S91

A111
A110501
A182
A236935
GENOCCHI
NJAS

A COMBINATORIAL INTERPRETATION OF THE SEIDEL GENERATION OF GENNOCHI NUMBERS

Annals of Discrete Math. 6 (1980), 77-87

D. Dumont and G. Viennot*

1. Introduction.

The Genocchi numbers G_{2n} are integers defined from the Bernoulli numbers by the relation:

(1) $$G_{2n} = 2(2^{2n} - 1)\, B_{2n} \qquad (n \geq 1).$$

They are related to the tangent numbers T_{2n-1} (or Euler numbers of the the second kind) by the relation:

(2) $$2^{2n-2} G_{2n} = nT_{2n-1} \qquad (n \geq 1).$$

*Work partially supported by NSF-CNRS Exchange Visitor Program n°G-05-0252

The first values of these sequences are given in the table 1. The exponential generating functions of these numbers (more exactly of $(-1)^n G_{2n}$ and $(-1)^{n+1} B_{2n}$) are respectively:

$$\frac{2t}{e^t+1} = t + \sum_{n\geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}$$

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{n\geq 1} (-1)^{n+1} B_{2n} \frac{t^{2n}}{(2n)!}$$

$$\tan t = \sum_{n\geq 1} T_{2n-1} \frac{t^{2n-1}}{(2n-1)!} \quad .$$

The relations (1) and (2) are easily derived from these expressions.

n	1	2	3	4	5	6	7
B_{2n}	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$
G_{2n}	1	1	3	17	155	2073	38227
T_{2n-1}	1	2	16	272	7936	353792	22368256

A110501
A182

Table 1.

Dumont gave in [4] the first combinatorial interpretation of the Genocchi numbers. It is related to a generation of these numbers conjectured in 1970 by Gandhi [8] and proved by Carlitz [2] and Riordan and Stein [12] by means of analytic calculus.

Here we give another interpretation related to the Seidel definition of the Bernoulli and Genocchi numbers [13]. Though being very natural and simple, this definition has been somewhat forgotten and is restated in section 4. The following identity can be deduced (where $G_0 = 0$ and $n \geq 1$ fixed)

(3) $$\sum_{0 \leq k \leq \frac{n}{2}} (-1)^k \binom{n}{2k} G_{2n-2k} = 0$$

This identity (called Seidel identity by Nielsen [11], pp 186-187) leads to the combinatorial interpretation with "alternating pistols" in section 2 and, in section 3, with a subclass of the well-known alternating permutations enumerated by the tangent numbers.

This combinatorial interpretation enables us to give a very convenient way to calculate the Genocchi (and hence Bernoulli) numbers, by constructing a tableau of positive integers with summations two by two of positive integers. These integers appear in Seidel's definition and, roughly speaking, are equivalent to that definition. In sections 4 and 5 we explain this fact by calculus and give some additional results, beyond the original paper of Seidel.

In section 6 we give a geometric correspondance between the interpretation [4] of the Gandhi generation and the permutations of section 3 interpreting the Seidel generation. This correspondance is briefly recalled as being a consequence of some techniques involving permutations introduced by Françon and Viennot in [7] (where one can find its complete definition).

2. Alternating pistols.

Notations. In this paper $[m]$ denote the set of integers $[1,m] = \{1, 2, \ldots, m\}$, and $|E|$ the cardinality of every finite set E. The map $\pi : \mathbb{N} \to \mathbb{N}$ is defined by the following condition

(4) $$\pi(1) = 1 \text{ and } \forall i \geq 1, \pi(2i) = \pi(2i - 1), \pi(2i + 1) = \pi(2i) + 1.$$

Definition 1. Let $m \geq 1$. A pistol on $[m]$ is a map $h:[m] \to [m]$ with $h(i) \leq \pi(i)$ for every $i \geq 1$.

Definition 2. A pistol h on $[m]$ is said to be alternating if the following condition holds:

(5) $$\forall i \in [m - 1], h(i) \geq h(i + 1) \text{ if } i \text{ is odd, and}$$
$$h(i) \leq h(i + 1) \text{ if is even.}$$

For example, the pistol $h = h(1) \ldots h(8) = 11213332$ is displayed in figure 1.

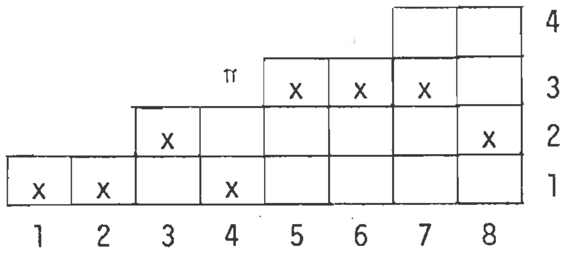

Fig 1.

We denote $\mathcal{P}_m$ (resp. $\mathcal{AP}_m$) the set of pistols (resp. alternating pistols) on $[m]$.

The next definition is introduced for two reasons: we need it in section 3, and the proof of proposition 3 below will be more clear.

Let $\mathcal{AP}'_m$ be the set of pistols h of $\mathcal{P}_m$ satisfying the following condition:

(5') $\forall i \in [m - 1],\ h(i) \geq h(i + 1)$ if i is odd, and

$h(i) < h(i + 1)$ if is is even.

For every pistol $h \in \mathcal{P}_m$, we define the map $\alpha(h) = h' : [m + 1] \rightarrow [m + 1]$ by the condition:

(6) $h'(1) = 1$ and for every i, $2 \leq 2i \leq m$,

$$h'(2i) = i + 1 - h(2i - 1)$$

$$h'(2i + 1) = i + 2 - h(2i)$$

For example, the pistol $h = 1121332$ gives the pistol $h' = 112131224$ displayed in figure 2:

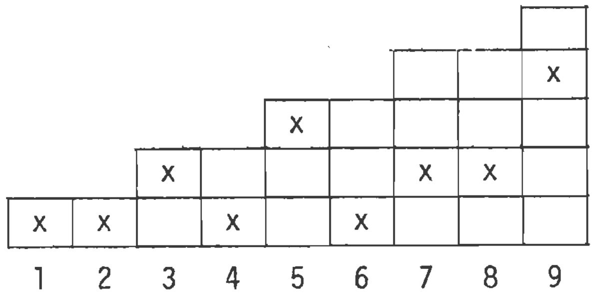

Fig 2.

It is easily checked that the map α is a bijection from $\mathcal{P}_m$ onto the set of pistols $h \in \mathcal{P}_{m+1}$ such that $h(i) \geq 2$ for every odd index $i \in [3, m]$. This bijection α maps $\mathcal{G}\mathcal{P}_m$ onto $\mathcal{G}\mathcal{P}'_{m+1}$.

Propsition 3. For every $n \geq 1$, <u>the number of alternating pistols on</u> $[2n]$ <u>is the Genocchi number</u> G_{2n+2}, <u>or</u>:

$$\forall i \geq 1, \quad |\mathcal{G}\mathcal{P}_{2n}| = |\mathcal{G}\mathcal{P}'_{2n+1}| = G_{2n+2}.$$

Denote $a_{2n} = |\mathcal{G}\mathcal{P}'_{2n+1}|$, and for $p \in [n]$, define a_{2n}^{2p} the number of pistols $h \in \mathcal{P}_{2n+1}$ such that the following holds:

$(5''_p)$ $\quad \forall i \in [2n+1],\ h(i) \geq h(i+1)$ if i is odd and $i \in [2p]$,

and $h(i) < h(i+1)$ in the opposite case.

It is easy to prove the relation:

(7) $\quad$ For $2 \leq p \leq n,\ a_{2n}^{2p} = \binom{n+1}{2n-2p+2} a_{2p-2} - a_{2n}^{2p-2}$.

A recursive application of (7) shows that the sequence $b_{2n} = a_{2n-2} = a_{2n-2}^{2n-2}$ $(n \geq 2)$ with $b_2 = 1$, satisfies the relation (3). This relation gives b_{2n} or G_{2n} as an expression involving the a_{2i} (resp. G_{2i}) for $1 \leq i < n$. The initial conditions $b_2 = G_2 = 1$ are the same, hence the two sequences are identical. Thus $a_{2n} = G_{2n+2}$, and the proposition results using the bijection α.

Q.E.D.

We give another way to enumerate these numbers. Define the tableau of integers h_m^k by $h_1^1 = 1$ and the following induction:

(8) $\quad$ for $m = 2n \geq 2$, $\ h_m^n = h_{m-1}^n$ and for $k \in [n-1]$, $h_m^k = h_m^{k+1} + h_{m-1}^k$

(8') $\quad$ for $m = 2n-1 \geq 3$, $\ h_m^1 = h_{m-1}^1$ and for $k \in [2, n-1]$,

$$h_m^k = h_m^{k-1} + h_{m}^k, \quad h_m^n = h_m^{n-1} .$$

The first values are given in table 2:

								155	155	5
						17	17	155	310	4
				3	3	17	34	138	448	3
		1	1	3	6	14	48	104	552	2
1	1	1	2	2	8	8	56	56	608	1
1	2	3	4	5	6	7	8	9	10	m \ k

Table 2.

Define $\mathcal{AP}_m^k$ to be the set of alternating pistols h with $h(m) = k$. It is easy to prove from (5) that the numbers $|\mathcal{AP}_m^k|$ satisfy the recurrences (8) and (8'), with same initial conditions. Thus, from proposition 3, we can state.

Corollary 4. <u>The Genocchi number</u> G_{2n+2} <u>is the sum of the</u> $2n^{\underline{th}}$ <u>column of the tableau</u> (h_n^k) <u>defined by</u> (8) <u>and</u> (8'):

$$G_{2n+2} = \sum_{k \in [n]} h_n^k \qquad (n \geq 1).$$

3. Combinatorial interpretation of Genocchi numbers with alternating permutations.

A permutation σ of $[2n+1]$ is said to be alternating if:

(9) $$\forall i \in [n], \quad \sigma(2i-1) \geq \sigma(2i) \leq \sigma(2i+1).$$

Since André [1], it is known that the number of such permutations is the tangent number T_{2n+1} defined in section 1.

Let $m \geq 1$ and σ a permutation of $[m]$. The inversion table (also called Lehmer code) is the map $\ell(\sigma) = f : [m] \to [0, m-1]$ defined by:

(10) $\forall i \in [m]$, $f(i)$ is the number of index j such that $1 \leq j < i$ and $\sigma(j) < \sigma(i)$.

The map f is a subexceedant function on $[m]$, that is a map $f : [m] \to [0, m-1]$ such that $f(i) < i$ for every $i \in [m]$. It is well known (see for example Knuth [9] p. 12) that the map $\ell : \sigma \to f$ is a bijection between the set $\mathfrak{S}_m$ of permutations on $[m]$ and the set $\mathfrak{F}_m$ of subexceedant functions on $[m]$. Furthermore, the following relation holds:

(11) $$\forall i \in [n], \quad \sigma(i) > \sigma(i+1) \text{ if } f(i) \geq f(i+1).$$

Hence, the set of alternating permutations on $m = 2n+1$ is in bijection by ℓ with the set of subexceedant functions on $[m]$ satisfying (5').

Now to every pistol h of $\mathcal{P}_{2n+1}$ we associate the map $\varphi(h) = f : [2n + 1] \rightarrow [0, 2n]$ by the following

$$(12) \qquad \forall i \in [2n + 1], \quad f(i) = 2(h(i) - 1).$$

Condition (4) implies $f \in \mathcal{F}_{2n+1}$. The correspondance φ is a bijection between $\mathcal{P}_{2n+1}$ and subexceedant functions of $\mathcal{F}_{2n+1}$ with even values. The pistol $h \in \mathcal{P}_{2n+1}$ satisfies (5') iff $\varphi(h)$ satisfies also (5'). Hence, from above, the map $\ell^{-1} \circ \varphi \circ \alpha$, defined by (6), (11), (12), is a bijection between the set $\mathcal{GP}_{2n+1}$ of alternating pistols, and the set of alternating permutations with inversion table having only even values.

For example, with alternating pistol $h = 11213332$ of section 2, we have successively $h' = \alpha(h) = 112131224$, $h'' = \varphi(h') = 002040226$, $\alpha = \ell^{-1}(h'') = 658291437$.

Proposition 3 becomes here:

Proposition 5. <u>For every</u> $n \geq 0$, <u>the Genocchi number</u> G_{2n+2} <u>is the number of alternating permutations</u> σ <u>of</u> $[2n + 1]$ <u>having an inversion table</u> $\ell(\sigma)$ (defined by (10)) <u>with even values</u>.

This proposition is essential for section 6.

4. <u>Seidel matrices</u>.

We show now the relation between the Seidel identity (3), the tableau defined by (8), (8') and the Seidel definition of Genocchi numbers.

Definition 6. A Seidel matrix is an infinite matrice $A = (a_n^k)_{n,k\geq 0}$, with elements in $\mathbb{Q}$, where the sequence of elements of the first row is given $a^0 = (a_0^0, a_1^0, \ldots, a_n^0, \ldots)$, the others elements being defined by the induction:

$$(13) \qquad \forall n \geq 0,\ \forall k \geq 1, \qquad a_n^k = a_n^{k-1} + a_{n+1}^{k-1} .$$

Following Seidel, we call the first row the initial sequence and the first column $a_0 = (a_0^0, a_0^1, \ldots)$ the terminal sequence. The following relations give the expression of every element of A from the initial sequence or from the terminal sequence.

$$(14) \qquad \forall k, n \geq 0, \qquad a_n^k = \sum_{i=0}^{k} \binom{k}{i} a_{n+i}^0$$

$$(14') \qquad \forall k, n \geq 0, \qquad a_n^k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_0^{k+n-i}$$

These relations are easily deduced by induction from (13) or the equivalent relation:

$$(13') \qquad \forall k \geq 0,\ \forall n \geq 1, \qquad a_n^k = a_{n-1}^{k+1} - a_{n-1}^k .$$

Let $a(t)$ the exponential generating function of the initial sequence:

$$a(t) = \sum_{n\geq 0} a_n^0 \frac{t^n}{n!}$$

Proposition 7. <u>The generating function</u> f <u>of the Seidel matrix</u> $A = (a_n^k)$ <u>is</u>:

$$\sum_{n,k\geq 0} a_n^k \frac{t^n}{n!} \frac{u^k}{k!} = e^u\, a(t+u) \tag{15}$$

From (14), the (exponential) generating function of the k^{th} row is:

$$\sum_{n\geq 0} a_n^k \frac{t^n}{n!} = \sum_{i=0}^{k} \binom{k}{i} a^{(i)}(t) \tag{16}$$

where $a^{(i)}(t)$ denote the i^{th} derivative of $a(t)$. With Leibnitz formula, we deduce

$$e^t\left(\sum_{n\geq 0} a_n^k \frac{t^n}{n!}\right) = [e^t\, a(t)]^{(k)} \tag{17}$$

Then, applying Taylor formula:

$$e^t\left(\sum_{n,k\geq 0} a_n^k \frac{t^n}{n!}\ \frac{u^k}{k!}\right) = e^{t+u}\ a(t+u)$$

By setting $t = 0$ in (15), we obtain the proposition of Seidel:

Corollary 8 (Seidel). In a Seidel matrix $A = (a_n^k)$, the generating function of the terminal sequence is the product by the exponential of the generating function of the initial sequence:

$$\sum_{k\geq 0} a_0^k \frac{u^k}{k!} = e^u\ a(u).$$

Hence, the transformation mapping the initial sequence onto the terminal sequence has no invariant. But, with Seidel, we can ask for "almost invariant" sequence in the following sense:

(18) $$b_0^0 = 1,\ b_o^1 = -\ b_1^0, \text{ and } \forall i \geq 2,\ b_0^i = b_i^0.$$

There exist a unique Seidel matrix $b = (b_n^k)$ satisfying (18). The initial sequence can be calculated: with (14) and (18), and form corollary 8, its (exponential) generating function is $b(t) = \dfrac{t}{e^t-1}$. Thus this initial sequence is the sequence of Bernoulli numbers:

$$\forall n \geq 1,\ b_{2n}^0 = (-1)^{n+1}B_{2n},\ b_0^0 = 1,\ b_1^0 = -\tfrac{1}{2},\ b_{2n+1}^0 = 0.$$

Similarly, we can ask for some "almost anti-invariant" of the correspondence defined by the Seidel process, that is:

(19) $$g_0^0 = 0,\ g_1^0 = g_0^1 = 1, \text{ and } \forall i \geq 2,\ g_0^i = -\ g_1^0.$$

Again, relation (14) with $n = 0$, defines by induction a unique initial sequence g_n^0 satisfying (19) and a unique Seidel matrix denoted by $G = (g_n^k)$. From corollary 8, its (exponential) generating function is $g(t) = \frac{2t}{e^t+1}$. The initial sequence is the sequence of Genocchi numbers:

$$\forall i \geq 1,\ g_{2n}^0 = (-1)^n\ G_{2n},\ g_0^0 = 0,\ g_1^0 = 1,\ g_{2n+1}^0 = 0.$$

The relations (18) and (19) are, what we call in this paper the Seidel definitions of Bernoulli and Genocchi numbers.

Remark 9. If one takes as initial sequence the tangent numbers $(-1)^n\ T_{2n-1}$ (with $t_0^0 = 1$), with exponential generating function $\frac{2}{e^{2t}+1}$, the terminal sequence is the sequence of <u>secant numbers</u> $(-1)^n\ E_{2n}$ (or Euler numbers of the first kind), with exponential generating function $\frac{1}{\mathrm{ch}\ t}$.

5. The Seidel matrix of Genocchi numbers.

The first values of the Seidel matrix $G = (g_n^k)$ defined above by (19) are given in table 3.

0	1	-1	0	1	0	-3	0	17	0	-155
1	0	-1	1	1	-3	-3	17	17	-155	
1	-1	0	2	-2	-6	14	34	-138		
0	-1	2	0	-8	8	48	-104			
-1	1	2	-8	0	56	-56				
0	3	-6	-8	56	0					
3	-3	-14	48	56						
0	-17	34	104							
-17	17	138								
0	155									
155										

Table 3.

The reader recognizes the integers of table 2. The following proposition makes explicit the signs and symetries of the integers g_n^k .

Proposition 10. In the Seidel matrix $G = (g_m^k)$ of Genocchi numbers defined by (19), the following properties hold:

(20) $$\forall m,\ k \geq 0,\ g_m^k = (-1)^{m+k-1}\ g_k^m$$

(21) $\forall m \geq k \geq 0$, let $m + k = 2n$ or $2n + 1$, then if g_m^k is not equal to 0, its sign is $(-1)^n$.

From (19) and the fact that the generating function $g(t) - t = t\,\frac{1-e^t}{1+e^t}$ is an even function, identity (20) holds for $k = 0$.

The general case is proved by induction, using (14) and (14') for $a_m^k = g_m^k$.

The relation (21) is proved by induction on n. From (19), it is true for $n = 0$. The reader will easily prove by induction from (14') and (14), that the sign of g_m^k changes when replacing "$m + k = 2n - 1$" by "$m + k = 2n$," and that the sign does not change when replacing "$m + k = 2n$" by "$m+k = 2n+1$."

Q.E.D.

A corollary of (20) is the relation:

(22) $$\forall n \geq 0, \quad g_n^n = 0 .$$

Taking the expression of g_n^n from (14) and using $g_{2n+1}^0 = 0$, the relation (22) is nothing but the Seidel identity (3). Lucas [10] p. 252, write also (22) or (3), under symbolic notations, by:

(23) $$G^n(G + 1)^n \equiv 0 \quad \text{where} \quad G^i \equiv (-1)^i G i.$$

With these notations, the relations (14') and (19) give the identity:

$$(G + 1)^n + G^n \equiv 1 \qquad (24)$$

Relations (13) and (21) show that the upper triangular matrix $(g_m^k)_{m \geq k \geq 1}$ is, under an appropriate bijection converting diagonals into rows, and by cancelling the signs given by (21), the triangle of integers h_m^k defined by (8) and (8').

Remark 11. The analogue of this transformation for the Seidel matrix of Euler numbers (Remark 9) gives the tableau of Entringer [5], that is the distribution of alternating permutations (for m odd or even) according to their last letters. We give in table 4 the first values of this Seidel matrix.

1	-1	0	2	0	-16	0	272
0	-1	2	2	-16	-16	272	
-1	1	4	-14	-32	256		
0	5	-10	-46	224			
5	-5	-56	178				
0	-61	122					
-61	61						
0							

A236935

Table 4.

6. Geometric relation with the Gandhi generation of Genocchi numbers.

Gandhi [8] defined, in a slight different way, the following polynomials:

(25) $$Q_0(t) = 1 \qquad \text{and}$$

$$\forall i \geq 0, \quad Q_{i+1}(t) = t^2 Q_i(t+1) - (t-1)^2 Q_i(t)$$

and conjectured that

(26) $$\forall n \geq 0, \quad Q_n(1) = G_{2n+2}.$$

This was proved by Carlitz [2] and Riordan and Stein [12]. Dumont gave in [4] the following geometric interpretation of this generation.

Define $\mathcal{SP}_{2n}$ the set of surjective pistols on $[2n]$, that is pistols $h \in \mathcal{P}_{2n}$ such that the map h, considered as a map $[2n] \to [n]$, is a surjection. Then G_{2n+2} is the number of such pistols. In fact the cofficients of the polynomials $Q_{n-1}(t+1)$ give the distribution of these pistols according to the number of $i \in [2n]$ with $h(i) = 1$.

In [4], the first author gave also a bijection between $\mathcal{SP}_{2n}$ and the set of permutations $\sigma \in \mathfrak{S}_{2n+1}$ satisfying the relation below:

(27) $$\forall i \in [2n], \quad \sigma(i) < \sigma(i+1) \ \text{ iff } \ \sigma(i) \text{ is even.}$$

We briefly recall some techniques of Françon and Viennot [7].

Define a contraction γ to be a map $[n] \to [n]$ satisfying $\gamma(1) = \gamma(n) = 1$ and $|\gamma(i+1) - \gamma(i)| \leq 1$ for every $i \in [n-1]$. A <u>colored contraction</u> γ_c is a contraction with possibility of coloring the <u>levels</u> (that is $\gamma(i+1) = \gamma(i)$) with two colors: blue and red. Let $\mathfrak{T}_n$ the set of pairs (γ_c, f) where γ_c is a colored contraction and f a map $[n] \to [n]$ with $f(i) \leq \gamma(i)$ for every $i \in [n]$. The set $\mathfrak{T}_n$ has $n!$ elements.

This fact is proved in [7] by defining a bijection $\theta : \mathfrak{T}_n \to \mathfrak{S}_n$ having the following property:

(28) for every $\sigma = \theta(\gamma_c, f) \in \mathfrak{S}_n$, and with convention $\sigma(0) = \sigma(n+1) = 0$,

$$\forall i \in [n-1],\ \gamma(i) > \gamma(i+1) \text{ iff } \sigma(i-1) < \sigma(i) > \sigma(i+1)$$

$$\gamma(i) < \gamma(i+1) \text{ iff } \sigma(i-1) > \sigma(i) < \sigma(i+1)$$

$$\gamma(i) = \gamma(i+1) \text{ with red, iff } \sigma(i-1) < \sigma(i) < \sigma(i+1)$$

$$\gamma(i) = \gamma(i+1) \text{ with blue, iff } \sigma(i-1) > \sigma(i) > \sigma(i+1)$$

From (28), the alternating permutations on $[2n+1]$ correspond to the pair (γ_c, f) where γ has no levels (and thus no color), called also <u>strict contractions</u> (and enumerated by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n})$. It is shown in [7] that the alternating permutations of proposition 5 correspond to the pairs (γ, f) where γ is a strict contraction and f a map with only odd values.

Furthermore, relation (28) shows that the permutations satisfying (27) correspond to the pairs (γ_c, f), where γ_c is an alternating contraction, that is a contraction satisfying condition (5), and with colors given by the rule:

(29) the level $\gamma(i) = \gamma(i + 1)$ is red for i even and blue for i odd.

This means in fact that the pair is defined by (γ, f) and we don't care about colors.

Thus the problem to find a geometric relation between the Gandhi and the Seidel generation of Genocchi numbers remains to find a bijection between the two sets of pairs (γ, f) defined above. A very simple one is given in [7]. Let (γ, f) a pair of $\mathcal{J}_{2n+1}$ with γ alternating contraction. Define $\rho(\gamma, f) = (\gamma', f')$ by the following relation:

(30) $\forall i \in [2n + 1], \gamma'(i) = 2\gamma(i) - 1, \quad f'(i) = 2f(i) - 1.$

Then $(\gamma', f') \in \mathcal{J}_{2n+1}$ with γ' strict contraction and f' function with only odd values. The map ρ is the desired bijection. The following diagram recalls bijections used to go from alternating pistols to surjective pistols:

Seidel generation		Gandhi generation
alternating pistols $\mathcal{AP}_{2n}$		surjective pistols $\mathcal{SP}_{2n}$
$\alpha \downarrow$ (6)		$\downarrow$ (see [4] or a special case of [14])
pistols $\mathcal{AP}'_{2n+1}$		
$\ell^{-1} \downarrow$ (10)		
alternating permutations σ with $\ell(\sigma)$ even values		permutations σ with (27)
$\theta \downarrow$ (see [7])		$\theta \downarrow$ (see [7])
(γ, f), γ strict contraction, f odd value	$\overset{\rho}{\longleftarrow}$ (30)	(γ, f), γ alternating contraction

Fig. 3.

A straightforward bijection between alternating and surjective pistols would be welcome. A simplification can be derived from [14] for the bijection between surjective pistols and permutations (25).

<u>Note</u>. In our lecture in Fort Collins, we give also more details about combinatorial interpretations of Euler numbers. They will be found in [15] where the second author gives, after the interpretation of André [1] (alterna-

ting permutations) and of Foata and Schützenberger [6] (André permutations), a third one (Jacobi permutations). This one leads to a combinatorial interpretation of Jacobi elliptic functions sn, cn and dn. The Seidel generation can also be interpreted as a subclass of Jacobi permutations.

REFERENCES

[1] D. André, Sur les permutations alternées, J. Math. Pures et Appl., vol 7, (1881) 167-184.

[2] L. Carlitz, A conjecture concerning Genocchi numbers, K. Norske Vidensk. Selsk. Sk., 9, (1972) 1-4.

[3] L. Comtet, Advanced Combinatorics, D. Reidel Publ. Comp., Dordrecht/ Boston, (1974).

[4] D. Dumont, Interprétations combinatoires des nombres de Genocchi, Duke Math. J., vol 41, 2(1974) 305-318.

[5] R.C. Entringer, A combinatorial interpretation of the Euler and Bernoulli numbers, Nieuw. Arch. Wisk., 14, (1966) 241-246.

[6] D. Foata and M.P. Schützenberger, Nombres d' Euler et permutations alternantes, in "A Survey of Combinatorial Theory", Sympos. Colorado State Univ, Fort Collins (1971), eds Srivastava and als, North-Holland, Amsterdam (1973).

[7] J. Françon and G. Viennot, Permutations selon les pics, creux, doubles montées et doubles descentes, nombres d' Euler et nombres de Genocchi, submitted for publication.

[8] J.M. Gandhi, A conjectured representation of Genocchi numbers, Amer. Math. Monthly, (1970) 505-506.

[9] D. Knuth, The Art of Computer Programming, vol 3, Addison Wesley, Reading, (1974).

[10] A. Lucas, Théorie des nombres, vol 1, Gauthiers-Villars, Paris, (1891).

[11] N. Nielsen, Traité élémentaire des nombres de Bernoulli, Gauthiers-Villars, Paris, (1923).

[12] J. Riordan and P.R. Stein, Proof of a conjecture on Genocchi numbers, Discrete Maths, 5 (1973) 381-388.

[13] L. Seidel, "Über eine einfache Entshehungsweise der Bernoullischen Zahlen und einiger verwandten Reihen", Sitzungsberichte der Münchener Akademie, Math. Phys. Classe (1877) 157-187.

[14] G. Viennot, Finite difference calculus and permutations with prescribed set of rises, submitted for publication.

[15] G. Viennot, Une interpretation combinatoire des coefficients des développements en série entière des fonctions elliptiques de Jacobi, submitted for publication.

Duminique Dumont
Université de Strasbourg
Département de Mathématiques
67084 Strasbourg - France

Gérard Viennot
University of California, San Diego
Department of Mathematics
La Jolla, California 92093, USA

The second author is on leave 77/78 from:

ENS - Centre de Mathématiques
45, rue d´ Ulm
75005, Paris, France