RECURRENCE RELATIONS

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(Last updated: October 30, 2003)

1. Recursive Definitions

A definition such that the object defined occurs in the definition is called a *recursive definition.* For instance the *Fibonacci sequence*

$$0, 1, 1, 2, 3, 5, 8, 13, \dots$$

is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$).

Other examples:

- Recursive definition of *factorial*:
  (1) $0! = 1$
  (2) $n! = n \cdot (n-1)! \qquad (n \geq 1)$
- Recursive definition of *power*:
  (1) $a^0 = 1$
  (2) $a^n = a^{n-1}\, a \qquad (n \geq 1)$

In all these examples we have:

(1) A *Basis*, where the function is explicitly evaluated for one or more values of its argument.
(2) A *Recursive Step*, stating how to compute the function from its previous values.

2. Recurrence Relations

When we considerer a recursive definition as an equation to be solved we call it *recurrence relation.* Here we will focus on *kth-order linear* *recurrence relations*, which are of the form

$$C_0\, x_n + C_1\, x_{n-1} + C_2\, x_{n-2} + \cdots + C_k\, x_{n-k} = b_n\,,$$

where $C_0 \neq 0$. If $b_n = 0$ the recurrence relation is called *homogeneous*. Otherwise it is called *non-homogeneous*. The coefficients C_i may depend on n, but here we will assume that they are constant unless stated otherwise.

The basis of the recursive definition is also called *initial conditions* of the recurrence. So, for instance, in the recursive definition of the Fibonacci sequence, the recurrence is

$$F_n = F_{n-1} + F_{n-2}$$

or

$$F_n - F_{n-1} - F_{n-2} = 0\,,$$

and the initial conditions are

$$F_0 = 0,\; F_1 = 1\,.$$

1. **Solving Recurrence Relations.** A solution of a recurrence relation is a sequence x_n that verifies the recurrence.

An important property of *homogeneous* linear recurrences ($b_n = 0$) is that given two solutions x_n and y_n of the recurrence, any linear combination of them $z_n = rx_n + sy_n$, where r, s are constant, is also a solution of the same recurrence, because

$$\sum_{i=0}^{k} C_i(rx_{n-i} + sy_{n-i}) = r\sum_{i=0}^{k} C_i x_{n-i} + s\sum_{i=0}^{k} C_i y_{n-i} = r \cdot 0 + s \cdot 0 = 0\,.$$

For instance, the Fibonacci sequence $F_n = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$ and the Lucas sequence $L_n = 2, 1, 3, 4, 7, 11, \ldots$ verify the same recurrence $x_n = x_{n-1} + x_{n-2}$, so any linear combination of them $aF_n + bL_n$, for instance their sum $F_n + L_n = 2, 2, 4, 6, 10, 16, \ldots$, is also a solution of the same recurrence.

If the recurrence is non-homogeneous then we have that the difference of any two solutions is a solution of the homogeneous version of the recurrence, i.e., if $\sum_{i=0}^{k} C_i x_{n-i} = b_n$ and $\sum_{i=0}^{k} C_i y_{n-i} = b_n$ then obviously $z_n = x_n - y_n$ verifies:

$$\sum_{i=0}^{k} C_i z_{n-i} = \sum_{i=0}^{k} C_i x_{n-i} - \sum_{i=0}^{k} C_i y_{n-i} = b_n - b_n = 0\,.$$

Some recurrence relations can be solved by *iteration*, i.e., by using the recurrence repeatedly until obtaining a explicit close-form formula. For instance consider the following recurrence relation:

$$x_n = r\,x_{n-1} \quad (n > 0)\,; \qquad x_0 = A\,.$$

By using the recurrence repeatedly we get:

$$x_n = r\,x_{n-1} = r^2\,x_{n-2} = r^3\,x_{n-3} = \cdots = r^n\,x_0 = A\,r^n\,,$$

hence the solution is $x_n = A\,r^n$.

Next we look at two particular cases of recurrence relations, namely first and second order recurrence relations, and their solutions.

2. **First Order Recurrence Relations.** The homogeneous case can be written in the following way:

$$x_n = r\,x_{n-1} \quad (n > 0)\,; \qquad x_0 = A\,.$$

Its general solution is

$$x_n = A\,r^n\,,$$

which is a *geometric sequence* with *ratio* r.

The non-homogeneous case can be written in the following way:

$$x_n = r\,x_{n-1} + c_n \quad (n > 0)\,; \qquad x_0 = A\,.$$

Using the summation notation, its solution can be expressed like this:

$$x_n = A\,r^n + \sum_{k=1}^{n} c_k\,r^{n-k}\,.$$

We examine two particular cases. The first one is

$$x_n = r\,x_{n-1} + c \quad (n > 0); \qquad x_0 = A\,.$$

where c is a constant. The solution is

$$x_n = A\,r^n + c\sum_{k=1}^{n} r^{n-k} = A\,r^n + c\,\frac{r^n - 1}{r - 1} \qquad \text{if } r \neq 1\,,$$

and

$$x_n = A + c\,n \qquad \text{if } r = 1\,.$$

The second particular case is for $r = 1$ and $c_n = c + d\,n$, where c and d are constant (so c_n is an arithmetic sequence):

$$x_n = x_{n-1} + c + d\,n \quad (n > 0)\,; \qquad x_0 = A\,.$$

The solution is now

$$x_n = A + \sum_{k=1}^{n} (c + d\,k) = A + c\,n + \frac{d\,n\,(n+1)}{2}\,.$$

3. **Second Order Recurrence Relations.** Now we look at the recurrence relation

$$C_0\,x_n + C_1\,x_{n-1} + C_2\,x_{n-2} = 0\,.$$

First we will look for solutions of the form $x_n = c\,r^n$. By plugging in the equation we get:

$$C_0\,c\,r^n + C_1\,c\,r^{n-1} + C_2\,c\,r^{n-2} = 0\,,$$

hence r must be a solution of the following equation, called the *characteristic equation* of the recurrence:

$$C_0\,r^2 + C_1\,r + C_2 = 0\,.$$

Let r_1, r_2 be the two (in general complex) roots of the above equation. They are called *characteristic roots*. We distinguish three cases:

(1) *Distinct Real Roots.* In this case the general solution of the recurrence relation is

$$x_n = c_1\,r_1^n + c_2\,r_2^n\,,$$

where c_1, c_2 are arbitrary constants.

(2) *Double Real Root.* If $r_1 = r_2 = r$, the general solution of the recurrence relation is

$$x_n = c_1\,r^n + c_2\,n\,r^n\,,$$

where c_1, c_2 are arbitrary constants.

(3) *Complex Roots.* In this case the solution could be expressed in the same way as in the case of distinct real roots, but in order to avoid the use of complex numbers we write $r_1 = r\,e^{\alpha i}$, $r_2 = r\,e^{-\alpha i}$, $k_1 = c_1 + c_2$, $k_2 = (c_1 - c_2)\,i$, which yields:[1]

$$x_n = k_1\,r^n\,\cos n\alpha + k_2\,r^n\,\sin n\alpha\,.$$

Example: Find a closed-form formula for the Fibonacci sequence defined by:

$$F_{n+1} = F_n + F_{n-1} \quad (n > 0)\,; \qquad F_0 = 0,\ F_1 = 1\,.$$

[1]Reminder: $e^{\alpha i} = \cos\alpha + i\sin\alpha$.

Answer: The recurrence relation can be written

$$F_n - F_{n-1} - F_{n-2} = 0\,.$$

The characteristic equation is

$$r^2 - r - 1 = 0\,.$$

Its roots are:[2]

$$r_1 = \phi = \frac{1+\sqrt{5}}{2}; \qquad r_2 = -\phi^{-1} = \frac{1-\sqrt{5}}{2}\,.$$

They are distinct real roots, so the general solution for the recurrence is:

$$F_n = c_1\,\phi^n + c_2\,(-\phi^{-1})^n\,.$$

Using the initial conditions we get the value of the constants:

$$\left\{ \begin{array}{lll} (n=0) & c_1 + c_2 & = \; 0 \\ (n=1) & c_1\,\phi + c_2\,(-\phi^{-1}) & = \; 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{lcr} c_1 & = & 1/\sqrt{5} \\ c_2 & = & -1/\sqrt{5} \end{array} \right.$$

Hence:

$$F_n = \frac{1}{\sqrt{5}}\left\{\phi^n - (-\phi)^{-n}\right\}\,.$$

[2] $\phi = \frac{1+\sqrt{5}}{2}$ is the *Golden Ratio.*