"Proceedings of the Royal Society of Edinburgh" Vol 59 1938-39 Part 2.

XV.—**On Non-Associative Combinations.** By **I. M. H. Etherington,** B.A.(Oxon.), Ph.D.(Edin.), Mathematical Institute, University of Edinburgh.

(MS. received October 18, 1938. Revised MS received March 24, 1939. Read May 1, 1939.)

CONTENTS.

§ 1. INTRODUCTION.

NUMEROUS combinatory problems arise in connection with a set of elements subject to a non-associative process of composition—let us say of multiplication—commutative or non-commutative.

Non-associative products may be classified according to their *shape*. By the shape of a product I mean the manner of association of its factors without regard to their identity. Shapes will be called *commutative* or *non-commutative* according to the type of multiplication under consideration. Thus if multiplication is non-commutative, the products (AB.C)D and (BA.C)D are distinct but have the same shape, while D(AB.C) has a different shape. The three expressions, however, have the same commutative shape. I confine attention to products (like these) in which the factors are combined only two at a time.

In § 2 I define addition and multiplication of shapes, and show that they may be regarded as the "positive integers" of a kind of non-associative arithmetic. With commutative multiplication this provides a convenient numerical notation by which shapes of great complexity can be easily specified.

A non-associative product or shape may be visualised as a *pedigree*, by which I mean a *tree* (Cayley, 1857) which (going from the root upwards, *i.e.* from the product to its factor elements) bifurcates at every knot (Cayley, 1859). Trees in general may quantifurcate arbitrarily at the knots, representing a more general kind of non-associative assemblage, which was also considered abstractly by Schröder (1870). A fourfold classification of shapes, arising partly out of this representation, is discussed in § 3.

Enumerative problems connected with non-associative combinations have been considered from various points of view by Catalan (1838, p. 515; etc.), Rodrigues (1838), Binet (1839), Schröder (1870): see Netto (1901, §§ 122–128) for a summary of their work; also by Cayley (1857, etc.), Wedderburn (1922). Some further enumerations are discussed here (§ 4); in particular, with the aid of the concept of *mutability*, defined in § 3, it is shown that the commutative and non-commutative cases can be treated simultaneously. Thus equation (33) below, with y put equal to 1, yields the known formulæ (25), (27) for the commutative case; putting $y=2$, the known results (24), (26), (28) follow for the non-commutative case.

§ 2. Arithmetic of Shapes.

To eliminate brackets in writing non-associative products, it is convenient to use groups of dots to separate the factors when necessary, fewness of dots implying precedence in multiplication. Thus A:.BC.AD²:E means $A\{[(BC)(A\overline{DD})]E\}$. (The notation is due to Peano.)

Products and shapes in which the factors are absorbed one at a time (*e.g.* A : BC.D :.E) will be called *primary*. The shapes generated by repeated squaring of an element, and products having such a shape (*e.g.* AB.CD : EF.GH), will be called *plenary*. It will be seen in § 3 that all other shapes are in a sense intermediate between these two extremes.

For the moment, confine attention to the case of commutative multiplication, where a primary shape is unique when the number of factors δ is given. A power having this shape will be denoted X^δ: *e.g.* X^4 means XX.X : X. All other powers can be represented by suitably partitioning the index, using brackets when necessary, with the following conventions: the product of two powers of the same element is indicated as a sum in the index, a power of a power as a product in the index, and an iterated power as a power in the index. Thus:

$$X^{2+3} = X^2X^3,$$
$$X^{2\cdot 3} = (X^2)^3, \qquad X^{3\cdot 2} = (X^3)^2,$$
$$X^{2^3} = ((X^2)^2)^2, \quad X^{3^2} = (X^3)^3,$$
$$X^{(2\cdot 2+3)+(2+3)^2} = X^2X^2.X^3 :: X^2X^3.X^2X^3 :. X^2X^3.X^2X^3 : X^2X^3.$$

Addition of indices, since it reflects non-associative multiplication of powers, is commutative but non-associative. On the other hand, multiplication of indices is non-commutative (as seen above), but associative, since $X^{ab.c}$ and $X^{a.bc}$ both mean $((X^a)^b)^c$, which can therefore be written X^{abc} unambiguously. This becomes X^{a^3} when $a=b=c$; and similarly with any number of factors in the index.

Further, $X^{a(b+c)}$ means $(X^a)^{b+c}$, *i.e.* $(X^a)^b(X^a)^c$, which is the same as X^{ab+ac}. Hence in the arithmetic of the indices

$$a(b+c) = ab + ac.$$

But in general

$$(b+c)a \neq ba + ca,$$

since $(X^bX^c)^a$ is not the same as $(X^b)^a(X^c)^a$. We may say therefore that in the arithmetic of the indices multiplication is predistributive with addition, but not in general postdistributive.

In these arguments a, b, c can be any expressions standing for complicated powers: they are not restricted to being simple integers indicating primary powers.

The notation provides an arithmetical method of specifying commutative shapes; for now the *shape* s of any commutative non-associative product can be redefined as the index of the corresponding power obtained by equating all the factors The product $AB.C^2:D$, for instance, has the same shape as the power $(X^2)^2X = X^{2.2+1}$, namely $s = 2.2+1$.

Consider what addition and multiplication of shapes mean when we are dealing with products in general instead of powers. Let Π_1, Π_2 be any two products with shapes s_1, s_2. Then $s_1 + s_2$ is the shape of the product $\Pi_1\Pi_2$, while s_1s_2 is the shape of the product formed by substituting Π_1 for each of the factor elements of Π_2.

The procedure of this § may be described as a representation of the set of all commutative non-associative continued products formed from given elements on a non-associative arithmetic, whose integers are commutative shapes a, b, c, . . . with the rules of combination

$$\left.\begin{array}{lll} a+b = b+a, & ab.c = a.bc, & a(b+c) = ab+ac, \\ ab \neq ba, & (a+b)+c \neq a+(b+c), & (b+c)a \neq ba+ca. \end{array}\right\} \quad . \qquad (1)$$

A similar representation is possible when multiplication of the original elements is non-commutative as well as non-associative. It is reflected as non-commutative addition of shapes, the other rules of combination (1) being unchanged. But the numerical specification of non-commutative shapes of increasing complexity rapidly becomes very complicated; to simplify it, some convention is required for distinguishing the $2^{\delta-2}$ distinct primary shapes of any given degree $\delta(>1)$.

§ 3. Classification of Shapes.

Shapes s will be classified by their *degree* $\delta(s)$, *altitude* $a(s)$, and *mutability* $\mu(s)$. Non-commutative shapes will be further classified by the commutative shapes with which they are *conformal.* These terms will now be defined.

The *degree* δ of a shape s means the number of factor elements in a product having this shape. It may be reckoned by evaluating s as if it were an integer in ordinary arithmetic.

Two non-commutative shapes s_1, s_2, which become the same shape s when multiplication is regarded as commutative, will be called *conformal* with each other and with s. Write $s_1 \sim s_2$ to indicate this. With commutative shapes, $s_1 \sim s_2$ means the same as $s_1 = s_2$, a commutative shape being conformal only with itself. The word is also applicable to products whose shapes are conformal. Thus

$$AB.C : D, \qquad A.BC : D, \qquad A : BC.D, \qquad A : B.CD$$

and their shapes

$$(2+1)+1, \qquad (1+2)+1, \qquad 1+(2+1), \qquad 1+(1+2)$$

are all conformal with the commutative power A^4 and its shape 4.

Let shapes be depicted as pedigrees (§ 1). Any non-associative product is then, so to speak, "descended from" its factors. The number of "generations" preceding the product itself is its *altitude* α (Cayley, 1875). At each knot in the pedigree two factors are united; the total number of knots is thus $\delta - 1$. Let a knot be called *balanced* if its two factors are conformal: then the number of unbalanced knots in a pedigree will be called its *mutability* μ. The various terms defined may be applied indiscriminately to the product, shape or pedigree.

If the mutability of any shape s (commutative or not) is μ, then there are evidently just 2^μ distinct non-commutative shapes which will become the same as s when multiplication is commutative. So μ could be defined alternatively as the logarithm to base 2 of the number of conformal non-commutative shapes.

If $s_1 \sim s_2$, then evidently

$$\delta(s_1) = \delta(s_2), \qquad \alpha(s_1) = \alpha(s_2), \qquad \mu(s_1) = \mu(s_2). \qquad (2)$$

The following formulæ are easily proved, r and s being any shapes, commutative or non-commutative, and ν an ordinary positive integer:—

$$\delta(r+s) = \delta(r) + \delta(s), \qquad (3)$$

$$\delta(rs) = \delta(r)\delta(s), \qquad (4)$$

$$\delta(s^\nu) = \delta(s)^\nu; \qquad (5)$$

$$\alpha(r+s) = 1 + \alpha(r) \quad \text{or} \quad 1 + \alpha(s) \quad \text{according as} \quad \alpha(r) > \text{ or } \leqq \alpha(s), \qquad (6)$$

$$\alpha(rs) = \alpha(r) + \alpha(s), \qquad (7)$$

$$\alpha(s^\nu) = \nu\alpha(s); \qquad (8)$$

$$\mu(r+s)=2\mu(s) \qquad \text{if } r\sim s, \quad \ldots \quad (9)$$

$$=1+\mu(r)+\mu(s) \text{ if not}, \quad \ldots \quad (10)$$

$$\mu(rs)=\delta(s)\mu(r)+\mu(s), \quad \ldots \quad (11)$$

$$\mu(s^{\nu})=(1+\delta+\delta^2+\ldots+\delta^{\nu-1})\mu(s), \quad \ldots \quad (12)$$

where

$$\delta=\delta(s).$$

The last result is proved by induction from the preceding one. It may also be written

$$\frac{\mu(s^{\nu})}{\mu(s)}=\frac{\tau(s^{\nu})}{\tau(s)}, \quad \ldots \quad (13)$$

where

$$\tau=\delta-1.$$

The degree, altitude and mutability can now be readily calculated for any given shape specified numerically. The table below gives all commutative shapes for which $a \leqslant 4$, $\delta \leqslant 6$.

TABLE OF COMMUTATIVE SHAPES.

a.	δ.	μ.	s.
0	1	0	1
1	2	0	2
2	3	1	3
2	4	0	2.2
3	4	2	4
3	5	1	2.2 + 1
3	5	2	3 + 2
3	6	1	2.3
3	6	2	3.2
3	7	2	2.2 + 3
3	8	0	2^3
4	5	3	5
4	6	2	(2.2 + 1) + 1
4	6	3	(3 + 2) + 1, 4 + 2
Etc.			

As the table suggests, we cannot construct a shape with a, δ, μ assigned arbitrarily. Certain relations must be satisfied, namely:

$$2^a \geqslant \delta \geqslant a+1; \quad \text{i.e.} \quad \delta-1 \geqslant a \geqslant \log_2 \delta. \quad \ldots \quad (14)$$

$$\delta \geqslant \mu+2, \quad \textit{except when} \quad \delta=1. \quad \ldots \quad (15)$$

$$\mu \leqslant 3.2^{a-3}-1; \quad \text{i.e. } a \geqslant 3+\log_2\frac{\mu+1}{3}, \quad \textit{except when} \quad a<3. \quad \ldots \quad (16)$$

δ is expressible as the sum of $\mu+1$ powers of 2, *not all alike if $\mu>0$.* (17)

(14) and (15) are easily proved by consideration of pedigrees. At one extreme, the equality $\delta=a+1$ holds only when s is primary; and the

same is true of $\delta = \mu + 2$. Similarly at the other extreme, $\delta = 2^a$, $\mu = 0$ occur when and only when s is plenary.

(17) is proved by induction from (3), (5), (9), (10); it being noted that when $\mu = 0$, s is of the form 2^a (plenary); when $\mu = 1$, $s = 2^a + 2^\beta$ $(a \neq \beta)$; and that two like powers of 2 can be combined if desired into a single power of 2.

To prove (16), let μ_a be the greatest possible mutability for a shape whose altitude a is given; it will be shown that for $a \geqslant 3$

$$\mu_a = 3.2^{a-3} - 1.$$

In view of (2) it will be sufficient to consider only commutative shapes. By inspection of the table of commutative shapes,

$$\mu_0 = \mu_1 = 0, \qquad \mu_2 = 1, \qquad \mu_3 = 2.$$

Now (see (6)) any shape of altitude $a+1$ is necessarily the sum of two shapes, one of altitude a and one of altitude $\beta \leqslant a$. By (9), (10), μ_{a+1} must be expressible either as $2\mu_a$ or as $1 + \mu_a + \mu_\beta$. Since $\mu_1 = 0$, $\mu_2 = 1$, it follows that

$$\mu_{a+1} > \mu_a \quad \text{for} \quad a > 0,$$

so that μ_a increases monotonically with a.

Now let a be any altitude (e.g. $a = 3$) for which there exist at least three distinct shapes s_1, s_2, s_3 with the maximum mutability μ_a. Then

$$\mu(s_1) = \mu(s_2) = \mu(s_3) = \mu_a > \mu(s),$$

where s is any shape of lower altitude. Hence for the altitude $a+1$ also there will exist at least three distinct shapes of maximum mutability; namely,

$$s_1 + s_2, \qquad s_2 + s_3, \qquad s_3 + s_1,$$

with the mutability given by (10)

$$\mu_{a+1} = 1 + 2\mu_a. \qquad (a \geqslant 3.)$$

It follows that

$$1 + \mu_{a+1} = 2(1 + \mu_a).$$

But

$$1 + \mu_3 = 3,$$

whence

$$1 + \mu_a = 3.2^{a-3},$$

or

$$\mu_a = 3.2^{a-3} - 1 \qquad \text{if} \quad a \geqslant 3.$$

This proves (16).

It will be seen that the equality in (16) is attained by N_a commutative shapes of altitude a, where

$$N_{a+1} = \tfrac{1}{2} N_a (N_a - 1), \qquad N_3 = 4. \qquad \qquad (16a)$$

§ 4. Enumeration of Shapes.

Let a_δ, p_α be the numbers of possible shapes of given degree δ and of given altitude α respectively, when multiplication is non-commutative and non-associative; and let b_δ, q_α be the corresponding numbers when multiplication is commutative and non-associative. Evidently

$$a_1 = b_1 = p_0 = q_0 = 1.$$

Remembering (3) and (6), and considering the different ways in which shapes of given degree or altitude can be formed from those of lower degree or altitude, we obtain the formulæ:

$$a_\delta = a_1 a_{\delta-1} + a_2 a_{\delta-2} + a_3 a_{\delta-3} + \ldots + a_{\delta-1} a_1, \qquad (18)$$

$$\left.\begin{aligned} b_{2\delta-1} &= b_1 b_{2\delta-2} + b_2 b_{2\delta-3} + \ldots + b_{\delta-1} b_\delta, \\ b_{2\delta} &= b_1 b_{2\delta-1} + b_2 b_{2\delta-2} + \ldots + b_{\delta-1} b_{\delta+1} + \tfrac{1}{2} b_\delta (b_\delta + 1), \end{aligned}\right\} \qquad (19)$$

$$p_{\alpha+1} = 2p_\alpha(p_0 + p_1 + p_2 + \ldots + p_{\alpha-1}) + p_\alpha^2, \qquad (20)$$

$$q_{\alpha+1} = q_\alpha(q_0 + q_1 + q_2 + \ldots + q_{\alpha-1}) + \tfrac{1}{2} q_\alpha(q_\alpha + 1). \qquad (21)$$

For $\delta = 1, 2, 3, \ldots$ and $\alpha = 0, 1, 2, \ldots$ the sequences start:

$$a_\delta = 1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862, \ldots$$
$$b_\delta = 1,\ 1,\ 1,\ 2,\ 3,\ 6,\ 11,\ 23,\ 46,\ 98, \ldots$$
$$p_\alpha = 1,\ 1,\ 3,\ 21,\ 651,\ 457653,\ 210065930571, \ldots$$
$$q_\alpha = 1,\ 1,\ 2,\ 7,\ 56,\ 2212,\ 2595782, \ldots$$

Let

$$F(x) = a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_\delta x^\delta + \ldots \qquad (22)$$

and

$$f(x) = -1 + b_1 x + b_2 x^2 + \ldots + b_\delta x^\delta + \ldots \qquad (23)$$

The following results are known:—

$$F(x)^2 - F(x) + x = 0, \qquad (24)$$
$$f(x)^2 + f(x^2) + 2x = 0; \qquad (25)$$
$$F(x) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4x}, \qquad (26)$$
$$f(x) = \lim_{n \to \infty} -\sqrt{-2x + \sqrt{-2x^2 + \sqrt{-2x^4 + \ldots + \sqrt{-2x^{2^n} + 1}}}}, \qquad (27)$$

where in (27) each $\sqrt{\ }$ covers all that follows it;

$$a_\delta = \frac{(2\delta - 2)!}{(\delta - 1)!\ \delta!} = \frac{1}{\delta}\cdot {}^{2\delta-2}C_{\delta-1}. \qquad (28)$$

Of those formulæ, (18), (28) were given by Catalan (1838). (Catalan pointed out that a_δ is the number of ways in which a convex polygon of $\delta + 1$ sides can be divided up into triangles by diagonals. (28), as a consequence of (18) with $a_1 = 1$, was first established from this point of view, and was known to other writers, apparently first to Euler. Several papers on this topic appear in the *Journ. de Math.*, 1838–39.) Binet (1839) introduced the generating function (22), and deduced (24), (26), (28) from (18). The calculations were repeated by Cayley (1859) from the pedigree point of view; by Schröder (1870); also by Wedderburn (1922), who discussed as well the commutative case, obtaining (19), (25), (27), and made a special study of the functional equation (25) and its more general solutions. (*Cf.* Etherington, 1937.)

It will now be shown that by introducing mutability we can discuss the commutative and non-commutative cases simultaneously and obtain a more general functional equation (33) which includes the two equations (24), (25) as special cases.

Let $c_{\delta\mu}$ be the number of possible commutative shapes of given degree δ and mutability μ, so that the corresponding number of non-commutative shapes will be

$$n_{\delta\mu} = 2^{\mu} c_{\delta\mu}. \qquad (29)$$

Then $n_{\delta\mu}$, $c_{\delta\mu}$ are defined for all integer values of δ, μ with $\delta > 1$, $\mu \geqslant 0$. For all other values of δ and μ, let $n_{\delta\mu}$, $c_{\delta\mu}$ be defined as zero.

Consider with the aid of (3), (9), (10) the different ways in which a non-commutative shape s of degree δ and mutability μ can be formed. Excluding $\delta = 1$, $\mu = 0$, $s = 1$, s must be of the form $s_1 + s_2$ where, by (3),

$$\delta(s_1) + \delta(s_2) = \delta.$$

If (10) held in all cases, we should have

$$1 + \mu(s_1) + \mu(s_2) = \mu,$$

and consequently

$$n_{\delta\mu} = \sum_{i,j,l,m} n_{il} n_{jm} \qquad (i + j = \delta, \quad 1 + l + m = \mu).$$

Subtracting the cases to which (10) does not apply, and adding those to which (9) does, we get as the correct formula

$$\left.\begin{aligned} n_{\delta\mu} &= \sum_{i,j,l,m} n_{il} n_{jm} - 2^{\frac{1}{2}(\mu-1)} n_{\frac{1}{2}\delta, \frac{1}{2}(\mu-1)} + 2^{\frac{1}{2}\mu} n_{\frac{1}{2}\delta, \frac{1}{2}\mu} \\ \text{where} \quad & i + j = \delta, \quad l + m = \mu - 1, \quad \delta \neq 1. \\ \text{Also} \quad & n_{10} = 1. \end{aligned}\right\} \qquad (30)$$

Putting $n_{\delta\mu} = 2^{\mu} c_{\delta\mu}$, and removing the factor 2^{μ},

$$\left.\begin{aligned} c_{\delta\mu} &= \tfrac{1}{2}\Big(\sum_{i,j,l,m} c_{il} c_{jm} - c_{\frac{1}{2}\delta, \frac{1}{2}(\mu-1)} \Big) + c_{\frac{1}{2}\delta, \frac{1}{2}\mu} \\ \text{where} \quad & i + j = \delta, \quad l + m = \mu - 1, \quad \delta \neq 1. \\ \text{Also} \quad & c_{10} = 1. \end{aligned}\right\} \qquad (31)$$

Now let

$$f(x, y) = \sum_{\delta, \mu} c_{\delta\mu} x^{\delta} y^{\mu}. \quad . \quad . \quad . \quad . \quad (32)$$

Substituting (31) in (32), we obtain the functional equation

$$f(x, y) = x + \tfrac{1}{2}y\{f(x, y)\}^2 + (1 - \tfrac{1}{2}y)f(x^2, y^2). \quad . \quad . \quad (33)$$

Now, from the definitions of a_δ, b_δ, $c_{\delta\mu}$, $n_{\delta\mu}$,

$$a_\delta = \sum_\mu n_{\delta\mu} = \sum_\mu 2^\mu c_{\delta\mu}, \qquad b_\delta = \sum_\mu c_{\delta\mu}.$$

Consequently, comparing (22), (23), (32),

$$\mathrm{F}(x) = f(x, 2), \qquad 1 + f(x) = f(x, 1). \quad . \quad . \quad . \quad (34)$$

It is readily verified that on putting $y = 2$ the equation (33) reduces to (24); and that on putting $y = 1$ it reduces to (25), as it should.

If on the right of (33) we substitute the first approximation

$$f(x, y) = x + \ldots, \qquad f(x^2, y^2) = x^2 + \ldots,$$

we obtain the second approximation

$$f(x, y) = x + x^2 + \ldots$$

Similarly the third approximation is

$$\begin{aligned} f(x, y) &= x + \tfrac{1}{2}y(x^2 + 2x^3 + x^4 + \ldots) + (1 - \tfrac{1}{2}y)(x^2 + x^4 + \ldots) \\ &= x + x^2 + x^4 + x^3 y + \ldots; \end{aligned}$$

and the process may be repeated to any required extent.

Alternatively, we may proceed in either of the following ways. Write

$$f(x, y) = xf_1(y) + x^2 f_2(y) + \ldots + x^\delta f_\delta(y) + \ldots \quad . \quad . \quad (35)$$

or

$$f(x, y) = g_0(x) + yg_1(x) + y^2 g_2(x) + \ldots + y^\mu g_\mu(x) + \ldots; \quad . \quad . \quad (36)$$

substitute in the functional equation (33), and equate coefficients. We obtain

$$\left.\begin{aligned} &f_1 = f_2 = 1, \quad f_3 = y, \quad f_4 = 1 + y^2, \\ &f_5 = y + y^2 + y^3, \quad f_6 = y + 2y^2 + 2y^3 + y^4, \ldots, \\ &f_{2\delta-1} = y(f_1 f_{2\delta-2} + f_2 f_{2\delta-3} + \ldots + f_{\delta-1} f_\delta), \\ &f_{2\delta} = y(f_1 f_{2\delta-1} + f_2 f_{2\delta-2} + \ldots + f_{\delta-1} f_{\delta+1} + \tfrac{1}{2}f_\delta^2) + (1 - \tfrac{1}{2}y) f_\delta(y^2); \end{aligned}\right\} . \quad (37)$$

$$\left.\begin{aligned} &g_0 = x(1 - x)^{-1}, \quad g_1 = x^3(1 - x)^{-1}(1 - x^2)^{-1}, \\ &g_2 = x^4(1 + x + 2x^2)(1 - x)^{-1}(1 - x^2)^{-1}(1 - x^4)^{-1}, \\ &g_3 = x^5(1 + x + 3x^2)(1 - x)^{-2}(1 - x^2)^{-1}(1 - x^4)^{-1}, \ldots, \\ &g_{2\mu-1} = g_0 g_{2\mu-2} + g_1 g_{2\mu-3} + \ldots + g_{\mu-2} g_\mu + \tfrac{1}{2} g_{\mu-1}^2 - \tfrac{1}{2} g_{\mu-1}(x^2), \\ &g_{2\mu} = g_0 g_{2\mu-1} + g_1 g_{2\mu-2} + \ldots + g_{\mu-1} g_\mu + g_\mu(x^2). \end{aligned}\right\} . \quad (38)$$

It will be observed that

$$f_\delta(2) = a_\delta, \qquad f_\delta(1) = b_\delta. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The first of these two methods is perhaps the quickest way of calculating many terms of the expansion of $f(x, y)$. By means of the second, we could find explicit formulæ for $c_{\delta 0}, c_{\delta 1}, c_{\delta 2}, c_{\delta 3}, \ldots$

With regard to the convergence of the various generating series, it may be observed that (22), since it is the expansion of (26), is absolutely convergent if $|x| < \frac{1}{4}$. Since $b_\delta \leqslant a_\delta$, it follows that (23) also converges absolutely if $|x| < \frac{1}{4}$; and since $f(x, 2) = \mathrm{F}(x)$, it follows that the double series (32) converges absolutely if $|x| < \frac{1}{4}$, $|y| \leqslant 2$.

Summary.

Non-associative combinations are classified and enumerated with the aid of a representation involving non-associative arithmetic.

REFERENCES TO LITERATURE.

Binet, M. J., 1839. "Réflexions sur le problème de déterminer le nombre de manières dont une figure rectiligne peut être partagée en triangles au moyens de ses diagonales," *Journ. de Math.* (1), vol. iv, pp. 79-90.

Catalan, E., 1838. "Note sur une équation aux différences finies," *Journ. de Math.*, (1) vol. iii, pp. 508–516. Also 1839, vol. iv, pp. 91–94, 95–99; 1841, vol. vi, p. 74.

Cayley, A., 1857, 1859. "On the theory of the analytical forms called trees," *Phil. Mag.*, vol. xiii, pp. 172–176; vol. xviii, pp. 374–378. Also 1875, *Rep. Brit. Assoc. Adv. Sci.*, pp. 257–305; 1881, *Amer. Journ. Math.*, vol. iv, pp. 266–268; 1889, *Quart. Journ. Pure App. Math.*, vol. xxiii, pp. 376–378. (*Collected Math. Papers*, vol. iii, no. 203; iv, 247; ix, 610; xi, 772; xiii, 895.)

Etherington, I. M. H., 1937. "Non-associate powers and a functional equation," *Math. Gaz.*, vol. xxi, pp. 36–39, 153.

Netto, E., 1901. *Lehrbuch der Combinatorik*, Leipzig.

Rodrigues, O., 1838. "Sur le nombre de manières d'effectuer un produit de n facteurs," *Journ. de Math.* (1), vol. iii, p. 549.

Schröder, E., 1870. "Vier combinatorische Probleme," *Zeits. Math.*, vol. xv, pp. 361–376.

Wedderburn, J. H. M., 1922. "The functional equation $g(x^2) = 2ax + [g(x)]^2$," *Ann. Math.* (2), vol. xxiv, pp. 121–140.

(*Issued separately June* 27, 1939.)