada in 1929 and attended the University of Toronto. After more than a decade of teaching matheamtics in Toronto, he took advantage of a sabbatical leave to continue his studies at the University of Waterloo, Canada. He joined its faculty in 1964 (Department of Combinatorics and Optimization) and has been there ever since. He is married, the father of three, and grandfather of three. He has published seven bestselling books with the Mathematical Association of America.

Here is a selection of reviews of Ross Honsberger's books.

"The book is designed to appeal to high school teachers and undergraduates particularly, but should find a much wider audience. The clarity of exposition and the care taken with all aspects of the explanations, diagrams and notation is of a very high standard." (about Mathematical Gems II)

K. E. Hirst in *Mathematical Reviews*

"Honsberger's purpose is simply to convey the excitement and delight of mathematical discovery at a fairly elementary level. The fact that elementary mathematics is alive and well, and even flourishing, is a salutary reminder for professional mathematicians....the results described in this book gave considerable pleasure to the discoverers and that can be shared by the readers of the book, thanks to the lucidity and enthusiasm of Honsberger's style." (about Mathematical Gems III)

N. L. Biggs in the *Bulletin of the London Mathematical Society*

"All (i.e., the articles in Mathematical Gems III) are written in the very clear style that characterizes the two previous volumes, and there is bound to be something here that will appeal to anyone, both student and teacher alike. For instructors, Mathematical Gems III is useful as a source of thematic ideas around which to build classroom lectures...Mathematical Gems III is to be warmly recommended, and we look forward to the appearance of a fourth volume in this series."

Joseph B. Dence, *Mathematics and Computer Education*

" Although the mathematics involved in quite elementary and requires little background, the [illegible] rial...Written in the author's usual cl[illegible] enthusiacally recommended to [illegible] ey be teachers or professional mat[illegible] Gems III)

[illegible] *in Mathematics Teacher*

ISB-N-0-88385-313-2
22 ~~510~~ 512 $21.00
MATH GEMS III
06/02/92

Cover design by Lisa M. Brzezniak for Barbieri and Green

ISBN 0-88385-313-2