FIBONACCI AND LUCAS NUMBERS AS TRIDIAGONAL MATRIX DETERMINANTS

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1. INTRODUCTION

There are many known connections between determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Strang [5, 6] presents a family of tridiagonal matrices given by:

$$M(n) = \begin{pmatrix} 3 & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{pmatrix}, \tag{1}$$

where $\boldsymbol{M}(n)$ is $n \times n$. It is easy to show by induction that the determinants $|\boldsymbol{M}(k)|$ are the Fibonacci numbers F_{2k+2}. Another example is the family of tridiagonal matrices given by:

$$\boldsymbol{H}(n) = \begin{pmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & i & 1 \end{pmatrix}, \tag{2}$$

described in [2] and [3] (also in [5], but with 1 and -1 on the off-diagonals, instead of i). The determinants $|\boldsymbol{H}(k)|$ are all the Fibonacci numbers F_k, starting with $k = 2$. In a similar family of matrices [1], the (1,1) element of $\boldsymbol{H}(n)$ is replaced with a 3. The determinants now generate the Lucas sequence L_k, starting with $k = 2$ (the Lucas sequence is defined by the second order recurrence $L_1 = 1, L_2 = 3, L_{k+1} = L_k + L_{k-1}, k \geq 2$).

In this article, we extend these results to construct families of tridiagonal matrices whose determinants generate any arbitrary linear subsequence $F_{\alpha k+\beta}$ or $L_{\alpha k+\beta}, k = 1, 2, \ldots$ of the Fibonacci or Lucas numbers. We then choose a specific linear subsequence of the Fibonacci numbers and use it to derive the following factorization:

$$F_{2mn} = F_{2m} \prod_{k=1}^{n-1} \left(L_{2m} - 2\cos\frac{\pi k}{n} \right). \tag{3}$$

This factorization is a generalization of one of the factorizations presented in [3]:

$$F_{2n} = \prod_{k=1}^{n-1} \left(3 - 2\cos\frac{\pi k}{n}\right).$$

In order to develop these results, we must first present a theorem describing the sequence of determinants for a general tridiagonal matrix. Let $A(k)$ be a family of tridiagonal matrices, where

$$A(k) = \begin{pmatrix} a_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & \\ & a_{3,2} & a_{3,3} & \ddots & \\ & & \ddots & \ddots & a_{k-1,k} \\ & & & a_{k,k-1} & a_{k,k} \end{pmatrix}.$$

Theorem 1: The determinants $|A(k)|$ can be described by the following recurrence relation:

$$|A(1)| = a_{1,1}$$
$$|A(2)| = a_{2,2}a_{1,1} - a_{2,1}a_{1,2}$$
$$|A(k)| = a_{k,k}|A(k-1)| - a_{k,k-1}a_{k-1,k}|A(k-2)|, \quad k \geq 3.$$

Proof: The cases $k = 1$ and $k = 2$ are clear. Now

$$|A(k)| = \det \begin{pmatrix} a_{1,1} & a_{1,2} & & & & \\ a_{2,1} & a_{2,2} & \ddots & & & \\ & \ddots & \ddots & a_{k-3,k-2} & & \\ & & a_{k-2,k-3} & a_{k-2,k-2} & a_{k-2,k-1} & \\ & & & a_{k-1,k-2} & a_{k-1,k-1} & a_{k-1,k} \\ & & & & a_{k,k-1} & a_{k,k} \end{pmatrix}.$$

By cofactor expansion on the last column and then the last row,

$$|A(k)| = a_{k,k}|A(k-1)| - a_{k-1,k} \det \begin{pmatrix} a_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & \ddots & & \\ & \ddots & \ddots & a_{k-3,k-2} & \\ & & a_{k-2,k-3} & a_{k-2,k-2} & a_{k-2,k-1} \\ & & & 0 & a_{k,k-1} \end{pmatrix}$$

$$= a_{k,k}|A(k-1)| - a_{k-1,k}a_{k,k-1}|A(k-2)|. \quad \square$$

2. FIBONACCI SUBSEQUENCES

Using Theorem 1, we can generalize the families of tridiagonal matrices given by (1) and (2) to construct, for every linear subsequence of Fibonacci numbers, a family of tridiagonal matrices whose successive determinants are given by that subsequence.

Theorem 2: The symmetric tridiagonal family of matrices $M_{\alpha,\beta}(k), k = 1, 2, \ldots$ whose elements are given by:

$$m_{1,1} = F_{\alpha+\beta},\ m_{2,2} = \lceil \frac{F_{2\alpha+\beta}}{F_{\alpha+\beta}} \rceil$$

$$m_{j,j} = L_\alpha,\ 3 \le j \le k,$$

$$m_{1,2} = m_{2,1} = \sqrt{m_{2,2}F_{\alpha+\beta} - F_{2\alpha+\beta}}$$

$$m_{j,j+1} = m_{j+1,j} = \sqrt{(-1)^\alpha}, 2 \le j < k,$$

with $\alpha \in Z^+$ and $\beta \in N$, has successive determinants $|M_{\alpha,\beta}(k)| = F_{\alpha k+\beta}$.

In order to prove Theorem 2, we must first present the following lemma:

Lemma 1: $F_{k+n} = L_n F_k + (-1)^{n+1} F_{k-n}$ for $n \ge 1$.

Proof: We use the second principle of finite induction on n to prove this lemma:

Let $n = 1$. Then the lemma yields $F_{k+1} = F_k + F_{k-1}$, which defines the Fibonacci sequence. Now assume that $F_{k+n} = L_n F_k + (-1)^{n+1} F_{k-n}$ for $n \le N$. Then

$$\begin{aligned} F_{k+N+1} &= F_{k+N} + F_{k+N-1} \\ &= L_N F_k + (-1)^{N+1} F_{k-N} + L_{N-1} F_k + (-1)^N F_{k-N+1} \\ &= (L_N + L_{N-1}) F_k + (-1)^{N+2} (F_{k-N+1} - F_{k-N}) \\ &= L_{N+1} F_k + (-1)^{N+2} F_{k-(N+1)}. \quad \square \end{aligned}$$

Now, using Theorem 1 and Lemma 1, we can prove Theorem 2.

Proof of Theorem 2: We use the second principle of finite induction on k to prove this theorem:

$$|M_{\alpha,\beta}(1)| = \det F_{\alpha+\beta} = F_{\alpha+\beta}$$

$$|M_{\alpha,\beta}(2)| = \det \begin{pmatrix} F_{\alpha+\beta} & \sqrt{m_{2,2}F_{\alpha+\beta} - F_{2\alpha+\beta}} \\ \sqrt{m_{2,2}F_{\alpha+\beta} - F_{2\alpha+\beta}} & \lceil \frac{F_{2\alpha+\beta}}{F_{\alpha+\beta}} \rceil \end{pmatrix} = F_{2\alpha+\beta}.$$

Now assume that $|M_{\alpha,\beta}(k)| = F_{\alpha k+\beta}$ for $1 \le k \le N$. Then by Theorem 1,

$$\begin{aligned} |M_{\alpha,\beta}(k+1)| &= m_{k,k}|M_{\alpha,\beta}(k)| - m_{k,k-1}m_{k-1,k}|M_{\alpha,\beta}(k-1)| \\ &= L_\alpha |M_{\alpha,\beta}(k)| - (-1)^\alpha |M_{\alpha,\beta}(k-1)| \\ &= L_\alpha F_{\alpha k+\beta} + (-1)^{\alpha+1} F_{\alpha(k-1)+\beta} \\ &= F_{\alpha+\alpha k+\beta} \quad \text{(by Lemma 1)} \\ &= F_{\alpha(k+1)+\beta} \quad \square \end{aligned}$$

Another family of matrices that satisfies Theorem 2 can be found by choosing the negative root for all of the super-diagonal and sub-diagonal entries. With Theorem 2, we can now construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of the Fibonacci numbers. For example, the determinants of:

$$\begin{pmatrix} 1 & 0 & & & & \\ 0 & 8 & 1 & & & \\ & 1 & 7 & 1 & & \\ & & 1 & 7 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 7 \end{pmatrix}, \begin{pmatrix} 8 & \sqrt{6} & & & & \\ \sqrt{6} & 5 & i & & & \\ & i & 4 & i & & \\ & & i & 4 & \ddots & \\ & & & \ddots & \ddots & i \\ & & & & i & 4 \end{pmatrix},$$

$$\text{and} \begin{pmatrix} 13 & -\sqrt{5} & & & & \\ -\sqrt{5} & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & -1 & 3 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 3 \end{pmatrix}$$

are given by the Fibonacci subsequences F_{4k-2}, F_{3k+3} and F_{2k+5}.

3. LUCAS SUBSEQUENCES

We can also generalize the families of tridiagonal matrices given by (1) and (2) to show a similar result for linear subsequences of Lucas numbers. We state this result as the following theorem:

Theorem 3: The symmetric tridiagonal family of matrices $T_{\alpha,\beta}(k), k = 1, 2, \ldots$ whose elements are given by:

$$t_{1,1} = L_{\alpha+\beta}, t_{2,2} = \lceil \frac{L_{2\alpha+\beta}}{L_{\alpha+\beta}} \rceil$$

$$t_{j,j} = L_{\alpha}, 3 \leq j \leq k,$$

$$t_{1,2} = t_{2,1} = \sqrt{t_{2,2}L_{\alpha+\beta} - L_{2\alpha+\beta}}$$

$$t_{j,j+1} = t_{j+1,j} = \sqrt{(-1)^{\alpha}}, 2 \leq j < k,$$

with $\alpha \in Z^+$ and $\beta \in N$, has successive determinants $|T_{\alpha,\beta}(k)| = L_{\alpha k+\beta}$.

Again we begin with a lemma; its proof imitates the proof of Lemma 1.

Lemma 2: $L_{k+n} = L_n L_k + (-1)^{n+1} L_{k-n}$ for $n \geq 1$.

Proof of Theorem 3: We use induction:

$$|T_{\alpha,\beta}(1)| = \det L_{\alpha+\beta} = L_{\alpha+\beta}.$$

$$|T_{\alpha,\beta}(2)| = \det \begin{pmatrix} L_{\alpha+\beta} & \sqrt{m_{2,2}L_{\alpha+\beta} - L_{2\alpha+\beta}} \\ \sqrt{m_{2,2}L_{\alpha+\beta} - L_{2\alpha+\beta}} & \lceil \frac{L_{2\alpha+\beta}}{L_{\alpha+\beta}} \rceil \end{pmatrix} = L_{2\alpha+\beta}.$$

Now assume that $|T_{\alpha,\beta}(k)| = L_{\alpha k+\beta}$ for $1 \le k \le N$. Then by Theorem 1,

$$\begin{aligned}|T_{\alpha,\beta}(k+1)| &= t_{k,k}|T_{\alpha,\beta}(k)| - t_{k,k-1}t_{k-1,k}|T_{\alpha,\beta}(k-1)| \\ &= L_\alpha|T_{\alpha,\beta}(k)| - (-1)^\alpha|T_{\alpha,\beta}(k-1)| \\ &= L_\alpha L_{\alpha k+\beta} + (-1)^{\alpha+1}L_{\alpha(k-1)+\beta} \\ &= L_{\alpha+\alpha k+\beta} \quad \text{(by Lemma 2)} \\ &= L_{\alpha(k+1)+\beta} \quad \square\end{aligned}$$

With Theorem 3, we can now construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of the Lucas numbers. For example, the determinants of:

$$\begin{pmatrix} 3 & 0 & & & & \\ 0 & 6 & -1 & & & \\ & -1 & 7 & -1 & & \\ & & -1 & 7 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 7 \end{pmatrix}, \begin{pmatrix} 18 & \sqrt{14} & & & & \\ \sqrt{14} & 5 & i & & & \\ & i & 4 & i & & \\ & & i & 4 & \ddots & \\ & & & \ddots & \ddots & i \\ & & & & i & 4 \end{pmatrix},$$

$$\text{and} \begin{pmatrix} 29 & \sqrt{11} & & & & \\ \sqrt{11} & 3 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 3 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 3 \end{pmatrix}$$

are given by the Lucas subsequences L_{4k-2}, L_{3k+3} and L_{2k+5}.

4. A FACTORIZATION OF THE FIBONACCI NUMBERS

In order to derive the factorization (3) given by $F_{2mn} = F_{2m}\prod_{k=1}^{n-1}\left(L_{2m} - 2\cos\frac{\pi k}{n}\right)$, we consider the symmetric tridiagonal matrices:

$$B_m(n) = \begin{pmatrix} L_{2m}F_{2m} & \sqrt{F_{2m}} & & & & \\ \sqrt{F_{2m}} & L_{2m} & 1 & & & \\ & 1 & L_{2m} & 1 & & \\ & & 1 & L_{2m} & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & L_{2m} \end{pmatrix}.$$

By Lemma 1, $F_{4m} = L_{2m}F_{2m}$, and $\lceil F_{6m}/F_{4m}\rceil = \lceil L_{2m} - (F_{2m}/F_{4m})\rceil = L_{2m}$. Furthermore, $\sqrt{\lceil F_{6m}/F_{4m}\rceil F_{4m} - F_{6m}} = \sqrt{L_{2m}F_{4m} - F_{6m}} = \sqrt{F_{2m}}$, so $B_m(n) = M_{2m,2m}(n)$ is a specific instance of the tridiagonal family of matrices described in Theorem 2. Therefore, by Theorem 2, $|B_m(n)| = F_{2m(n+1)}$.

By using the property of determinants that $|AB| = |A \| B|$, and by defining $\boldsymbol{e}_j$ to be the j^{th} column of $n \times n$ identity matrix $\boldsymbol{I}$, we have $|B_m(n)| = F_{2m}|C_m(n)|$, where:

$$C_m(n) = \left(\boldsymbol{I} + \left(\frac{1}{F_{2m}} - 1\right)\boldsymbol{e}_1\boldsymbol{e}_1^T\right)B_m(n).$$

The determinant is the product of the eigenvalues. Therefore, let $\lambda_k, k = 1, 2, \ldots, n$ be the eigenvalues of $C_m(n)$ (with associated eigenvectors $\boldsymbol{x}_k$), so $|C_m(n)| = \prod_{k=1}^n \lambda_k$. Letting $G_m(n) = C_m(n) - L_{2m}\boldsymbol{I}$, we see that $G_m(n)\boldsymbol{x}_k = C_m(n)\boldsymbol{x}_k - L_{2m}\boldsymbol{I}\boldsymbol{x}_k = \lambda_k\boldsymbol{x}_k - L_{2m}\boldsymbol{x}_k = (\lambda_k - L_{2m})\boldsymbol{x}_k$. Then $\gamma_k = \lambda_k - L_{2m}$ are the eigenvalues of $G_m(n)$.

An eigenvalue γ of $G_m(n)$ is a root of the characteristic polynomial $|G_m(n) - \gamma\boldsymbol{I}| = 0$. Note that $|G_m(n) - \gamma\boldsymbol{I}| = |\left(\boldsymbol{I} + \left(\sqrt{F_{2m}} - 1\right)\boldsymbol{e}_1\boldsymbol{e}_1^T\right)(G_m(n) - \gamma\boldsymbol{I})\left(\boldsymbol{I} + \left(1/\sqrt{F_{2m}} - 1\right)\boldsymbol{e}_1\boldsymbol{e}_1^T\right)|$, so γ is also a root of the polynomial:

$$\begin{vmatrix} -\gamma & 1 & & & & \\ 1 & -\gamma & 1 & & & \\ & 1 & -\gamma & 1 & & \\ & & 1 & -\gamma & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & -\gamma \end{vmatrix} = 0.$$

This polynomial is a transformed Chebyshev polynomial of the second kind [4], with roots $\gamma_k = -2\cos\frac{\pi k}{n+1}$. Therefore,

$$F_{2m(n+1)} = |B_m(n)| = F_{2m}|C_m(n)| = F_{2m}\prod_{k=1}^n \lambda_k = F_{2m}\prod_{k=1}^n \left(L_{2m} - 2\cos\frac{\pi k}{n+1}\right).$$

(3) follows by a simple change of variables.

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