A PRIMER ON STERN'S DIATOMIC SEQUENCE

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PART I: HISTORY

1. Eisenstein's Function

In 1850, F. M. G. Eisenstein, a brilliant mathematician and disciple of Gauss, wrote a treatise [1] on number theoretic functions of a reciprocating nature. In this paper he discusses the following sequence as part of another discussion.

For positive integers λ, u, and v:

1) $x_{u,v} = x_{u,u+v} + x_{u+v,v} \pmod{\lambda}$, for $u + v < \lambda$;
2) $x_{u,v} = \emptyset$, for $u + v > \lambda$;
3) $x_{u,v} = v$, for $u + v = \lambda$.

On February 18, 1850, M. A. Stern, who taught theory of equations at the University of Göttingen, attended a conference on Mathematical Physics where Eisenstein mentioned that the function described in his paper was too complex and did not lend itself to elementary study. Within two years of that conference, Eisenstein would die prematurely at the age of 29, but the study of Stern numbers had been born, and research was in progress.

2. Stern's Version

In a paper written in 1858, Stern presented an extensive discussion [2] on what may be characterized as "Generalized Stern Numbers." Many important results were generated in this paper, some of more importance than others. The authors will attempt to present a synopsis of these results, translated from German, as they were presented.

(1) Stern provided the following definition as his specialization of Eisenstein's function. The sequence is a succession of rows, each generated from a previous row starting with two numbers, m and n.

$$\begin{array}{ccccc} & m & & n & \\ & m & m+n & n & \\ m & 2m+n & m+n & m+2n & n \\ & & \text{etc.} & & \end{array}$$

Stern also provided some special terms for the elements of the rows.

Definition: ARGUMENT—The starting terms, m and n, are called ARGUMENTS of the sequence.

Definition: GRUPPE—In each successive row every other term is from the previous row and the terms in between are the sum of the adjacent two. Any three successive elements within a row are called a GRUPPE.

Definition: STAMMGLIED—In each GRUPPE, the two numbers which were from the previous row are termed STAMMGLIED.

Definition: SUMMENGLIED—In each GRUPPE, the middle term, the summed element, is termed SUMMENGLIED.

Some results are immediately obvious. The first SUMMENGLIED, $m+n$, is always the center element in succeeding rows. The arguments m and n always straddle the row. The row is symmetric about the center if $m = n$; even so, if a SUMMENGLIED is of the form $km + ln$, then $lm + kn$ appears reflected about the center element (MITTELGLIED).

(2) If there are k elements in a given row, then there are $2(k-1) + 1$ elements in the next row; if the first row has three elements, the pth row has $2^p + 1$ elements. Also, if we let $S_p(m,n)$ denote the sum of the elements in each row, then

$$S_p(m,n) = \frac{3^p + 1}{2}(m + n).$$

Note that $S_p(m,n)$ is reflexive or that $S_p(m,n) = S_p(n,m)$. Stern also observed that

$$\frac{S_p(m',n')}{S_p(m,n)} = \frac{m' + n'}{m + n},$$

and

$$S_p(m + m',n + n') = S_p(m,n) + S_p(m',n').$$

This latter result led to

$$\lim_{n\to\infty} \frac{S_p(F_n,F_{n+1})}{S_p(F_{n-1},F_n)} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}} = \alpha, \text{ the golden ratio,}$$

a nice Fibonacci result.[1]

(3) Stern observed next that some properties concerning odd and even numbers as they occur, or more precisely, Stern numbers mod 2. He noted that, in any three successive rows, the starting sequence of terms is

odd, even, odd
odd, odd, even
odd, even, odd

(4) Given a GRUPPE a, b, c, where b is a SUMMENGLIED in row p, the number will appear also in row $p - k$, where

$$k = \frac{a + b - c}{2b}.$$

Also, if b is in position

$$2^{t-1}(2l - 1) + 1$$

in row p, then it occurs also in row $p - (t - 1)$ in position $2l$. Related to this, Stern noted that with two GRUPPEs a, b, c and d, e, f in different rows, but in the same columns, that

[1]*This is a generalization of what was actually presented. The study of Fibonacci numbers as such was not yet in play. Hoggatt notes that this result is also true for generalized Fibonacci numbers.*

$$\frac{a + c}{b} = \frac{d + f}{e}.$$ [2]

(5) No two successive elements in a given row may have a common factor. Furthermore, in a GRUPPE a, b, c ($b = a + c$), a and c are relatively prime.

(6) Two sequential elements a, b cannot appear together, in the same order, in two different rows or in the same row. When $m = n = 1$ (the starting elements) then a group a, b may never occur again in any successive row.

(7) The GRUPPEs a, b, c and c, b, a may not occur together in the first (or last, because of the symmetry) half of a row.

(8) In the simple Stern sequence using $m = n = 1$, all positive integers will occur and all relatively prime pairs a, c will occur. For all elements of this same sequence that appear as SUMMENGLIED, that same element will be relatively prime to all smaller-valued elements that are STAMMGLIED. Stern pointed out that this is also a result of (6).

(9) The last row in which the number n will occur as a SUMMENGLIED is row $n - 1$. The number n will occur only $n - 1$ more times.

(10) Given a relatively prime pair b, c (or c, b) of a GRUPPE, the row in which that pair of elements will occur may be found by expansion of b/c into a continued fraction. That is, if

$$\frac{b}{c} = (k,\ k',\ k'',\ \ldots,\ k_m,\ r_{m-1}),$$

then b, c occurs in row

$$(k + k' + k'' + \cdots + k_m + r_{m-1} - 1),$$

and the pair $(1, r_{m-1})$ occurs in a row $(k + k' + \cdots + k_m)$.

(11) Let (m,n) denote row p generated by the Generalized Stern Sequence starting with m and n. Then

$$(m,n)_p \pm (m',n')_p = (m \pm m', n \pm n')_p,$$

which says that the element-by-element addition of the same row of two sequences is equal to row p of a sequence generated by the addition, respectively, of the starting elements.

In particular, an analysis of $(\emptyset,1)_p$ generates an interesting result. The first few rows are:

p	(0, 1)
∅	0, 1
1	0, 1, 1
2	0, 1, 1, 2, 1
3	0, 1, 1, 2, 1, 3, 2, 3, 1

Interestingly enough, all the nonzero elements in row k appear in the same position in every row thereafter. Stern observed also that in any given column of $(1, 1)_p$ the column was an arithmetic progression whose difference was equal to the value occurring in the same relative column of $(0, 1)_p$.

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[2] *The authors note that $(a+c)/b$ being an integer is not surprising, but the fact that this ratio is the same within columns is not immediately obvious.*

(12) From the last result in (4), we recall that

$$\frac{a+c}{b} = \frac{d+f}{e}$$

where a, b, c; and d, e, f are GRUPPEs and in the same column positions, but perhaps different rows, then

$$|db - ae| = |p_1 - p_2|,$$

where p_1 and p_2 are the row numbers.

(13) The next special case of interest is the examination of row $(1,n)_p$, for $n > 1$. The first noteworthy result is that all elements of the row $(1,n)_p$ appear at the start of the row $(1, 1)_{p+n-1}$. Also, all terms are of the form $k + ln$ or $l + kn$.

(14) Moving right along, the rows $(1,n)_p$ may be written as

$1 + \emptyset n, 1 + 1n, \emptyset + 1n$

$1 + \emptyset n, 2 + 1n, 1 + 1n, 1 + 2n, \emptyset + 1n$

$1 + \emptyset n, 3 + 1n, 2 + 1n, 3 + 2n, 1 + 1n, 2 + 3n, 1 + 2n, 1 + 3n, \emptyset + 1n$

etc.

Notice that the constant coefficients are the elements of $(1, 1)_{p-2}$, and that the coefficients of n are the elements of $(\emptyset, 1)_{p-1}$. Note also that the difference between any two successive elements, $k + ln$ and $k' + l'n$, within a row is

$$|kl' - k'l| = 1,$$

and no element may have the form

$$hk + h'kn.$$

(15) With k and k' in (14), k and k' are relatively prime. Correspondingly, l and l' are also relatively prime.

(16) Given $N > n$ in the sequence $(1,n)$ and of the form $N = K - Ln$, then K and L are relatively prime; N and n are relatively prime; L and N are relatively prime as well as K and N. Numbers between $\emptyset$ and N/n that occur in $(1,n)$ will be relatively prime to all N whenever N is a SUMMENGLIED.

(17) In order to proceed symmetrically, Stern next examined the sequence of rows $(n,1)_p$. The first immediately obvious result is that $(n,1)_p$ is reflexively symmetric to $(1,n)_p$ about the center element. When m and n of (m,n) are relatively prime and p is the largest factor of m or n, then for (m',n'), where $m = pm'$ and $n = pn'$, each element of (m',n') multiplied by p yields the respective element of (m,n). Stern noted at this point that all sequences (m,n) appear as a subset somewhere in $(1, 1)$.

(18) Given that N occurs in (m,n) and

$$N = mk + nl$$

for k and l relatively prime, Stern reported that a theorem of Eisenstein's says that N is relatively prime to elements between $(n_0/n)N$ and $(m_0/m)N$ where m_0 and n_0 are such that $|nm_0 - mn_0| = 1$; N is a SUMMENGLIED. When $m = m_0 = 1$ and $n_0 = n - 1$, N is relatively prime to elements between $(n - 1/n)N$ and N.

(19) Given again that $N = mk + ln$ and N relatively prime to elements between $(n_0/n)N$ and $(m_0/m)N$ and, further, that we are given a GRUPPE

$$k'm + l'n,\ N,\ k''m + l''n, \text{ then } (k' + k'')(k''m + l''n) \equiv n \pmod{N}.$$

Eisenstein stated that for a GRUPPE α, N, β where

$$\beta = k'm + l'n + sN$$

and

$$\beta \equiv k''m + l''n + TN,$$

then

$$\beta \equiv k''m + l''n \pmod{N}.$$

(20) Eisenstein continued to contribute to Stern's analysis hoping to arrive at the more complex function he had originally proposed. Stern stated that in the analysis of row $(1, 2)_p$ and N that are SUMMENGLIED, that N is relatively prime to elements between $\emptyset$ and $N/2$. Further, since $(1,1)_p$ occurs in the first half of $(1, 2)_p$, that N occurring in the $(1,1)_p$ portion are relatively prime to the rest of the system [not in $(1, 1)_p$] mod N. And last, but not least, Eisenstein commented that if N is relatively prime to numbers between $(n_0/n)N$ and $(m_0/m)N$ then it is also relatively prime to numbers between $(m - m_0/m)N$ and $(n - n_0/n)N$.

(21) Let us now examine rows $(m,n)_p$ and SUMMENGLIED of the form $kn + ln$. Let the GRUPPE be

$$k'm + l'n,\ km + ln,\ k''m + l''n,$$

then

$$1)\quad k'l - kl' = 1$$

and

$$2)\quad k''l - kl'' = -1.$$

Now presume that the continued fraction

$$\frac{k}{l} = (a, a_1, a_2, \ldots, a_m)$$

and that $k' = k_0$ and $l' = l_0$ or $k'' = k_0$ and $l'' = l_0$ (at the reader's option). Eisenstein states that the following is true:

$$\frac{k'}{k''} = a_m + (-1, a_{m-1}, \ldots, a_1, a);$$

and, consequently, that

$$p = a + a, + \cdots + a_m - 1.$$

This result is, of course, similar to the result (1) observed by Stern.

(22) Now with some of Stern's sequence theory under our belts, we can analyze Eisenstein's function:

(a) $f(m,n) = f(m,m + n) + f(m + n,n)$ when $m + n < \lambda$;

(b) $f(m,n) = n$ when $m + n = \lambda$;

(c) $f(m,n) = \emptyset$ when $m + n > \lambda$;

where m and n are positive numbers and λ is prime.

Note the relationship to Stern numbers when expanding $f(m,n)$:

$$\begin{aligned} f(m,n) &= f(m,m+n) + f(m+n,n) \\ &= f(m,2m+n) + f(2m+n,m+n) \\ &\qquad + f(m+n,m+2n) + f(m+2n,n) \end{aligned}$$

The arguments of the function are generalized Stern numbers. The following conclusion can now be drawn concerning Eisenstein's function.

1. For any given $f(km + ln, k'm + l'n)$, that $(k + k')m + (l + l')n = \lambda$.
2. If $m = 1$ and $n = 2$, then (16) implies that $f(1, 2)$ can be composed of elements of the form $f(\alpha, \lambda - \alpha)$ and that
$$f(1, 2) = \lambda - \alpha + \lambda - \alpha' + \lambda - \alpha'' + \cdots$$
3. For whole numbers "r" such that $\frac{\lambda + 1}{2} \le r \le \lambda - 1$,
$$f(1, 2) \equiv \sum \frac{1}{r} \pmod{\lambda}.$$
4. For whole numbers "r" such that, as in (18),
$$\frac{n_0\lambda}{n} \le r \le \frac{m_0\lambda}{m}$$
then
$$f(m,n) \equiv \sum \frac{1}{r} \pmod{\lambda}.$$

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2. Herrn Stern, "Ueber eine zahlentheoretische Funktion," *Journal für die reine und Angewandte Mathematik in zwanglosen heften* (Göttingen), Vol. 55, pp. 193-220.

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A MULTINOMIAL GENERALIZATION OF A BINOMIAL IDENTITY

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1. The binomial identity which we wish to generalize is the following:

$$(1) \qquad (x + y)^n = \sum_{k=1}^{n} \binom{2n-k-1}{n-1}\left(x^k + y^k\right)\left(\frac{xy}{x+y}\right)^{n-k}.$$

It can be found and is proved in [2]. Let us begin by giving a demonstration suitable to a generalization to more than two variables. Symbolizing $C_{t^n}f(t)$ for the coefficient a_n of t^n in any power series $f(t) = \sum_{n \ge 0} a_n t^n$, it is easily shown that the second number of (1) is: