# Enumerative Formulae for Some Functions on Finite Sets

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#### Abstract

In the first section of this paper we give a matrix interpretation of well-known inclusion-exclusion principle. A result concerning (0,1)-matrices is also proved.

In the next two section we apply these results to count the number of specified functions from a finite set into an another finite set.

In second section we first find a formula for the number of functions whose images contain a fixed subset. This gives a combinatorial interpretation for finite differences of the function  $n^m$ . We also obtain an extension of the well-known relation for Stirling numbers of the second kind. Two combinatorial identities are also proved. The first concerns the power function, and the second involves Stirling numbers of the second kind. We then prove two results for the number of function which specifically map not particular elements, but particular subsets. Several special cases of these formulae are stated.

In the third section we investigate the set of injective maps from a finite set into a finite set. We first prove a formula for the number of permutations which change all elements of a fixed set. This formula gives a combinatorial meanings for finite differences for factorial function. As a special case, the formula for the number of derangements is obtained. Then two combinatorial identities are proved. The first one deals with factorials, and the second with derangements.

Then we consider a more general situation and obtain some extensions of derangements.

More then 200 sequences in well-known Sloan's Encyclopedia of Integer sequences are generated by the functions stated in the paper.

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# 1 Inclusion-Exclusion Principle

In this section we shall introduce a matrix interpretation of well-known Inclusion-Exclusion method.

For the proof of the main theorem we need the following simple result:

$$\sum_{I} (-1)^{|I|} = 0, \tag{1}$$

where I run over all subsets of a finite set (empty set included). This may be easily proved by induction or using Binomial theorem.

Let A be an  $m \times n$  rectangular matrix filled with elements from a set  $\Omega$ . By the i-column of A we shall mean each column of A that is equal to  $[c_1, c_2, \ldots, c_m]^T$ , where  $c_1, c_2, \ldots, c_m$  of  $\Omega$  are given. We shall denote the number of i-columns of A by  $\nu_A(c)$  or simply by  $\nu(c)$ .

For  $I = \{i_1, i_2, \dots, i_k\} \subset [m]$ , by A(I) will be denoted the maximal number of columns j of A such that

$$a_{ij} \neq c_i$$
, (for all  $i \in I$ ).

We also define

$$A(\emptyset) = n.$$

**Theorem 1.1.** The number  $\nu(c)$  of i-columns of A is equal

$$\nu(c) = \sum_{I} (-1)^{|I|} A(I), \tag{2}$$

where summation is taken over all subsets I of [m].

**Proof.** Theorem will be proved by the standard combinatorial method, counting the contribution of each column of A in the sum on the right side of (2). Write this formula in the form

$$\nu(c) = n + \sum_{I \neq \emptyset} (-1)^{|I|} A(I).$$

Let  $c = [c_1, \ldots, c_m]^T$  be the i-column of A. It is clear that its contribution to the number A(I),  $(I \neq \emptyset)$  is equal zero. Hence, its contribution to the right side of the preceding equation is equal 1.

If b is not an i-column then there is  $i_0 \in [m]$  such that  $b_{i_0} \neq c_{i_0}$ . Let  $I_0$  be the set of all such indices. The contribution of b to the number A(I) is equal 1 if and only if  $I \subseteq I_0$ . Hence, its contribution to the whole sum is

$$1 + \sum_{I \subset I_0, \ I \neq \emptyset} (-1)^i = 0,$$

according to (1).

**Theorem 1.2.** If, in the condition of Theorem 1.1, the number A(I) depends not of I, but only on |I| then

$$\nu(c) = \sum_{i=0}^{m} (-1)^{i} {m \choose i} A(i), \tag{3}$$

where |I| = i.

**Proof.** We only need to collect terms in (2) in which I is fixed.

Let c be the i- column of a (0,1) matrix A. Take  $I_0 \subseteq [m], |I_0| = k$  such that

$$c_i = \begin{cases} 1 & i \in I_0 \\ 0 & i \notin I_0 \end{cases} \tag{4}$$

Then A(I) is equal to the number of columns of A having 0's in the rows whose indices lie in  $I \cap I_0$ , and 1's in the rows whose indices belong to the set  $I \setminus I_0$ . **Theorem 1.3.** Given a (0,1) matrix A of the format  $m \times n$ . Suppose that  $c = [c_1, \ldots, c_n]^T$  is the i-column of A, and  $I_0 \subseteq [m]$ ,  $|I_0| = k$  such that

$$c_i = \begin{cases} 1 & i \in I_0 \\ 0 & i \notin I_0 \end{cases}$$

If A(I) does not depend on elements of  $I \cap I_0$ ,  $I \setminus I_0$ , but only on its numbers  $|I \cap I_0| = i_1$ ,  $|I \setminus I_0| = i_2$ , then we have

$$\nu(c) = \sum_{i_1=0}^{k} \sum_{i_2=0}^{m-k} (-1)^{i_1+i_2} \binom{k}{i_1} \binom{m-k}{i_2} A(i_1, i_2).$$
 (5)

**Proof.** Write arbitrary  $I \subseteq [m]$  in the form

$$I = (I \cap I_0) \cup (I \setminus I_0).$$

If we collect terms on the right side of (2) in which  $I \cap I_0$ , and  $I \setminus I_0$  are fixed, we obtain (5).

## 2 Functions from a finite set into a finite set

Let  $X = \{x_1, \ldots, x_m\}$ ,  $Y = \{y_1, \ldots, y_n\}$  be two finite sets. Label all functions  $f: X \to Y$  by the numbers  $1, 2, \ldots, n^m$  arbitrary and form an  $n \times n^m$  matrix A in the following way:

$$a_{ij} = \begin{cases} 1, & y_i \notin Im(f_j) \\ 0, & y_i \in Im(f_j) \end{cases}.$$

Take  $I_0 \subset [n]$ ,  $|I_0| = k$ , and consider the submatrix B of A consisting of those rows of A whose indices belong to  $I_0$ .

Suppose that i-column c of B consists of 0's. Then  $\nu(c)$  is equal to the number of functions which images contain the set  $Y_0 = \{y_i : i \in I_0\}$ . For  $I \subset I_0$ 

the number B(I) is equal to the number of functions whose images do not intersect  $\{y_i : i \in I\}$ . There are  $(n - |I|)^m$  such functions. It follows that the formula (3) may be applied to obtain

**Theorem 2.1.** If  $k \leq n$ , m are arbitrary nonnegative integers then the number F(m, n, k) the functions from an m-set into an n-set, whose images contain fixed k elements of Y, is

$$F(m,n,k) = \sum_{i=0}^{k} (-1)^{i} {k \choose i} (n-i)^{m}.$$
 (6)

Note first that in the trivial case m = n = k = 0 holds J(0,0,0) = 0. In the following corollary we give the basic properties of the function F(m,n,k). Corollary 2.1.

1° If S(m,n) is the number of surjections of an [m]-set onto an [n]-set then

$$F(m, n, n) = \mathcal{S}(m, n).$$

 $2^{\circ}$  Among  $n^m$  numbers from 0 to  $n^m-1$  written in base n there are F(m,n,k) of them in which k fixed ciphers occur.

 $3^{\circ}$ 

$$F(m, n+k, k) = \Delta^k n^m,$$

where  $\Delta^k$  is the finite difference of the order k.

**Proof.** 1° is clear true. 2° holds according to the fact that for  $Y = \{0, 1, \ldots, n-1\}$  functions  $f : [m] \to Y$  can be regarded as numbers with m digits written in base n. The equation 3° is in fact the basic formula for the finite difference of the order kof the function  $n^m$ .

The function F(m, n, k) generates several sequences which appear in [1].

In the case  $m \ge n$  we may express F(m, n, k) in terms of number of surjections S(m, n) of an m- set onto an n-set, that is, in terms of Stirling numbers of the second kind.

Namely, it holds

$$F(m,n,k) = \sum_{Z \subseteq Y \setminus Y_0} S(m,|I_0 \cup Z|) = \sum_{i=0}^{n-k} {n-k \choose i} S(m,k+i).$$

We thus have.

**Corollary 2.2.** For positive integers m, n and nonnegative integer k,  $(k \le n)$  holds

$$F(m,n,k) = \sum_{i=k}^{n} \binom{n-k}{i-k} \mathcal{S}(m,i).$$

Taking k = 0 in the preceding equation, according to the fact that  $F(m, n, 0) = n^m$ , a well-known formula for Stirling numbers S(m, n) of the second is obtained.

Corollary 2.3. If m, n are positive integers then

$$n^{m} = \sum_{i=0}^{n} n(n-1)\cdots(n-i+1)S(m,i).$$

We shall now count the number of i-columns of the matrix B consisting of 1's. This number is equal to the number of functions which map X into  $Y \setminus Y_0$ . There is obviously  $(n-k)^m$  such functions.

On the other A(I) is equal to the number of functions which images contain  $\{y_i : i \in I\}$ . This number may be obtained from (6). Replacing n - k by n we derive

**Corollary 2.4.** For positive integers m, n and nonnegative integer k,  $(k \le n)$  holds

$$n^{m} = \sum_{i_{1}=0}^{k} \sum_{i_{2}=0}^{i_{1}} (-1)^{i_{1}+i_{2}} \binom{k}{i_{1}} \binom{i_{1}}{i_{2}} (n+k-i_{2})^{m}.$$

Suppose now that c is arbitrary i-column of A, and denote  $J_0 = \{i \in [n] : c_i = 0\}$ ,  $|J_0| = l$ . The number  $\nu(c)$  is equal to the number of functions which images contain  $\{y_i : i \in J_0\}$ , and that do not meet  $\{y_i : i \in [m] \setminus J_0\}$ . Thus  $\nu(c) = \mathcal{S}(m, l)$ . The number A(I) corresponds to the functions which images do not intersect  $\{y_i : i \in I \cap J_0\}$ , and contain  $\{y_i : i \in I \setminus J_0\}$ . It follows that these functions send elements of X into  $Y \setminus \{y_i : i \in I \cap J_0\}$ . Since its images must contain the set  $\{y_i : i \in I \setminus J_0\}$  it follows from (6) that the number of such function is equal

$$\sum_{i_1=0}^{|I\setminus I_0|} (-1)^{i_3} \binom{|I\setminus I_0|}{i_3} (n-|I\cap I_0|-i_3)^m.$$

Since this expression depends only on  $|I \cap I_0|$ , and  $|I \setminus I_0|$  we may apply (5) to obtain

**Corollary 2.5.** For a positive integer m, and arbitrary nonnegative integers l and n holds.

$$S(m,l) = \sum_{i_1=0}^{l} \sum_{i_2=0}^{n} \sum_{i_3=0}^{i_2} (-1)^{i_1+i_2+i_3} \binom{l}{i_1} \binom{n}{i_2} \binom{i_2}{i_3} (n+l-i_1-i_3)^m.$$

In the following results we count functions that map specifically not particular elements, but particular subsets.

**Theorem 2.2.** Suppose that  $X_1, X_2, ..., X_n$  are blocks of a finite set X, and  $Y_1, Y_2, ..., Y_n$  are subsets of a finite set Y. The number  $N_1$  of function  $f: X \to Y$  such that

$$f(X_i) \not\subseteq Y_i, (i = 1, 2, ..., n)$$

is

$$N_1 = \sum_{I \subseteq [n]} (-1)^{|I|} |Y|^{|X \setminus \bigcup_{i \in I} X_i|} \cdot \prod_{i \in I} |Y_i|^{|X_i|}.$$
 (7)

**Proof.** Form an  $n \times |Y|^{|X|}$  matrix A such that  $a_{ij} = 0$  if for the function  $f_j : X \to Y$  labelled by j holds  $f_j(X_i) \not\subseteq Y_i$ , and  $a_{ij} = 1$  otherwise.

According to Theorem 1.1, the number A(I) is the the number of functions  $f: X \to Y$  such that  $f(X_i) \subseteq Y_i$ ,  $(i \in I)$ . This number is clearly equal to

$$|Y|^{|X\setminus \cup_{i\in I}X_i|}\cdot \prod_{i\in I}|Y_i|^{|X_i|}.$$

In a similar way we obtain the following:

**Theorem 2.3.** Suppose that  $X_1, X_2, ..., X_n$  are blocks of a finite set X and  $Y_1, Y_2, ..., Y_n$  are subsets of a finite set Y. The number  $N_2$  of the functions  $f: X \to Y$  such that  $f(X_i) \neq Y_i$ , (i = 1, 2, ..., n) is

$$N_2 = \sum_{I \subset [n]} (-1)^{|I|} |Y|^{|X \setminus \bigcup_{i \in I} X_i|} \cdot \prod_{i \in I} |Y_i|! S(X_i, Y_i).$$

Depending on the number of elements of  $X_1, \ldots, X_n; Y_1, \ldots, Y_n$  it is possible to obtain a number of different enumerative formulae. Consider first the simplest case when each  $X_1, \ldots, X_n; Y_1, \ldots, Y_n$  consists of one element. Then

$$A(I) = |Y|^{|X|-|I|},$$

so that Theorem 1.2. may be applied. We thus obtain the following: **Corollary 2.6.** Given distinct  $x_1, \ldots, x_n$  in X and arbitrary  $y_1, \ldots, y_n$  in Y, then the number  $N_3$  of functions  $f: X \to Y$  such that

$$f(x_i) \neq y_i, (i = 1, 2, \dots, n),$$

is equal

$$N_3 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} |Y|^{|X|-i}.$$

According to Newton binomial formula we have

$$N_3 = |Y|^{|X|-n} (|Y|-1)^n.$$

Suppose that

$$|X_1| = |X_2| = \dots = |X_n| = 1, |Y_1| = |Y_2| = \dots = |Y_k| = 2.$$

Then

$$A(I) = 2^{i}|Y|^{|X|-|I|}.$$

We may again apply Theorem 1.2 to obtain

**Corollary 2.7.** Given distinct  $x_1, \ldots, x_n$  in X and arbitrary 2-subsets  $Y_1, \ldots, Y_n$  of Y, then the number  $N_4$  of functions  $f: X \to Y$  such that

$$f(x_i) \notin Y_i, (i = 1, 2, ..., n),$$

is equal

$$N_4 = \sum_{i=0}^{n} (-2)^i \binom{|Y|}{i} |Y|^{|X|-i}.$$

From Newton binomial theorem we get

$$N_4 = |Y|^{|X|-n} (|Y| - 2)^n.$$

If 
$$|X_1| = \cdots = |X_k| = 2$$
;  $|Y_1| = \cdots |Y_k| = 1$  then

$$A(I) = |Y|^{|X|-2|I|},$$

which yields:

**Corollary 2.8.** Suppose that  $X_1, \ldots, X_n$  are 2-blocks of X and  $y_1, \ldots, y_n$  arbitrary elements in Y, then the number  $N_5$  of functions  $f: X \to Y$  such that

$$f(X_i) \neq \{y_i\}, (i = 1, 2, \dots, n)$$

is equal

$$N_5 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} |Y|^{|X|-2i}.$$

Newton binomial theorem yields

$$N_5 = |Y|^{|X|-2n} (|Y|^2 - 1)^n.$$

Take finally the case  $|X_i| = |Y_i| = 2$ , (i = 1, 2, ..., k). We have now

$$A(I) = 4^{|I|} \cdot |Y|^{|X|-2|I|}.$$

We thus obtain the following consequence of Theorem 2.3.

**Corollary 2.9.** Let  $X_1, \ldots, X_n$  be 2-blocks of X, and  $Y_1, \ldots, Y_n$  arbitrary 2-subsets of Y, then the number  $N_5$  of functions  $f: X \to Y$  such that

$$f(X_i) \neq Y_i, (i = 1, 2, ..., n)$$

is equal

$$N_6 = \sum_{i=0}^{n} (-4)^i \binom{n}{i} |Y|^{|X|-2i}.$$

Newton binomial theorem yields

$$N_6 = |Y|^{|X|-2n} (|Y|^2 - 4)^n.$$

# 3 Injections from a finite set into a finite set

Consider now the set of all permutation of a finite set  $X = \{x_1, x_2, \dots, x_n\}$ . Label all permutations with numbers  $1, 2, \dots, n!$  arbitrary and form an  $n \times n!$ -matrix A such that  $a_{ij} = 0$  if  $\pi_j(x_i) \neq x_i$ , and  $a_{ij} = 1$  otherwise, where  $\pi_j$  is the permutation labelled by j.

Consider the submatrix B of A consisting of rows of A whose indices belong to the set  $I_0 \subseteq [n]$ ,  $|I_0| = k$ . The i-columns of B consisting of 0's are made of permutations that change all elements of  $x_i$ ,  $(i \in I_0)$ . We denote this number by  $P(n, I_0)$ . If  $I \subseteq I_0$  then the number B(I) corresponds to those permutations which left all elements  $x_i$ ,  $(i \in I)$  fixed. There are (n - |I|)! such permutations. We thus may apply (3) to obtain

$$P(n, I_0) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n-i)!.$$

It is clear that  $P(n, I_0)$  depends only of the number k of elements of  $I_0$ . Thus, if we denote by P(n, k) the number of permutations that change k elements of X we get

**Theorem 3.1.** If  $k, n, (k \le n)$  are nonnegative integers then

$$P(n,k) = \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} (n-i)!.$$
 (8)

Note that in trivial case n = k = 0 holds J(n, 0) = n!. Taking n + k instead of n to obtain

Corollary 3.1. If n, k are positive integers then

$$P(n+k,k) = \Delta^k n!$$

where  $\Delta^k$  is the finite difference of the order k.

**Proof.** The formula follows immediately from the basic formula for finite difference of n!..

Taking specially k = n we obtain.

Corollary 3.2. For the number D(n) derangements of  $S_n$  holds

$$D(n) = P(n, n).$$

The i-columns of B consisting of 1's correspond to those permutations which remain all elements  $x_i$ ,  $(i \in I_0)$  fixed. There are (n-k)! such permutations. In this case the number B(I) is obtained of permutations that change all elements of I. Applying Theorem 1.2. we obtain.

Corollary 3.3. If n is positive and  $k \leq n$  nonnegative integer then

$$(n-k)! = \sum_{i_1=0}^{k} \sum_{i_2=0}^{i_1} (-1)^{i_1+i_2} \binom{k}{i_1} \binom{i_1}{i_2} (n-i_2)!. \tag{9}$$

The following identity is a special case of the preceding, when k = n. Corollary 3.4. For positive integer n holds

$$1 = \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1} (-1)^{i_1+i_2} \binom{n}{i_1} \binom{i_1}{i_2} (n-i_2)!.$$

Suppose finally that c is arbitrary i-column of A, and  $I_0 = \{i \in [n] : c_i = 1\}$ ,  $|I_0| = k$ . The number  $\nu(c)$  is equal to the number of permutations of  $S_n$  which remain fixed only elements  $x_i$ ,  $(i \in I_0)$ , which yields that  $\nu(c) = D(n-k)$ .

The number A(I) is equal to the number of permutation which change elements  $x_i$ ,  $(i \in I \cap I_0)$ , while the elements  $x_i$ ,  $(i \in I \setminus I_0)$  remain fixed. It follows from (8) that

$$A(I) = \sum_{i_3=0}^{|I\cap I_0|} (-1)^{i_3} \binom{|I\cap I_0|}{i_3} (n-|I\setminus I_0|-i_3)!.$$

We see that the right side of this equation depends only on  $|I \cap I_0|$  and  $|I \setminus I_0|$ . Using Theorem 1.3. we obtain the following identity.

Corollary 3.5. If  $k, n, (k \le n)$  are positive integer then

$$D(n) = \sum_{i_1=0}^{k} \sum_{i_2=0}^{n} \sum_{i_3=0}^{i_1} (-1)^{i_1+i_2+i_3} \binom{k}{i_1} \binom{n}{i_2} \binom{i_1}{i_3} (n+k-i_2-i_3)!.$$

We shall now consider injective functions from a finite set X into a finite set Y,  $|Y| \ge |X|$ . We start with the following:

**Theorem 3.2.** Let  $X_1, X_2, \ldots, X_n$  be blocks in X and  $Y_1, Y_2, \ldots, Y_n$  blocks in Y such that

$$|X_i| = |Y_i|, (i = 1, 2, \dots, n).$$

Then the number I(m, n, k) of injection  $f: X \to Y$  such that

$$f(X_i) \neq Y_i, (i = 1, 2, ..., n)$$

is equal

$$I(m,n,k) = \sum_{I \subset [n]} (-1)^{|I|} (|Y| - \sum_{i \in I} |Y_i|)^{(|X| - \sum_{i \in I} |X_i|)} \cdot \prod_{i \in I} |Y_i|^{(|X_i|)}.$$

Here, as usual, we denote by  $r^{(s)}$  the falling factorials, that is,

$$r^{(s)} = r(r-1)\cdots(r-s+1).$$

**Proof.** In this case we have

$$A(I) = (|Y| - |\cup_{i \in I} Y_i|)^{(|X| - \sum_{i \in I} |X_i|)} \cdot \prod_{i \in I} |Y_i|^{(|X_i|)}!,$$

so that theorem follows from Theorem 1.1.

We shall also state some particular cases of this theorem. Suppose first that

$$|X_i| = |Y_i| = 1, (i = 1, ..., n).$$

For the number A(I) in this case we have

$$A(I) = (|Y| - |I|)^{(|X| - |I|)},$$

so that Theorem 1.2 may be applied to obtain

**Corollary 3.5.** For mutually disjoint  $x_1, \ldots, x_n$  in X and mutually disjoint  $y_1, \ldots, y_n$  in Y, the number  $I_1(m, n, k)$  of injections  $f: X \to Y$  such that

$$f(x_i) \neq y_i, (i = 1, 2, ..., n)$$

is equal

$$I_1(n, X, Y) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (|Y| - i)^{(|X| - i)}.$$

Since obviously holds D(n) = I(n, n, n), where D(n) is the number of derangements of n elements, this function is an extension of derangements.

As a special case we also have the following generalization of derangements. Corollary 3.6. If  $X_1, X_2, ..., X_n$  is a partition of [kn] such that

$$|X_i| = k, (i = 1, 2, \dots, n),$$

then the number D(n,k) of permutations f of [kn] such that  $f(X_i) \neq X_i$ , (i = 1, 2, ..., n) is equal

$$D(n,k) = \sum_{i=0}^{n} (-1)^{i} (k!)^{i} (nk - ik)!.$$

For k = 1 we obtain the standard formula for derangements.

#### References

[1] N. J. A. Sloane: The On-Line Encyclopedia of Integer Sequences, njas@research.att.com