the equivalence $T^n(T - x)^m = T^{m-1}(T + x)^n$ which follows from iteration of $T^n(T - x) = (T + x)^n$. In general if $n = 2m$, the generating function is $T^{m-1}(T + x)^m$ for both black and white; if $n = 2m + 1$, the generating functions are $T^{m-1}(T + x)^{m+1}$ and $T^m(T + x)^m$. For $n = 8$, the computation may be set forth with detached coefficients as follows:

1	28	266	1050	1701	966	127	1	1
	1	21	140	350	301	63	1	4
		1	15	65	90	31	1	6
			1	10	25	15	1	4
				1	6	7	1	1
1	32	356	1704	3532	2816	632	16	
1	x	x^2	x^3	x^4	x^5	x^6	x^7	

A88960
A274105
black
A274106
white

That is, $T^3(T + x)^4 = 1 + 32x + 356x^2 + 1704x^3 + 3532x^4 + 2816x^5 + 632x^6 + 16x^7$, e.g., the number of ways of putting 4 bishops on the black squares in 3532. These results are in agreement with Perott [12] and Ahrens [1], whose methods are considerably more elaborate.

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Moment recurrence relations given in §6 also apply without change for specifications of this type. The mean and variance are

$$m_1 = \frac{1}{2}(n-1) + \frac{k_1}{n},$$

$$\mu_2 = \frac{1}{12}(n+1) - \frac{k_1^2}{n^2} + \frac{k_2}{n(n-1)}$$

with k_1 and k_2 fixed by initial conditions.

Finally it should be noted that all this is readily extended to problems of restricted position involving mutilated triangles of sides $n - c$.

8. **Bi-triangular permutations.** We give here an application of some of the results above to a problem which arose in war work. This may be stated briefly as: how many permutations are there such that no element is in a forbidden position given by the following array (or any of its equivalents)?

```
              x
              x x
              . . .
m             .   .   .
              .     .
              x x . . . x
                            x
                            x x
                            . . .
n - m                       .   .   .
                            . . .   .
                            x x x . . x

                   m             n - m
```

It is evident that the array is composed of disjunct triangles of sides m and $n - m$ and hence has polynomial $T_m T_{n-m}$. Without loss of generality m may be taken not greater than $[\frac{1}{2}n]$, the brackets indicating integral part, and

$$T_m T_{n-m} = \sum_i S(m+1-i,\ m+1)\, x^i T_{n-m}. \tag{17}$$

The corresponding generating function for permutations with r violations of the given conditions, say $B_{n,m}(y)$, may then be written as

$$B_{n,m}(y) = \sum_i S(m+1-i,\ m+1)(y-1)^i A_{n-i,m-i}(y), \tag{18}$$

where $A_{n,c}$ is the function defined in §6.

The number desired is $B_{n,m,0}$ where

$$\begin{aligned} B_{n,m,0} &= \sum_i S(m+1-i,\ m+1)(-)^i A_{n-i,m-i,0} \\ &= \sum_i S(m+1-i,\ m+1)(-)^i (m-i)^{n-m}(m-i)! \\ &= \sum_i S(i+1,\ m+1)(-)^{m-i} i^{n-m} i! \end{aligned} \tag{19}$$

the last step by the easily proved result: $A_{n,c,0} = c^{n-c}c!$.

A short table of the numbers $B_{n,m,0}$ follows:

n/m	1	2	3	4	5	6	7	8
2	1							
3	1	1						
4	1	5	1					
5	1	13	13	1				
6	1	29	73	29	1			
7	1	61	301	301	61	1		
8	1	125	1081	2069	1081	125	1	
9	1	253	3613	11581	11581	3613	253	1

Column 2 = A36563
Δ = A272644

col 2
= A36563
Δ =
A272644

9. **The problem of the bishops.** In how many ways can k bishops be placed on an n by n chessboard so that no two attack each other? It is evident that the board decomposes into two disjoint boards: the black and white squares. Thus the problem reduces at once to the corresponding problem on the white or black squares, and in fact the generating function for the whole board is simply the product of the two generating functions for the halves. Let us now give the board a 45° turn; the problem becomes one of rooks on a diamond shaped board, e.g., in the 6 by 6 case:

```
    x x
  x x x x
x x x x x x
  x x x x
    x x
```

If we rearrange and supply the three indicated dots as follows:

```
.
x x
x x .
x x x x
x x x x .
x x x x x x
```

we see that the board is a special case of "Simon Newcomb's problem" described in §7. Its generating function is $T^3(T-x)^3$ or $T^2(T+x)^3$, if we use non-descending) runs when not all the integers are considered to be distinct. Thus if we identify i and j $(i < j)$, we no longer reckon ji as an instance of descent, and in the triangular board of §4 the point (i, j) is deleted. Following MacMahon, we shall confine ourselves to the case where only blocks of successive integers are identified. Then the identification of i and $i + 1$ removes a point from the diagonal, that of i, $i + 1$, and $i + 2$ tears a triangle of side 2 from the diagonal, etc.

We shall indicate the specification of the elements in the usual fashion by the symbol $(1^{p_1}2^{p_2} \cdots s^{p_s})$, where $\sum ip_i = n$, meaning that there are p_1 single elements, p_2 distinct pairs of identified elements, p_3 triples of identified elements, etc.

The specification does not uniquely determine the identifications. For example, specification (12^23) might correspond to the identifications (1) (23) (45) (678) or (12) (3) (456) (78) or many others. The mutilated triangles arising from these two are respectively (the dots indicate the requisite deletions):

x	·
x ·	x x
x x x	x x x
x x x ·	x x x ·
x x x x x	x x x · ·
x x x x x ·	x x x x x x
x x x x x · ·	x x x x x x ·

However it has been proved by MacMahon [9] (and may also be proved by modification of the proof of (15) below) that these two boards are equivalent for the problem of the rooks, and hence the specification does determine uniquely the quantities in which we are interested.

Specification (1^n) means all distinct elements and gives a triangular board of side $n - 1$, whose generating function we have found to be $\sum S(n - k, n)\, x^k$. Let us write T_{n-1} for this polynomial.

Suppose next that elements 1 to p are identical and the remaining $n - p$ unlike: specification $(1^{n-p}p)$. The board for this is the trapezoid of §5 with the parameters q and a given by $q = n - p$, $a = 1$. By (7) the polynomial for this trapezoid may be written

$$\sum_i s(i, p)\, x^{p-i}\, T_{n-p-1+i}\,, \tag{14}$$

where $s(i, p)$ is a Stirling number of the first kind, defined by $(x)_p = \sum s(i, p)x^i$. With the definition

$$(T)_p = T(T - x) \cdots [T - (p - 1)x] = \sum_i s(i, p)\, x^{p-i}\, T_i\,,$$

(14) may be written more briefly in the symbolic form $T^{n-p-1}(T)_p$.

This is perhaps sufficient to suggest the general result: for specification $(1^{p_1}2^{p_2} \cdots s^{p_s})$, or $[s]$ for brevity, the generating function $U[s]$ is given by

$$U[s] = T^{p_1-1}(T)_2^{p_2} \cdots (T)_s^{p_s}. \tag{15}$$

The proof is by induction on p_s. To avoid complications which add nothing to the idea, we shall carry it out for a special case. Suppose (15) is known for $[s'] = (12^23)$ and $[s''] = (2^23)$; we shall prove it for $[s] = (23^2)$. For $[s']$ we may take the figure on the left above, for $[s'']$ the same with the first column deleted, and for $[s]$ the same with the top two members of the first column deleted. Then the number of ways of putting k non-attacking rooks on $[s']$ evidently equals the number of ways of putting them on $[s]$ plus twice the number of ways of putting $k - 1$ rooks on $[s'']$. That is,

$$\begin{aligned} U[s] &= U[s'] - 2x\, U[s''] \\ &= (T)_2^2\, (T)_3 - 2x\, T^{-1}\, (T)_2^2\, (T)_3 \\ &= T^{-1}\, (T)_2\, (T)_3^2\,, \end{aligned}$$

as desired.

The corresponding generating function for the numbers under consideration may be expressed concisely in the following symbolic form

$$(1!)^{p_1}\, (2!)^{p_2} \cdots (s!)^{p_s}\, A_{[s]}(y) = A^{p_1}\, (A)_2^{p_2} \cdots (A)_s^{p_s} \tag{16}$$

with $(A)_s = A[A - (y - 1)] \cdots [A - (s - 1)\,(y - 1)]$ and $A^n \equiv A_n \equiv A_{n,1}$ of §6.

Equation (16) contains a host of recurrences for the numbers. As one example, take $[s] = [21^{n-2}]$; then

$$2A_{21^{n-2}}\,(y) = A^{n-2}\,(A)_2 = A_n - (y - 1)\, A_{n-1}$$

and

$$2A_{21^{n-2},r} = A_{n,r} + A_{n-1,r} - A_{n-1,r-1}\,,$$

which has been given by MacMahon. Others given by MacMahon are derived with similar ease, but of course do not exhaust the possibilities.

Again, with p_2 to p_s fixed, $A_{[s]}(y)$ has recurrence (11) (with $c = 1$) and the corresponding numbers $A_{[s],r}$ have recurrence (12), which may be used to build up tables of which the following is a sample.

Specification (321^{n-5})

n/r	*1*	*2*	*3*	*4*	*5*	*6*
5	1	6	3			
6	1	17	33	9		
7	1	40	184	168	27	
8	1	87	792	1592	807	81

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non-descending) runs when not all the integers are considered to be distinct. Thus if we identify i and j $(i < j)$, we no longer reckon ji as an instance of descent, and in the triangular board of §4 the point (i, j) is deleted. Following MacMahon, we shall confine ourselves to the case where only blocks of successive integers are identified. Then the identification of i and $i + 1$ removes a point from the diagonal, that of i, $i + 1$, and $i + 2$ tears a triangle of side 2 from the diagonal, etc.

We shall indicate the specification of the elements in the usual fashion by the symbol $(1^{p_1}2^{p_2} \cdots s^{p_s})$, where $\sum ip_i = n$, meaning that there are p_1 single elements, p_2 distinct pairs of identified elements, p_3 triples of identified elements, etc.

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```
x                  .
x .                x x
x x x              x x x
x x x .            x x x .
x x x x x          x x x . .
x x x x x .        x x x x x x
x x x x x . .      x x x x x x .
```

However it has been proved by MacMahon [9] (and may also be proved by modification of the proof of (15) below) that these two boards are equivalent for the problem of the rooks, and hence the specification does determine uniquely the quantities in which we are interested.

Specification (1^n) means all distinct elements and gives a triangular board of side $n - 1$, whose generating function we have found to be $\sum S(n - k, n)\, x^k$. Let us write T_{n-1} for this polynomial.

Suppose next that elements 1 to p are identical and the remaining $n - p$ unlike: specification $(1^{n-p}p)$. The board for this is the trapezoid of §5 with the parameters q and a given by $q = n - p$, $a = 1$. By (7) the polynomial for this trapezoid may be written

$$\sum_i s(i, p)\, x^{p-i}\, T_{n-p-1+i}\,, \tag{14}$$

where $s(i, p)$ is a Stirling number of the first kind, defined by $(x)_p = \sum s(i, p) x^i$. With the definition

$$(T)_p = T(T - x) \cdots [T - (p - 1)x] = \sum_i s(i, p)\, x^{p-i}\, T_i\,,$$

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as desired.

The corresponding generating function for the numbers under consideration may be expressed concisely in the following symbolic form

$$(1!)^{p_1}\, (2!)^{p_2} \cdots (s!)^{p_s}\, A_{[s]}(y) = A^{p_1}\, (A)_2^{p_2} \cdots (A)_s^{p_s} \tag{16}$$

with $(A)_s = A[A - (y - 1)] \cdots [A - (s - 1)(y - 1)]$ and $A^n \equiv A_n \equiv A_{n,1}$ of §6.

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$$2A_{21^{n-2}}(y) = A^{n-2}\, (A)_2 = A_n - (y - 1)\, A_{n-1}$$

and

$$2A_{21^{n-2},r} = A_{n,r} + A_{n-1,r} - A_{n-1,r-1}\,,$$

which has been given by MacMahon. Others given by MacMahon are derived with similar ease, but of course do not exhaust the possibilities.

Again, with p_2 to p_s fixed, $A_{[s]}(y)$ has recurrence (11) (with $c = 1$) and the corresponding numbers $A_{[s],r}$ have recurrence (12), which may be used to build up tables of which the following is a sample.

Specification (321^{n-5})

n/r	1	2	3	4	5	6
5	1	6	3			
6	1	17	33	9		
7	1	40	184	168	27	
8	1	87	792	1592	807	81

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as follows from (3) by induction and the identity

$$x^q(x)_p = x^{q-1}(x)_{p+1} + px^{q-1}(x)_p .$$

In particular

$$u_k(0, q, 1) = \Delta^{q-k}0^q/(q - k)! = S(q - k, q), \tag{8}$$

the latter being a Stirling number of the second kind. Thus we have an interesting combinatorial interpretation of these Stirling numbers: *the number of ways of putting k non-attacking rooks on a right-angled isosceles triangle of side $q - 1$ is the Stirling number $S(q - k, q)$.*

It is worth noting in passing that (7) may also be written as an Appell polynomial in the Stirling numbers:

$$u_k(p, q, 1) = [S(q - k) + p]^q = \sum_i {}_qC_i\, p^i S(q - k, q - i).$$

6. **Triangular permutations.** We shall now apply the results of the preceding section to those problems described in §§3, 4 which give rise to triangular boards. The classification of permutations by ascending runs gives us a triangular board of side $n - 1$, as already observed. A triangle of side n arises if, for $i = 1, \cdots, n$, we forbid to i the positions $i, i + 1, \cdots, n$. For brevity we call these triangular permutations. To encompass both of these cases, as well as others, let us consider a triangle of side $n - c$ $(c \geq 0)$. Let us write $A(n, r, c)$ for the number of permutations violating r of the restrictions and $A_{n,c}$ for the generating function $\sum_r A(n, r, c)y^r$. Then by (1) and (8)

$$A_{n,c} = \sum_i S(i - c + 1, n - c + 1)(y - 1)^{n-i}\, i!.$$

More generally, let us suppose we have

$$A_n = \sum_i f(i, n)(y - 1)^{n-i}\, i!$$

with

$$f(i, n) = (i - c + 1)f(i, n - 1) + f(i - 1, n - 1), \tag{9}$$

$$f(i, n) = 0 \qquad (i > n); \tag{10}$$

(9) and (10) being well-known properties of the Stirling numbers. Then

$$A_{n+1} = y\sum_i f(i, n)(y - 1)^{n-i}(i + 1)! - c(y - 1)A_n .$$

Also, if A_n' denotes the derivative with respect to y,

$$(y - 1)A_n' = (n + 1)A_n - \sum_i f(i, n)(y - 1)^{n-i}\,(i + 1)!.$$

Hence

$$A_{n+1} = y(n + 1)A_n - c(y - 1)A_n - y(y - 1)A_n' . \tag{11}$$

Taking the coefficient of y^r in (11), we obtain

$$\begin{aligned} A(n + 1, r, c) &= (r + c)A(n, r, c) \\ &\quad + (n + 2 - r - c)A(n, r - 1, c). \end{aligned} \tag{12}$$

Equation (12) shows that $A(n, r - c, c)$ and $A(n, r, 0)$ enjoy the same recurrence formula, and that this formula is a direct algebraic consequence of the corresponding one for the Stirling numbers. It was first given by MacMahon [9], and was used by Moore and Wallis [11] to compile tables for significance tests.

Returning to the case $f(i, n) = S(i - c + 1, n - c + 1)$, we may verify that $A(n, r, 0) = A(n, r - 1, 1)$ holds for $n = 1$ and hence for all n. Thus classification by triangular permutations and by ascending runs yield the very same numbers; and moreover the recurrence and boundary condition also serve to identify $A(n, r, 0)$ with Dwyer's cumulative numbers.

From (2) and (8) we see that the factorial moments of the distribution are precisely Stirling numbers divided by certain binomial coefficients. With the aid of a table of Stirling numbers it would thus be possible to compute factorial and other moments quite rapidly. We can also obtain direct recurrence formulas for ordinary and central moments, most expeditiously by using (11) and the exponential generating functions

$$A_n\,(e^y)/n! = \exp ym(n) = \exp y[\mu(n) + m_1(n)].$$

Here m_i, μ_i denote respectively ordinary and central moments of order i, and after expansion of the exponentials, exponents of m and μ are degraded to subscripts. The recurrence for $\mu_i(n)$ obtained in this way has the following symbolic form:

$$\begin{aligned} [\mu(n + 1) + f]^i &= (1 - f)\mu_i(n) + f[\mu(n) + 1]^i + (n + 1)^{-1} \\ &\quad \cdot [\mu_{i+1}(n) - \mu(n)(\mu(n) + 1)^i], \end{aligned} \tag{13}$$

where $f \equiv f(n, c) = m_1(n + 1) - m_1(n)$. For $c = 1$ all odd moments vanish and (13) may be reduced to the simpler recurrence given by Mann [10].

For the more general case of permutations subjected to a trapezoidal set of restrictions, the mean and variance are in effect given by (4) and (5). These may be conveniently used too in the triangular cases: $p = 1$, $a = 1$, $q = n - 1$ and n with the results:

$$q = n - 1: m_1 = (n - 1)/2, \qquad \mu_2 = (n + 1)/12,$$

$$q = n \quad : m_1 = (n + 1)/2, \qquad \mu_2 = (n + 1)/12.$$

7. **Simon Newcomb's problem.** MacMahon gave this name to a problem equivalent to enumerating permutations of $1, \cdots, n$ by ascending (or better

If A is an m by n rectangle, then $u_k(A) = (m)_k(n)_k/k!$, where $(m)_k = m(m-1) \cdots (m-k+1)$ is the Jordan factorial notation.

3. **Application to permutations with restricted positions.** Consider permutations of $1, 2, \cdots, n$ subjected to a set of conditions of the form i is not j-th, k is not l-th, etc. Let it be required to find N_0, the number of permutations fulfilling these conditions, or more generally N_r, the number violating precisely r of them. From the prescribed set of conditions select any subset of k. The number of permutations satisfying these k conditions is 0 if the conditions are incompatible, or $(n-k)!$ if they are compatible. Let t_k be the number of ways of choosing k compatible conditions. Then it is known [4; (39)] that the generating function $N(y) = \sum_r N_r y^r$ is given by

$$N(y) = \sum_k t_k \, (n-k)! \, (y-1)^k. \tag{1}$$

Thus our problem has been reduced to finding the numbers t_k. We now observe that this is an instance of the problem of the rooks. For let A be the board formed by the points $(i, j), (k, l), \cdots$, where $i, k, \cdots$ are elements and $j, l, \cdots$ are the corresponding forbidden positions. Then incompatibility of conditions translates into points of A lying in the same row or column. Hence $t_k = u_k(A)$, as defined in §2. Moreover it follows from (1) that $N(y)$ can be obtained from the generating function $U(A)$ if it is agreed that x^k is a symbol for $(n-k)! \, (y-1)^k$. We also note that the i-th factorial moment of the distribution, defined by

$$n! \, M_{(i)} = \sum_r (r)_i \, N_r$$

is given by [4; (48)]

$$M_{(i)} = u_i(A)/{}_nC_i \, . \tag{2}$$

Particularly simple is the case where the conditions are such that A decomposes into disjoint rectangles. Such a decomposition in fact occurs in the problem of matching two decks of cards [5]. The generalization [6] to an arbitrary number of decks merely requires analogous consideration of higher dimensional boards.

4. **Application to runs.** Permutations of $1, 2, \ldots, n$ may be classified according to the number of ascending runs, or equivalently the number of instances in which i immediately succeeds j with $i < j$. The former always exceeds the latter by one. Thus in 256413, we have the three ascending runs 256, 4, and 13, and the two instances of descent 64 and 41.

More generally we may subject the permutation to an arbitrary set of prescriptions of the form i does not immediately succeed j $(i < j)$, and inquire how many permutations violate precisely r of the conditions. If we form a board A from the points (i, j), we find that the discussion of §3 applies verbatim, so that (1) and (2) are again valid.

In particular the board which corresponds to the admission of all points (i, j) with $i < j$ is a right-angled isosceles triangle (of side $n - 1$). This is the board of primary interest in the discussion to follow; however, it is convenient to take first the more general case of the trapezoid.

5. **Trapezoidal boards.** Consider the trapezoid (p, q, a) given by the following scheme:

$$\begin{array}{c|l}
 & \longleftarrow p \longrightarrow \\
\uparrow & \mathrm{xxx} \cdots \mathrm{xxx} \\
 & \qquad\qquad\quad \longleftarrow a \longrightarrow \\
q & \mathrm{xxx} \cdots \mathrm{xxxxxx} \cdots \mathrm{xxx} \\
 & \qquad\qquad \cdots \\
\downarrow & \mathrm{xxx} \qquad\quad \cdots \qquad\qquad\quad \mathrm{xxx.}
\end{array}$$

The first row has p points, each succeeding row has a more till the q-th which has $p + (q-1)a$.

A recurrence formula for $u_k(p, q, a)$ can be obtained by separation of the selections into those which do and those which do not contain a point of the first row:

$$u_k(p, q, a) = u_k(p + a, q - 1, a) + p \, u_{k-1}(p + a - 1, q - 1, a). \tag{3}$$

From (3), or directly, we find

$$u_1(p, q, a) = pq + a \, {}_qC_2 \, , \tag{4}$$

$$\begin{aligned} u_2(p, q, a) = {}_qC_2[p(p-1) + ap(q-1) \\ - (q-2)a/3] + {}_qC_3(3q-1)a^2/4. \end{aligned} \tag{5}$$

There seems to be no simple general result. However the case $a = 2$ has a remarkably simple answer:

$$u_k(p, q, 2) = (q)_k \, (p + q - 1)_k/k!, \tag{6}$$

as follows readily from (3) by induction. Equation (6) may be rendered as follows: *for the purpose of the problem of the rooks, the trapezoid with $a = 2$ is equivalent to a rectangle of the same height and area.*

For $a = 1$ we have

$$u_k(p, q, 1) = \frac{[\Delta^{p+q-k}(x^q(x)_p)]_{x=0}}{(p+q-k)!}, \tag{7}$$

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THE PROBLEM OF THE ROOKS AND ITS APPLICATIONS

By Irving Kaplansky and John Riordan

1. **Introduction.** The classification of permutations by ascending runs was first studied by MacMahon in [8; I, §IV, Chapters IV, V] and [9]. Applications of his results to significance tests in statistics have been made recently by Moore and Wallis [11] and Mann [10].

It does not seem to be generally known that the numbers arising in this study have occurred several times in other contexts. Rather extensive references to older literature (going back to Laplace) are given by v. Schrutka in [13], who however was unaware of MacMahon's work. More recently the numbers have appeared in papers [2], [3] by Dwyer, who calls them "cumulative numbers", and in papers [14], [15] by Toscano on summation of series.

In this paper we shall endeavor to show that these results may be unified and generalized by the study of a chessboard recreation: in how many ways can a given number of rooks be placed on a chessboard so that no two attack each other? We shall not identify solutions which are equivalent under the symmetry group of the board (cf. [7; 240–247] for a study of this version of the problem), so that for us the problem on a rectangular board is trivial. The board of primary interest is the trapezoid, and in particular the triangle. A triangle mutilated so as to become an irregular ramp-staircase is shown in §7 to be associated with what MacMahon called "Simon Newcomb's problem", for which we are able to give a compact solution. In §8 a board composed of two triangles is studied; the application to bi-triangular permutations there described arose in connection with war work. Finally Ahrens's problem of the bishops [1; 140–151] is solved in §9 by an application of the results of §7.

As noted in §3, this study is closely connected with the card-matching problem treated by symbolic methods in [5] and [6]. While there is no fundamental difference between the two methods, we believe that in the present context the terminology of the chessboard is conducive to clarity and simplicity of proof.

2. **Chessboards.** A general chessboard A may be formally defined as a set of points (x, y) with integral coordinates. Let $u_k(A)$ denote the number of ways of choosing k points in A so that no two are in the same row or column, i.e., no two have the same x- or y-coordinate. In the language of the chessboard, $u_k(A)$ is the number of ways of placing k non-attacking rooks on A. We also use the generating function $U(A) = \sum_{k=0}^{\infty} u_k(A)x^k$.

Two boards A and B may be said to be *disjoint*, if no point of A is in the same row or column as a point of B. If the board C can be decomposed into two disjoint sub-boards A and B, then evidently $U(C) = U(A)U(B)$.

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