An Explicit Formula for the Euler zigzag numbers (Up/down numbers) from power series

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In this paper, I will derive an explicit formula for the Euler zigzag numbers (Up/down numbers). Euler zigzag number is the number of alternating permutation in a set. Therefore the explicit formula of Euler numbers(Secant numbers) and Bernoulli numbers are found as well. The formula involves two finite sum.

Content

Introduction

Euler zigzag numbers, A_n, is the number of alternating permutation of the set {1,2,...,n}. And it is well known that:

$$\sec x + \tan x = \sum_{n=0}^{\infty} \frac{A_n}{n!} x^n$$

In the following section, I am going to derive an explicit formula of A_n by using power series expansion.

Integrand of sec(x) + tan(x)

Let's consider the integrand of sec(x) + tan(x):

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$$\sum_{n=0}^{\infty} \frac{A_n}{n!} \int_0^x y^n \, dy = \int_0^x (\sec y + \tan y) \, dy$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{A_{n-1}}{n!} x^n &= \int_0^x \frac{1+\sin y}{\cos y} \, dy \\
&= \int_0^x \frac{(1+\sin y)(1-\sin y)}{\cos y(1-\sin y)} \, dy \\
&= \int_0^x \frac{\cos y dy}{1-\sin y} \\
&= -\ln(1-\sin x)
\end{aligned}$$

Let:

$$f(x) = -\ln(1-\sin x)$$

I will do a power expansion of the function f(x), and a finite sum explicit formula for A_n can be found by some simplification.

Power Series Expansion of f(x)

$$\begin{aligned}
f(x) &= -\ln(1-\sin y) \\
&= \sum_{n=1}^{\infty} \frac{\sin^n y}{n} \\
&= \sum_{n=1}^{\infty} \frac{\left(e^{iy} - e^{-iy}\right)^n}{2^n i^n n} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{C_k^n e^{(n-k)iy} e^{-iky} (-1)^k}{2^n i^n n} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{C_k^n e^{(n-2k)iy} (-1)^k}{2^n i^n n} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{C_k^n (-1)^k}{2^n i^n n} \sum_{j=0}^{\infty} \frac{(n-2k)^j i^j y^j}{j!} \\
&= \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{C_k^n (n-2k)^j i^j (-1)^k}{2^n i^n j! n} y^j
\end{aligned}$$

Therefore, by equating coefficients, we have:

$$A_{j-1} = i^j \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{C_k^n (n-2k)^j (-1)^k}{2^n i^n n}$$

or

$$A_j = i^{j+1} \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{C_k^n (n-2k)^{j+1} (-1)^k}{2^n i^n n}$$

Now, I am going to simplify it, and show that it is actually

e^-lambda/2 e^A e^B if [A,B]=lambda
Solving linear non-homogeneous ordinary differential equation with variable coefficients with operator method
Solving a partial difference equation in 2 variables with operator method
Solving recurrence equation with indexes from negative infinity to positive infinity
Reducing a partial difference equation into a partial differential equation and solving for the generating function using method of characteristics
Binomial Expansion for non-commutative elements (A+B)^n where [A, B] = lambda
Finding nth derivative of the function sec x + tan x and partial difference equation
A procedure to list all derangements of a multiset
A free ebook about generating function, "generatingfunctionology"

equals to:

$$A_j = i^{j+1} \sum_{n=1}^{j+1} \sum_{k=0}^{n} \frac{C_k^n (n-2k)^{j+1} (-1)^k}{2^n i^n n}$$

Simplification

In order to reduce the infinite sum to a finite sum, I first let:

$$B_n^j = \sum_{k=0}^n C_k^n (n-2k)^j (-1)^k$$

So that:

$$A_j = i^{j+1} \sum_{n=1}^{\infty} \frac{B_n^{j+1}}{2^n i^n n}$$

I observed that:

$$B_n^j = 0 \text{ if } n > j$$

To show that, let's define the translational operator D such that:

$$\begin{cases} D a_n = a_{n+1} \\ D^{-1} a_n = a_{n-1} \end{cases}$$

Then:

$$\begin{aligned} B_n^j &= \left(\sum_{k=0}^{n} C_k^n (-1)^k D^{-2k} \right) n^j \\ &= \left(1 - D^{-2}\right)^n n^j \\ &= \left(1 + D^{-1}\right)^n \left(1 - D^{-1}\right)^n n^j \end{aligned}$$

Here, $\nabla = 1 - D^{-1}$ is the backward difference operator.

Now, consider:

$$\begin{aligned} \nabla n^j &= n^j - (n-1)^j \\ &= n^j - \sum_{k=0}^{j} C_k^j (-1)^{j-k} n^k \\ &= -\sum_{k=0}^{j-1} C_k^j (-1)^{j-k} n^k \end{aligned}$$

The result is a polynomial of degree $j-1$. We can see that, if we apply backward difference operator to a polynomial, its degree decreases by 1. Therefore, if we apply the backward difference operator for a number of time which is larger than the degree of a polynomial, the result is zero.

As a result, we have $B_n^j = 0 \text{ if } n > j$. Therefore:

$$A_j = i^{j+1} \sum_{n=1}^{j+1} \sum_{k=0}^{n} \frac{C_k^n (n-2k)^{j+1} (-1)^k}{2^n i^n n}$$

Explicit Formula for Euler number

Euler number E_n is given by the generating function:

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} \\ = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n$$

And it is given by:

$$\begin{cases} E_{2n} & = i \sum_{k=1}^{2n+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (k-2j)^{2n+1}}{2^k i^k k} \\ E_{2n+1} & = 0 \end{cases}$$

Explicit Formula for Bernoulli Numbers

From Wikipedia, we know:

$$B_{2n} = \frac{(-1)^{n-1} 2n}{4^{2n} - 2^{2n}} A_{2n-1}$$

Therefore:

$$\begin{cases} B_0 = 1 \\ B_1 = -\frac{1}{2} \\ B_{2n} = \frac{2n}{2^{2n} - 4^{2n}} \sum_{k=1}^{2n} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (k-2j)^{2n}}{2^k i^k k} & \text{for } n > 0 \\ B_{2n+1} = 0 & \text{for } n > 0 \end{cases}$$

Conclusion

I have found out an simple formula for the Euler zigzag number:

$$A_n = i^{n+1} \sum_{k=1}^{n+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (k-2j)^{n+1}}{2^k i^k k}$$

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