FEM Software Automation, with a case study on the Stokes Equations

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Masters Presentation

Outline

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State of Scientific Software

Partial Differential Equation (PDE) Software needs to:

- Solve large problems
- Solve interesting problems
- Use the best methods

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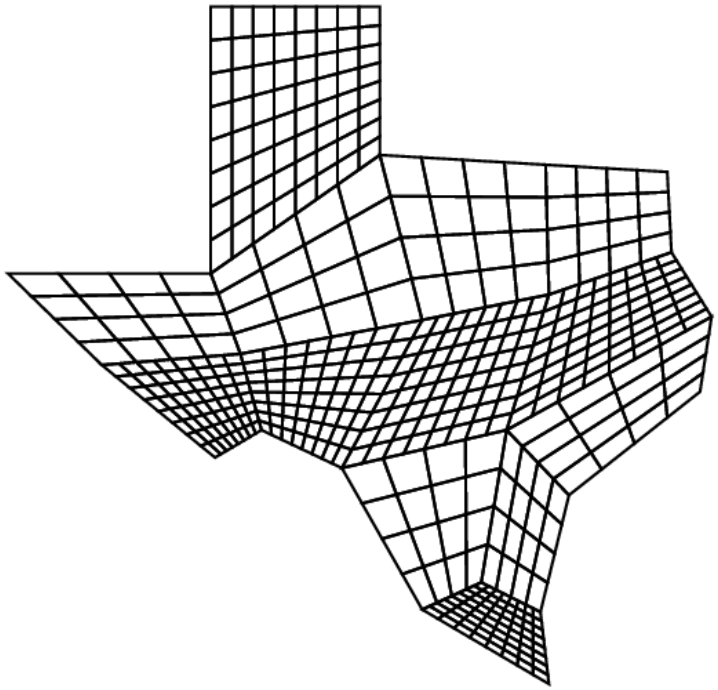

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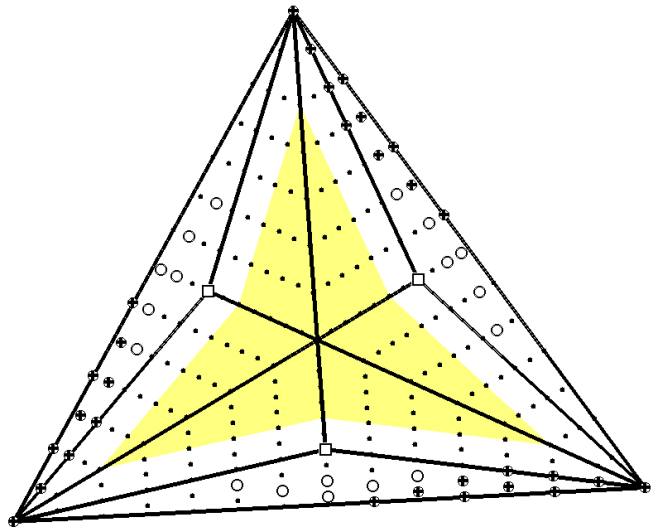

The Galerkin Method

Given a PDE:

$$-\frac{d^2u}{dx^2} = f \text{ in } (0,1),$$
$$u(0) = 0, \qquad u'(1) = 0$$

The problem can be characterized by a weak formulation:

$$u \in V \text{ such that } a(u,v) = (f,v) \qquad \forall v \in V, \text{ where}$$
$$V = \{v \in L^2(0,1) : a(v,v) < \infty \text{ and } v(0) = 0$$

Ritz-Galerkin Approximation:

$$u_S \in S \text{ such that } a(u_S,v) = (f,v) \qquad \forall v \in S \text{ where}$$
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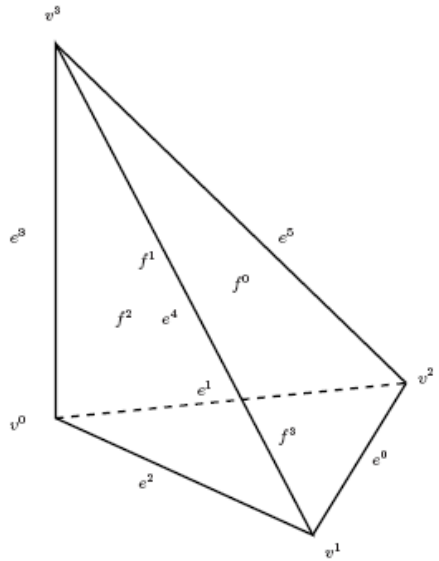
The Local Finite Element

- A reference element, K
- A space of shape functions, $\mathcal{P}$
- A basis, $\mathcal{N}$



To get the global space we use a mapping to change the coordinates into the global element. Then using our given method we need to solve some matrix equations:

$$AU = F$$

Why Automate FEM?

- Ensure Correctness:
  Complicated error prone `mathematical` process
  Complicated error prone `programming` process
- Reduce Programming Hours:
  Gives ability to quickly change `models`
  Gives ability to quickly change `elements`
  Gives ability to quickly change `methods`
- Optimize Computation:
  Allow a non-expert programmer to make efficent calculations

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Why are we NOT Automated?

- Different mathematical and algorithmic abstractions
  - The local finite element is understood and automated
  - Mathematical framework for global-local interactions needs developing.
    - Face Directions
    - Links to other elements
- Hand coding is very attractive (“If you want it done right...”)
- Quite difficult to switch between elements, solvers, and methods.

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What Are the Parts Needed for FEM Software?

- Mesh Generation
- Function Spaces
- Equation Description
- Discrete Equation Solver
- Parallel Computing Support

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- uniform meshes,
- general geometry,
- adaptive meshes,
- unstructured meshes

What Are the Parts Needed for FEM Software?

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- linears,
- menu of options,
- arbitrary order,
- tabulator

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- menu,
- language,
- derived forms,
- error estimators,
- constraints

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- Parallel Computing Support
  - parallel linear solve,
  - parallel assembly,
  - load balancing

Types of Software

Simulation Engine
Holds the pieces together.

Tabulator
Tabulates the Finite Element

Linear Solver
Solves the linear equation and more if you let it.

Some Major Projects

Simulation Engines

- Sundance
- FFC/Dolfin
- Deal.II
- ComSol
- Analysa
- FreeFEM
- GetDP

Tabulators

- FIAT
- SyFi

Linear Solvers

- UMFPack
- PETSc
- Trilinos

Math v. Software: The Problem Domain

- Mathematics
  - Distinguish what problem lies where
  - Adaptively refine on important parts of the domain
  - Hook up with domains of other problems effortlessly
- Software
  - Use some mesh description.
    - Allow coarsening (usually only uniform)
    - Use set theoretic operators to filter different parts (Sundance)
    - Ultimately gives some iterator for assembly process

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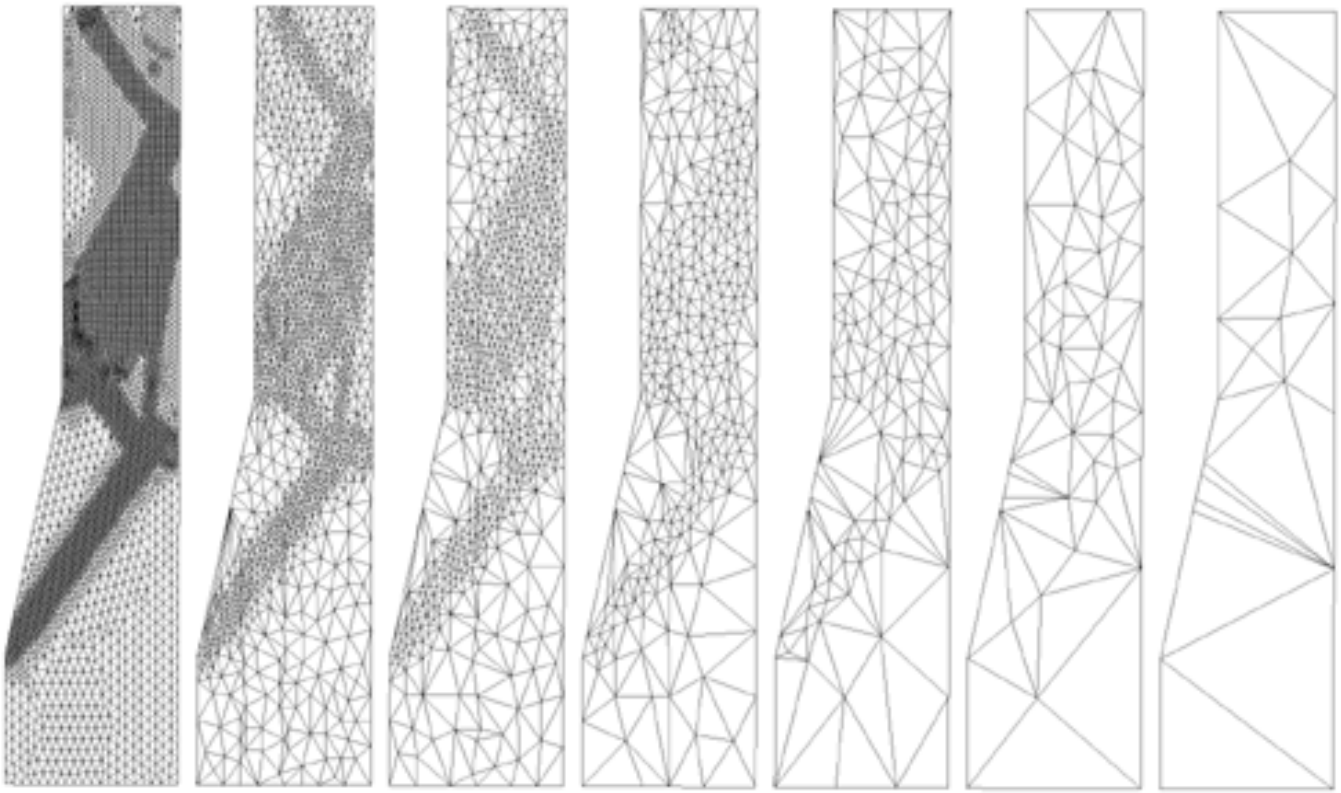
What meshes can we handle?

None of these software packages are giving us great tools for multigrid adaptivity.



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  - Many ways to describe a problem.
  - Often problems are split or reformulated.
  - Conceivably we should be able to use any well-formed formula (PDE, ODE, ...)
- Software Approaches
  - GUI Strong Form (ComSol)
  - The Variational Form (FFC and Sundance)
  - The Brute Force Method (Deal.II)

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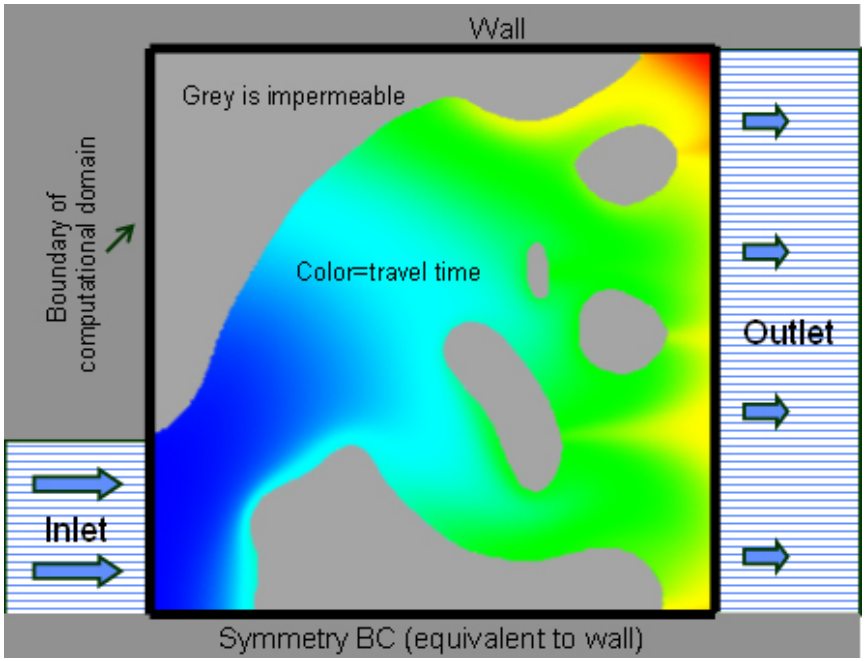
How about optimization problems?

- Use Automatic Differentiation tools on code - expensive on user side
- Create a symbolics engine that can give derivatives - expensive on developer side

Example Problem in Microfluidic Devices

- To optimize flow, change channel geometry
- Most effective methods, use level set methods



Math v Software: Role of Symbolics

The use of the variational form can be motivation for supporting a larger symbolics engine that can then be used for differentiation.

The Sundance Symbolics Engine

- Environment with large number of calculations
- Not “symbols” but numbers
- Graph relations to implement chain rule

The SyFi Symbolics Engine

- Preprocessing environment, less calculations
- Fully symbolic.

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Math v Software: The Assembly Process

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  - Rote application of algebra and calculus
- Software
  - The most computationally demanding process.
  - Do I have to touch the process at all? (Declarative or procedural?)
  - Can I get my matrix to play with before sending it to the solver?
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Math v Software: Solvers

- Mathematics
  - Just solve $Au = f$, what's so hard?
- Software
  - De facto standard is to use some library. Either Trilinos, PETSc, uBlas, UMFPack, ...
  - Is there more we can do here?
    - Adaptively choose our precision.
    - Pre-solve important blocks.

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What hasn't been done?

In general, some important pieces are not being implemented:

- Usually only Lagrange
- Parallel assembly
- Adaptive/unstructured grids
- Error estimators or optimization loops
- Boundary Condition calculus or embedded geometries

Outline

Introduction to Stokes Flow

The Stokes equations are a model for steady incompressible flow:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla \mathbf{p} &= f \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

Important Features:

- Coupling of pressure and velocity
- Well studied problem
- Numerous methods for solving

Intro to Mixed Method

Let $V = H^1(\Omega)^n$ and $\Pi = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$. Given $F \in V'$, find functions $\mathbf{u} \in V$ and $p \in \Pi$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= F(\mathbf{v}) \quad \forall \mathbf{v} \in V \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in \Pi \end{aligned}$$

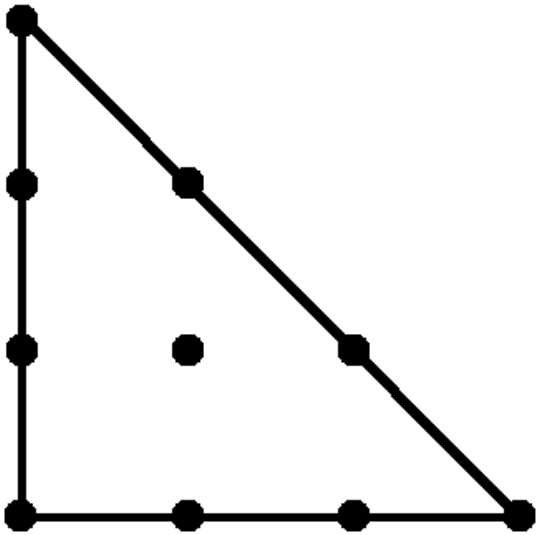
Where,

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx, \\ b(\mathbf{v}, q) &:= \int_\Omega (\nabla \cdot \mathbf{v}) q dx \end{aligned}$$

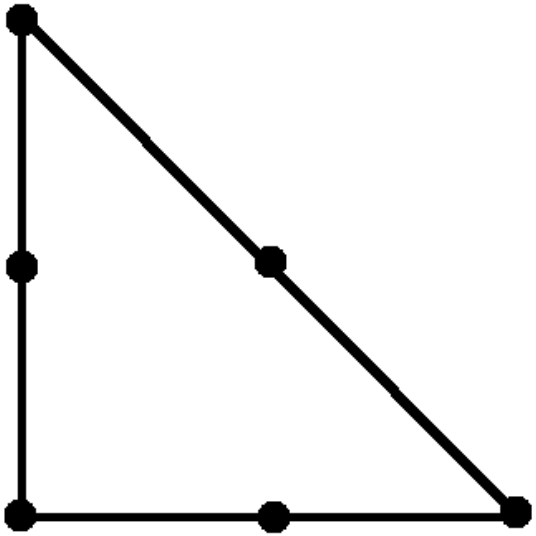
Important Feature

- Two discrete spaces, V and Π

Taylor - Hood Elements



(a) P_3 for V

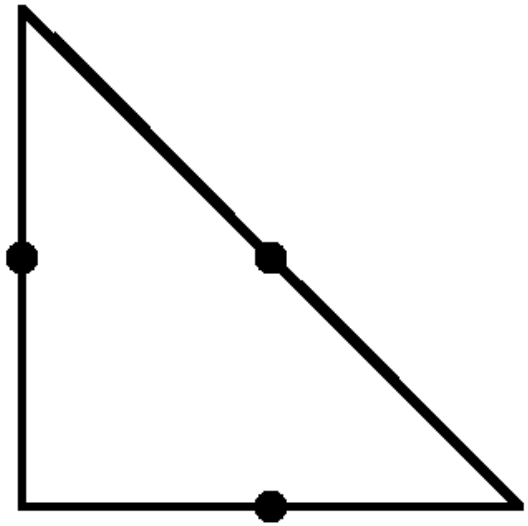


(b) P_2 for Π

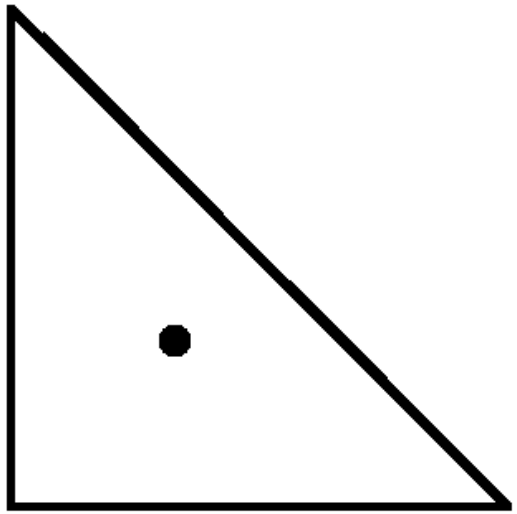
Important Features:

- Available using any P_k elements, extendable to 3d,
- Built from standard Lagrange elements,
- Easily extendable to arbitrary order, and
- Widely used

Crouzeix - Raviart Elements



(c) Crouzeix-Raviart for V



(d) P_0 for Π

Important Features:

- Non Conforming
- Low Order

$C^0 P_i C^{-1} P_{i-1}$ Elements

Use a Continuous Lagrange element P_i for V and a Discontinuous Lagrange element P_{i-1} for Π

Important Features:

- May not satisfy inf sup condition

Iterated Penalty

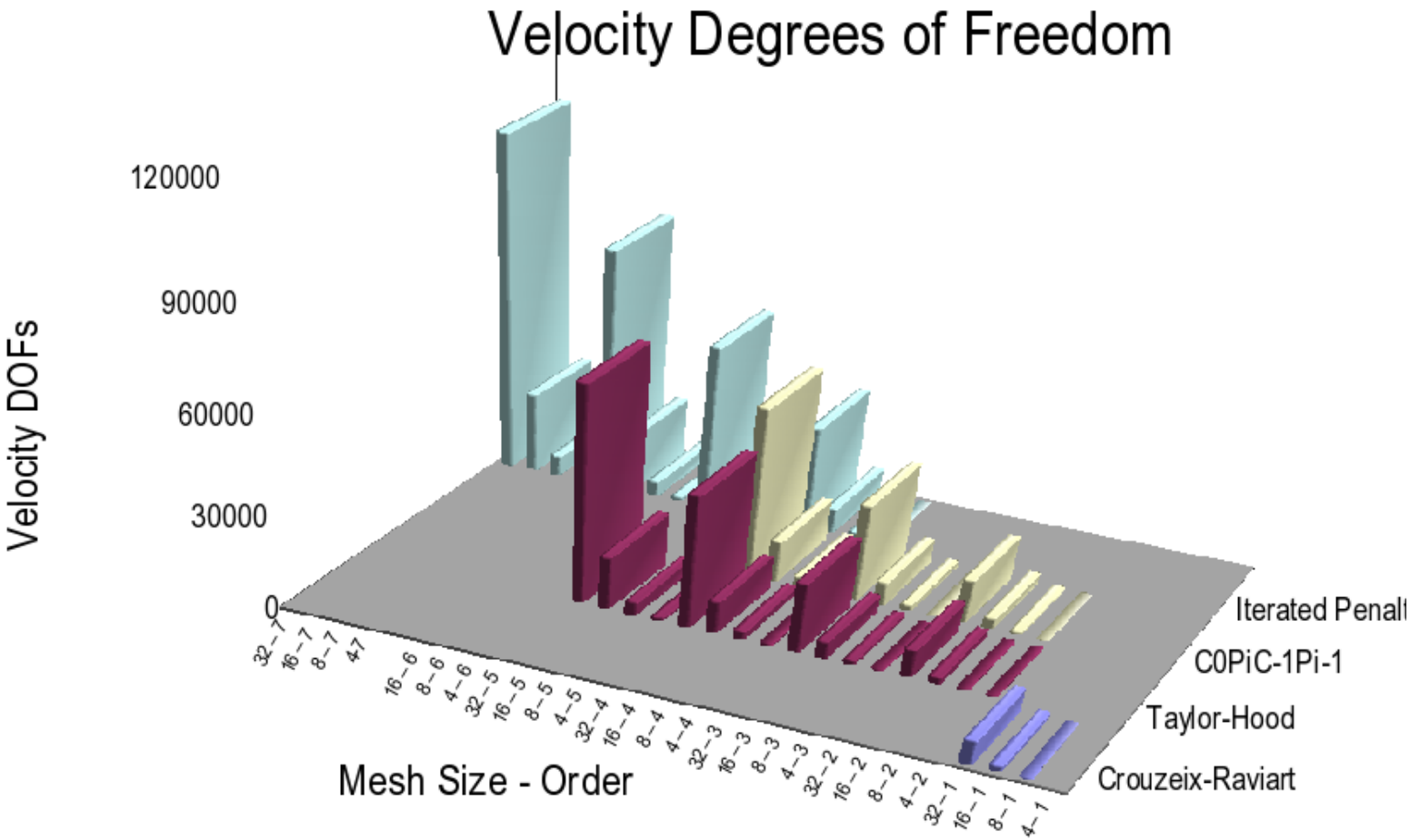
Let $r \in \mathbb{R}$ and $\rho > 0$ define u^n and $p = w^n$ by

$$\begin{aligned} a(\mathbf{u^n}, \mathbf{v}) + r(\nabla \cdot \mathbf{u^n}, \nabla \cdot \mathbf{v}) &= F(\mathbf{v}) - (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{w^n}) \\ \mathbf{w}^{n+1} &= \mathbf{w}^n + \rho \mathbf{u}^n \end{aligned}$$

Important Features

- One discrete spaces, V
- Use higher order finite elements, P_4 and above
- Use $||\nabla \cdot \mathbf{u^n}||_{\mathbf{V}} < \epsilon$ as stopping criteria
- Iteration count highly effected by choice of ρ and r, for our experiments choose $\rho = -r = 1.0e-3$.

Order vs Degrees of Freedom



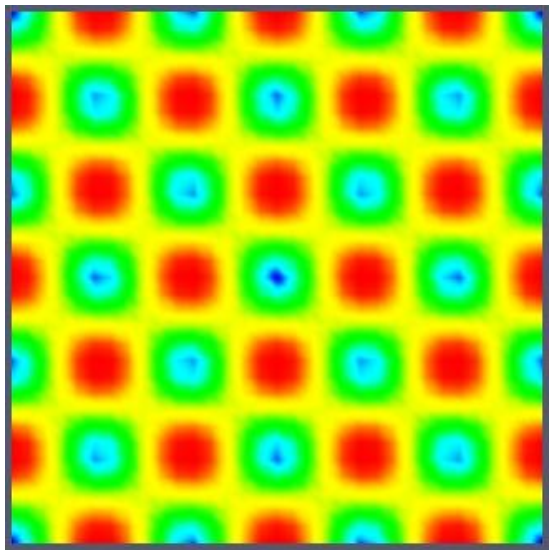
Problem statement

Using a $n \times n$ uniform mesh for a domain [0,1]$\times$[0,1] solve these problems.

Case 2:

$$\mathbf{u} = \begin{bmatrix} \sin(3\pi x)\cos(3\pi y) \\ -\cos(3\pi x)\sin(3\pi y) \end{bmatrix}$$

$$p = \sin(3\pi x)\sin(3\pi y)$$



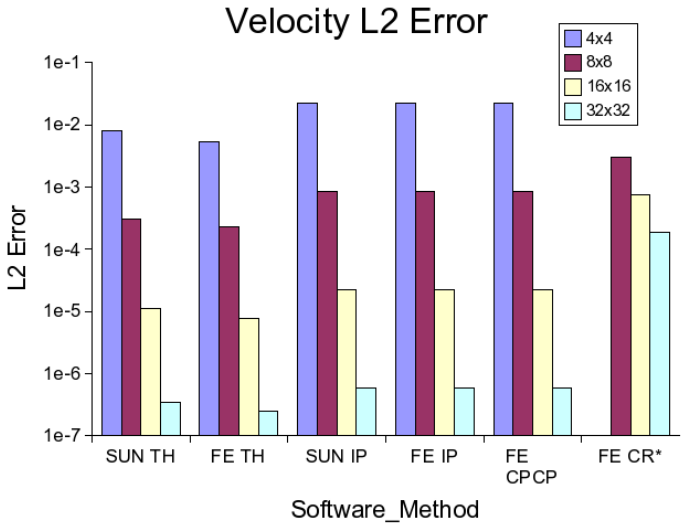
Software and Solvers

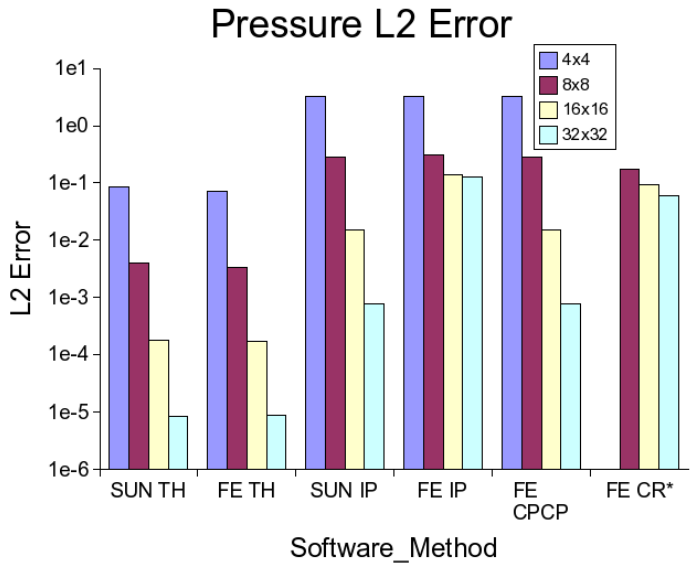
	Sundance	FEniCS
Taylor-Hood	X	X
Crouzeix-Raviart	-	X
$C^0 P^i C^{-1} P^{i-1}$	-	X
Iterated Penalty	X	X

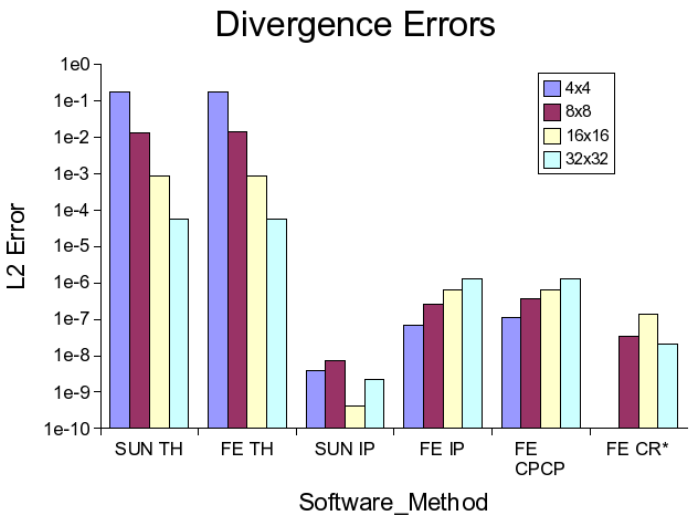
For each of these methods, we use UMFPACK LU Direct solver. Other iterative solvers from the Trilinos or PETSc Toolkits would also work and make the code parallel, but not the focus of this work.

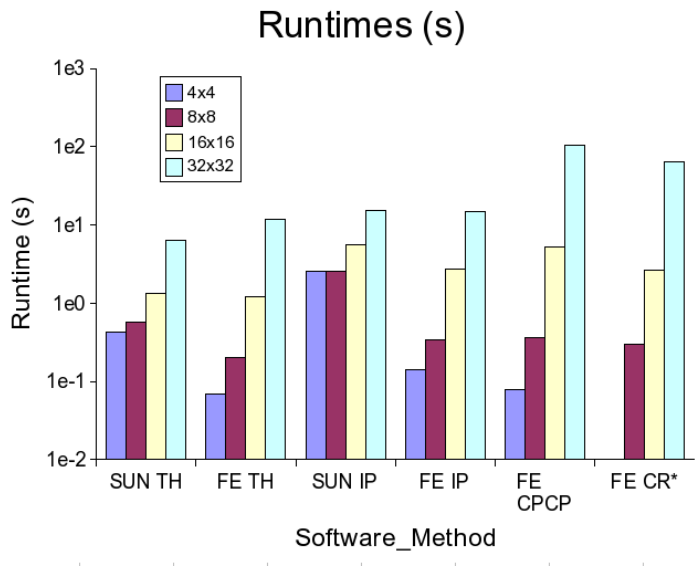
Outline

4th Order Numbers

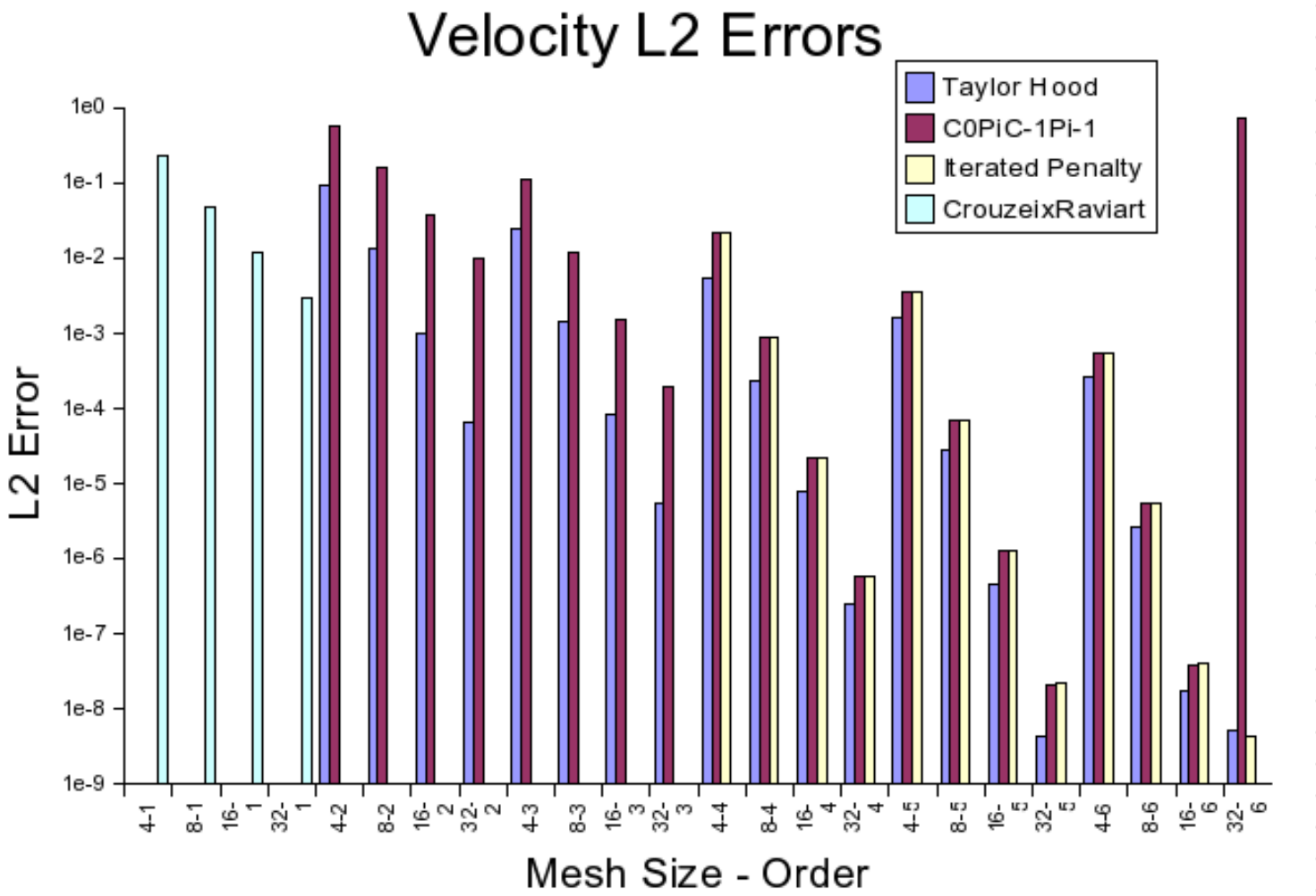




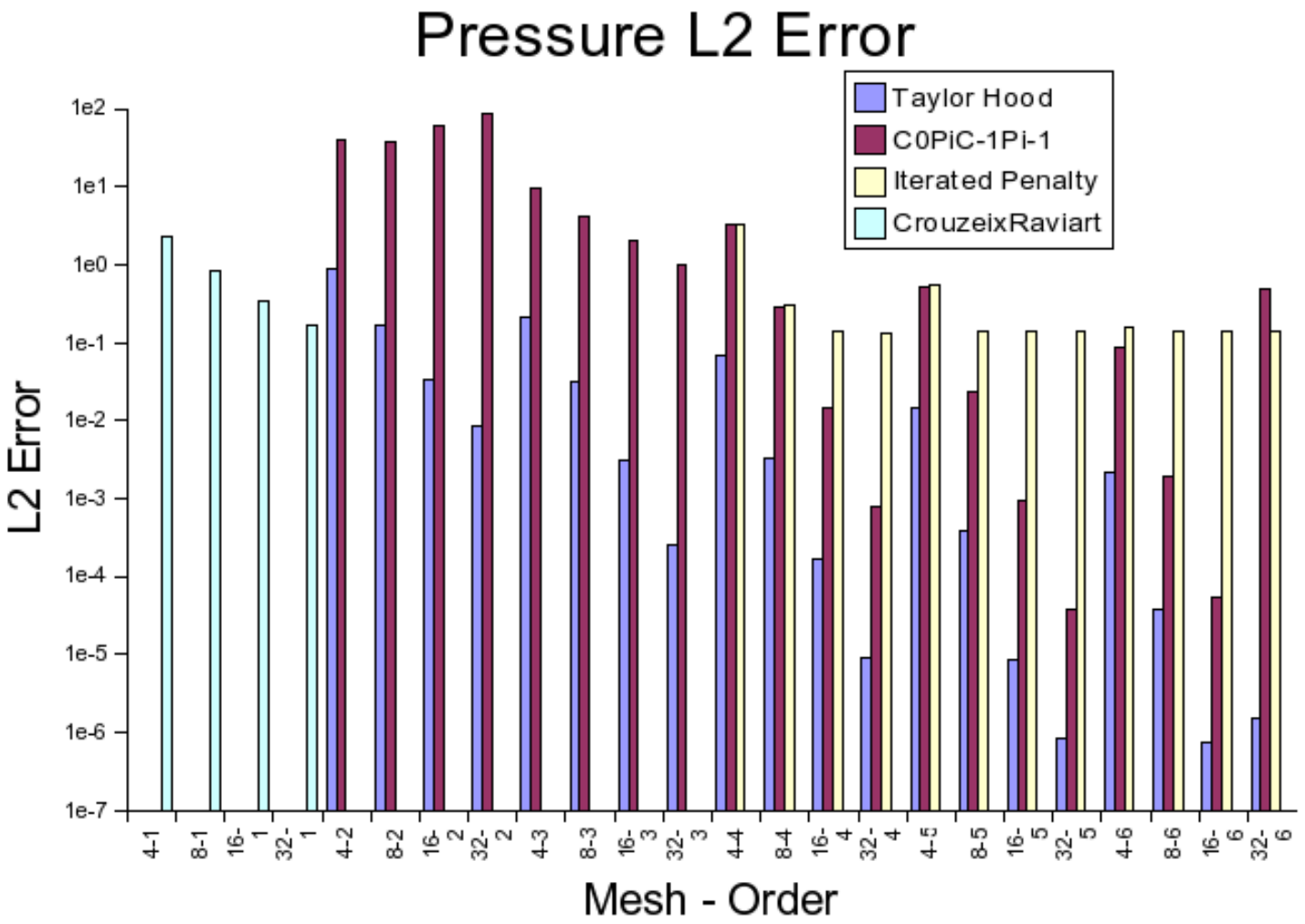




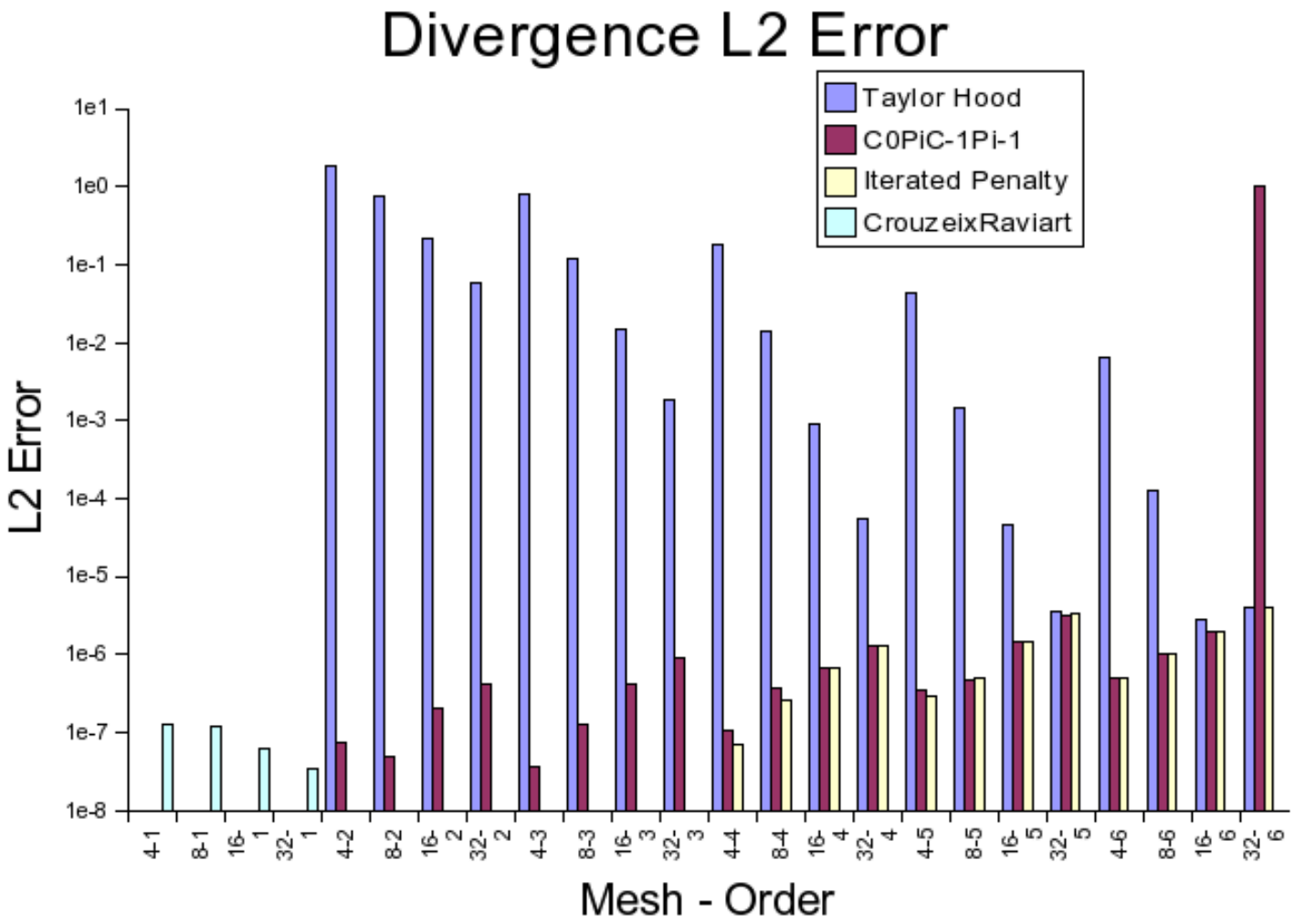
FEniCS Velocity Errors



FEniCS Pressure Errors



FEniCS Divergence Errors



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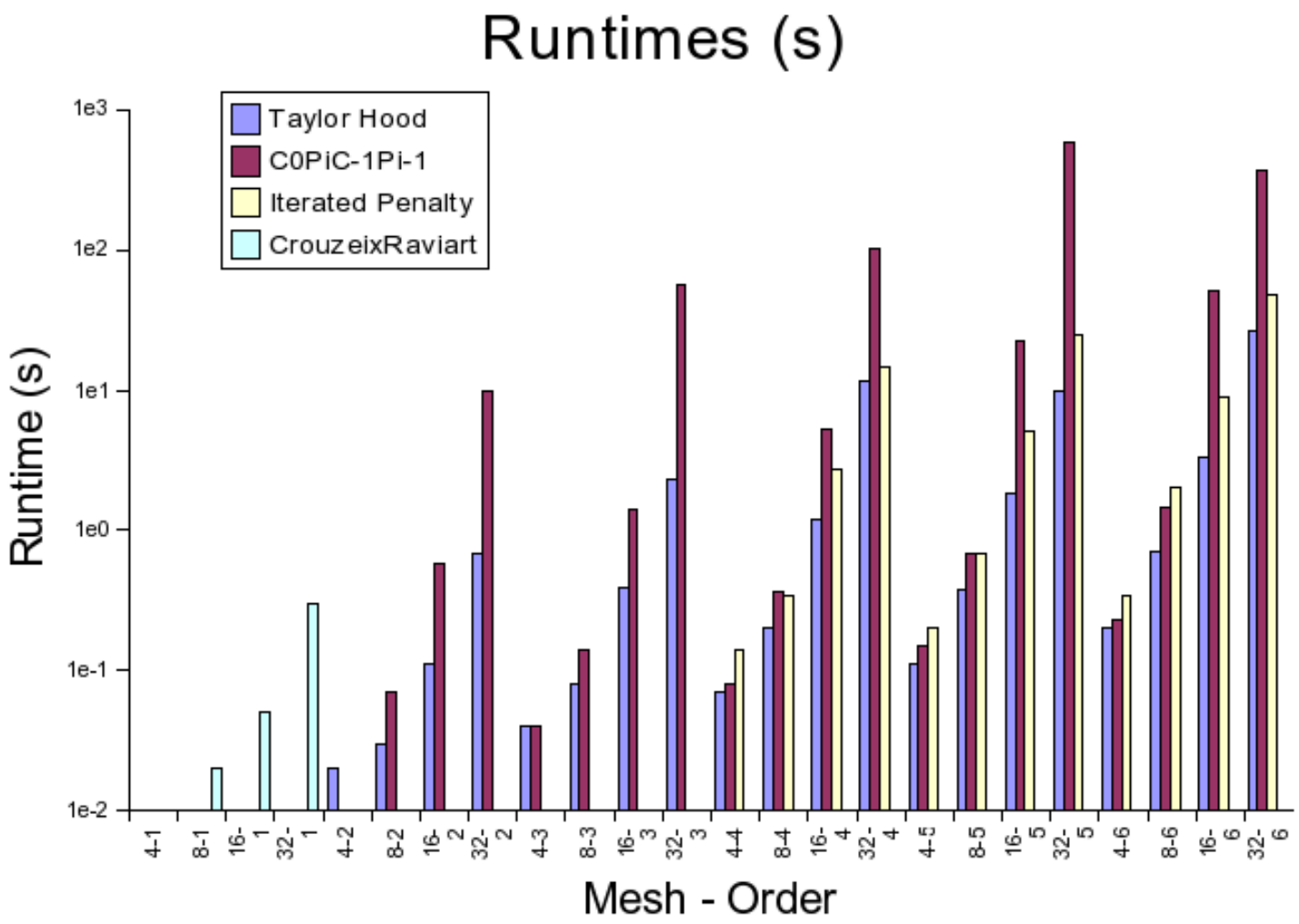

Comparisons between Software Packages

- Sundance and Dolfin treat assembly very similarly
- FIAT a common interface for defining elements
- Coding time almost identical
- Both still very active development

More Notes

Notes about FEniCS

- A much smaller code to comprehend - hence easier to make changes if it does not currently have a feature
- Interface less like a scripting language in so much as development requires multiple tools to run
- Problems handling fancy things

Notes about Sundance

- Seemless scripting style code
- Ability to handle fancier things
- Closer to a production quality code

Coding Challenges in this project

- Have to read the code
  - Often documentation is either incomplet, or inakurate
  - To see how things were really done, had to read code
- Possible Methods
  - FEniCS did not have mixed methods
  - Sundance did not have higher order methods
- Integration Bugs
  - FEniCS operators did not check for underflow
  - FEniCS Div operator is not as accurate
  - Sundance bug with volume
- Iterated Penalty has typo in Brenner Scott and many papers.

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Conclusions

- FEM Automation enables flexiblity in simulation software
- Mathematics $\Leftrightarrow$ Software Abstractions
- Meaningful test simulations (not just Poisson)

- Outlook
  - Explore mathematical abstractions for global-local interactions
  - Compare Grade 2 and Oldroyd-B fluid model

Do It Yourself

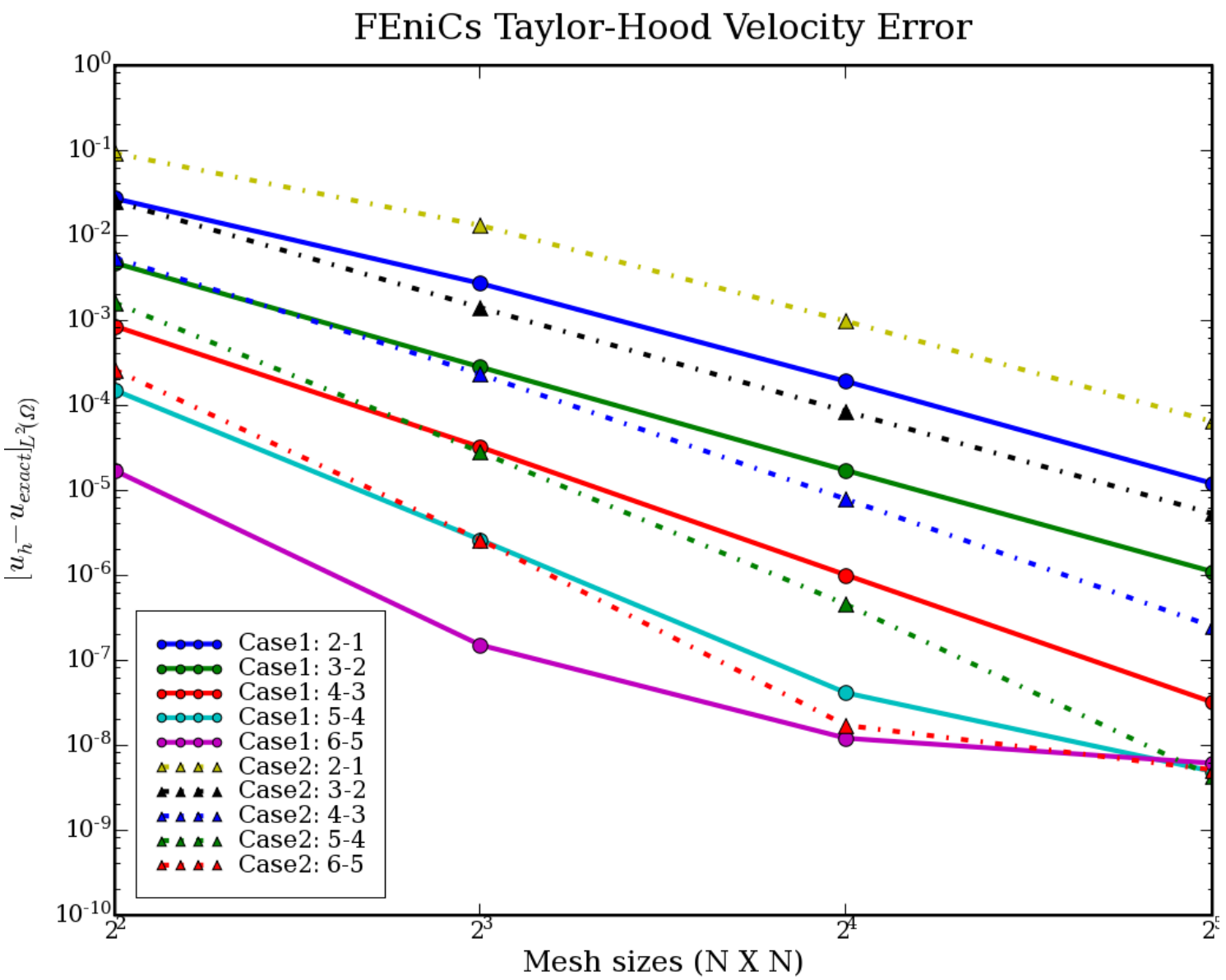
Where to get the code:

Sundance - http://software.sandia.gov/sundance/
FEniCS Project (FIAT, FFC, DOLFIN) - http://www.fenics.org
Masters Thesis - http://people.cs.uchicago.edu/~aterrel/Masters

Any Questions

aterrel@uchicago.edu

Taylor-Hood Error



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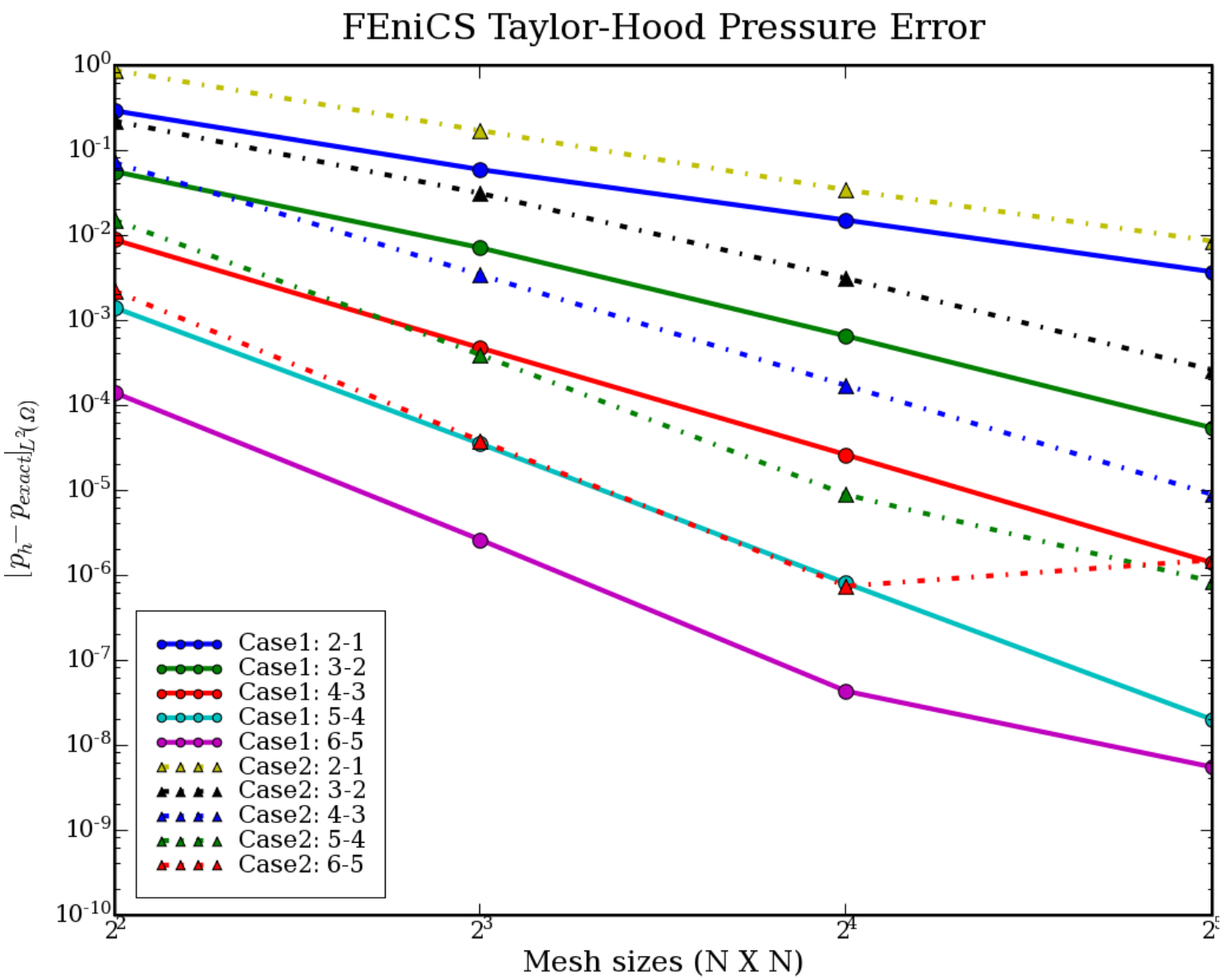

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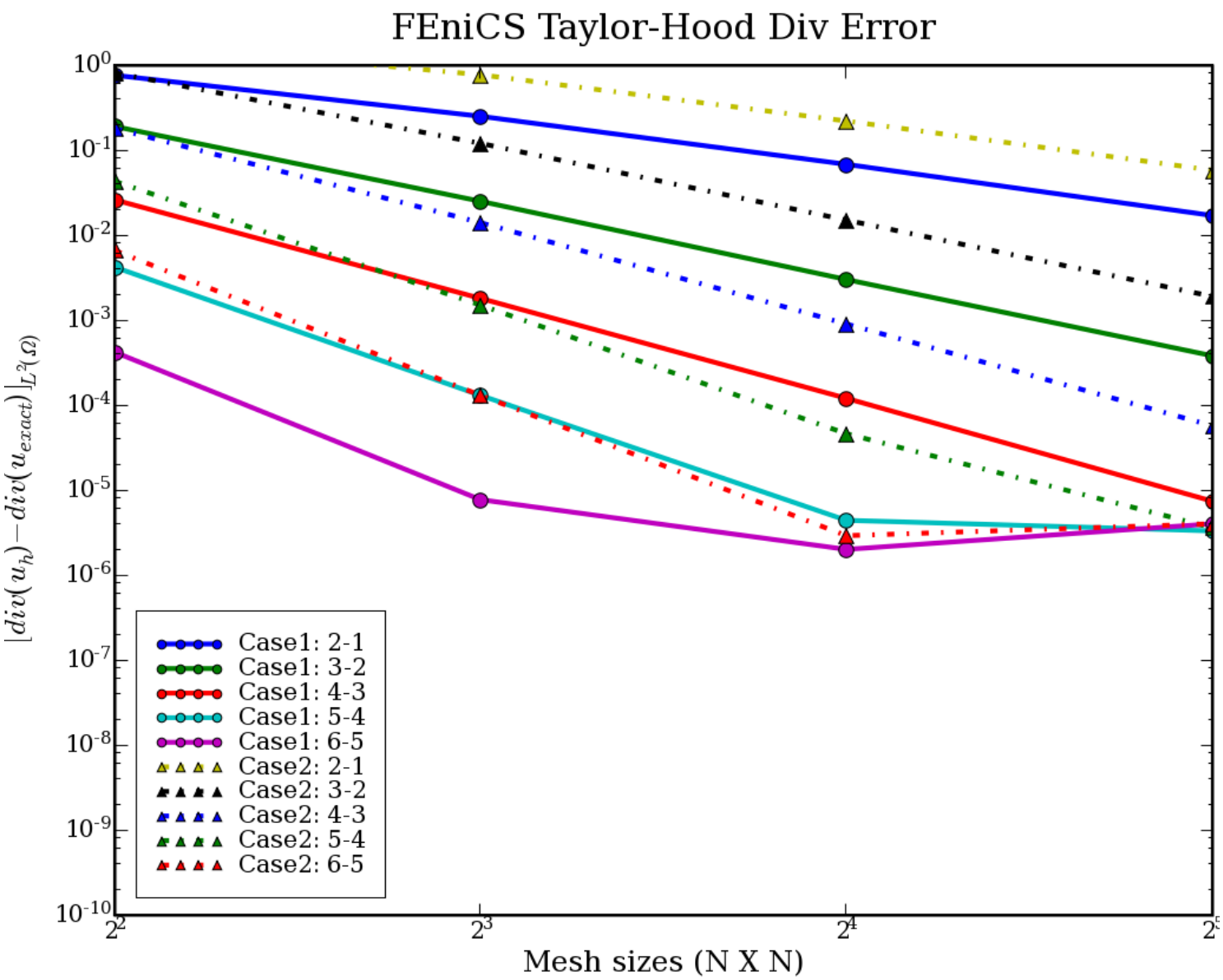
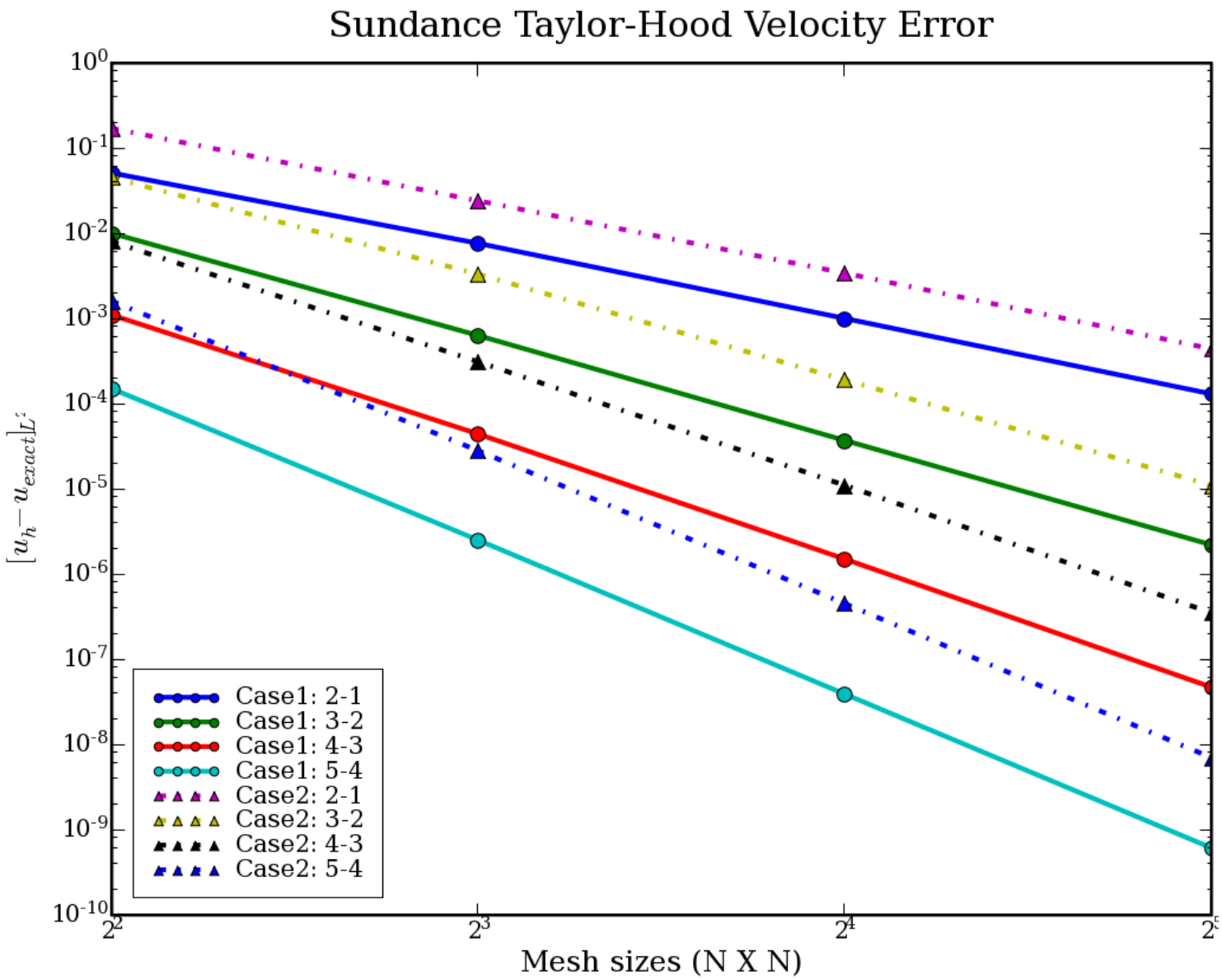
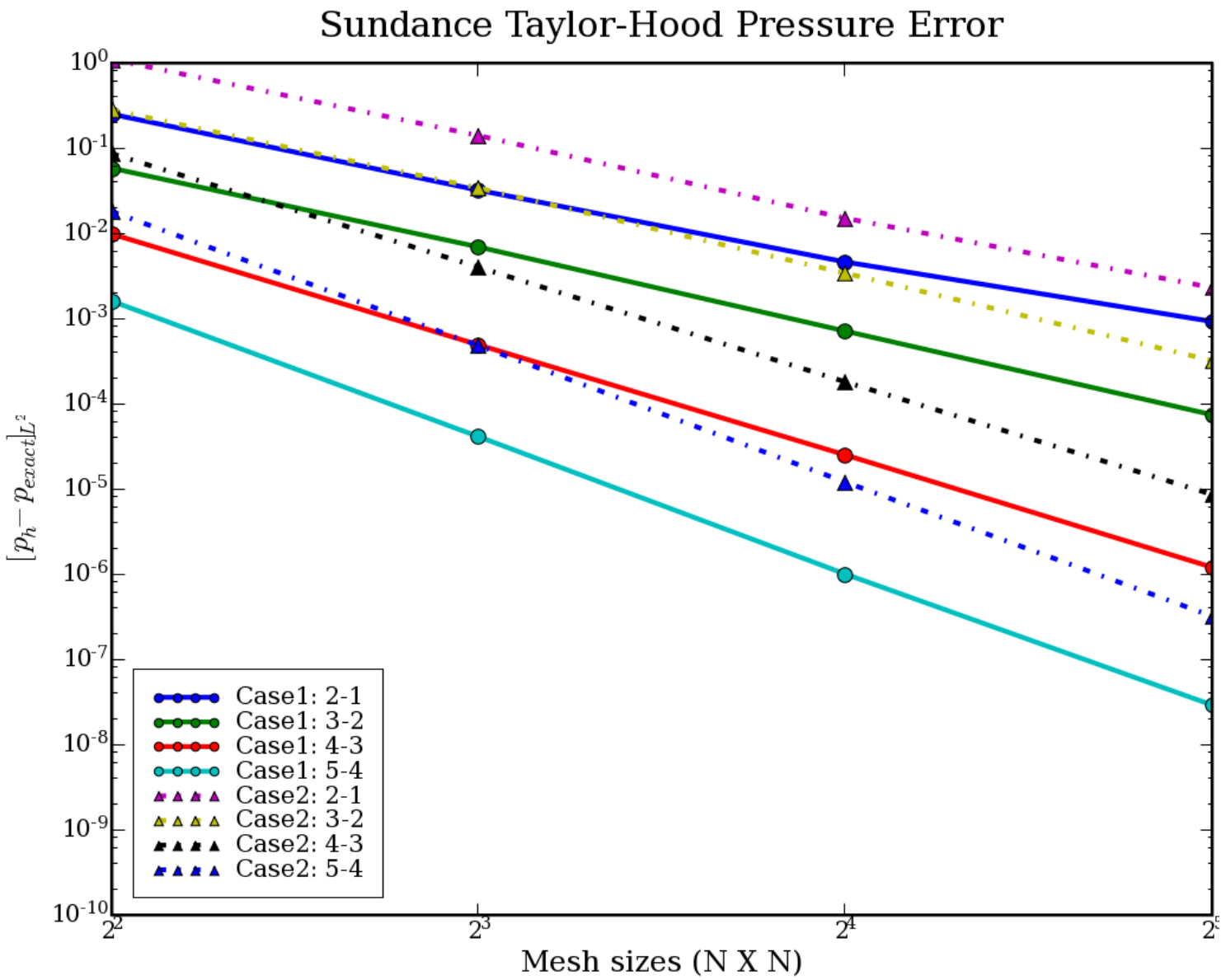


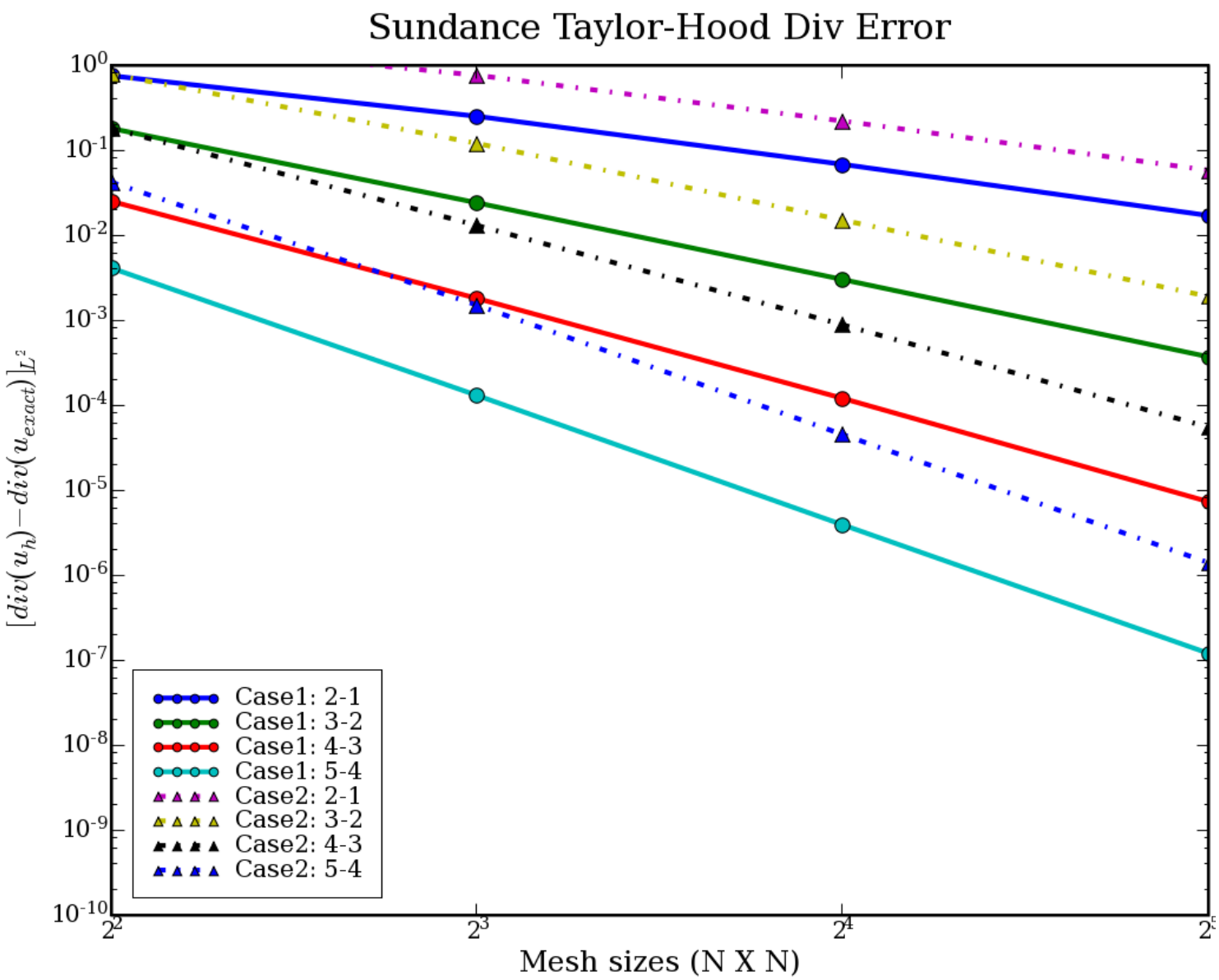
Taylor-Hood Error



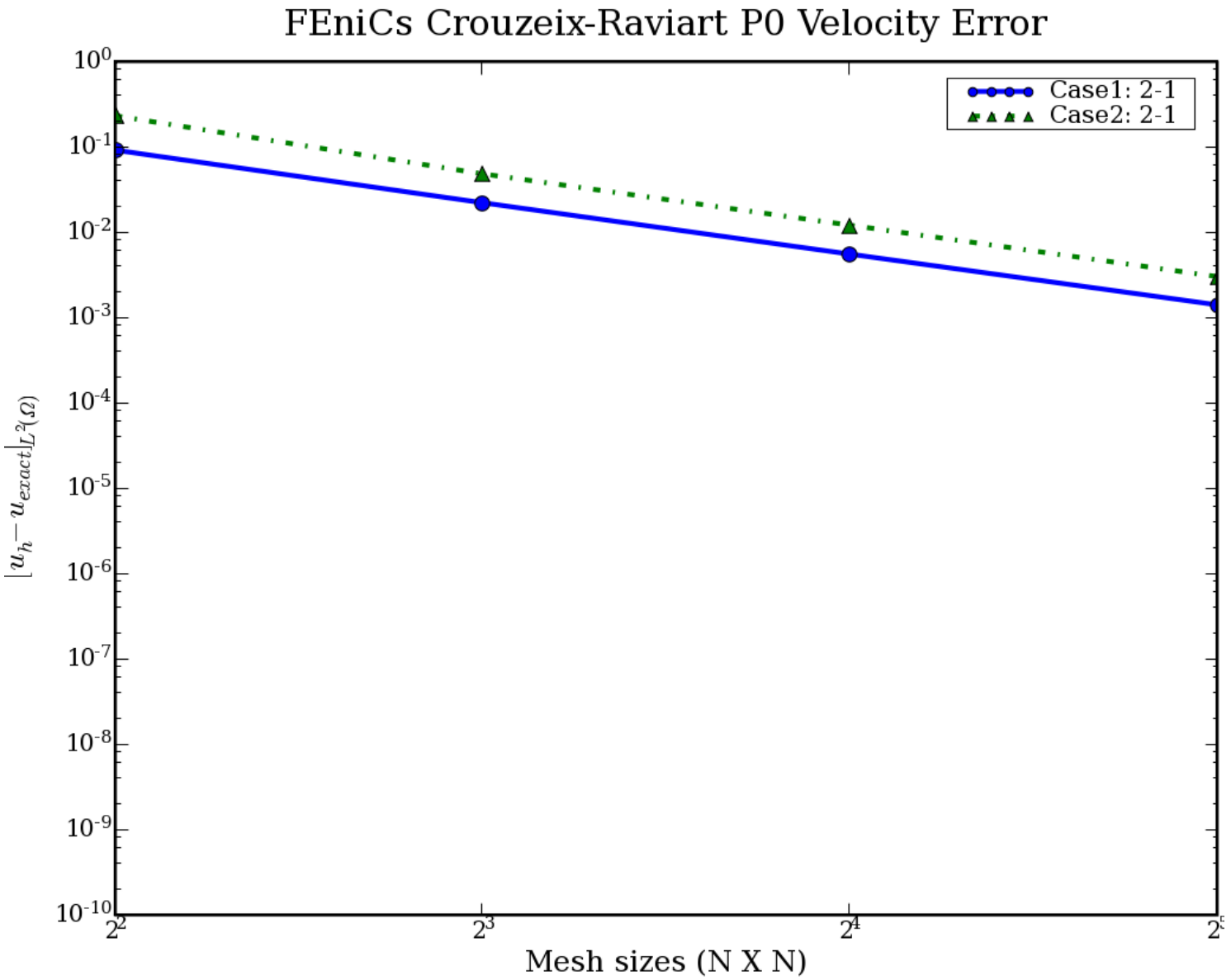
Taylor-Hood Error



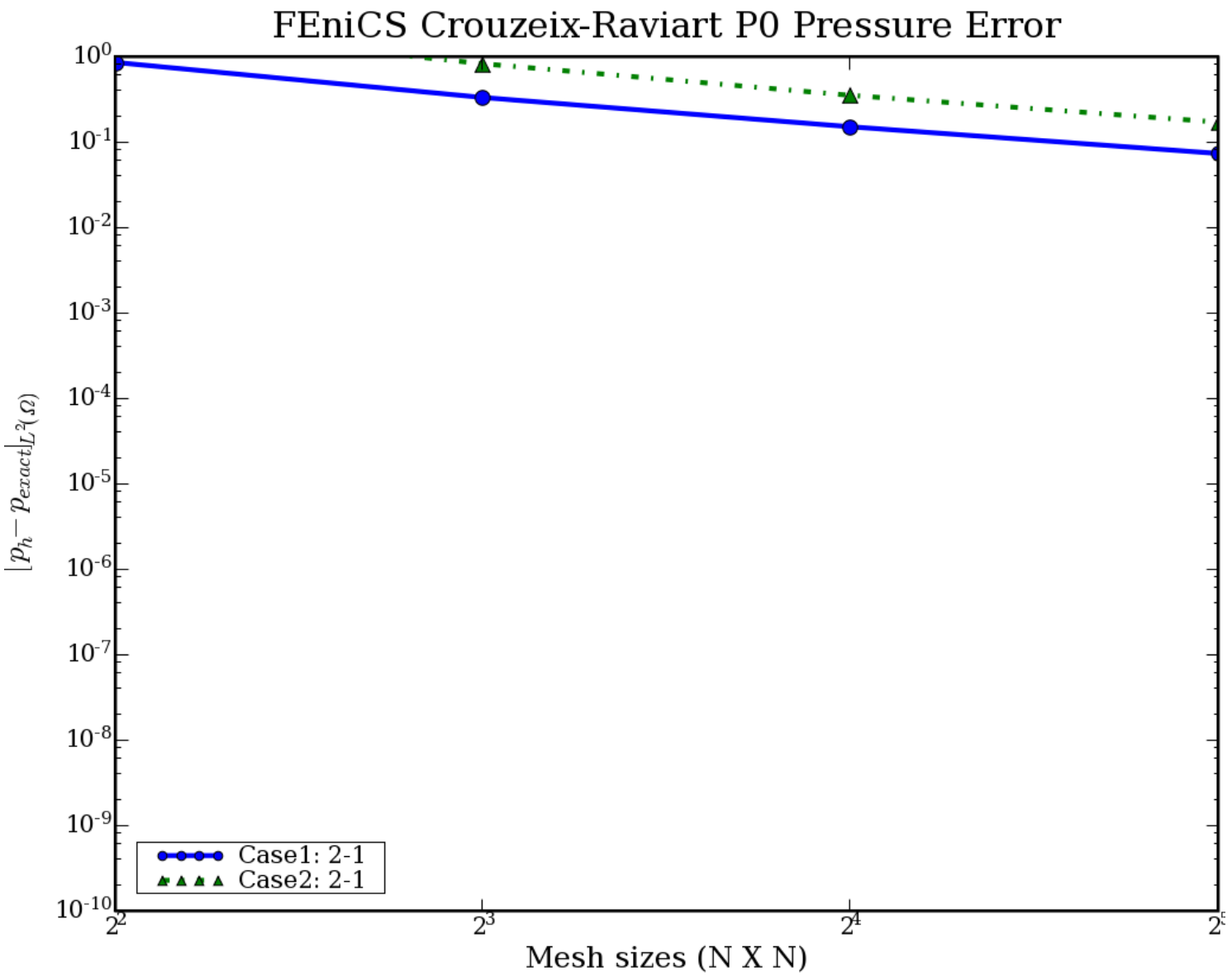
Taylor-Hood Error



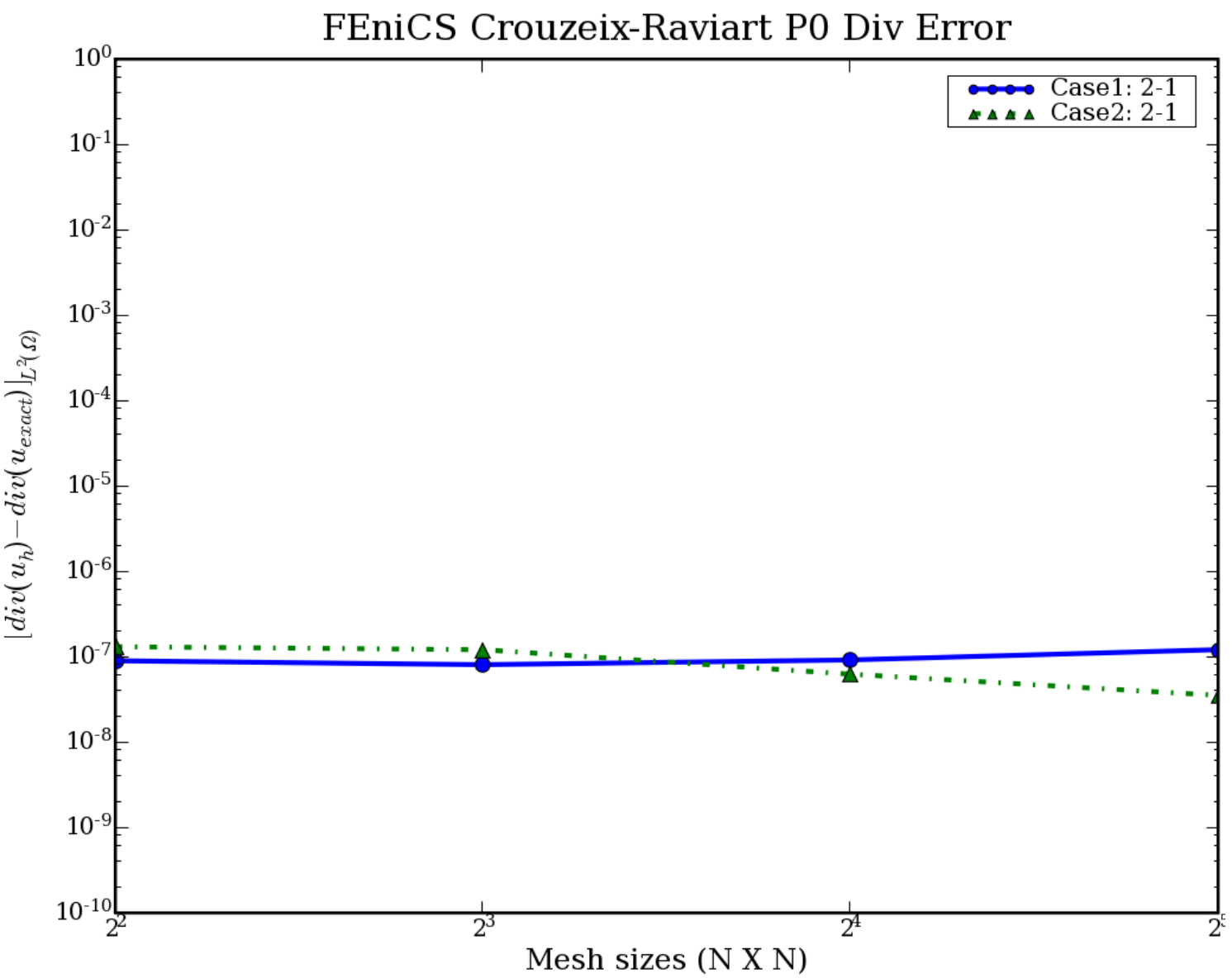
Crouzeix-Raviart P0 Error



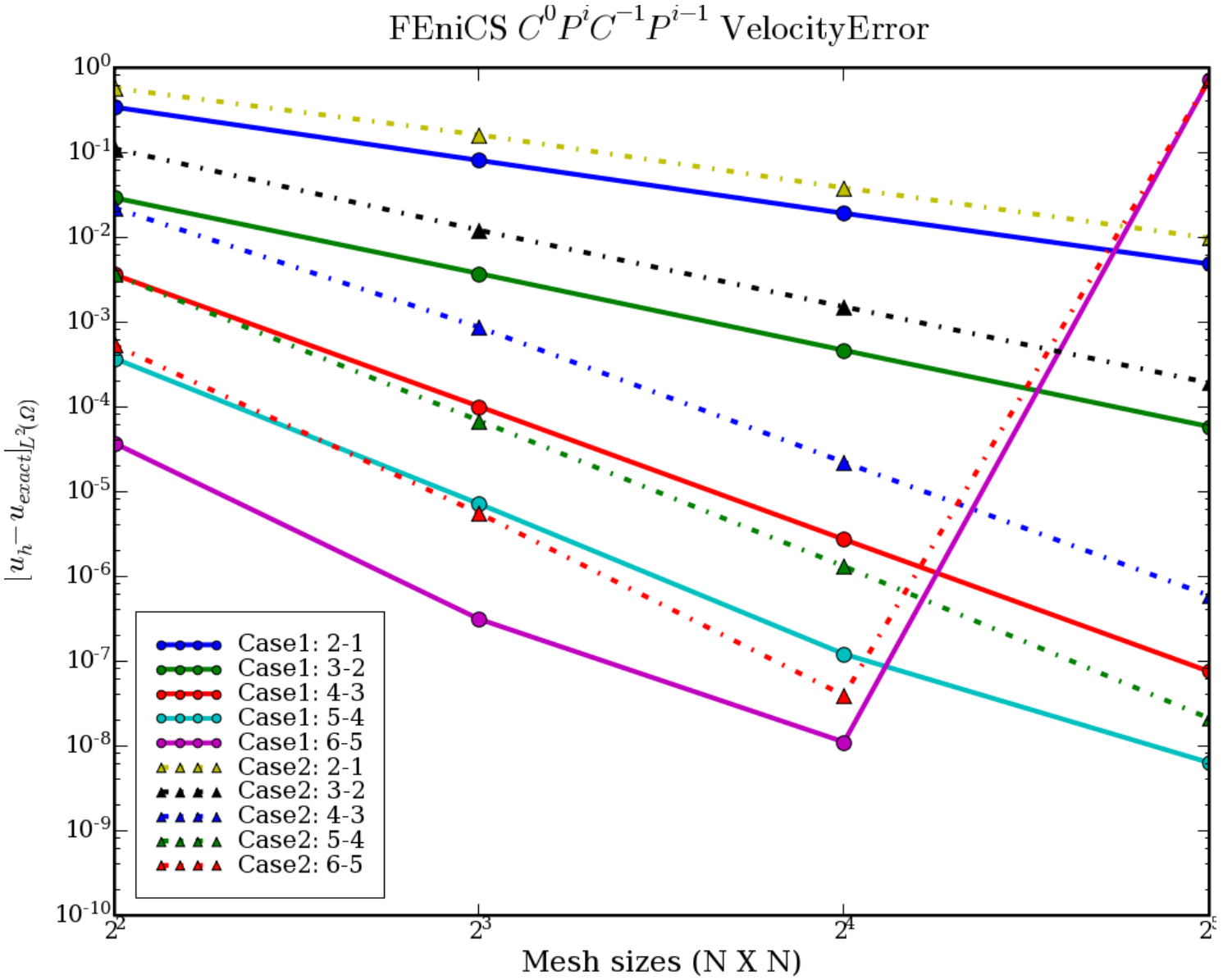
Crouzeix-Raviart P0 Error



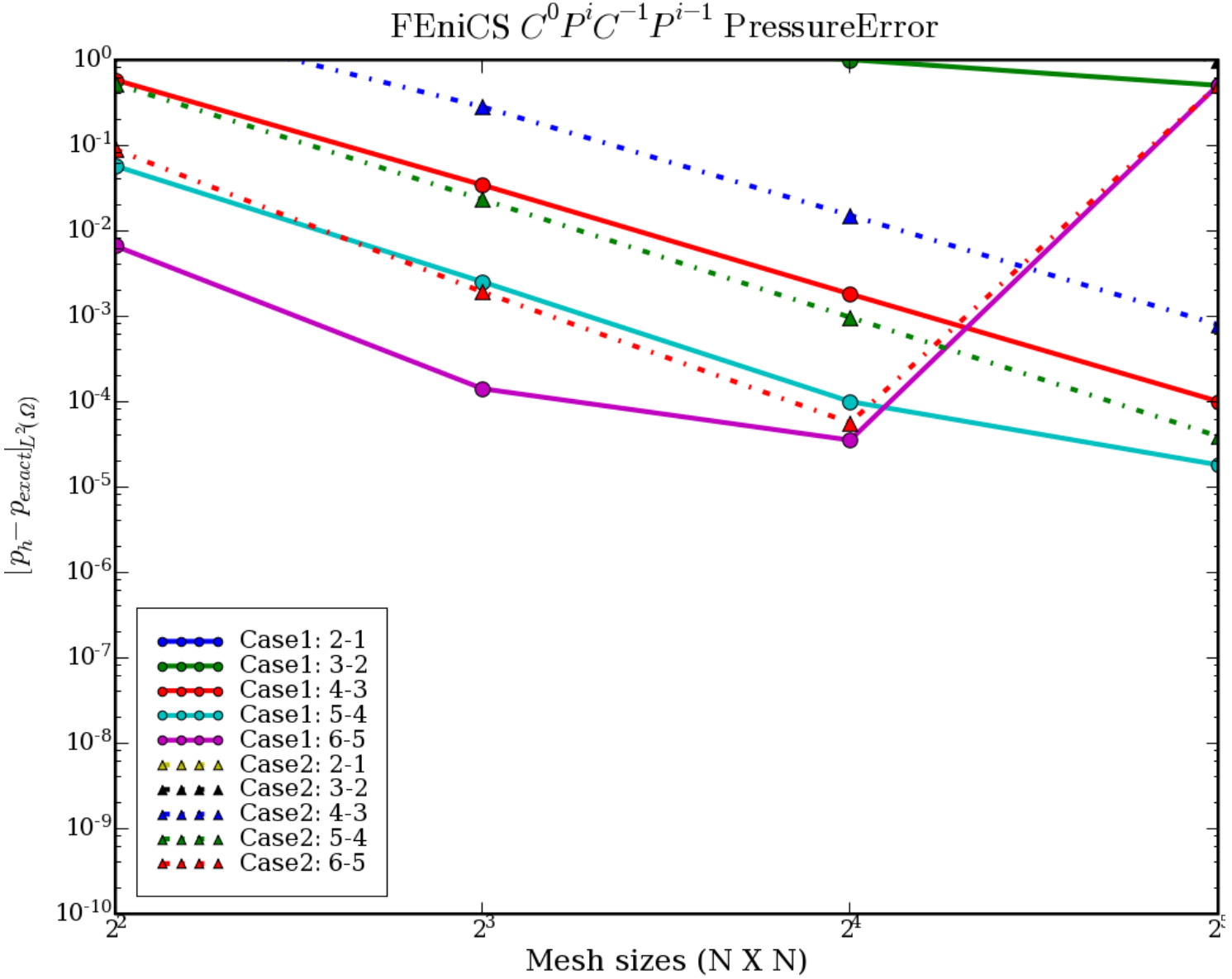
Crouzeix-Raviart P0 Error



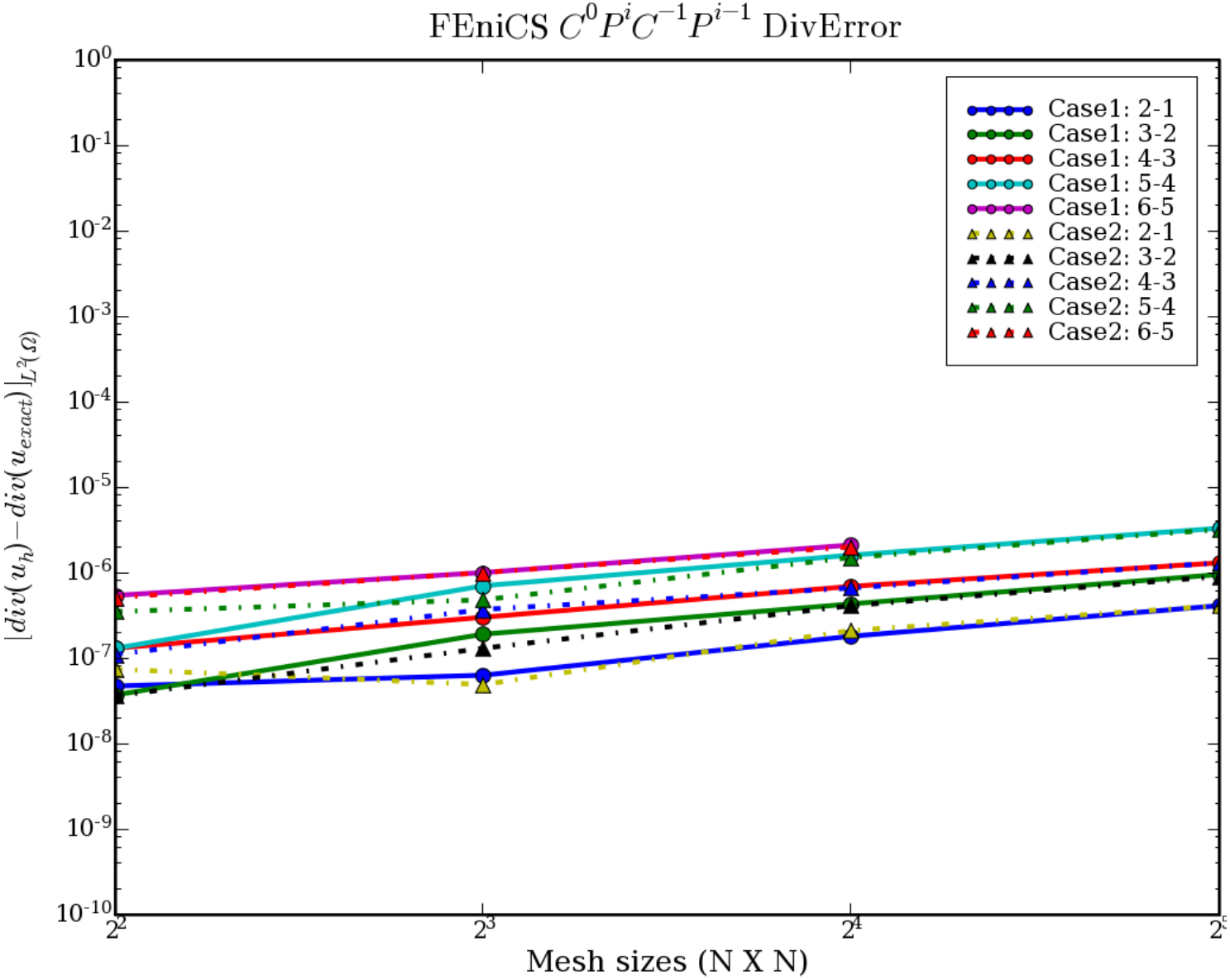
$C^0 P^i C^{-1} P^{i-1}$ Error



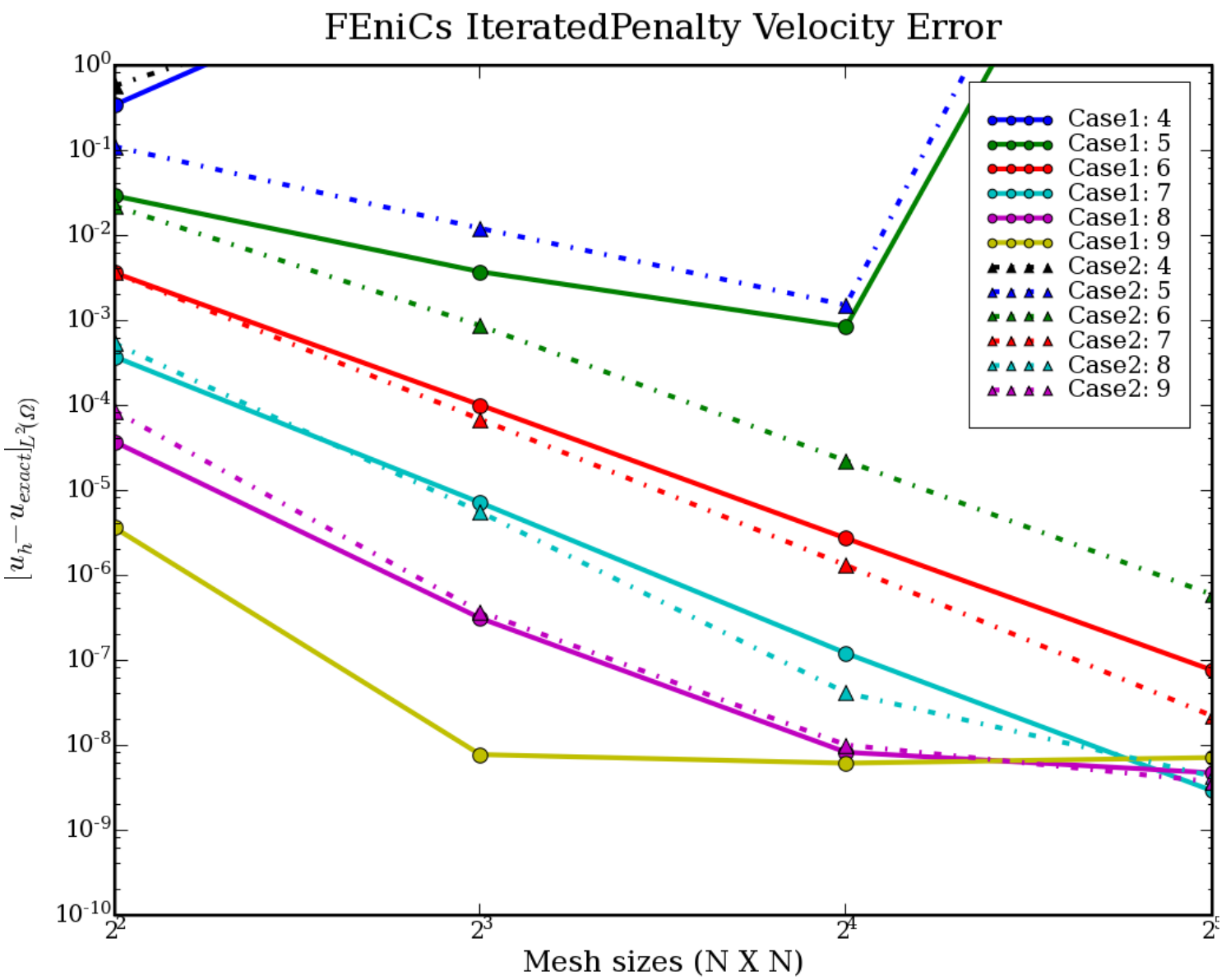
$C^0 P^i C^{-1} P^{i-1}$ Error



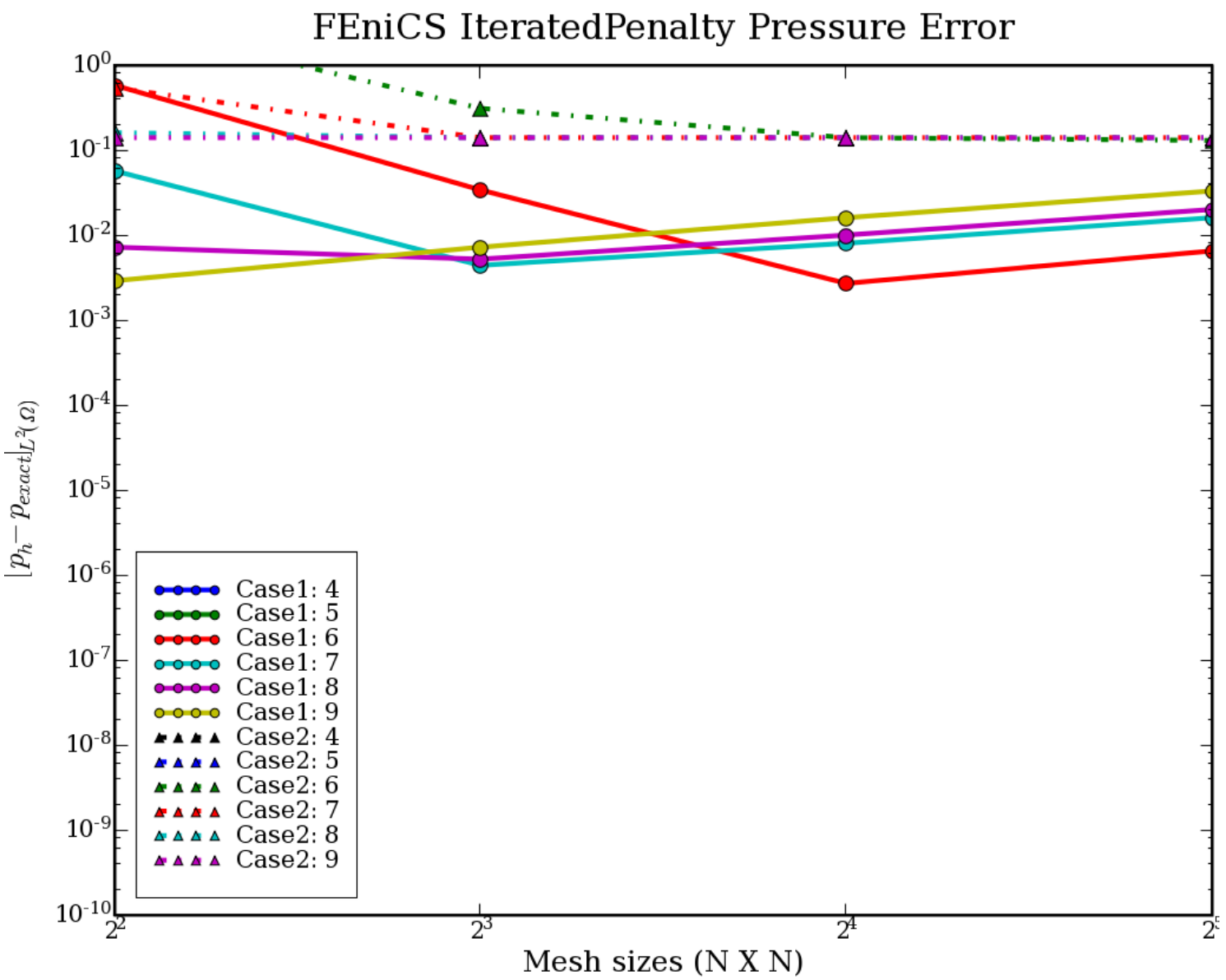
$C^0 P^i C^{-1} P^{i-1}$ Error



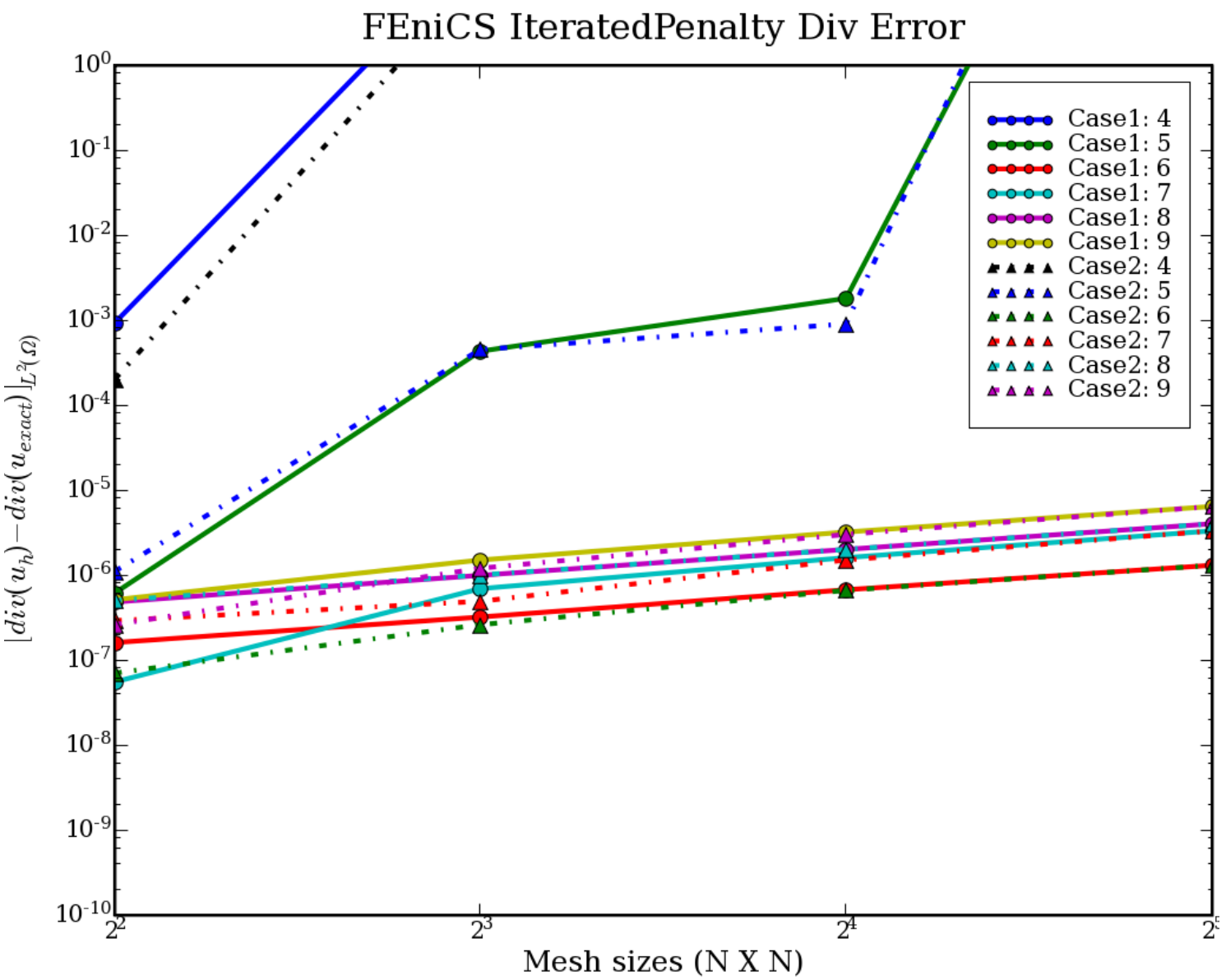
Iterated Penalty Error



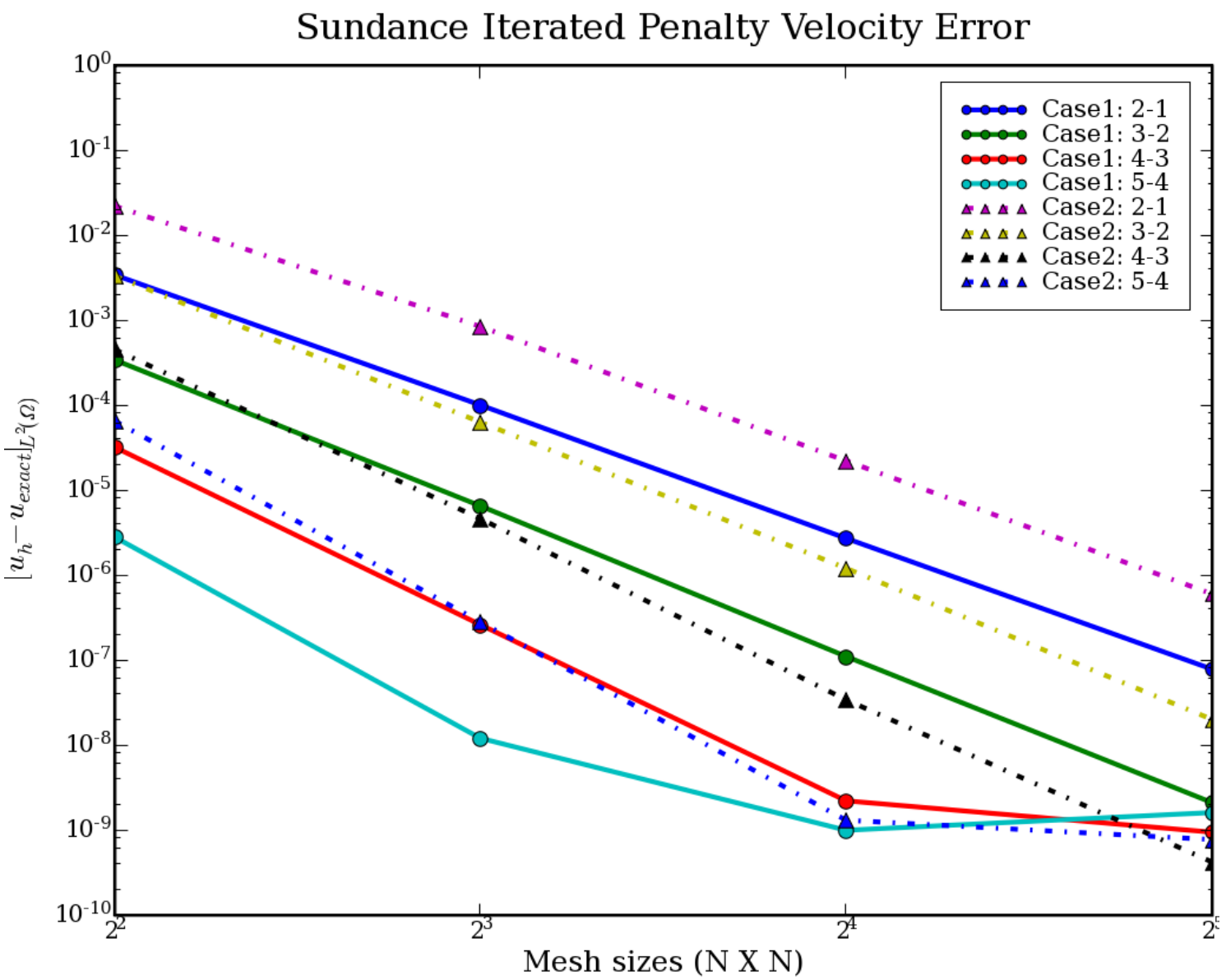
Iterated Penalty Error



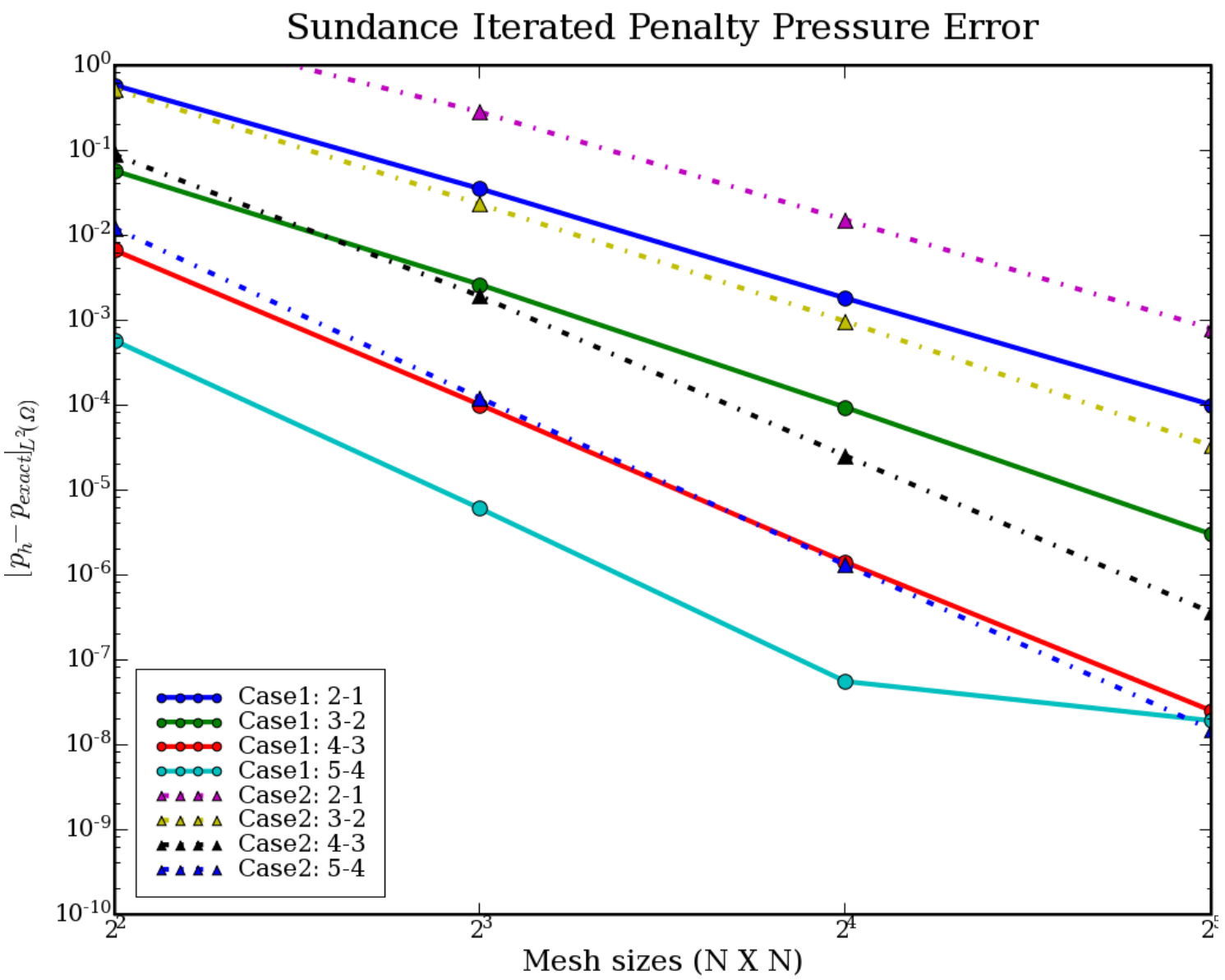
Iterated Penalty Error



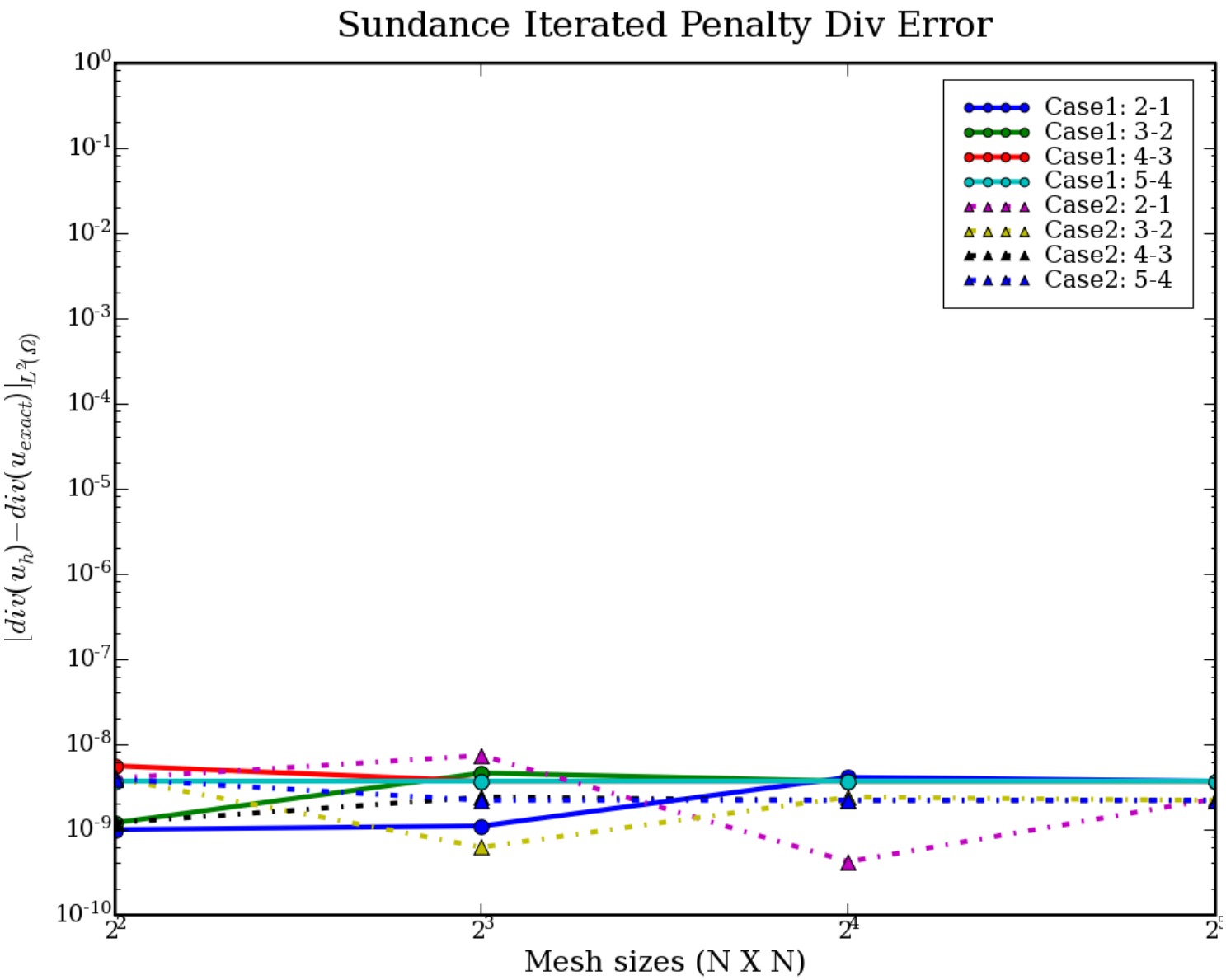
Iterated Penalty Error



Iterated Penalty Error



Iterated Penalty Error



Taylor-Hood Error

FEniCS TaylorHoodCase0

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
2 - 1	4	205	5.4e-05	0.00	1.1e-03	0.00	1.2e-02	0.00	0.01	7.7 - 3.3
	8	677	3.3e-06	-4.02	1.3e-04	-2.99	3.1e-03	-1.93	0.04	7.7 - 3.3
	16	2485	2.1e-07	-4.00	1.7e-05	-3.00	7.9e-04	-1.98	0.10	7.7 - 3.3
	32	9557	1.3e-08	-3.95	2.1e-06	-3.00	2.0e-04	-2.00	0.66	7.7 - 3.3
3 - 2	4	463	3.8e-09	0.00	4.8e-10	0.00	4.2e-08	0.00	0.04	3.9 - 2.9
	8	1583	5.4e-09	0.51	4.0e-09	3.07	1.3e-07	1.57	0.07	3.9 - 2.9
	16	5935	4.0e-09	-0.42	4.1e-09	0.04	2.3e-07	0.85	0.38	3.9 - 2.9
	32	23087	3.3e-09	-0.29	1.4e-09	-1.52	4.4e-07	0.95	2.20	3.9 - 2.9
4 - 3	4	829	7.1e-09	0.00	4.6e-09	0.00	9.5e-08	0.00	0.07	6.0 - 5.6
	8	2885	2.6e-09	-1.45	2.6e-09	-0.81	1.1e-07	0.19	0.20	6.0 - 5.6
	16	10933	6.2e-09	1.23	3.5e-09	0.40	2.7e-07	1.29	1.16	6.0 - 5.6
	32	42773	4.5e-09	-0.44	1.5e-09	-1.20	6.3e-07	1.24	10.97	6.0 - 5.6
5 - 4	4	1303	1.1e-08	0.00	7.1e-09	0.00	2.0e-07	0.00	0.11	9.1 - 10.7
	8	4583	7.3e-09	-0.53	1.6e-09	-2.16	3.8e-07	0.90	0.38	9.1 - 10.7
	16	17479	7.6e-09	0.06	6.1e-09	1.95	5.7e-07	0.58	1.76	9.1 - 10.7
	32	68615	4.2e-09	-0.84	7.4e-09	0.26	1.1e-06	0.99	9.65	9.1 - 10.7
6 - 5	4	1885	7.3e-09	0.00	9.3e-09	0.00	3.8e-07	0.00	0.20	15.3 - 21.5
	8	6677	8.8e-09	0.27	7.7e-09	-0.26	5.2e-07	0.45	0.69	15.3 - 21.5
	16	25573	6.6e-09	-0.43	2.6e-09	-1.58	8.0e-07	0.61	3.31	15.3 - 21.5
	32	100613	6.5e-09	-0.01	2.0e-08	2.94	1.5e-06	0.94	25.46	15.3 - 21.5

Taylor-Hood Error

FEniCS TaylorHoodCase1

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
2 - 1	4	205	2.7e-02	0.00	2.9e-01	0.00	7.6e-01	0.00	0.02	3.2 - 2.0
	8	677	2.7e-03	-3.34	5.9e-02	-2.28	2.5e-01	-1.60	0.03	3.2 - 2.0
	16	2485	1.9e-04	-3.82	1.5e-02	-2.00	6.8e-02	-1.88	0.12	3.2 - 2.0
	32	9557	1.2e-05	-3.96	3.7e-03	-1.98	1.7e-02	-1.97	0.67	3.2 - 2.0
3 - 2	4	463	4.7e-03	0.00	5.6e-02	0.00	1.9e-01	0.00	0.04	3.7 - 2.9
	8	1583	2.8e-04	-4.08	7.1e-03	-2.97	2.5e-02	-2.92	0.07	3.7 - 2.9
	16	5935	1.7e-05	-3.99	6.5e-04	-3.47	3.0e-03	-3.02	0.40	3.7 - 2.9
	32	23087	1.1e-06	-3.98	5.4e-05	-3.59	3.8e-04	-3.01	2.22	3.7 - 2.9
4 - 3	4	829	8.5e-04	0.00	8.8e-03	0.00	2.6e-02	0.00	0.07	5.6 - 5.6
	8	2885	3.2e-05	-4.73	4.7e-04	-4.24	1.8e-03	-3.80	0.20	5.6 - 5.6
	16	10933	1.0e-06	-4.94	2.6e-05	-4.20	1.2e-04	-3.95	1.15	5.6 - 5.6
	32	42773	3.2e-08	-5.01	1.4e-06	-4.21	7.4e-06	-4.02	11.05	5.6 - 5.6
5 - 4	4	1303	1.5e-04	0.00	1.4e-03	0.00	4.2e-03	0.00	0.11	9.2 - 10.8
	8	4583	2.6e-06	-5.91	3.5e-05	-5.33	1.3e-04	-4.98	0.37	9.2 - 10.8
	16	17479	4.1e-08	-5.96	8.0e-07	-5.43	4.4e-06	-4.92	1.77	9.2 - 10.8
	32	68615	4.9e-09	-3.07	2.0e-08	-5.35	3.3e-06	-0.40	9.75	9.2 - 10.8
6 - 5	4	1885	1.7e-05	0.00	1.4e-04	0.00	4.2e-04	0.00	0.20	15.3 - 21.3
	8	6677	1.5e-07	-6.75	2.6e-06	-5.79	7.7e-06	-5.75	0.69	15.3 - 21.3
	16	25573	1.2e-08	-3.70	4.3e-08	-5.91	2.0e-06	-1.95	3.36	15.3 - 21.3
	32	100613	6.1e-09	-0.97	5.5e-09	-2.97	4.0e-06	0.99	26.41	15.3 - 21.3

Taylor-Hood Error

FEniCS TaylorHoodCase2

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
2 - 1	4	205	9.2e-02	0.00	8.7e-01	0.00	1.9e+00	0.00	0.02	6.5 - 3.6
	8	677	1.3e-02	-2.86	1.7e-01	-2.37	7.7e-01	-1.31	0.03	6.5 - 3.6
	16	2485	9.7e-04	-3.69	3.4e-02	-2.30	2.2e-01	-1.79	0.11	6.5 - 3.6
	32	9557	6.4e-05	-3.92	8.5e-03	-2.01	5.8e-02	-1.94	0.68	6.5 - 3.6
3 - 2	4	463	2.5e-02	0.00	2.2e-01	0.00	8.1e-01	0.00	0.04	3.8 - 2.9
	8	1583	1.4e-03	-4.17	3.1e-02	-2.84	1.2e-01	-2.74	0.08	3.8 - 2.9
	16	5935	8.4e-05	-4.05	3.1e-03	-3.30	1.5e-02	-2.99	0.39	3.8 - 2.9
	32	23087	5.3e-06	-3.99	2.6e-04	-3.55	1.9e-03	-3.01	2.27	3.8 - 2.9
4 - 3	4	829	5.4e-03	0.00	7.1e-02	0.00	1.8e-01	0.00	0.07	6.6 - 5.6
	8	2885	2.3e-04	-4.53	3.4e-03	-4.36	1.4e-02	-3.74	0.20	6.6 - 5.6
	16	10933	7.8e-06	-4.90	1.7e-04	-4.32	9.0e-04	-3.93	1.21	6.6 - 5.6
	32	42773	2.5e-07	-4.98	8.9e-06	-4.28	5.7e-05	-3.98	11.75	6.6 - 5.6
5 - 4	4	1303	1.6e-03	0.00	1.5e-02	0.00	4.3e-02	0.00	0.11	9.2 - 10.8
	8	4583	2.8e-05	-5.78	3.9e-04	-5.30	1.5e-03	-4.86	0.37	9.2 - 10.8
	16	17479	4.5e-07	-5.96	8.8e-06	-5.48	4.6e-05	-5.01	1.82	9.2 - 10.8
	32	68615	4.3e-09	-6.71	8.4e-07	-3.39	3.6e-06	-3.70	9.88	9.2 - 10.8
6 - 5	4	1885	2.6e-04	0.00	2.2e-03	0.00	6.6e-03	0.00	0.20	15.6 - 21.1
	8	6677	2.6e-06	-6.65	3.8e-05	-5.87	1.3e-04	-5.71	0.70	15.6 - 21.1
	16	25573	1.7e-08	-7.20	7.4e-07	-5.69	2.9e-06	-5.46	3.35	15.6 - 21.1
	32	100613	5.1e-09	-1.78	1.5e-06	1.04	4.0e-06	0.49	26.90	15.6 - 21.1

Taylor-Hood Error

Sundance TaylorHoodCase0

Ord	Mesh	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run
2 - 1	4	5.1e-04	0.00	1.1e-03	0.00	1.2e-02	0.00	0.17
	8	6.7e-05	-2.94	1.4e-04	-2.99	3.1e-03	-1.93	0.20
	16	8.4e-06	-2.99	1.8e-05	-3.00	7.9e-04	-1.98	0.35
	32	1.1e-06	-3.00	2.2e-06	-3.00	2.0e-04	-2.00	1.17
3 - 2	4	1.4e-12	0.00	1.3e-11	0.00	4.1e-12	0.00	0.23
	8	1.4e-12	0.00	1.3e-11	-0.00	4.1e-12	-0.00	0.31
	16	1.4e-12	-0.00	1.3e-11	-0.00	4.1e-12	-0.00	0.67
	32	1.4e-12	-0.00	1.3e-11	-0.00	4.1e-12	-0.00	2.57
4 - 3	4	1.4e-12	0.00	1.3e-11	0.00	4.1e-12	0.00	0.38
	8	1.4e-12	0.00	1.3e-11	-0.00	4.1e-12	-0.00	0.52
	16	1.4e-12	0.00	1.3e-11	-0.00	4.1e-12	0.00	1.31
	32	1.4e-12	0.00	1.3e-11	0.00	4.1e-12	0.00	8.50
5 - 4	4	1.4e-12	0.00	1.3e-11	0.00	4.1e-12	0.00	0.66
	8	1.4e-12	0.00	1.3e-11	0.00	4.1e-12	0.00	0.95
	16	1.4e-12	0.00	1.3e-11	0.01	4.1e-12	0.00	2.31
	32	1.4e-12	0.01	1.3e-11	0.06	4.1e-12	0.01	27.99

Taylor-Hood Error

Sundance TaylorHoodCase1

Ord	Mesh	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run
2 - 1	4	5.1e-02	0.00	2.5e-01	0.00	7.5e-01	0.00	0.33
	8	7.6e-03	-2.75	3.2e-02	-3.00	2.5e-01	-1.60	0.25
	16	1.0e-03	-2.89	4.6e-03	-2.78	6.8e-02	-1.87	0.40
	32	1.3e-04	-2.97	9.2e-04	-2.33	1.7e-02	-1.96	1.22
3 - 2	4	1.0e-02	0.00	5.8e-02	0.00	1.8e-01	0.00	0.28
	8	6.3e-04	-4.03	6.9e-03	-3.06	2.4e-02	-2.92	0.36
	16	3.7e-05	-4.09	7.1e-04	-3.29	3.0e-03	-3.00	0.72
	32	2.2e-06	-4.04	7.4e-05	-3.26	3.7e-04	-3.01	2.62
4 - 3	4	1.1e-03	0.00	9.8e-03	0.00	2.5e-02	0.00	0.43
	8	4.4e-05	-4.70	4.9e-04	-4.31	1.8e-03	-3.79	0.58
	16	1.5e-06	-4.90	2.5e-05	-4.33	1.2e-04	-3.96	1.39
	32	4.7e-08	-4.97	1.2e-06	-4.37	7.3e-06	-4.00	6.53
5 - 4	4	1.5e-04	0.00	1.6e-03	0.00	4.1e-03	0.00	0.71
	8	2.5e-06	-5.93	4.1e-05	-5.29	1.3e-04	-4.98	0.98
	16	3.9e-08	-6.01	1.0e-06	-5.28	3.9e-06	-5.03	2.38
	32	6.1e-10	-6.01	2.9e-08	-5.18	1.2e-07	-5.02	12.46

Taylor-Hood Error

Sundance TaylorHoodCase2

Ord	Mesh	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run
2 - 1	4	1.7e-01	0.00	1.1e+00	0.00	2.0e+00	0.00	0.21
	8	2.4e-02	-2.82	1.4e-01	-3.05	7.6e-01	-1.40	0.25
	16	3.4e-03	-2.86	1.5e-02	-3.17	2.2e-01	-1.78	0.40
	32	4.4e-04	-2.95	2.3e-03	-2.69	5.8e-02	-1.93	1.22
3 - 2	4	4.5e-02	0.00	2.8e-01	0.00	7.8e-01	0.00	0.28
	8	3.3e-03	-3.79	3.4e-02	-3.05	1.2e-01	-2.73	0.35
	16	1.9e-04	-4.10	3.4e-03	-3.33	1.5e-02	-2.98	0.71
	32	1.1e-05	-4.07	3.2e-04	-3.40	1.9e-03	-3.00	2.62
4 - 3	4	8.0e-03	0.00	8.6e-02	0.00	1.8e-01	0.00	0.43
	8	3.1e-04	-4.68	4.0e-03	-4.44	1.3e-02	-3.71	0.58
	16	1.1e-05	-4.84	1.8e-04	-4.42	8.8e-04	-3.93	1.35
	32	3.5e-07	-4.95	8.6e-06	-4.42	5.5e-05	-3.99	6.40
5 - 4	4	1.6e-03	0.00	1.8e-02	0.00	4.2e-02	0.00	0.70
	8	2.8e-05	-5.77	4.8e-04	-5.22	1.5e-03	-4.85	0.99
	16	4.5e-07	-5.99	1.2e-05	-5.33	4.5e-05	-5.02	2.37
	32	7.0e-09	-6.01	3.2e-07	-5.23	1.4e-06	-5.02	11.17

Crouzeix-Raviart P0 Error

FEniCS Crouzeix-RaviartCase0

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
1 - 0	4	160	6.8e-03	0.00	6.8e-02	0.00	1.4e-08	0.00	0.01	8.4 - 7.6
	8	560	2.2e-03	-1.60	2.6e-02	-1.36	5.6e-08	2.02	0.01	8.4 - 7.6
	16	2128	6.2e-04	-1.87	1.1e-02	-1.33	4.3e-08	-0.38	0.06	8.4 - 7.6
	32	8336	1.6e-04	-1.96	4.6e-03	-1.19	2.5e-08	-0.79	0.29	8.4 - 7.6

Crouzeix-Raviart P0 Error

FEniCS Crouzeix-RaviartCase1

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
1 - 0	4	160	9.1e-02	0.00	8.5e-01	0.00	8.9e-08	0.00	0.01	2.8 - 1.3
	8	560	2.2e-02	-2.08	3.3e-01	-1.35	8.0e-08	-0.15	0.02	2.8 - 1.3
	16	2128	5.5e-03	-1.98	1.5e-01	-1.14	9.1e-08	0.17	0.05	2.8 - 1.3
	32	8336	1.4e-03	-1.99	7.3e-02	-1.04	1.2e-07	0.45	0.30	2.8 - 1.3

Crouzeix-Raviart P0 Error

FEniCS Crouzeix-RaviartCase2

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
1 - 0	4	160	2.3e-01	0.00	2.3e+00	0.00	1.3e-07	0.00	0.01	2.6 - 1.3
	8	560	4.8e-02	-2.26	8.2e-01	-1.46	1.2e-07	-0.07	0.02	2.6 - 1.3
	16	2128	1.2e-02	-2.03	3.5e-01	-1.22	6.2e-08	-0.94	0.05	2.6 - 1.3
	32	8336	3.0e-03	-1.99	1.7e-01	-1.03	3.5e-08	-0.82	0.30	2.6 - 1.3
	64	33040	7.4e-04	0.00	9.4e-02	0.00	1.4e-07	0.00	2.65	7.4 - 6.2
	128	131600	1.9e-04	-2.00	6.1e-02	-0.62	2.1e-08	-2.76	64.76	7.4 - 6.2

$C^0 P^i C^{-1} P^{i-1}$ Error

FEniCS C0PiC-1Pi-1Case0

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
2 - 1	4	286	1.0e-02	0.00	2.4e+00	0.00	1.9e-08	0.00	0.03	3.2 - 2.0
	8	990	3.6e-03	-1.48	4.1e+00	0.74	5.9e-08	1.62	0.06	3.2 - 2.0
	16	3742	1.1e-03	-1.70	5.8e+00	0.51	9.1e-08	0.63	0.57	3.2 - 2.0
	32	14622	3.2e-04	-1.79	7.6e+00	0.38	1.5e-07	0.68	10.50	3.2 - 2.0
3 - 2	4	590	9.0e-09	0.00	5.8e-05	0.00	2.5e-08	0.00	0.04	3.8 - 2.9
	8	2078	2.3e-09	-1.96	5.3e-05	-0.13	1.4e-07	2.44	0.14	3.8 - 2.9
	16	7934	4.6e-09	1.00	3.3e-05	-0.67	2.2e-07	0.66	1.43	3.8 - 2.9
	32	31166	1.6e-09	-1.50	5.7e-05	0.77	4.5e-07	1.07	63.18	3.8 - 2.9
4 - 3	4	1002	1.1e-08	0.00	5.5e-05	0.00	8.8e-08	0.00	0.08	5.7 - 6.0
	8	3562	3.9e-09	-1.56	4.4e-05	-0.32	1.2e-07	0.50	0.36	5.7 - 6.0
	16	13674	4.4e-09	0.19	1.0e-04	1.21	3.1e-07	1.32	5.54	5.7 - 6.0
	32	53866	4.0e-09	-0.13	1.0e-04	-0.04	6.3e-07	1.03	111.34	5.7 - 6.0
5 - 4	4	1522	9.0e-09	0.00	4.1e-05	0.00	2.1e-07	0.00	0.15	10.2 - 11.2
	8	5442	3.0e-09	-1.59	7.1e-06	-2.53	3.3e-07	0.64	0.69	10.2 - 11.2
	16	20962	5.0e-09	0.73	1.3e-05	0.86	5.9e-07	0.84	24.27	10.2 - 11.2
	32	82722	4.2e-09	-0.24	5.8e-06	-1.16	1.1e-06	0.91	618.19	10.2 - 11.2
6 - 5	4	2150	1.1e-08	0.00	3.0e-04	0.00	4.4e-07	0.00	0.23	29.0 - 26.9
	8	7718	5.7e-09	-1.01	3.1e-04	0.03	5.5e-07	0.31	1.50	29.0 - 26.9
	16	29798	6.1e-09	0.10	3.4e-04	0.16	7.8e-07	0.51	51.77	29.0 - 26.9
	32	117734	3.7e-01	25.84	4.1e-01	10.21	0.0e+00	-inf	376.88	29.0 - 26.9

$C^0 P^i C^{-1} P^{i-1}$ Error

FEniCS C0PiC-1Pi-1Case1

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
2 - 1	4	286	3.4e-01	0.00	3.4e+01	0.00	4.7e-08	0.00	0.01	8.4 - 5.3
	8	990	8.0e-02	-2.08	3.5e+01	0.05	6.3e-08	0.44	0.07	8.4 - 5.3
	16	3742	1.9e-02	-2.04	3.6e+01	0.03	1.8e-07	1.50	0.58	8.4 - 5.3
	32	14622	4.8e-03	-2.01	3.6e+01	0.01	4.1e-07	1.20	9.94	8.4 - 5.3
3 - 2	4	590	2.9e-02	0.00	3.9e+00	0.00	3.7e-08	0.00	0.04	4.0 - 2.9
	8	2078	3.7e-03	-3.00	2.0e+00	-0.96	1.9e-07	2.40	0.14	4.0 - 2.9
	16	7934	4.6e-04	-2.99	1.0e+00	-1.00	4.3e-07	1.14	1.39	4.0 - 2.9
	32	31166	5.8e-05	-3.00	5.0e-01	-1.01	9.5e-07	1.15	57.21	4.0 - 2.9
4 - 3	4	1002	3.6e-03	0.00	5.8e-01	0.00	1.3e-07	0.00	0.08	5.7 - 5.6
	8	3562	1.0e-04	-5.13	3.4e-02	-4.08	3.0e-07	1.18	0.37	5.7 - 5.6
	16	13674	2.7e-06	-5.24	1.8e-03	-4.23	6.9e-07	1.19	5.37	5.7 - 5.6
	32	53866	7.5e-08	-5.16	1.0e-04	-4.13	1.3e-06	0.92	104.74	5.7 - 5.6
5 - 4	4	1522	3.7e-04	0.00	5.7e-02	0.00	1.3e-07	0.00	0.15	9.1 - 10.8
	8	5442	7.1e-06	-5.70	2.5e-03	-4.48	7.0e-07	2.38	0.69	9.1 - 10.8
	16	20962	1.2e-07	-5.86	9.9e-05	-4.68	1.6e-06	1.15	22.72	9.1 - 10.8
	32	82722	6.3e-09	-4.28	1.8e-05	-2.43	3.3e-06	1.09	597.36	9.1 - 10.8
6 - 5	4	2150	3.7e-05	0.00	6.6e-03	0.00	5.4e-07	0.00	0.23	25.9 - 27.1
	8	7718	3.1e-07	-6.90	1.4e-04	-5.55	1.0e-06	0.92	1.48	25.9 - 27.1
	16	29798	1.1e-08	-4.89	3.5e-05	-2.01	2.1e-06	1.02	52.33	25.9 - 27.1
	32	117734	7.1e-01	26.00	5.0e-01	13.80	0.0e+00	-inf	376.89	25.9 - 27.1

$C^0 P^i C^{-1} P^{i-1}$ Error

FEniCS C0PiC-1Pi-1Case2

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Run	Comp
2 - 1	4	286	5.7e-01	0.00	4.1e+01	0.00	7.4e-08	0.00	0.01	7.9 - 5.0
	8	990	1.6e-01	-1.85	3.8e+01	-0.09	4.9e-08	-0.59	0.07	7.9 - 5.0
	16	3742	3.8e-02	-2.07	6.0e+01	0.66	2.1e-07	2.08	0.58	7.9 - 5.0
	32	14622	9.7e-03	-1.96	9.0e+01	0.57	4.1e-07	0.97	9.80	7.9 - 5.0
3 - 2	4	590	1.1e-01	0.00	9.6e+00	0.00	3.6e-08	0.00	0.04	4.4 - 3.2
	8	2078	1.2e-02	-3.24	4.1e+00	-1.23	1.3e-07	1.86	0.14	4.4 - 3.2
	16	7934	1.5e-03	-2.97	2.1e+00	-0.98	4.1e-07	1.66	1.39	4.4 - 3.2
	32	31166	1.9e-04	-2.99	1.0e+00	-1.00	9.0e-07	1.12	57.22	4.4 - 3.2
4 - 3	4	1002	2.2e-02	0.00	3.3e+00	0.00	1.1e-07	0.00	0.08	5.7 - 5.6
	8	3562	8.6e-04	-4.67	2.8e-01	-3.55	3.7e-07	1.68	0.36	5.7 - 5.6
	16	13674	2.2e-05	-5.32	1.5e-02	-4.27	6.6e-07	0.85	5.29	5.7 - 5.6
	32	53866	5.9e-07	-5.20	7.9e-04	-4.21	1.3e-06	1.00	102.86	5.7 - 5.6
5 - 4	4	1522	3.6e-03	0.00	5.2e-01	0.00	3.5e-07	0.00	0.15	9.1 - 10.8
	8	5442	6.8e-05	-5.74	2.3e-02	-4.53	4.8e-07	0.45	0.68	9.1 - 10.8
	16	20962	1.3e-06	-5.71	9.6e-04	-4.56	1.5e-06	1.65	22.26	9.1 - 10.8
	32	82722	2.1e-08	-5.92	3.9e-05	-4.63	3.2e-06	1.09	590.72	9.1 - 10.8
6 - 5	4	2150	5.3e-04	0.00	8.9e-02	0.00	5.1e-07	0.00	0.23	32.6 - 27.3
	8	7718	5.5e-06	-6.60	1.9e-03	-5.56	1.0e-06	0.99	1.46	32.6 - 27.3
	16	29798	3.9e-08	-7.14	5.6e-05	-5.07	2.0e-06	1.02	50.78	32.6 - 27.3
	32	117734	7.1e-01	24.12	5.0e-01	13.13	0.0e+00	-inf	373.76	32.6 - 27.3

Iterated Penalty Error

FEniCS IteratedPenaltyCase0

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Iters	Run	Comp
2	4	178	1.0e-02	0.00	2.4e+00	0.00	2.0e-05	0.00	5	0.04	4.5
	8	594	3.5e+00	8.45	4.1e+03	10.70	3.1e+00	17.23	5	0.10	4.5
	16	2194	2.8e+00	-0.31	9.1e+03	1.16	8.2e+00	1.40	5	0.36	4.5
	32	8466	3.2e+00	0.18	2.4e+04	1.42	2.3e+01	1.50	5	1.84	4.5
3	4	374	6.4e-08	0.00	1.0e-04	0.00	9.8e-08	0.00	2	0.07	4.8
	8	1286	1.0e-07	0.69	4.7e-04	2.18	1.6e-07	0.70	2	0.15	4.8
	16	4838	1.1e-04	10.12	3.6e-01	9.59	4.2e-04	11.38	5	1.14	4.8
	32	18854	6.6e-04	2.53	4.6e+00	3.67	4.3e-03	3.35	5	5.97	4.8
4	4	642	8.2e-09	0.00	1.2e-04	0.00	7.1e-08	0.00	2	0.11	8.6
	8	2242	1.3e-09	-2.64	2.7e-04	1.13	1.4e-07	0.93	2	0.34	8.6
	16	8514	2.0e-09	0.62	9.0e-04	1.74	2.8e-07	1.06	2	1.64	8.6
	32	33346	3.8e-09	0.92	3.2e-03	1.82	6.5e-07	1.19	5	14.61	8.6
5	4	982	2.6e-09	0.00	5.7e-04	0.00	2.3e-07	0.00	2	0.21	16.7
	8	3462	1.3e-08	2.31	1.1e-03	0.94	3.1e-07	0.40	2	0.68	16.7
	16	13222	6.0e-09	-1.08	3.0e-03	1.44	5.8e-07	0.92	5	5.07	16.7
	32	51942	2.2e-09	-1.46	5.3e-03	0.83	1.1e-06	0.90	5	25.34	16.7
6	4	1394	9.6e-09	0.00	1.3e-03	0.00	4.0e-07	0.00	2	0.34	34.6
	8	4946	6.5e-09	-0.55	1.6e-03	0.24	4.6e-07	0.23	2	1.21	34.6
	16	18962	5.3e-09	-0.30	4.1e-03	1.38	8.4e-07	0.86	5	8.90	34.6
	32	74642	4.3e-09	-0.30	7.1e-03	0.80	1.4e-06	0.73	5	47.11	34.6
7	4	1878	1.5e-08	0.00	1.1e-03	0.00	4.1e-07	0.00	2	0.55	82.1
	8	6694	1.3e-08	-0.18	3.8e-03	1.74	8.1e-07	0.97	5	3.37	82.1
	16	25734	5.1e-09	-1.36	7.3e-03	0.93	1.4e-06	0.83	5	14.70	82.1
	32	101446	3.7e-01	26.11	3.0e-03	-1.29	0.0e+00	-inf	1	33.67	82.1

Iterated Penalty Error

FEniCS IteratedPenaltyCase1

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Iters	Run	Comp
2	4	178	3.4e-01	0.00	3.4e+01	0.00	9.2e-04	0.00	5	0.04	10.0
	8	594	2.7e+01	6.33	3.1e+04	9.86	2.4e+01	14.64	5	0.10	10.0
	16	2194	2.4e+01	-0.15	7.7e+04	1.30	6.9e+01	1.56	5	0.37	10.0
	32	8466	9.2e+02	5.25	6.9e+06	6.48	6.6e+03	6.57	5	1.90	10.0
3	4	374	2.9e-02	0.00	3.9e+00	0.00	6.1e-07	0.00	5	0.09	5.0
	8	1286	3.7e-03	-3.00	2.0e+00	-0.93	4.3e-04	9.46	5	0.26	5.0
	16	4838	8.4e-04	-2.13	3.5e+00	0.78	1.8e-03	2.05	5	1.17	5.0
	32	18854	4.1e+04	25.54	2.9e+08	26.28	2.9e+05	27.26	5	6.20	5.0
4	4	642	3.6e-03	0.00	5.8e-01	0.00	1.6e-07	0.00	3	0.14	8.9
	8	2242	1.0e-04	-5.13	3.4e-02	-4.08	3.2e-07	1.03	2	0.35	8.9
	16	8514	2.7e-06	-5.24	2.7e-03	-3.65	6.7e-07	1.08	5	2.77	8.9
	32	33346	7.6e-08	-5.15	6.5e-03	1.25	1.3e-06	0.95	5	14.83	8.9
5	4	982	3.7e-04	0.00	5.7e-02	0.00	5.5e-08	0.00	2	0.21	15.6
	8	3462	7.1e-06	-5.70	4.4e-03	-3.69	6.9e-07	3.67	5	1.14	15.6
	16	13222	1.2e-07	-5.85	8.0e-03	0.85	1.6e-06	1.21	5	5.16	15.6
	32	51942	2.9e-09	-5.42	1.6e-02	1.05	3.3e-06	1.05	5	25.61	15.6
6	4	1394	3.7e-05	0.00	7.2e-03	0.00	4.8e-07	0.00	4	0.56	30.3
	8	4946	3.1e-07	-6.89	5.2e-03	-0.48	9.9e-07	1.04	5	2.05	30.3
	16	18962	8.2e-09	-5.24	1.0e-02	0.98	2.0e-06	1.03	5	9.00	30.3
	32	74642	4.7e-09	-0.81	2.0e-02	1.00	4.0e-06	0.98	5	48.15	30.3
7	4	1878	3.6e-06	0.00	2.9e-03	0.00	5.1e-07	0.00	5	0.93	79.4
	8	6694	7.7e-09	-8.86	7.2e-03	1.30	1.5e-06	1.51	5	3.43	79.4
	16	25734	6.1e-09	-0.33	1.6e-02	1.12	3.2e-06	1.13	5	15.04	79.4
	32	101446	7.1e-09	0.23	3.3e-02	1.06	6.4e-06	1.00	5	91.30	79.4

Iterated Penalty Error

FEniCS IteratedPenaltyCase2

Ord	Mesh	DOFs	Vel Err	Vp	Pre Err	Pp	Div	Dp	Iters	Run	Comp
2	4	178	5.7e-01	0.00	4.1e+01	0.00	2.0e-04	0.00	5	0.04	18.4
	8	594	1.4e+01	4.58	1.6e+04	8.60	1.2e+01	15.84	5	0.08	18.4
	16	2194	8.4e+02	5.96	2.7e+06	7.42	2.4e+03	7.68	5	0.37	18.4
	32	8466	5.1e+03	2.61	3.9e+07	3.84	3.7e+04	3.93	5	1.90	18.4
3	4	374	1.1e-01	0.00	9.6e+00	0.00	1.1e-06	0.00	5	0.09	4.8
	8	1286	1.2e-02	-3.24	4.1e+00	-1.23	4.5e-04	8.68	5	0.25	4.8
	16	4838	1.5e-03	-2.93	2.7e+00	-0.61	9.0e-04	0.98	5	1.14	4.8
	32	18854	3.9e+05	27.94	2.7e+09	29.90	2.7e+06	31.50	5	6.07	4.8
4	4	642	2.2e-02	0.00	3.3e+00	0.00	7.0e-08	0.00	3	0.14	8.7
	8	2242	8.6e-04	-4.67	3.1e-01	-3.40	2.6e-07	1.86	2	0.34	8.7
	16	8514	2.2e-05	-5.32	1.4e-01	-1.21	6.6e-07	1.36	5	2.75	8.7
	32	33346	5.9e-07	-5.20	1.3e-01	-0.01	1.3e-06	0.99	5	14.74	8.7
5	4	982	3.6e-03	0.00	5.4e-01	0.00	2.9e-07	0.00	2	0.20	15.9
	8	3462	6.8e-05	-5.74	1.4e-01	-1.97	4.9e-07	0.73	2	0.68	15.9
	16	13222	1.3e-06	-5.71	1.4e-01	-0.02	1.5e-06	1.63	5	5.08	15.9
	32	51942	2.2e-08	-5.91	1.4e-01	0.01	3.3e-06	1.11	5	25.17	15.9
6	4	1394	5.3e-04	0.00	1.6e-01	0.00	5.0e-07	0.00	2	0.34	30.1
	8	4946	5.5e-06	-6.60	1.4e-01	-0.26	1.0e-06	1.04	5	1.99	30.1
	16	18962	4.1e-08	-7.07	1.4e-01	0.00	2.0e-06	0.95	5	8.86	30.1
	32	74642	4.3e-09	-3.24	1.4e-01	0.01	4.0e-06	1.02	5	47.47	30.1
7	4	1878	8.5e-05	0.00	1.4e-01	0.00	2.6e-07	0.00	2	0.58	78.5
	8	6694	3.7e-07	-7.85	1.4e-01	-0.00	1.2e-06	2.24	5	3.37	78.5
	16	25734	1.0e-08	-5.20	1.4e-01	0.01	3.0e-06	1.33	5	14.70	78.5
	32	101446	3.7e-09	-1.44	1.4e-01	0.03	6.4e-06	1.08	5	90.66	78.5

Iterated Penalty Error

Sundance IteratedPenaltyCase0

Ord	Mesh	Vel Err	Vp	Pre Err	Pp	Div	Dp	Iters	Run
4	4	1.4e-09	0.00	3.2e-08	0.00	5.0e-09	0.00	2	1.20
	8	1.4e-09	-0.01	2.2e-08	-0.51	4.9e-09	-0.02	2	1.74
	16	1.4e-09	-0.00	2.2e-08	-0.04	4.9e-09	-0.01	2	4.06
	32	1.4e-09	-0.00	2.1e-08	-0.01	4.9e-09	-0.00	2	14.79
5	4	1.4e-09	0.00	2.2e-08	0.00	4.9e-09	0.00	2	2.53
	8	1.4e-09	-0.00	2.1e-08	-0.03	4.9e-09	-0.01	2	3.15
	16	1.4e-09	-0.00	2.1e-08	-0.02	4.9e-09	-0.00	2	7.80
	32	1.4e-09	0.00	2.1e-08	-0.01	4.9e-09	-0.00	2	31.51
6	4	1.4e-09	0.00	2.2e-08	0.00	4.9e-09	0.00	2	3.77
	8	1.4e-09	-0.00	2.1e-08	-0.01	4.9e-09	-0.00	2	5.73
	16	1.4e-09	-0.00	2.2e-08	0.01	4.9e-09	-0.00	2	14.48
	32	1.4e-09	0.00	2.2e-08	0.04	4.9e-09	-0.00	2	62.11
7	4	1.4e-09	0.00	2.1e-08	0.00	4.9e-09	0.00	2	6.80
	8	1.4e-09	-0.00	2.1e-08	-0.03	4.9e-09	-0.00	2	10.37
	16	1.4e-09	0.00	2.0e-08	-0.07	4.9e-09	-0.00	2	26.68
	32	1.4e-09	0.01	1.7e-08	-0.21	4.9e-09	-0.00	2	126.95

Iterated Penalty Error

Sundance IteratedPenaltyCase1

Ord	Mesh	Vel Err	Vp	Pre Err	Pp	Div	Dp	Iters	Run
4	4	3.4e-03	0.00	5.8e-01	0.00	1.0e-09	0.00	4	2.52
	8	1.0e-04	-5.10	3.5e-02	-4.08	1.1e-09	0.12	3	2.49
	16	2.7e-06	-5.21	1.8e-03	-4.23	4.1e-09	1.90	2	4.33
	32	7.8e-08	-5.13	1.0e-04	-4.17	3.7e-09	-0.12	2	15.26
5	4	3.4e-04	0.00	5.7e-02	0.00	1.2e-09	0.00	3	3.18
	8	6.5e-06	-5.69	2.6e-03	-4.49	4.6e-09	1.89	2	3.29
	16	1.1e-07	-5.84	9.3e-05	-4.78	3.7e-09	-0.30	2	8.04
	32	2.1e-09	-5.78	3.0e-06	-4.94	3.7e-09	-0.00	2	30.41
6	4	3.2e-05	0.00	6.6e-03	0.00	5.6e-09	0.00	2	3.85
	8	2.6e-07	-6.91	1.0e-04	-6.02	3.7e-09	-0.58	2	5.87
	16	2.2e-09	-6.92	1.4e-06	-6.16	3.7e-09	-0.00	2	14.66
	32	9.4e-10	-1.21	2.5e-08	-5.80	3.7e-09	-0.00	2	56.45
7	4	2.8e-06	0.00	5.7e-04	0.00	3.7e-09	0.00	2	6.84
	8	1.2e-08	-7.82	6.0e-06	-6.57	3.7e-09	-0.00	2	10.53
	16	9.9e-10	-3.64	5.5e-08	-6.76	3.7e-09	-0.00	2	26.82
	32	1.6e-09	0.71	1.9e-08	-1.58	3.7e-09	-0.00	2	119.75

Iterated Penalty Error

Sundance IteratedPenaltyCase2

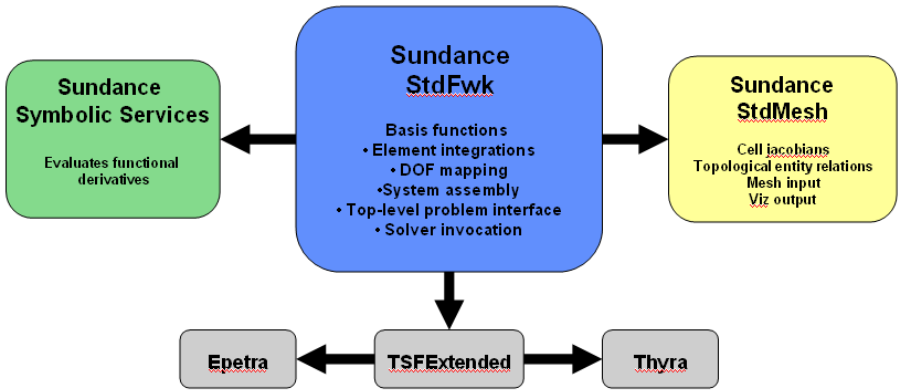
Ord	Mesh	Vel Err	Vp	Pre Err	Pp	Div	Dp	Iters	Run
4	4	2.2e-02	0.00	3.3e+00	0.00	4.0e-09	0.00	4	2.58
	8	8.4e-04	-4.69	2.8e-01	-3.55	7.4e-09	0.88	3	2.54
	16	2.2e-05	-5.28	1.5e-02	-4.27	4.2e-10	-4.14	3	5.51
	32	6.0e-07	-5.17	7.9e-04	-4.21	2.3e-09	2.42	2	15.27
5	4	3.3e-03	0.00	5.2e-01	0.00	4.0e-09	0.00	3	2.97
	8	6.3e-05	-5.72	2.3e-02	-4.51	6.2e-10	-2.72	3	4.26
	16	1.2e-06	-5.70	9.6e-04	-4.57	2.4e-09	1.96	2	8.08
	32	2.0e-08	-5.88	3.3e-05	-4.84	2.2e-09	-0.13	2	30.50
6	4	4.5e-04	0.00	8.8e-02	0.00	1.2e-09	0.00	3	5.06
	8	4.6e-06	-6.61	1.9e-03	-5.56	2.4e-09	1.00	2	5.93
	16	3.4e-08	-7.08	2.5e-05	-6.21	2.2e-09	-0.14	2	14.74
	32	4.2e-10	-6.34	3.6e-07	-6.14	2.2e-09	-0.00	2	56.60
7	4	6.5e-05	0.00	1.2e-02	0.00	3.9e-09	0.00	2	6.88
	8	2.8e-07	-7.84	1.2e-04	-6.64	2.2e-09	-0.84	2	10.60
	16	1.3e-09	-7.76	1.3e-06	-6.62	2.2e-09	-0.00	2	26.97
	32	7.7e-10	-0.76	1.5e-08	-6.38	2.2e-09	-0.00	2	130.80

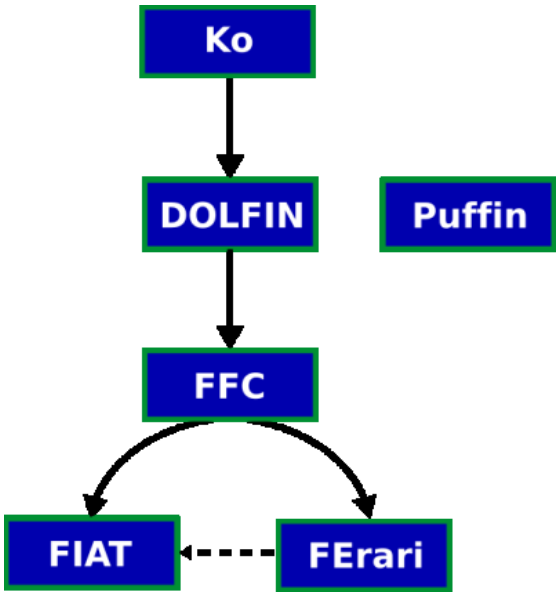
What about code complexity?

One bad estimate is lines of code:

- Dolfin + FFC $\sim$ 50K lines
- Sundance $\sim$ 100K lines
- DEAL $\sim$ 400K lines

Some Organization Charts





More Detailed Dependencies

An example of Dependencies for Sundance (not especially different from others)