Generalized Action Graphs

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Abstract

We discuss the combinatorics of *generalized action graphs*, which are an extension of a sequence of graphs called *action graphs*. Action graphs arise naturally in the study of rooted category actions in the work of Bergner and Hackney. Alvarez, Bergner, and Lopez prove that the number of vertices added to form the n^{th} action graph is the n^{th} Catalan number, C_n. We introduce a new sequence of graphs called *generalized action graphs* and prove that the number of vertices added to form the generalized action graph, $T_{n,k}$, is the Fuss–Catalan number $C_{n,k}$. We also establish a bijection between the vertices of the generalized action graph $T_{n,k}$ and full $(k+1)$-ary trees with at most $n(k+1)+1$ vertices.

Preliminary Definitions

An $\mathbb{N}$-**labeled directed graph** is a triple $G = (V, E, \lambda)$ such that (V, E) is a directed graph and $\lambda : V \to \mathbb{N}$ is a function. For a vertex $v \in V$, we say that v is $\lambda(v)$-labeled or that v has the label $\lambda(v)$. We refer to the function λ as the labeling.

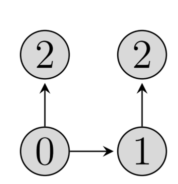


Fig. 1: Example of a Labeled Directed Graph

The **Fuss–Catalan numbers** $C_{n,k}$ are defined by the recurrence relation

$$C_{n,k} = \sum_{n_1+n_2+\cdots+n_{k+1}=n-1} \prod_{i=1}^{k+1} C_{n_i,k}$$

with initial conditions $C_{0,k} = 1$. More explicitly, the Fuss–Catalan numbers satisfy the formula

$$C_{n,k} = \frac{1}{kn+1}\binom{n(k+1)}{n}.$$

The sequence $(C_{n,1})_{n\geq 0}$ is the sequence $(C_n)_{n\geq 0}$ of Catalan numbers [3, A000108].

Most of the objects which the Fuss–Catalan numbers count are more general versions of objects counted by the Catalan numbers.

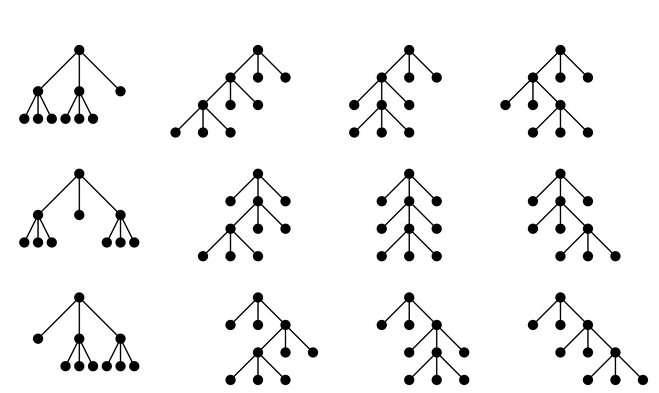


Fig. 2: The set of full ternary trees (3-ary trees) with 10 vertices, counted by $C_{3,2}$.

Of particular interest to us are the class of **full k-ary trees**, which are a structure counted by the Fuss–Catalan numbers. Either they are a 1-element tree $T = *$, or they are recursively defined of the type $T = (*, S)$ where S is a k-tuple of full k-ary trees. We refer to the tree $*$ as the identity tree and denote it by id. The tree id has 1 vertex. Otherwise, the number of vertices of a tree $T = (*, S)$ is defined recursively, analogous to the above definition.

Generalized Action Graphs

A **path** p of length ℓ in a directed graph $G = (V, E)$ is an $(\ell+1)$-tuple of distinct vertices $(x_0, \ldots, x_\ell) \in V^{\ell+1}$ such that, for each $i \in \{1, \ldots, \ell\}$, we have $(x_{i-1}, x_i) \in E$. We say that x_0 is the **source vertex** of the path p, denoted $s(p)$, and that x_ℓ is the **target vertex** of the path p, denoted $t(p)$. For $0 \leq n \leq \ell$, we will write $v_p(n) = x_n$ for the n^{th} vertex in the path p. We will write $\ell(p)$ for the length of the path p.

We define paths in this way so we can encode the notion of a trivial path (x_0) consisting of only one vertex for which its length $\ell = 0$.

Informally, the Generalized Action Graph is defined as follows. First, we let $T_{0,k}$ be the labeled directed tree with one vertex with label 0. Then inductively, the graph $T_{n+1,k}$ is constructed as follows: For each path p with target vertex with label n in $T_{n,k}$, we adjoin $\binom{k+\ell(p)-1}{\ell(p)}$ new vertices to the **source** vertex of the path p in $T_{n,k}$. We label these new vertices with label $n+1$.

Below is an example of the first few action graphs with $k = 2$.

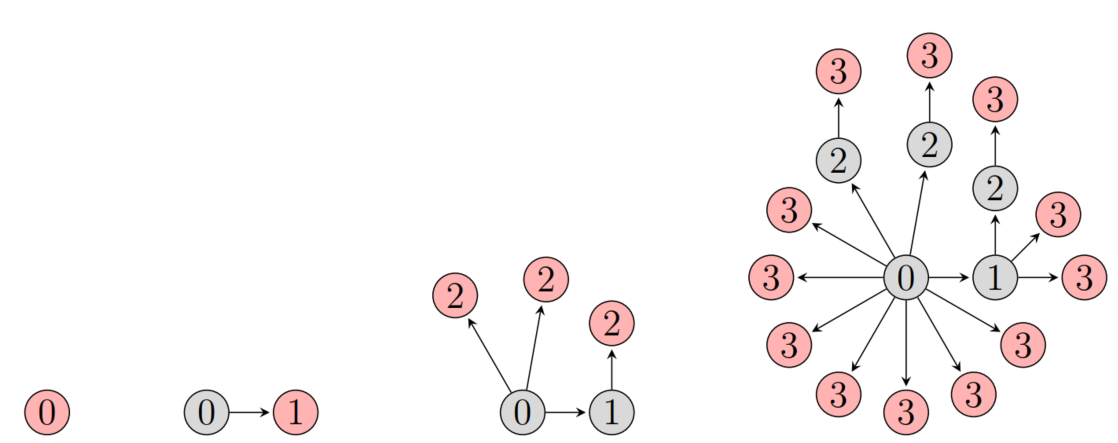


Fig. 3: The first few generalized action graphs of parameter $k = 2$

Some Essential Properties

- All the generalized action graphs $T_{n,k}$ are trees (that is, they contain no cycles). As a result, it makes sense to think of $T_{n,k}$ as a rooted tree with root vertex the sole vertex labeled 0.
- Realizing $T_{n,k}$ as a directed tree, in the graph $T_{n,k}$, any rooted subtree with root vertex labeled m and containing all the children of that vertex is isomorphic to $T_{n-m,k}$. So in a sense the generalized action graphs are **self-similar**. The visual below illustrates this property.

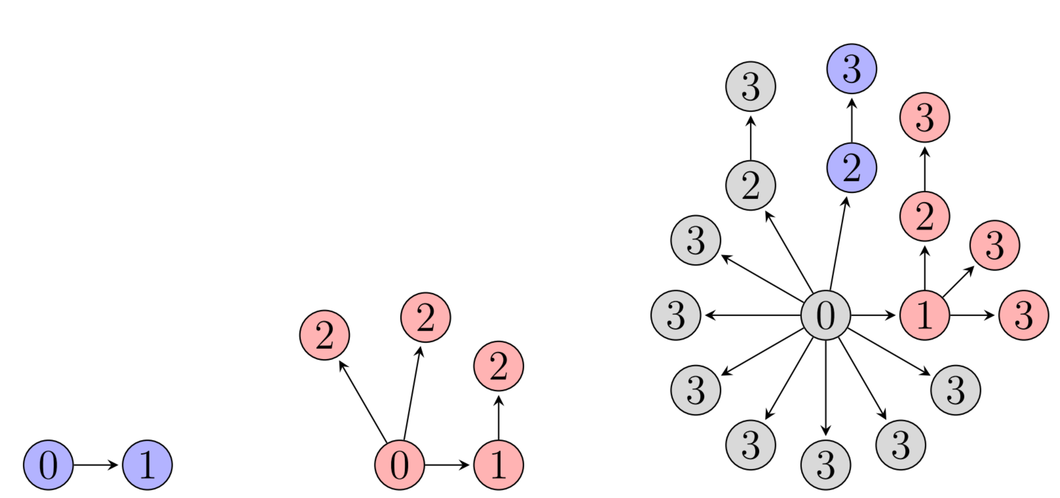


Fig. 4: An illustration of self-similarity for parameter $k = 2$

Main Results

The main result we proved was the following:

In the generalized action graph $T_{n,k}$, the number of vertices labeled n is the Fuss–Catalan number $C_{n,k}$.

We proved this in two ways:

- Using a counting argument, we proved this result more or less directly.
- We exhibit a natural bijection between the vertices in $T_{n,k}$ and the collection of full $(k+1)$-ary trees with at most $n(k+1)+1$ vertices.

Both methods rely on the following elegant recharacterization of the addition of $\binom{\ell(p)+k-1}{\ell(p)}$ vertices for any given path p: This is the number of *order-preserving maps*

$$g : \{1, \ldots, \ell(p)\} \to \{1, \ldots, k\}.$$

Here is the idea for the proof of the first method. For any vertex of label n, we can associate with it the path p it was created from and an order-preserving map $g : \{1, \ldots, \ell(p)\} \to \{1, \ldots, k\}$. But given this information, we can use the order-preserving map to divide the path into various sections, as illustrated below:

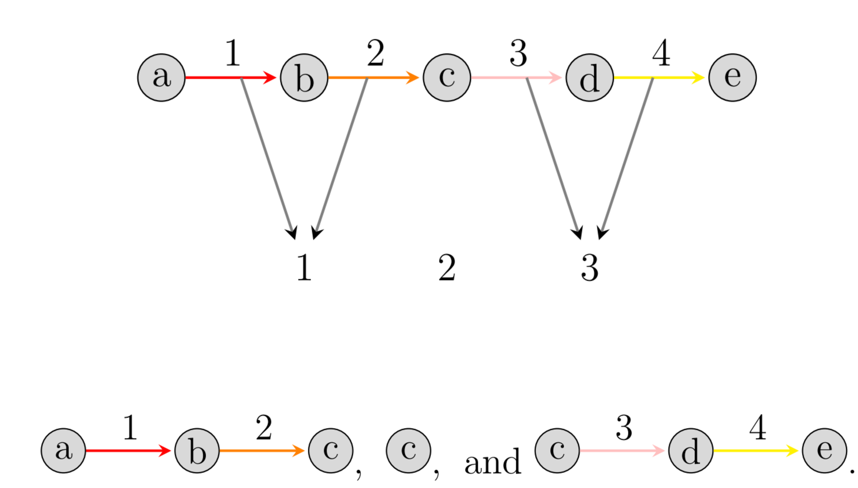


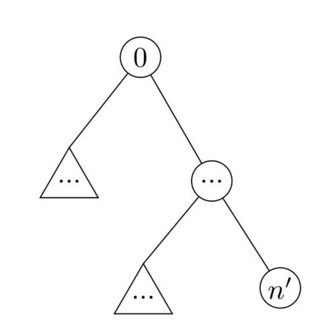
Fig. 6: An example path subdivision by an order-preserving map

From the path subdivision, we can deduce from the self similarity of the generalized action graphs that there are $\prod_{i=1}^{k+1} f(n_i, k)$ paths with the same labeling and order-preserving map. Summing over all these paths, we get the recurrence

$$f(n, k) = \sum_{n_1+\cdots+n_{k+1}=n-1} \prod_{i=1}^{k+1} f(n_i, k),$$

which is exactly the recurrence for the Fuss–Catalan numbers $C_{n,k}$.

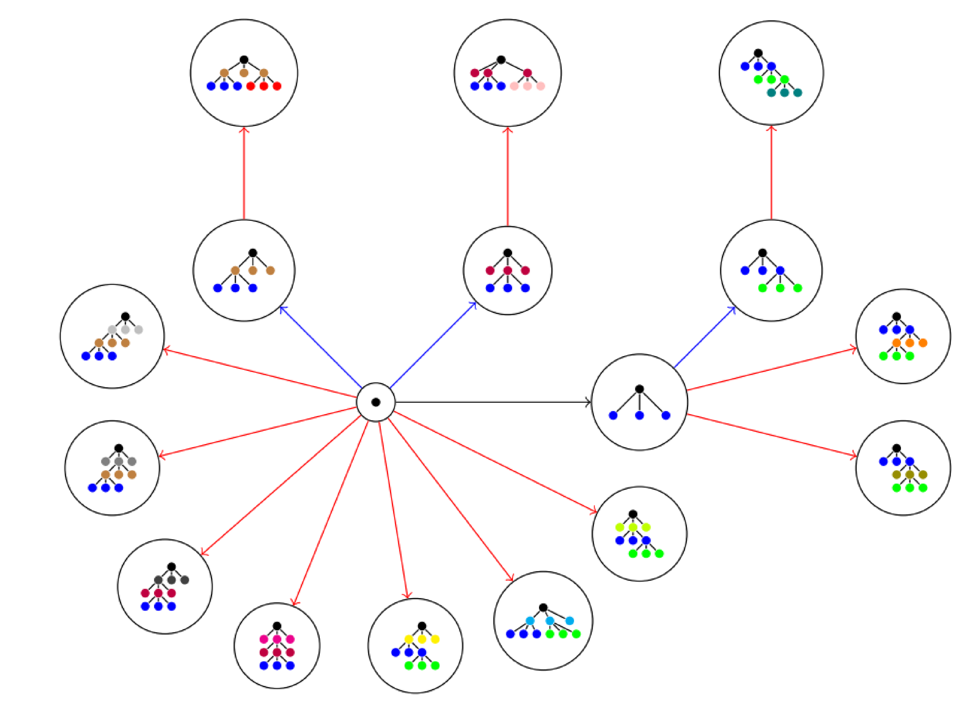
The main idea of the bijection is as follows. First, note that for each vertex v with label n in action graph $T_{n,k}$, there is a unique path from 0 to v. If we consider the set of complete $(k+1)$-ary trees with $n(k+1)+1$ nodes labelled by preorder traversal, each tree will be of the form



where $n' = n(k+1)$ and every node on the path from 0 to v is of the form $a(k+1)$, with $a \in \mathbb{N}$ and $a < n$. Additionally, each triangle represents k complete $(k+1)$-ary trees.

If we take this path from 0 to n' in each of our trees and divide the label of each vertex by $(k+1)$, we are able to find a unique path in our action graph $T_{n,k}$ to which it corresponds.

Illustration of Bijection



Further Questions

- We defined the Generalized Action Graph in such a manner that the number of new vertices added was precisely the Fuss–Catalan number $C_{n,k}$. It might be interesting to think about another definition that encodes a different generalization of the Catalan numbers.
- First introduced by Alvarez, Bergner, and Lopez [1], action graphs are a sequence of directed graphs $(A_n)_{n\geq 0}$ that generate the category $\mathcal{D}_{n,0}$, as defined in the context of rooted category actions by Bergner and Hackney [2]. It might be interesting to think about if there is some "natural" notion of a category action that encodes the generalized action graphs.

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