Catalan lattices and realizers of triangulations

Olivier Bernardi - Centre de Recerca Matemàtica

Joint work with Nicolas Bonichon

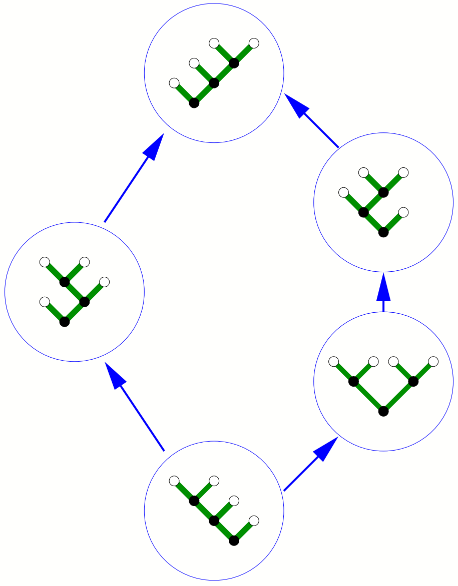
CRM, April 2007

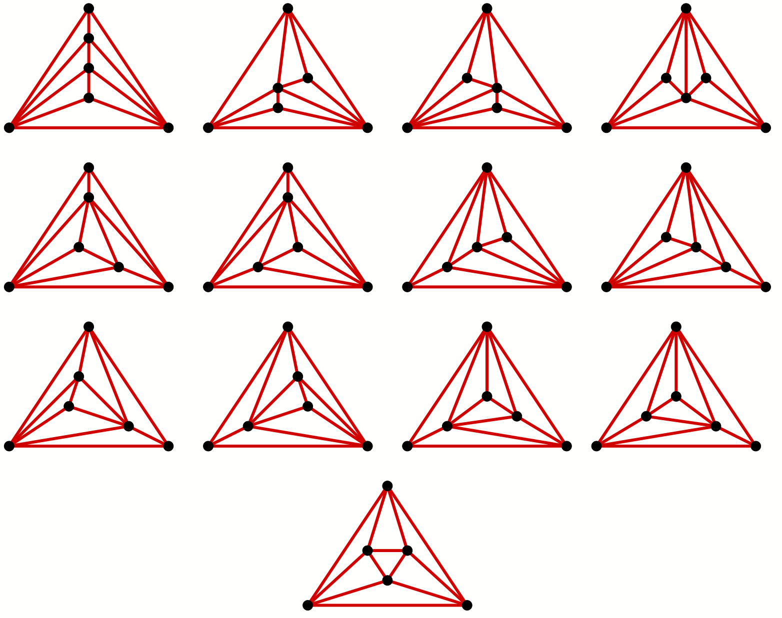
Question [Chapoton] :

Why does the number of intervals in the Tamari lattice on binary trees of size n equals the number of triangulations of size n ?

Question [Chapoton] :

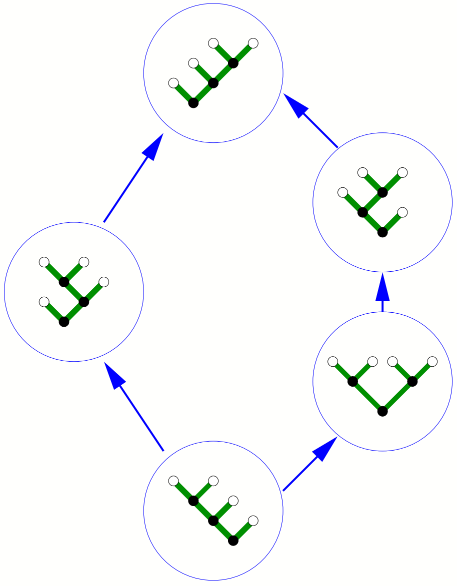
Why does the number of intervals in the Tamari lattice on binary trees of size n equals the number of triangulations of size n ?

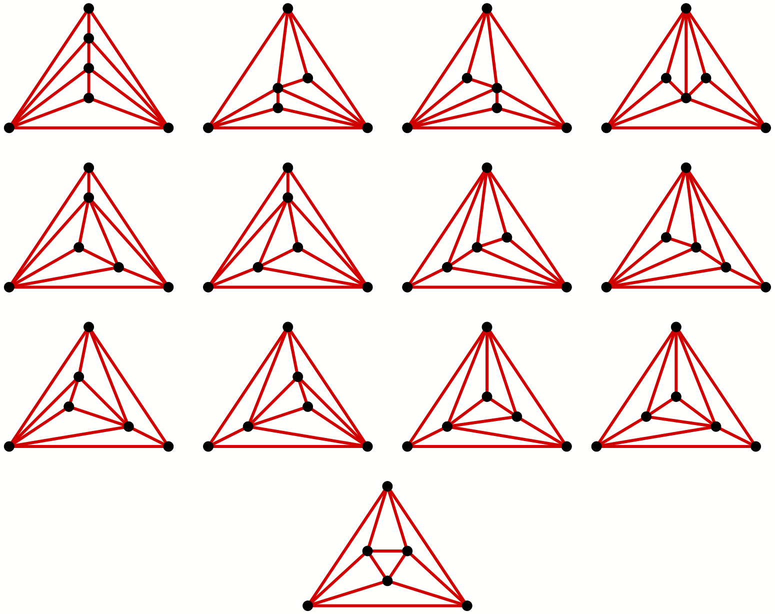




Question [Chapoton] :

Why does the number of intervals in the Tamari lattice on binary trees of size n equals the number of triangulations of size n ?





$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

[Chapoton 06] **[Tutte 62, Poulalhon & Schaeffer 03]**

Broader picture :

Stanley intervals $\Longleftrightarrow$ Realizers of triangulations.

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

Tamari intervals $\Longleftrightarrow$ Triangulations.

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

Kreweras intervals $\Longleftrightarrow$ Stack triangulations.

$$\frac{1}{2n+1}\binom{3n}{n}$$

Catalan lattices and realizers

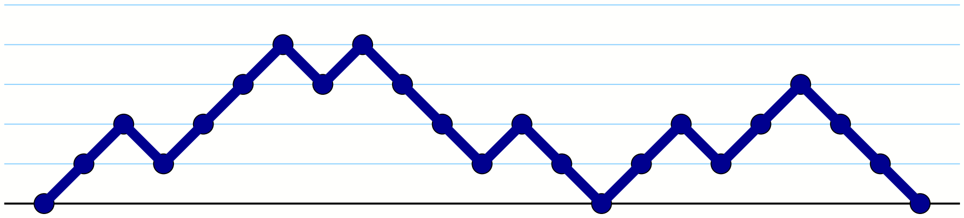
- Catalan lattices : Stanley, Tamari, Kreweras.
- Triangulations and realizers.
- Bijections:

Stanley intervals	$\Longleftrightarrow$	Realizers.
Tamari intervals	$\Longleftrightarrow$	Minimal realizers.
Kreweras intervals	$\Longleftrightarrow$	Minimal and maximal realizers.

Catalan lattices

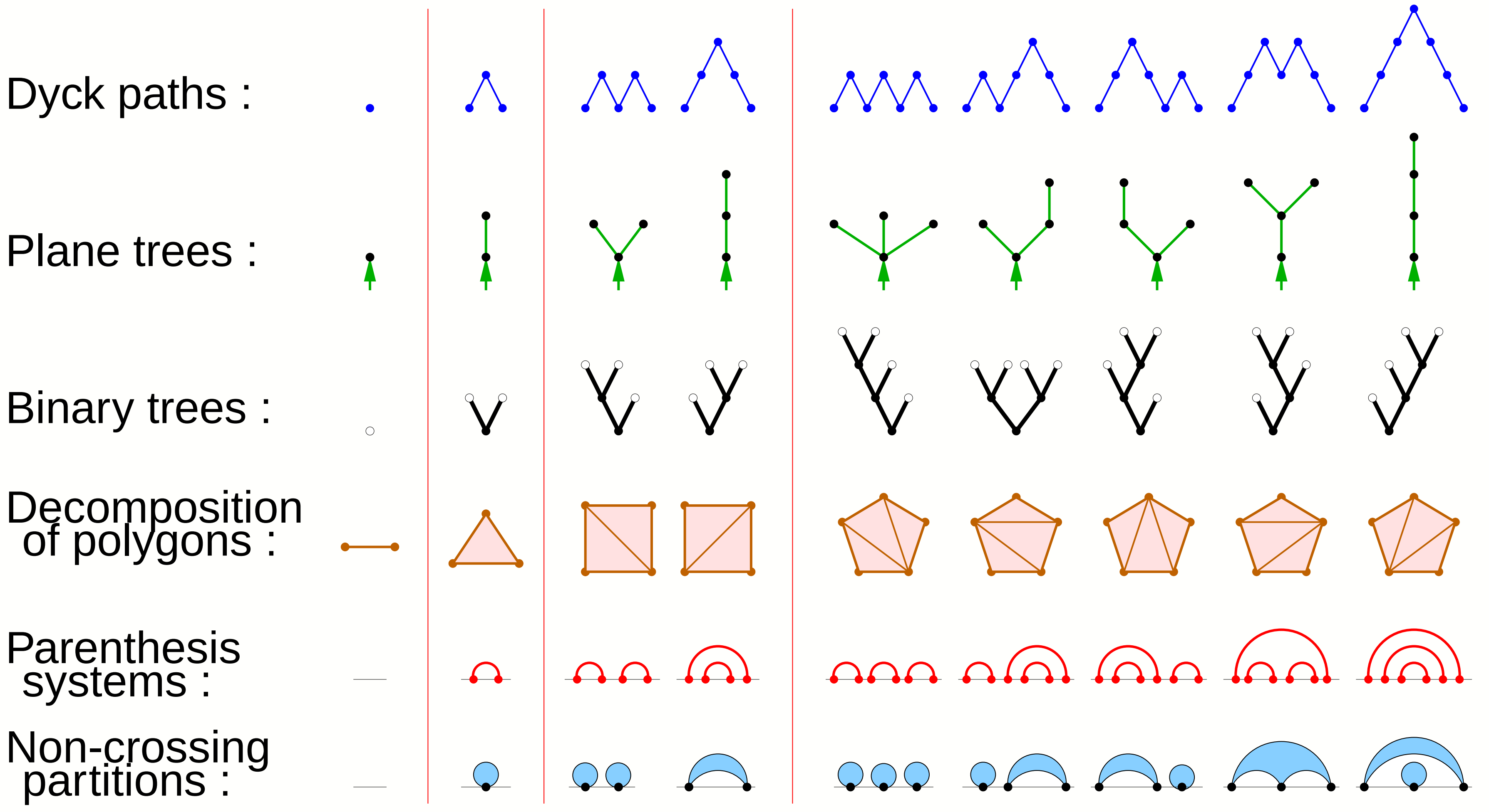
Dyck paths

A Dyck path is made of +1, -1 steps, starts from 0, remains non-negative and ends at 0.



There are $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ Dyck paths of size n (length $2n$).

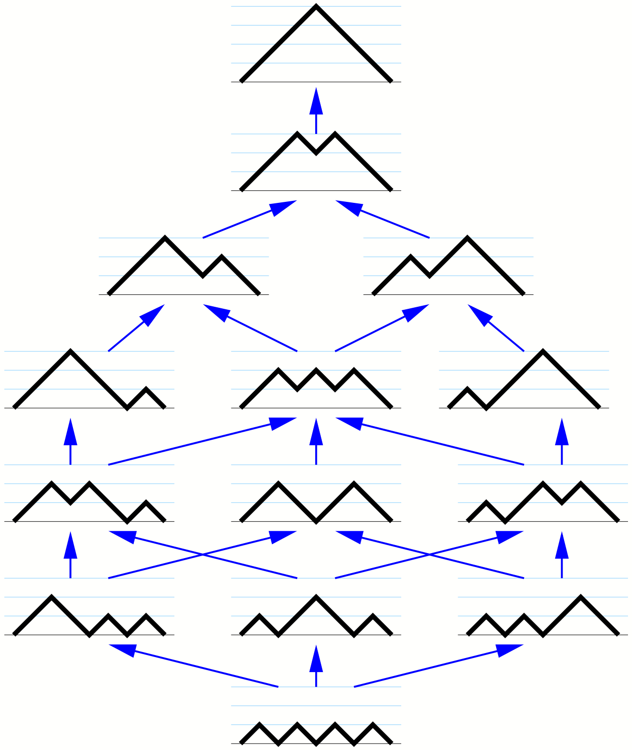
Catalan objects



Stanley lattice

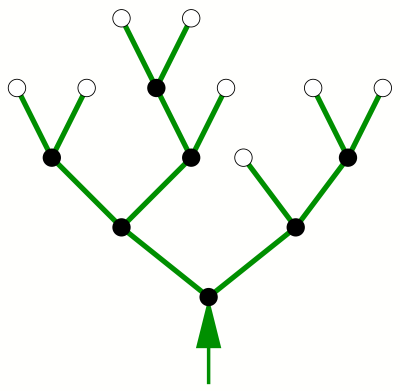
The relation of being above defines the Stanley lattice on the set of Dyck paths of size n.

Hasse Diagram $n = 4$:



Tamari lattice

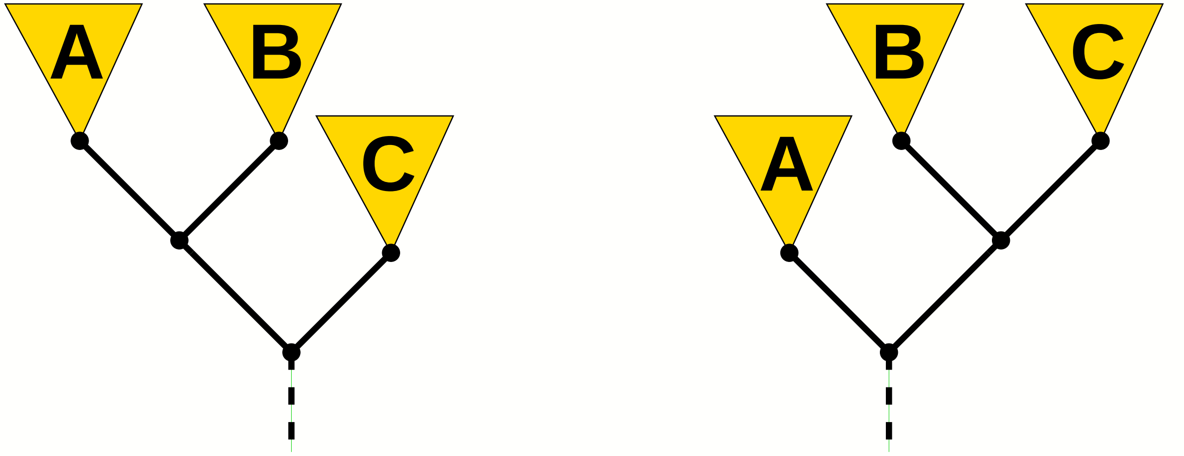
The Tamari lattice is defined on the set of binary trees with n nodes.



Tamari lattice

The Tamari lattice is defined on the set of binary trees with n nodes.

The covering relation corresponds to right-rotation.

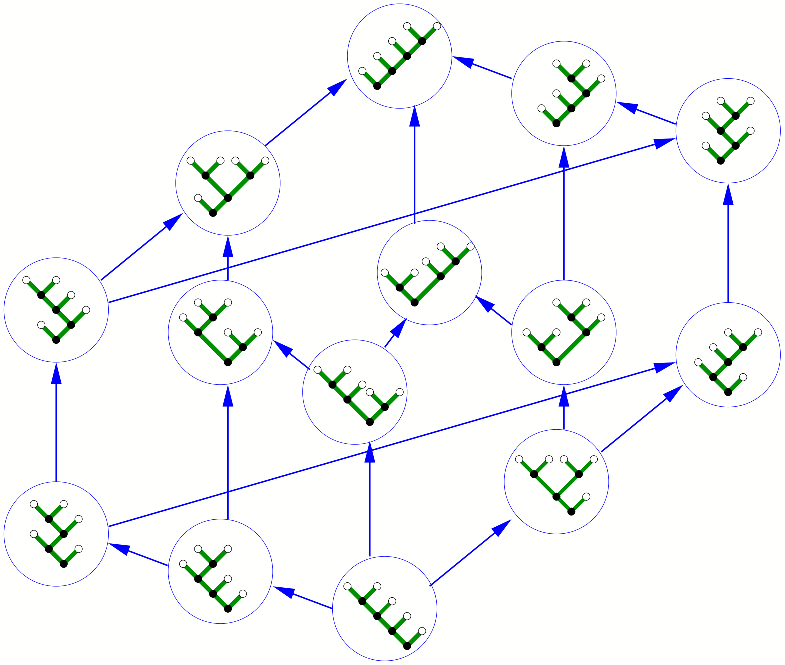


Tamari lattice

The Tamari lattice is defined on the set of binary trees with n nodes.

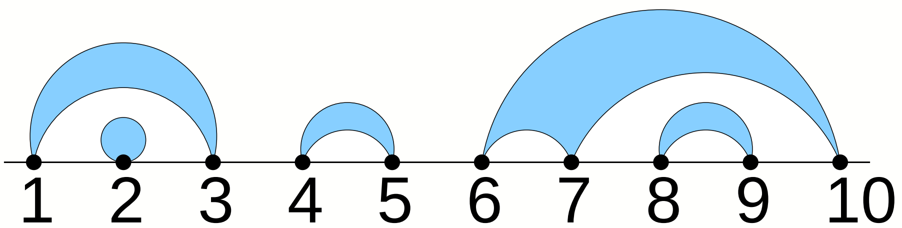
The covering relation corresponds to right-rotation.

Hasse Diagram $n = 4$:



Kreweras lattice

The Kreweras lattice is defined on the set of non-crossing partitions of $\{1, \ldots, n\}$.

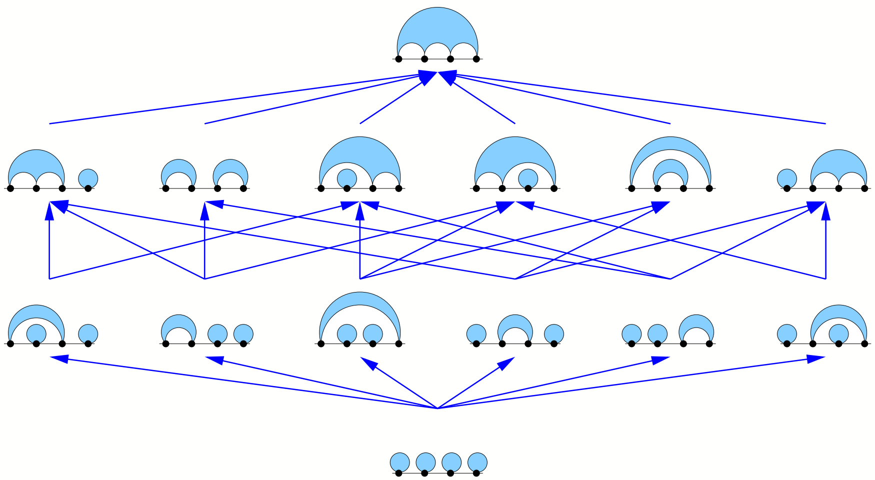


Kreweras lattice

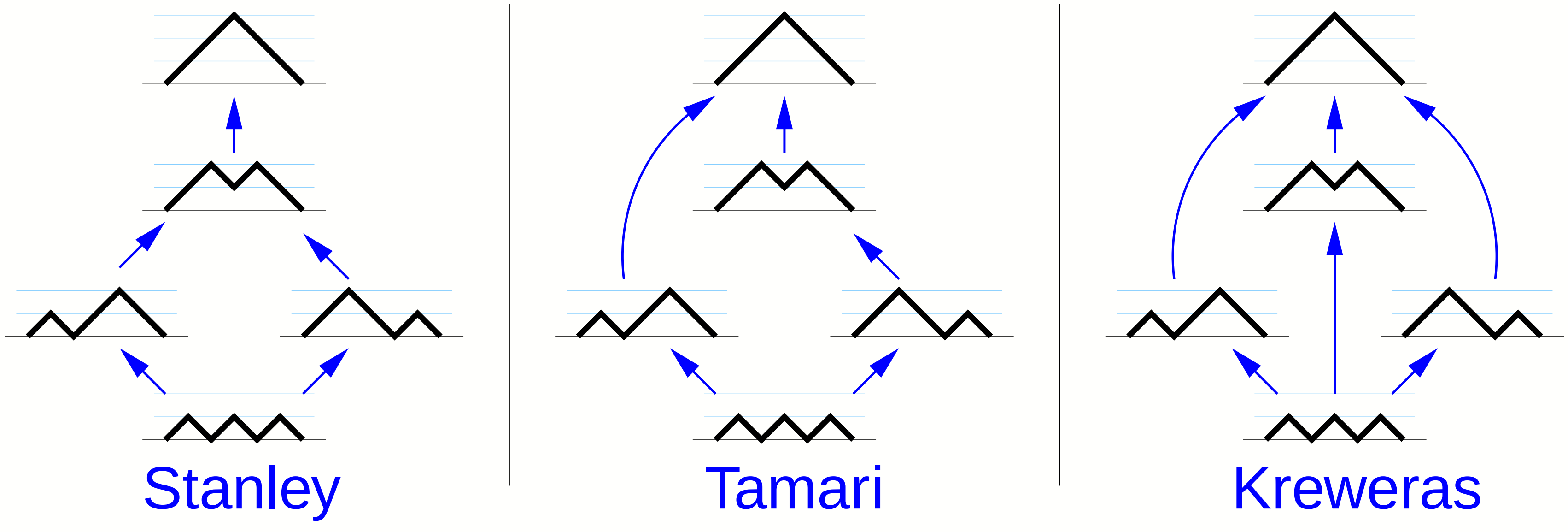
The Kreweras lattice is defined on the set of non-crossing partitions of $\{1, \ldots, n\}$.

Kreweras relation corresponds to refinement.:

Hasse Diagram $n = 4$**:**



Stanley, Tamari and Kreweras

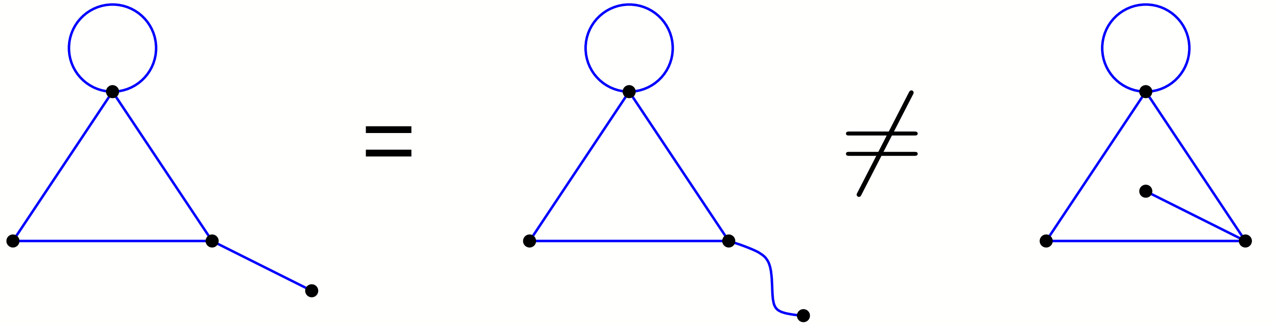


[Knuth 06] The Stanley lattice is an extension of the Tamari lattice which is an extension of the Kreweras lattice.

Triangulations and realizers

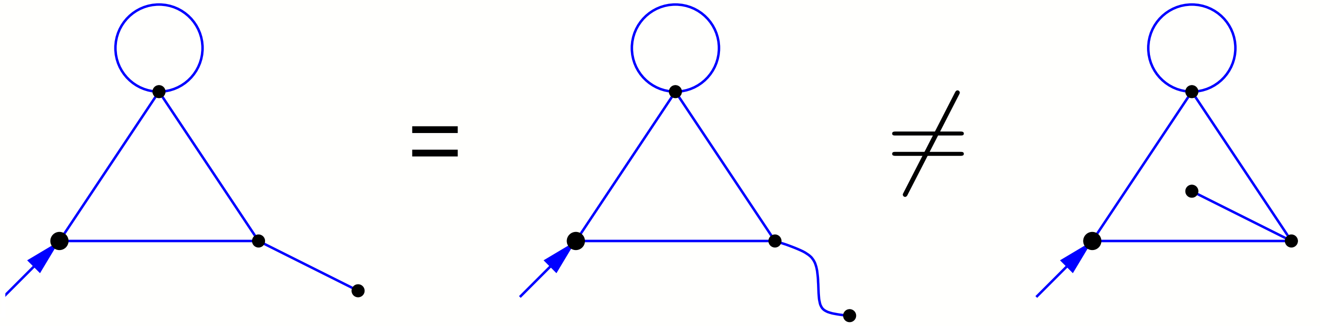
Maps

A map is a connected planar graph properly embedded in the sphere.
The map is considered up to homeomorphism.



Maps

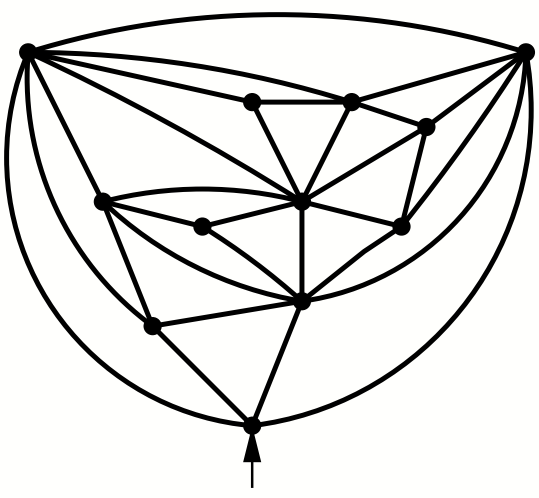
A map is a connected planar graph properly embedded in the sphere.
The map is considered up to homeomorphism.



A map is rooted if a half-edge is distinguished as the root.

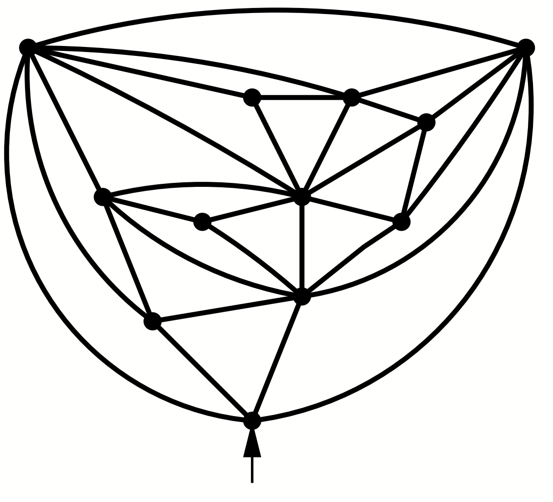
Triangulations

A triangulation is a 3-connected map in which every face has degree 3.



Triangulations

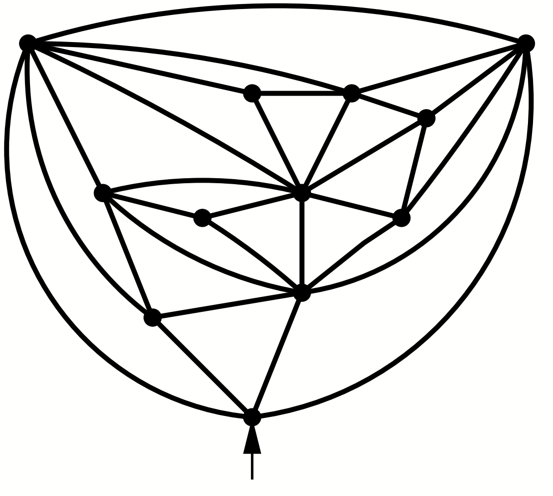
A triangulation is a 3-connected map in which every face has degree 3.



A triangulation of size n has n internal vertices, $3n$ internal edges, $2n+1$ internal triangles.

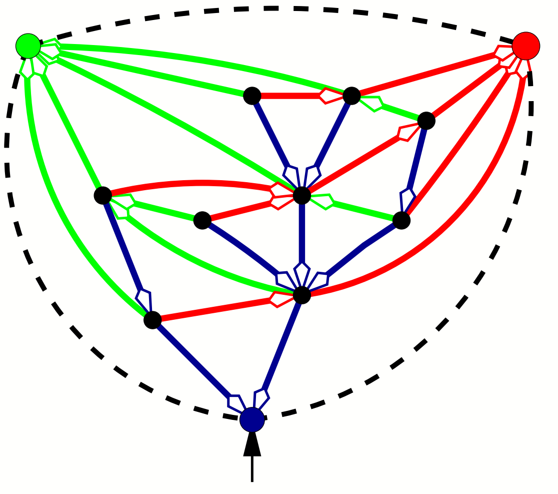
Realizers [Schnyder 89,90]

Example:



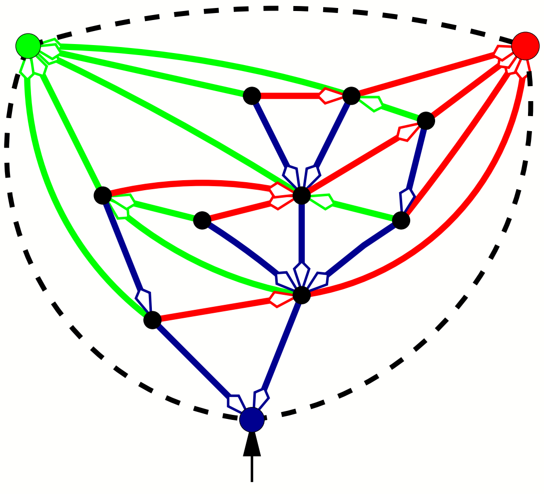
Realizers [Schnyder 89,90]

Example:

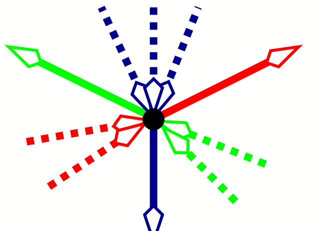


Realizers [Schnyder 89,90]

Example:



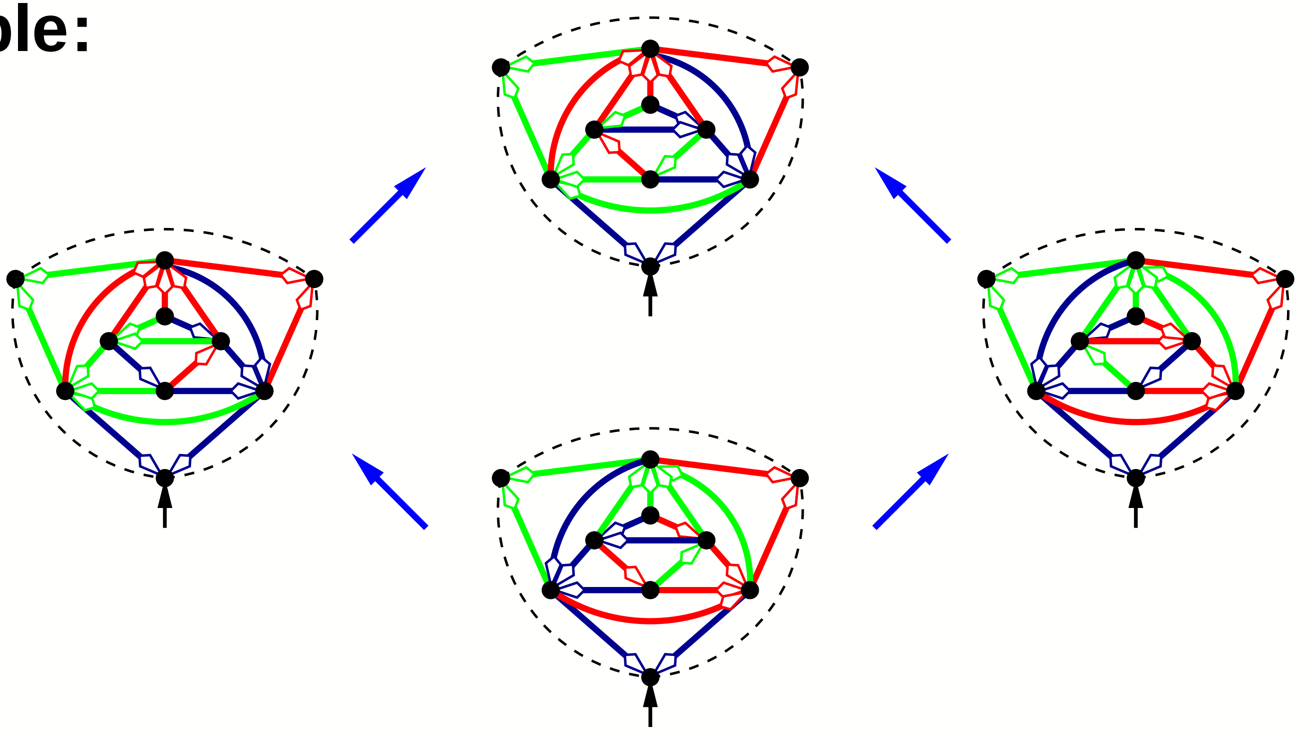
A realizer is a partition of the internal edges in 3 trees satisfying the Schnyder condition:



Prop [Schnyder 89], [Propp 93, Ossona de Mendez 94]:
For any triangulation, the set of realizers is non-empty and can be endowed with a lattice structure.

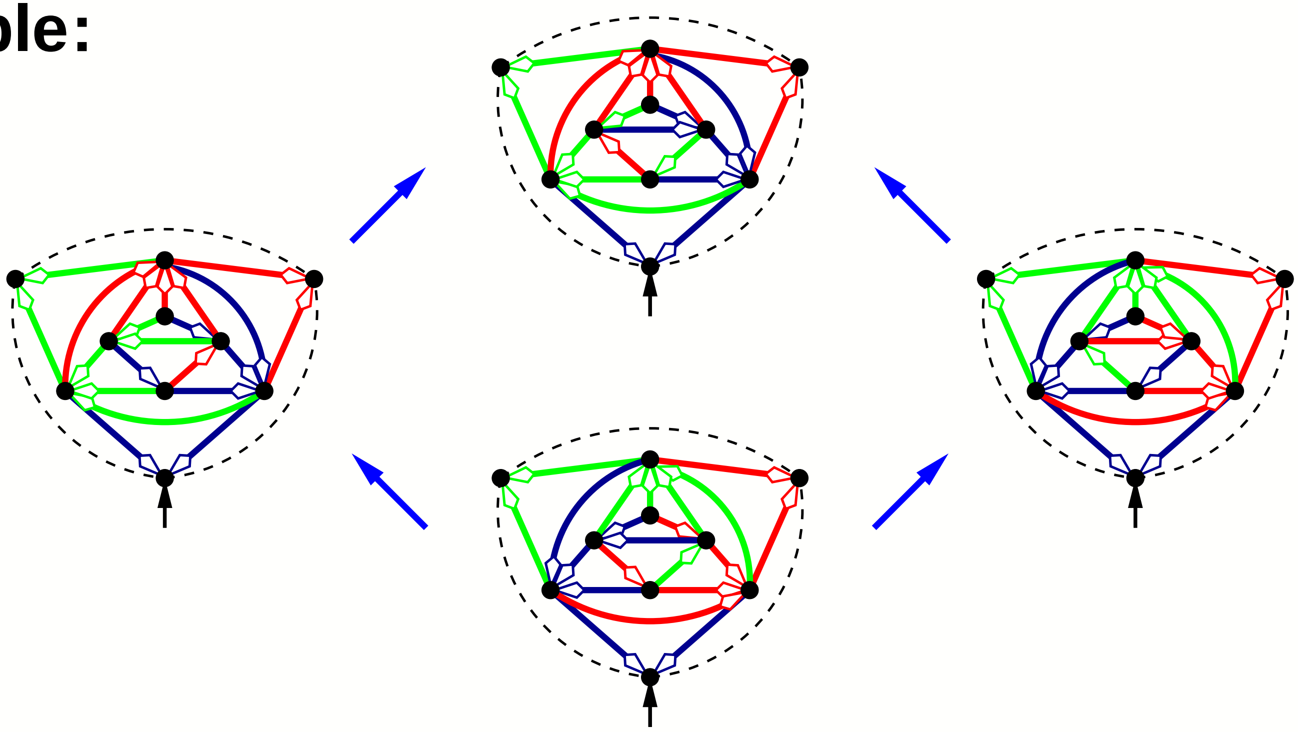
Prop [Schnyder 89], [Propp 93, Ossona de Mendez 94]:
For any triangulation, the set of realizers is non-empty and can be endowed with a lattice structure.

Example:

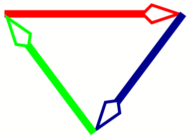


Prop [Schnyder 89], [Propp 93, Ossona de Mendez 94]:
For any triangulation, the set of realizers is non-empty and can be endowed with a lattice structure.

Example:

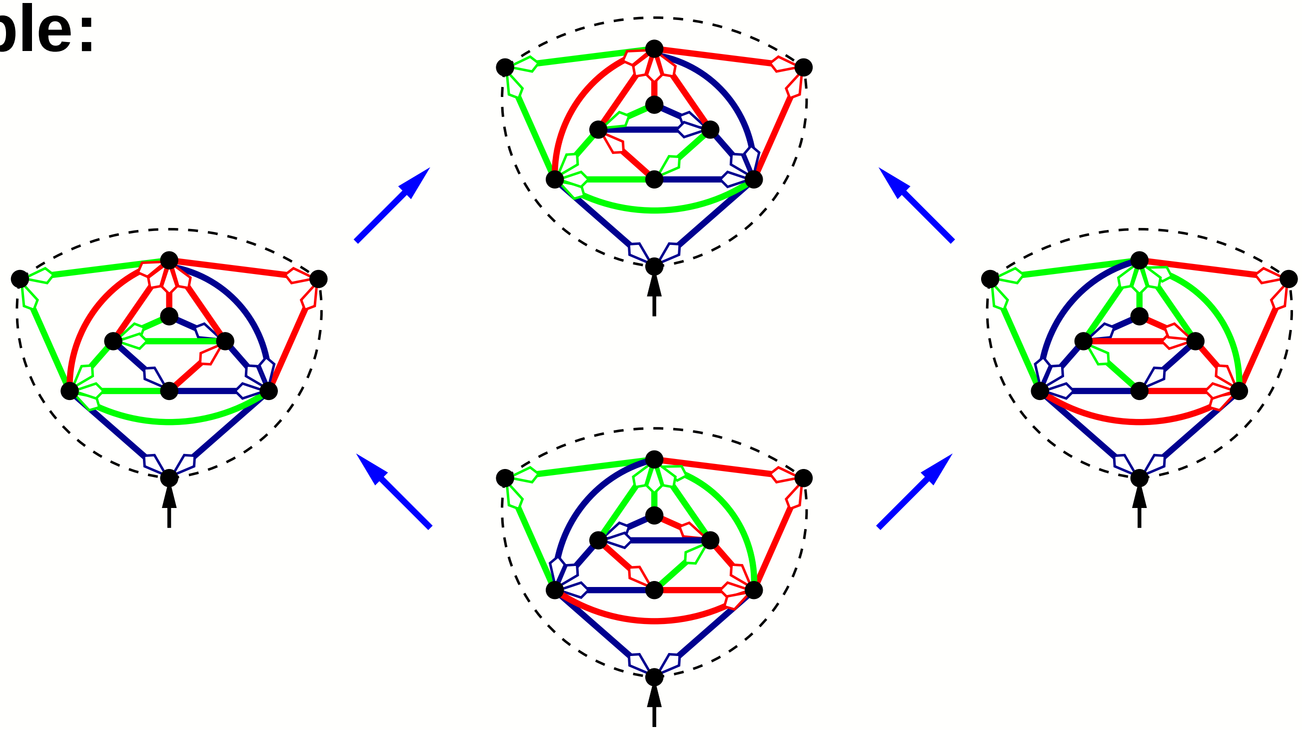


The minimal element for this lattice is the realizer containing no clockwise triangle.

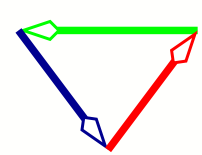


Prop [Schnyder 89], [Propp 93, Ossona de Mendez 94]:
For any triangulation, the set of realizers is non-empty and can be endowed with a lattice structure.

Example:

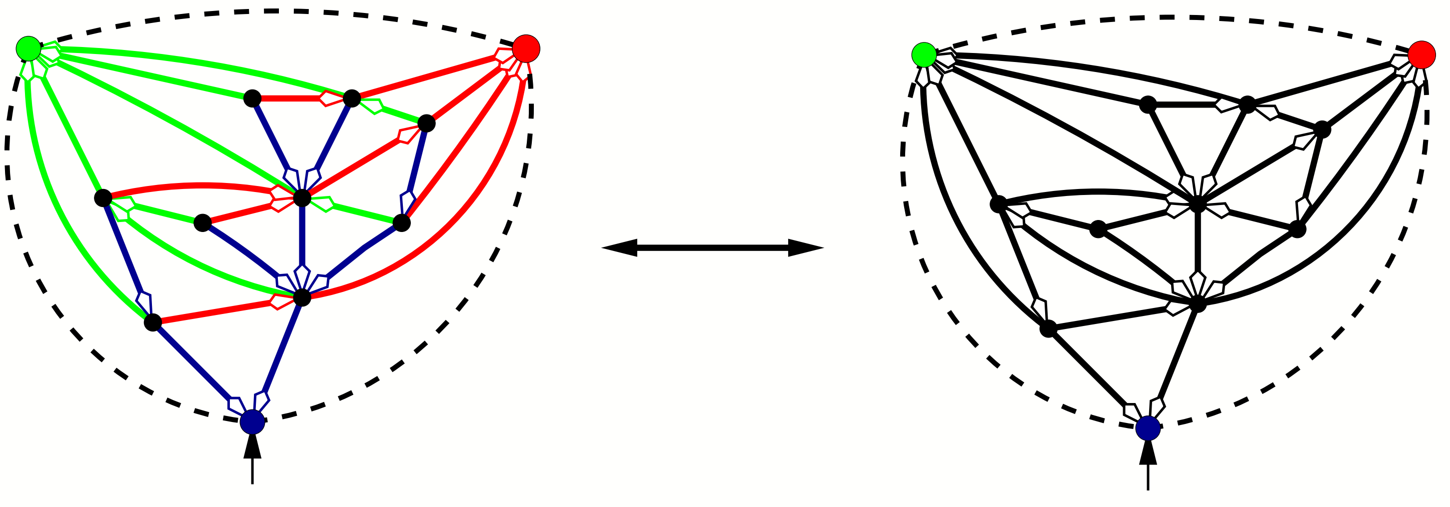


The maximal element for this lattice is the realizer containing no counterclockwise triangle.



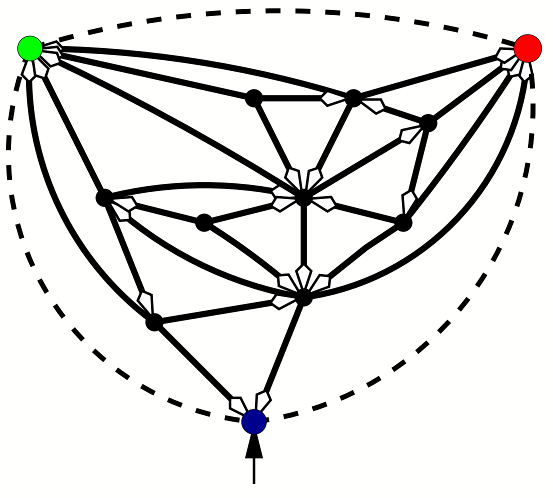
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



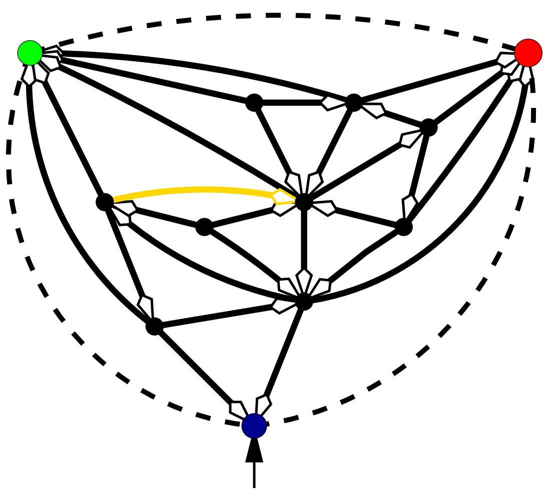
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



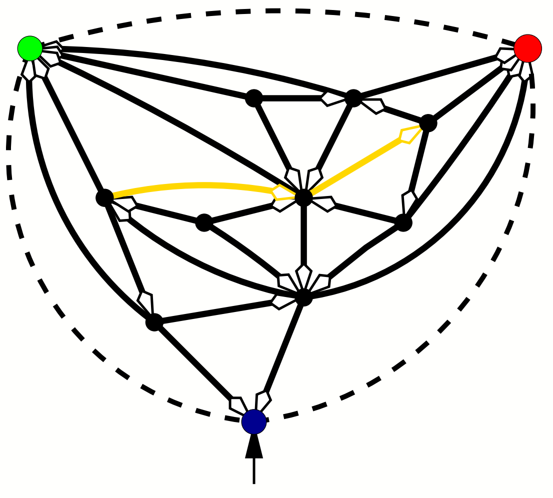
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



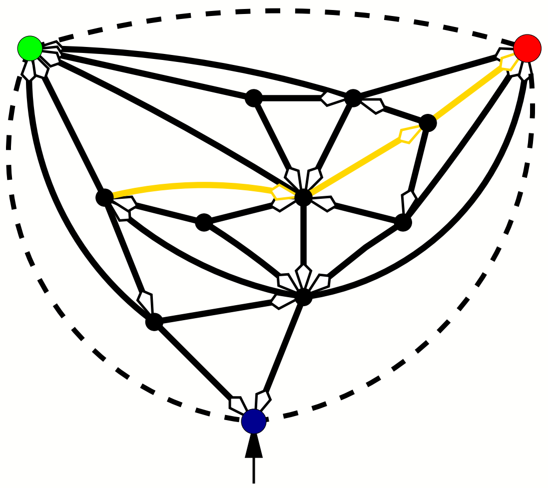
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



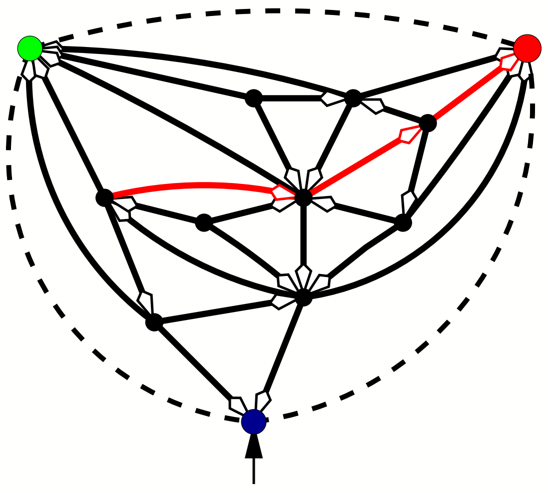
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



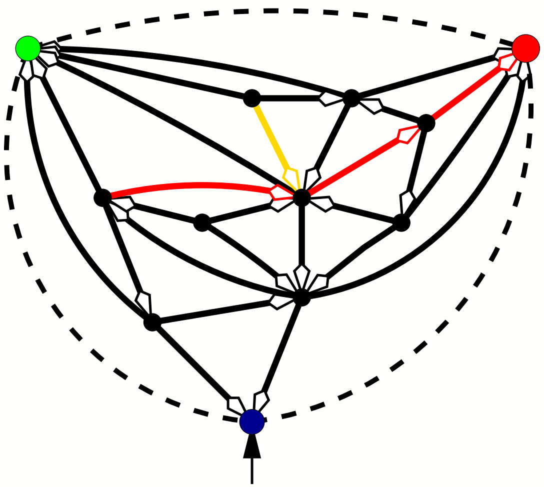
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



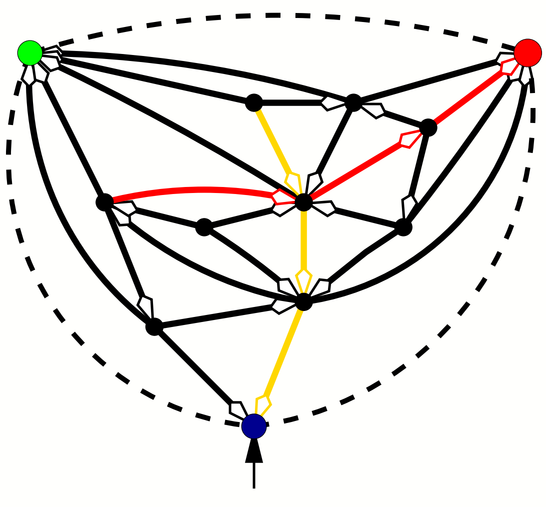
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



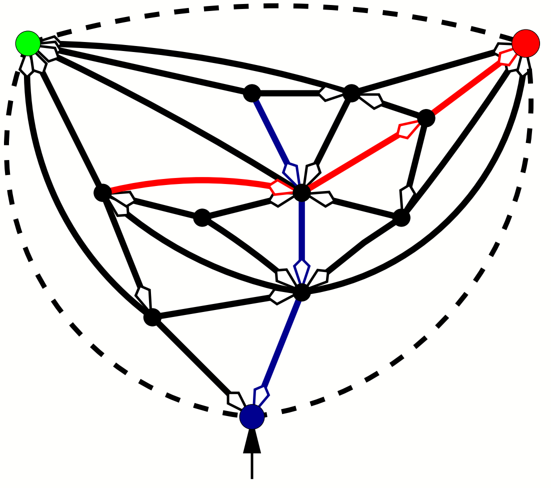
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



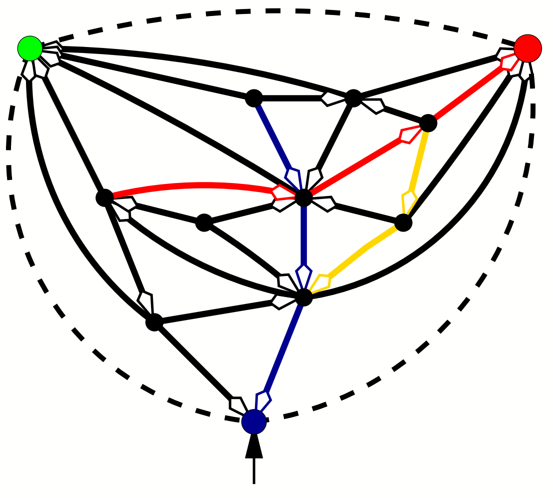
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



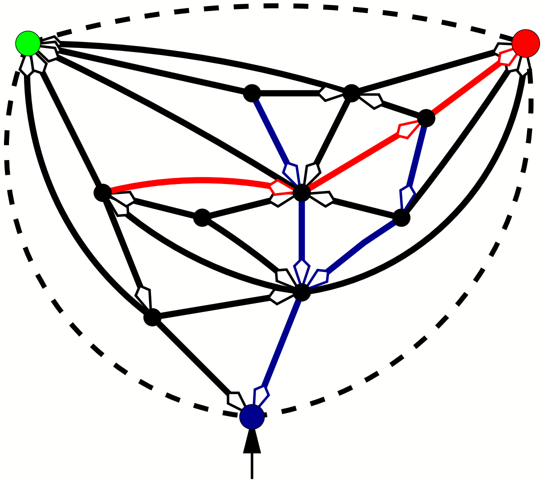
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



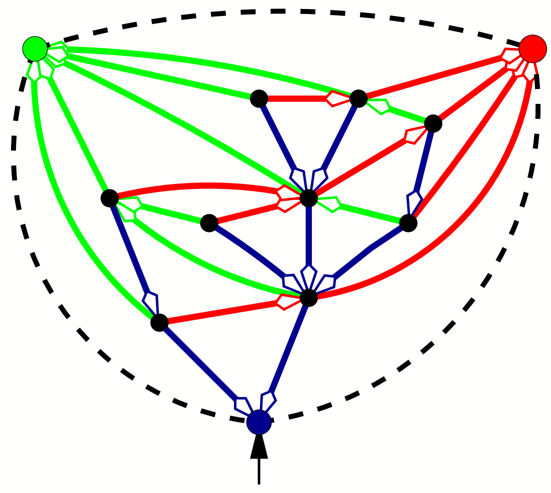
Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.



Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.

Proposition: Any triangulation has a 3-orientation. (Characterization of score vectors **[Felsner 04]**).

Digression : lattice of realizers

Proposition [Schnyder 90]: The realizers are in one-to-one correspondence with the 3-orientations.

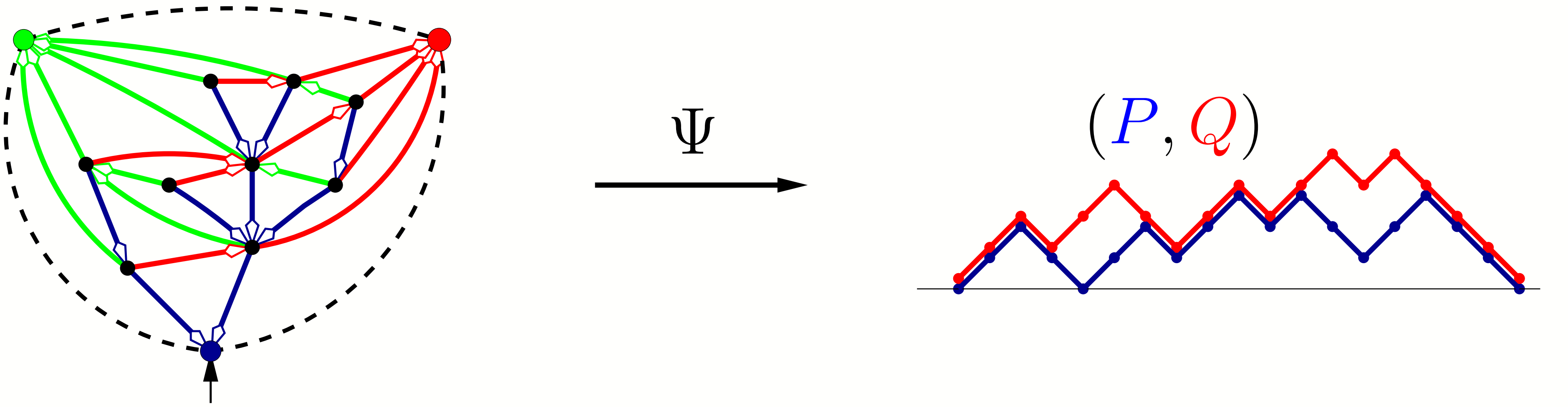
Proposition: Any triangulation has a 3-orientation. (Characterization of score vectors **[Felsner 04]**).

Proposition [Propp 93, Felsner 04]: For any score vector $\alpha : V \mapsto \mathbb{N}$, the set of α-orientations of a planar map can be endowed with a lattice structure.

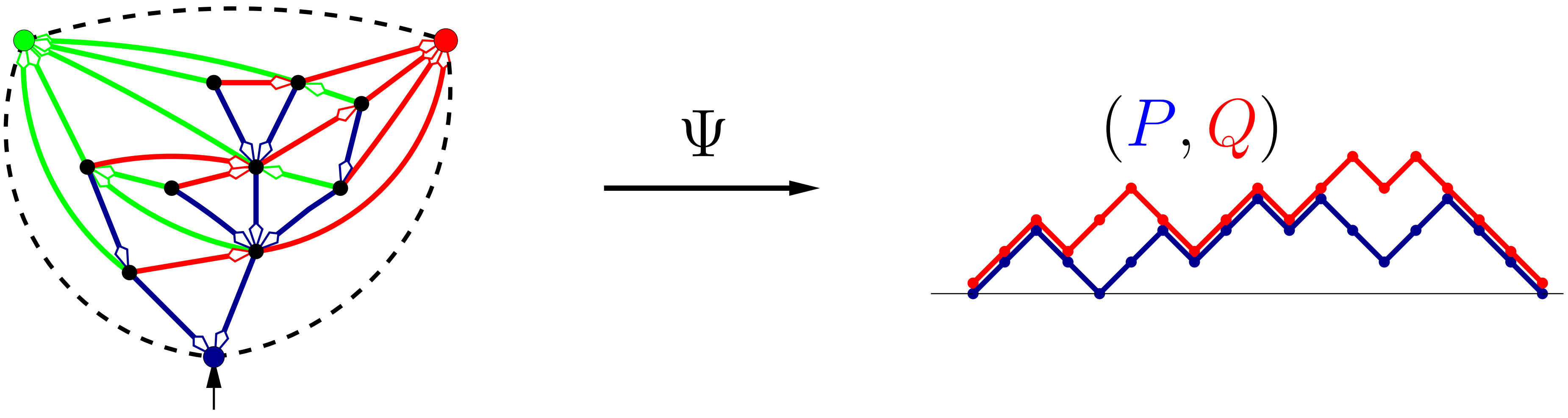
Main results

Stanley intervals	$\Longleftrightarrow$	Realizers.
Tamari intervals	$\Longleftrightarrow$	Minimal realizers.
Kreweras intervals	$\Longleftrightarrow$	Minimal and maximal realizers.

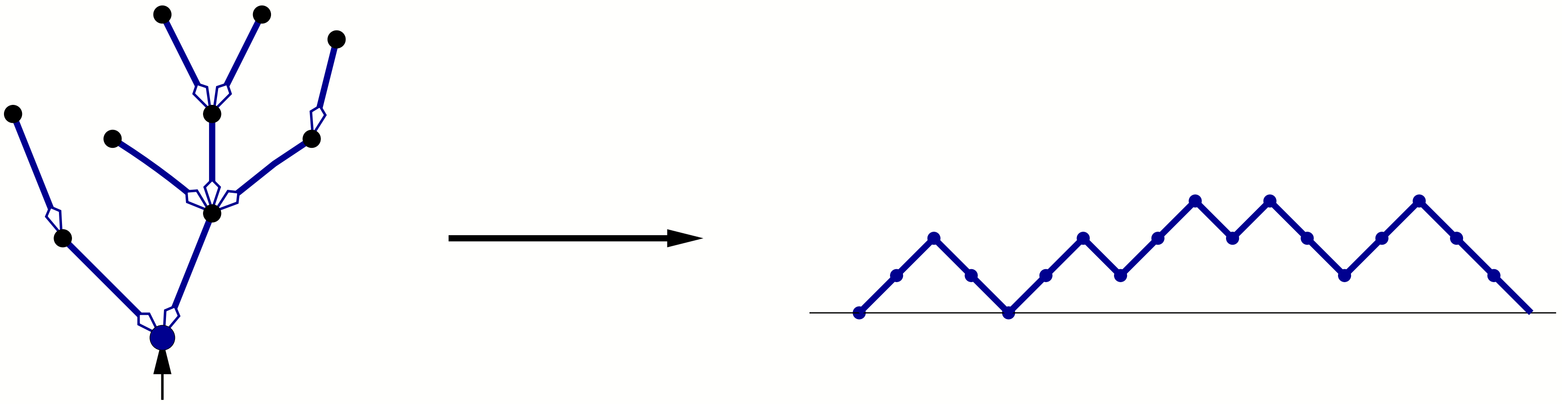
From realizers to pairs of Dyck paths



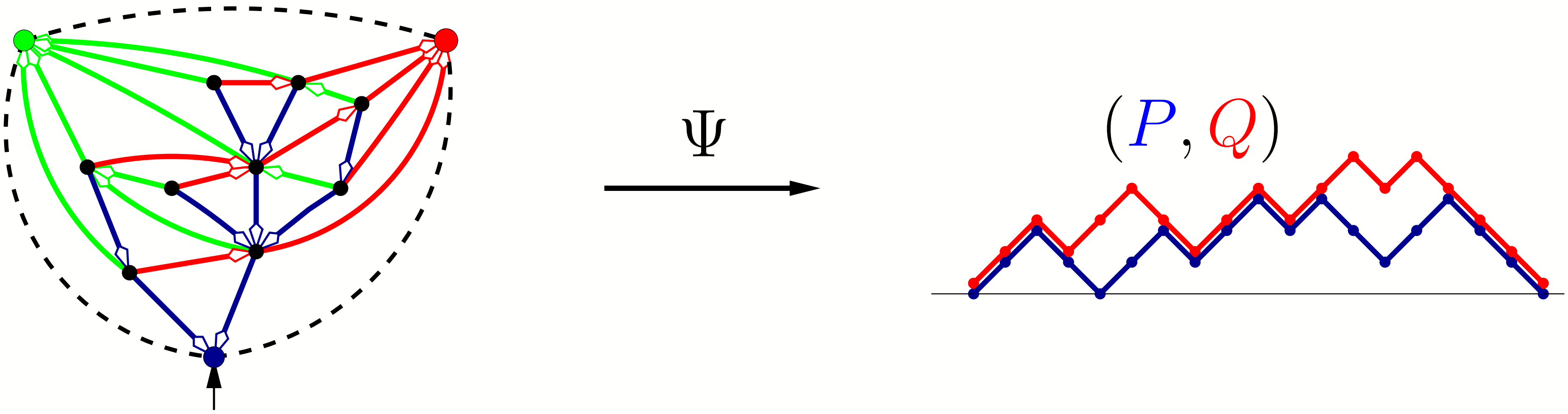
From realizers to pairs of Dyck paths



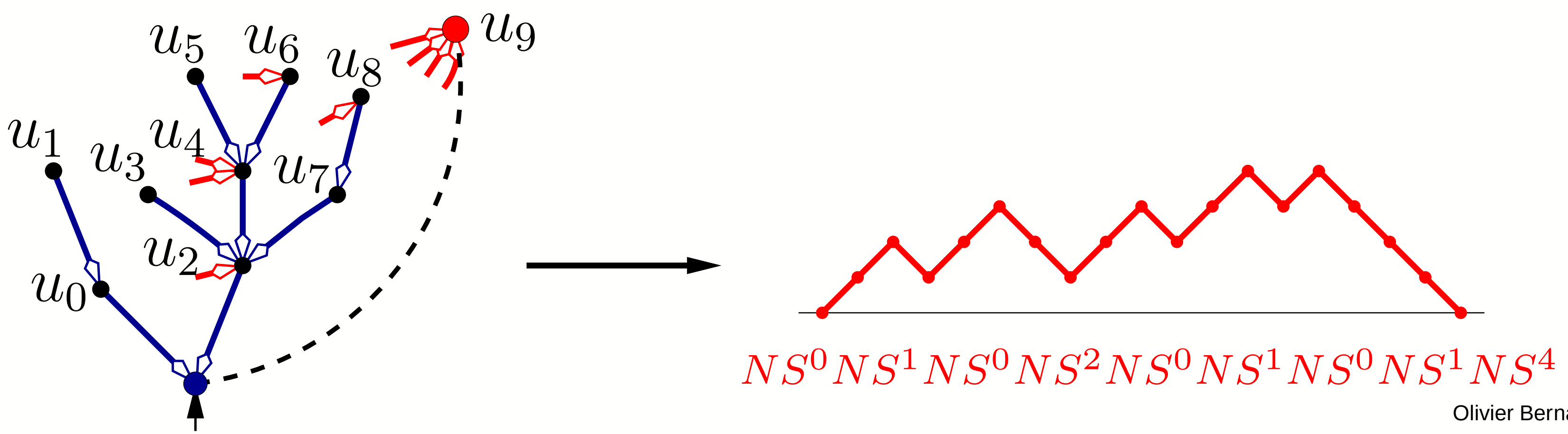
- P is the Dyck path associated to the blue tree.



From realizers to pairs of Dyck paths

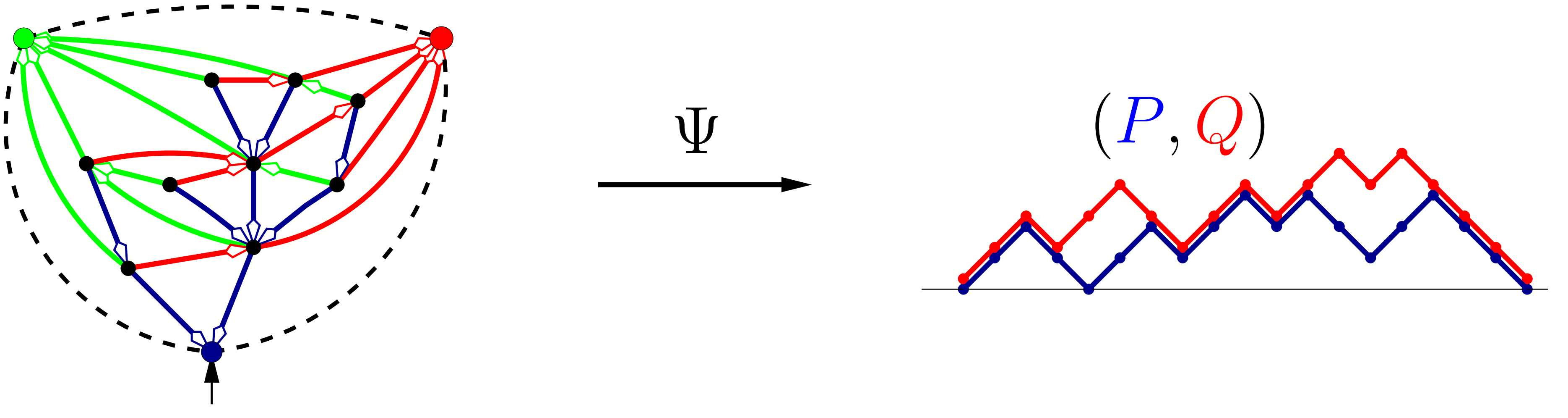


- Q is the Dyck path $NS^{\beta_1} \ldots NS^{\beta_n}$, where β_i is the number of red heads incident to the vertex u_i.



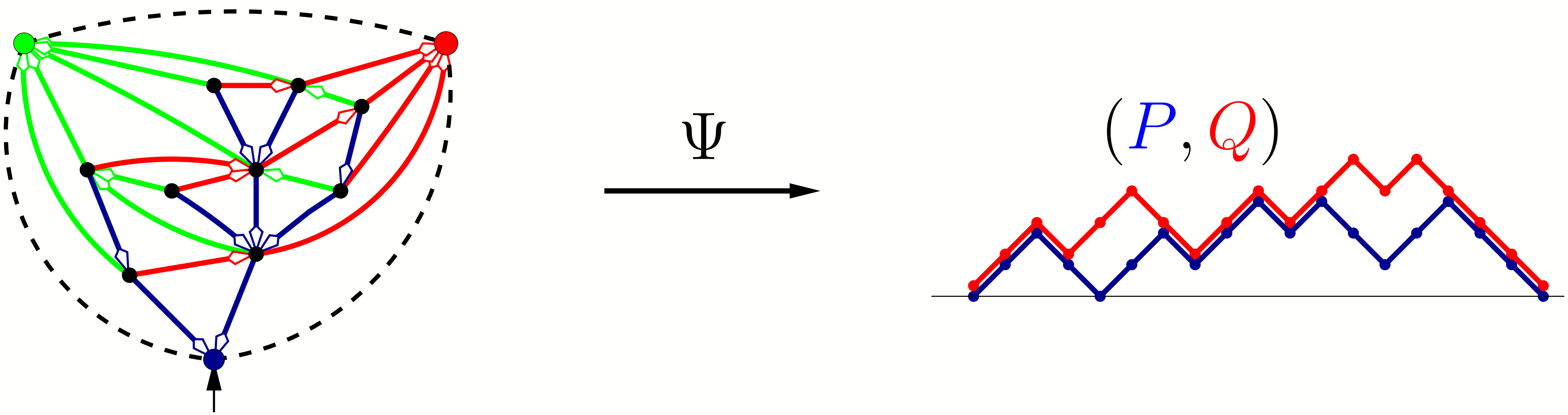
Main results

Theorem: The mapping Ψ is a bijection between realizers of size n and pairs of non-crossing Dyck paths of size n.



Main results

Theorem: The mapping Ψ is a bijection between realizers of size n and intervals in the n^{th} Stanley lattice.



Main results : Tamari

Theorem: The mapping Ψ induces a bijection between minimal realizers of size n and intervals in the n^{th} Tamari lattice.

Main results : Tamari

Theorem: The mapping Ψ induces a bijection between minimal realizers of size n and intervals in the n^{th} Tamari lattice.

Corollary: We obtain a bijection between triangulations of size n and intervals in the n^{th} Tamari lattice.

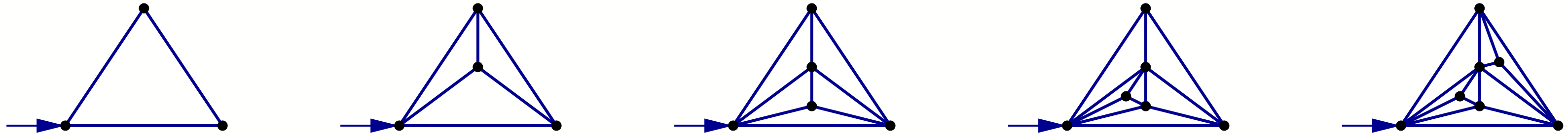
Main results : Kreweras

Theorem: The mapping Ψ induces a bijection between minimal and maximal realizers of size n and intervals in the n^{th} Kreweras lattice.

Main results : Kreweras

Theorem: The mapping Ψ induces a bijection between minimal and maximal realizers of size n and intervals in the n^{th} Kreweras lattice.

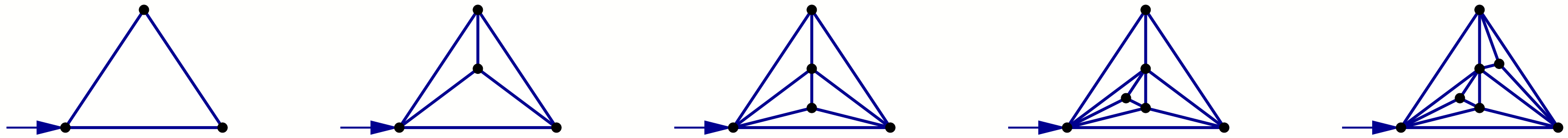
Proposition: A triangulation has a unique realizer if and only if it is stack.



Main results : Kreweras

Theorem: The mapping Ψ induces a bijection between minimal and maximal realizers of size n and intervals in the n^{th} Kreweras lattice.

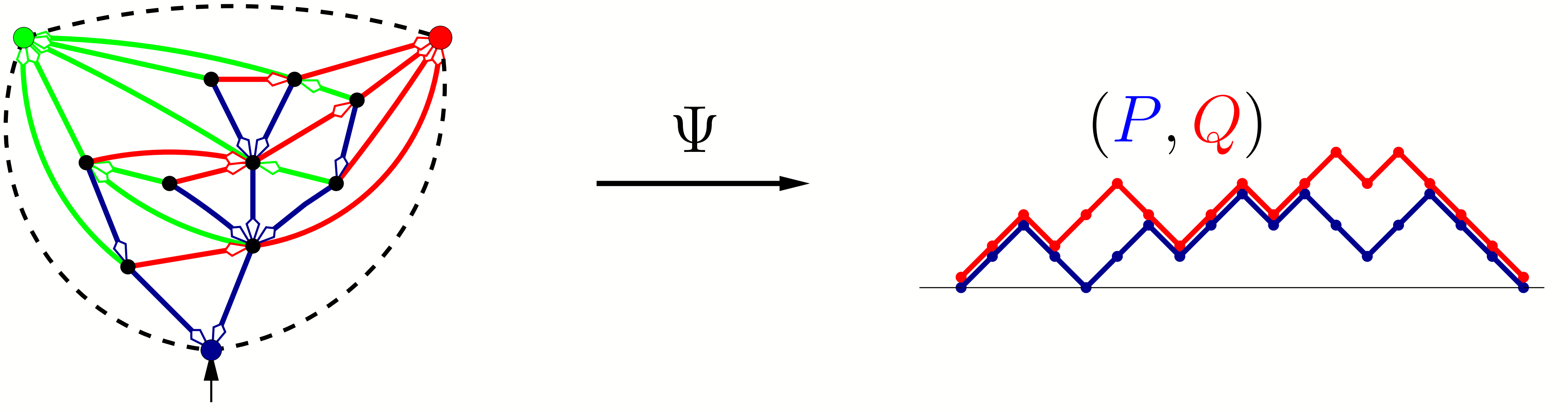
Proposition: A triangulation has a unique realizer if and only if it is stack.



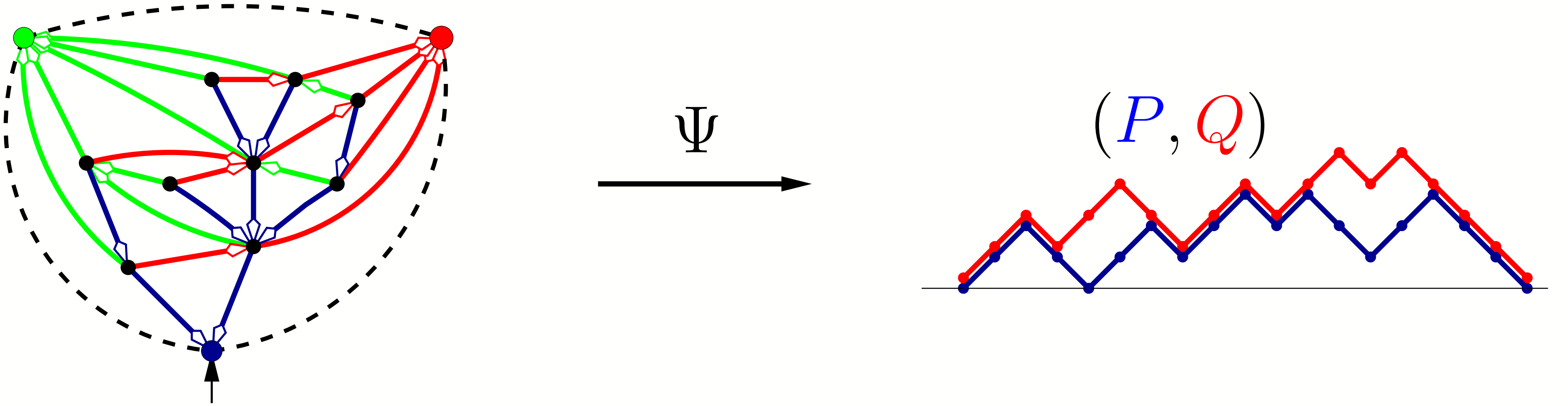
Corollary: We obtain a bijection between stack triangulations ($\Leftrightarrow$ ternary trees) of size n and intervals in the n^{th} Kreweras lattice.

Elements of proofs

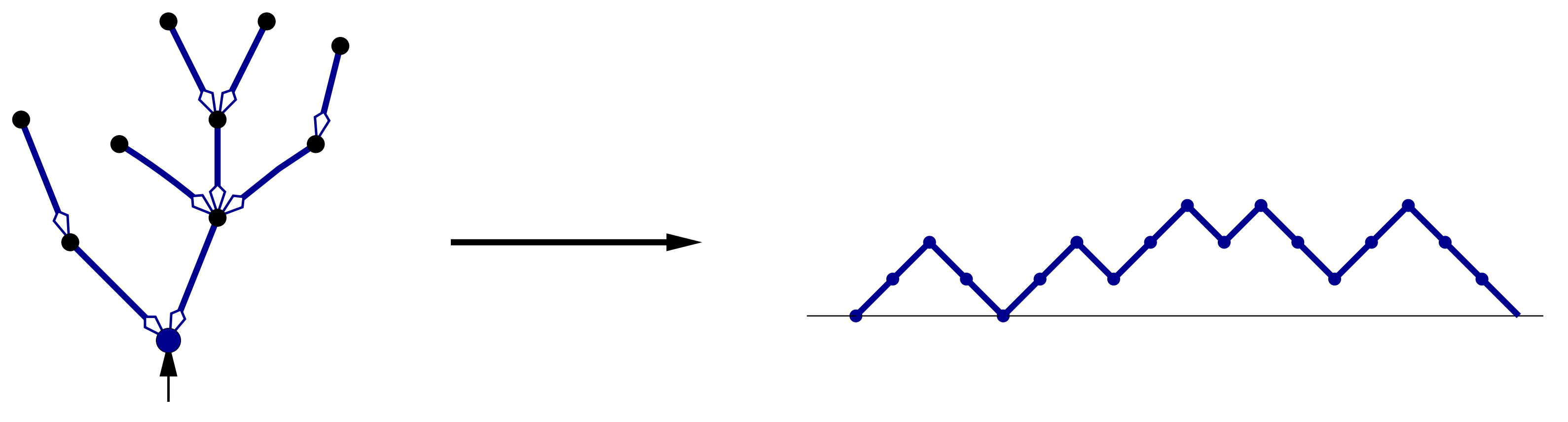
Claim : The image of any realizer is a pair of non-crossing Dyck paths.



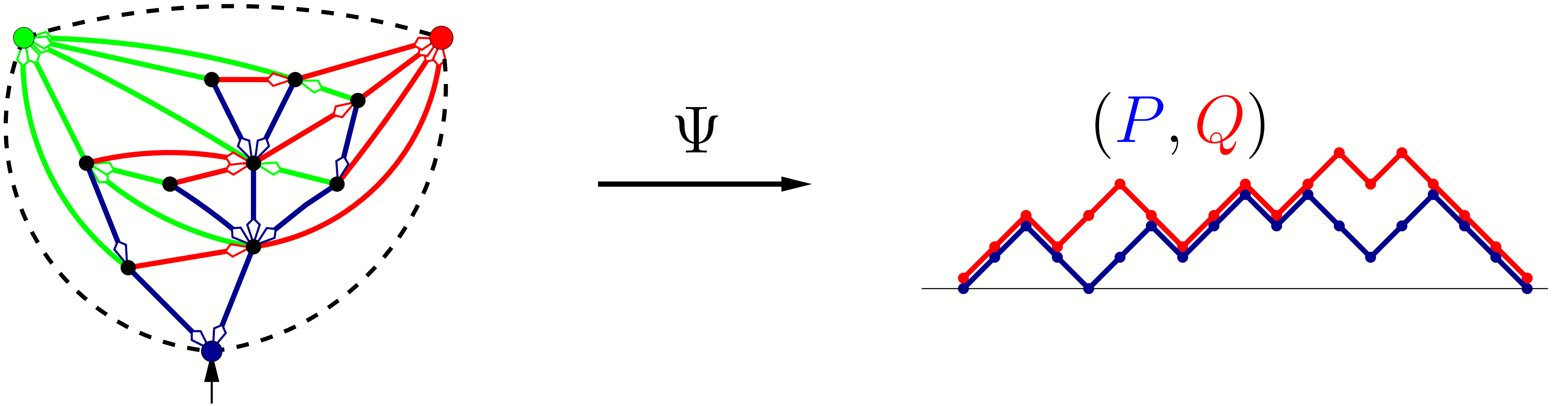
Claim : The image of any realizer is a pair of non-crossing Dyck paths.



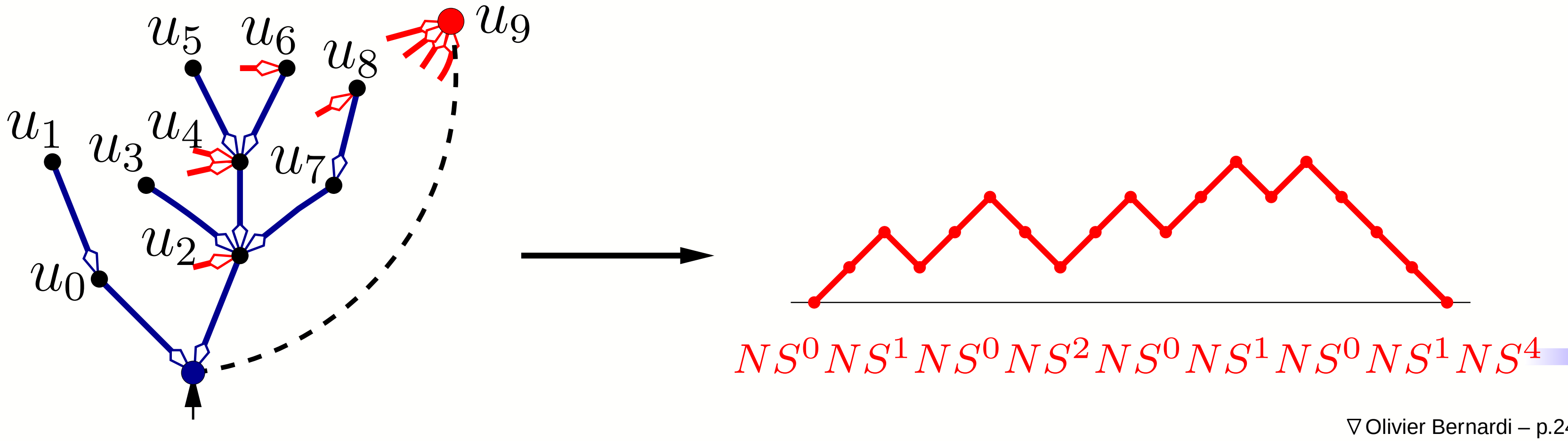
- P is a Dyck path.



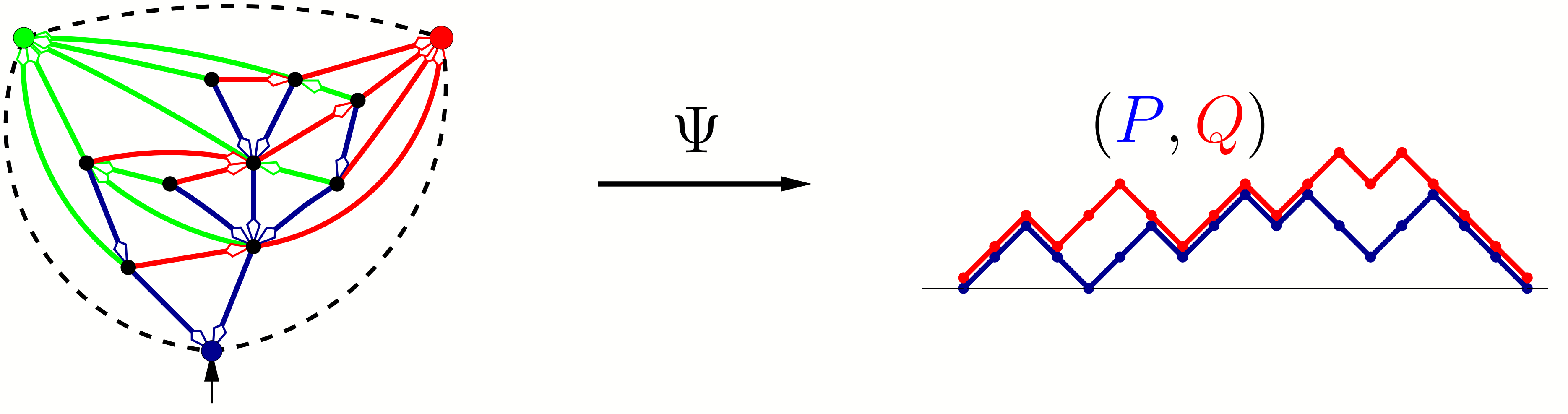
Claim : The image of any realizer is a pair of non-crossing Dyck paths.



- P is a Dyck path.
- Q returns to the 0.

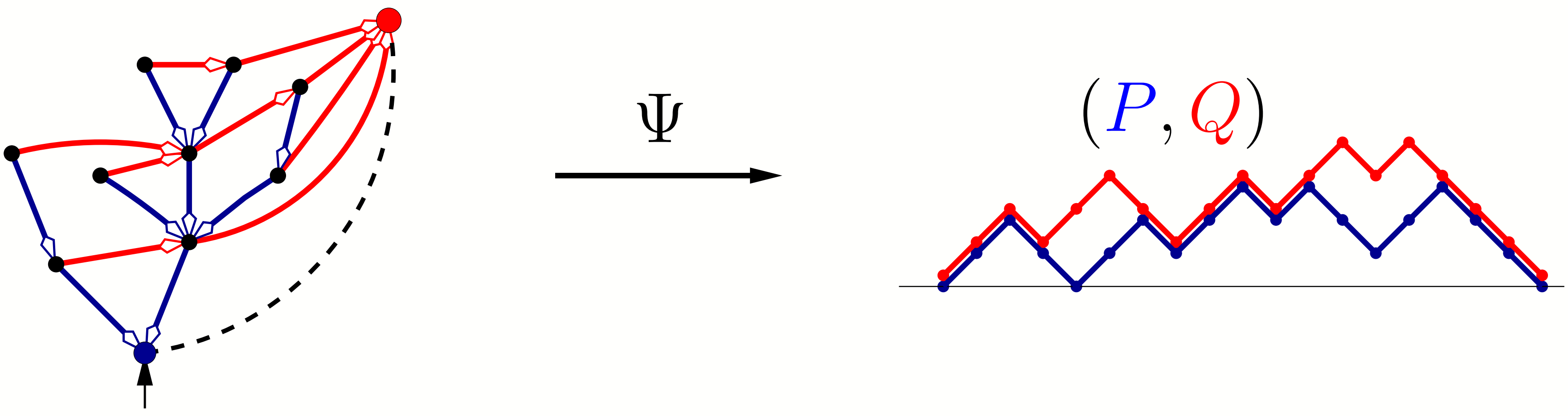


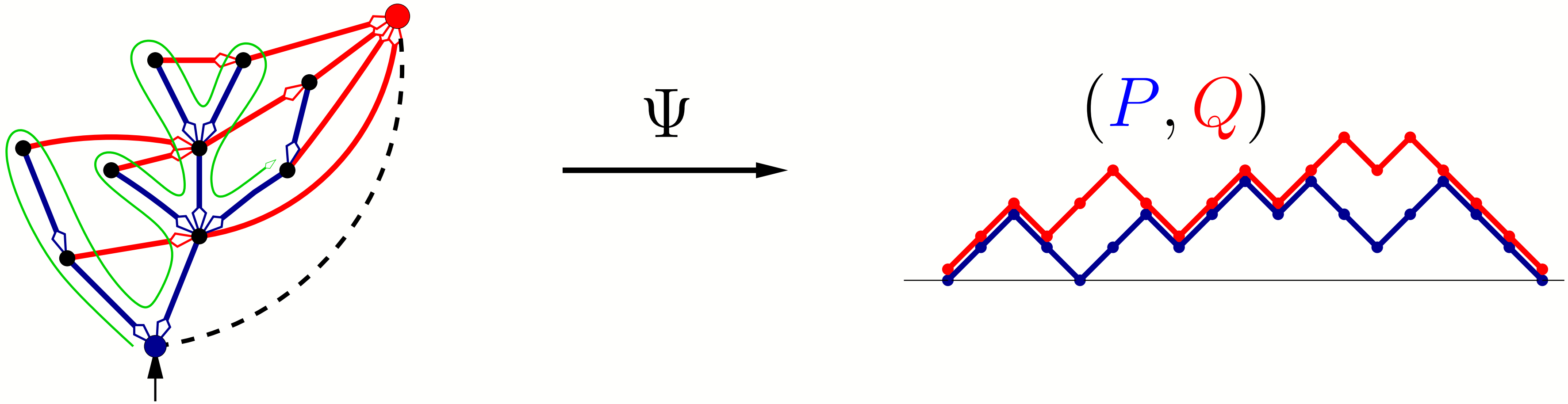
Claim : The image of any realizer is a pair of non-crossing Dyck paths.



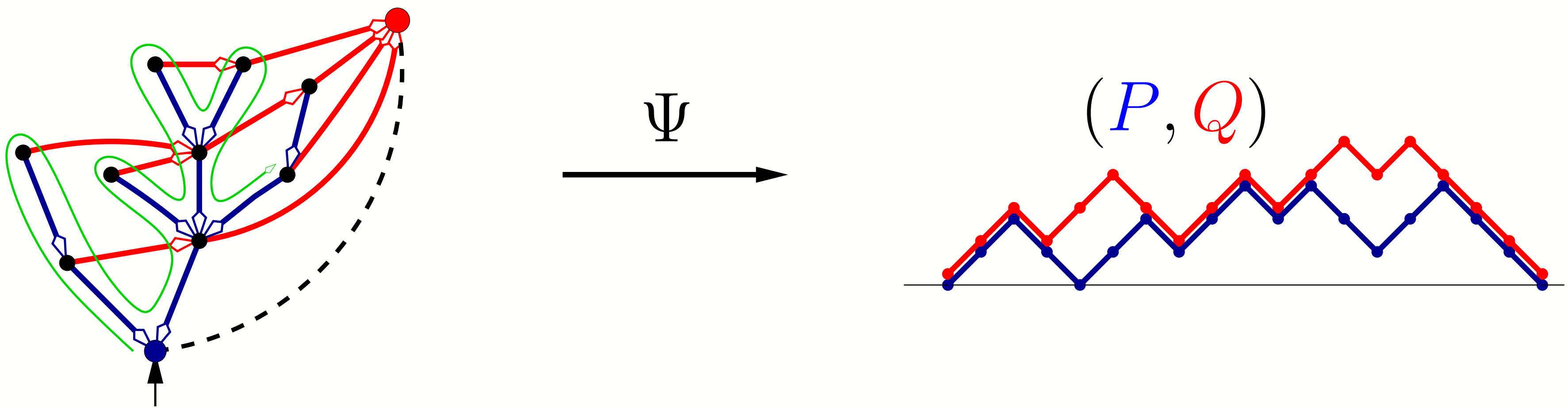
- P is a Dyck path.
- Q returns to the 0.

It only remains to show that the path Q stays above P.


Ψ
(P, Q)



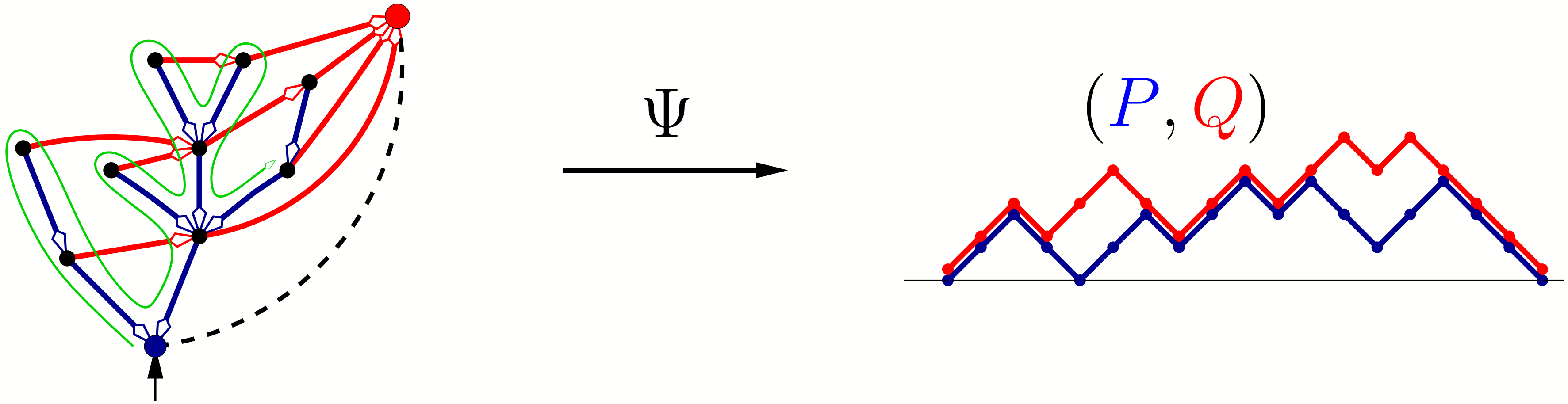
• For any red edge, the tail appears before the head around the blue tree.
⇒ The sequence of heads and tails is a *Dyck path*.



- For any red edge, the tail appears before the head around the blue tree.

$\Rightarrow$ The sequence of heads and tails is a *Dyck path*.

- The sequence of heads and tails is $T^{\alpha_1} H^{\beta_1} \dots T^{\alpha_n} H^{\beta_n}$, where $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$.



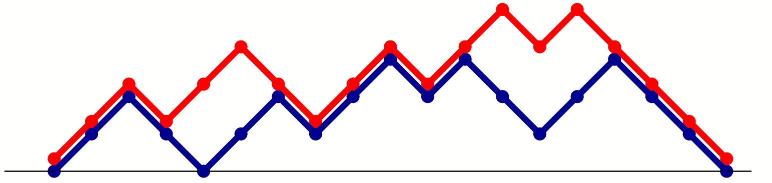
• For any red edge, the tail appears before the head around the blue tree.
$\Rightarrow$ The sequence of heads and tails is a *Dyck path*.

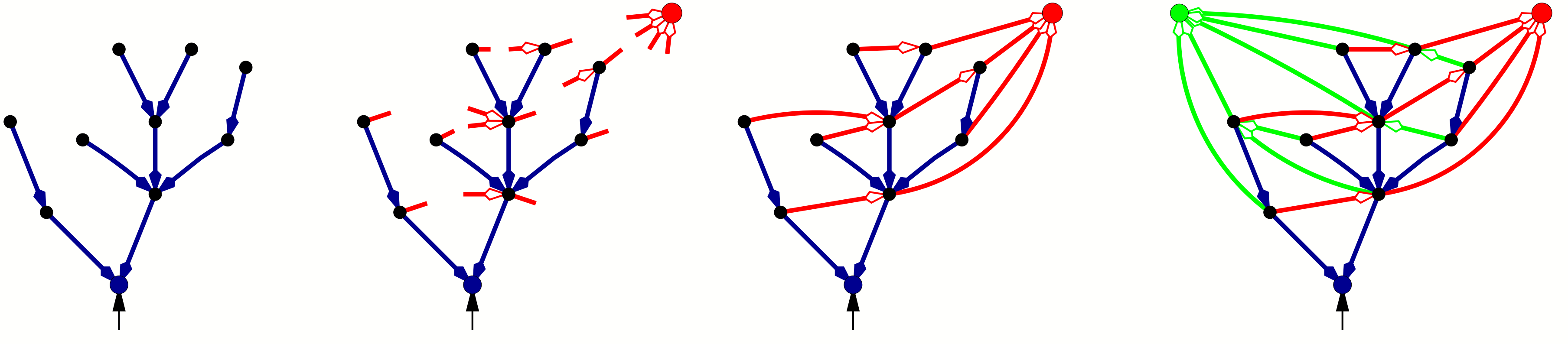
• The sequence of heads and tails is $T^{\alpha_1}H^{\beta_1}\ldots T^{\alpha_n}H^{\beta_n}$, where $P = NS^{\alpha_1}\ldots NS^{\alpha_n}$ and $Q = NS^{\beta_1}\ldots NS^{\beta_n}$.

$\Longrightarrow$ The path Q stays above P.

Inverse mapping

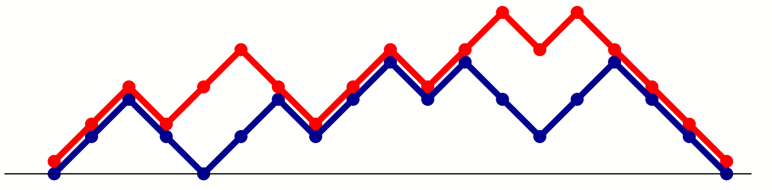
(P, Q)



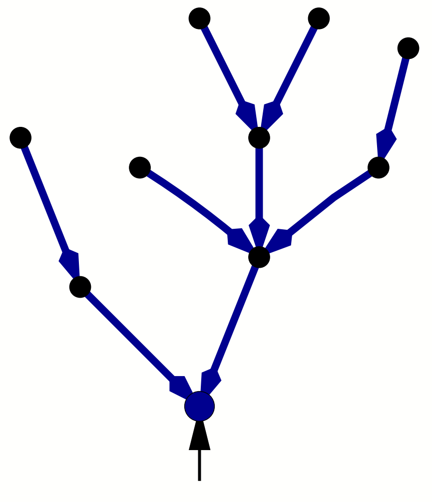


Inverse mapping

(P, Q)

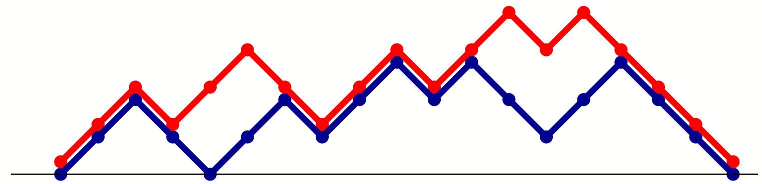


Step 1: Construct the blue tree (using P).

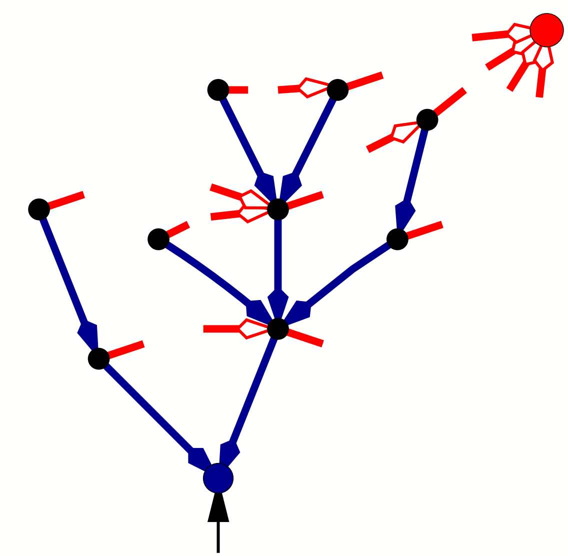


Inverse mapping

(P, Q)

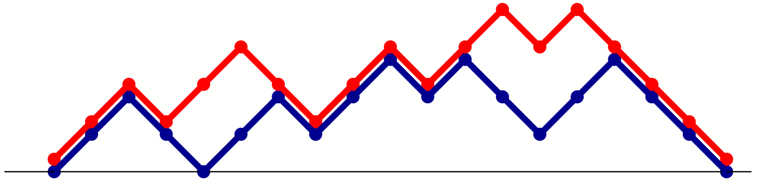


Step 2: Add red tails and heads (using Q).

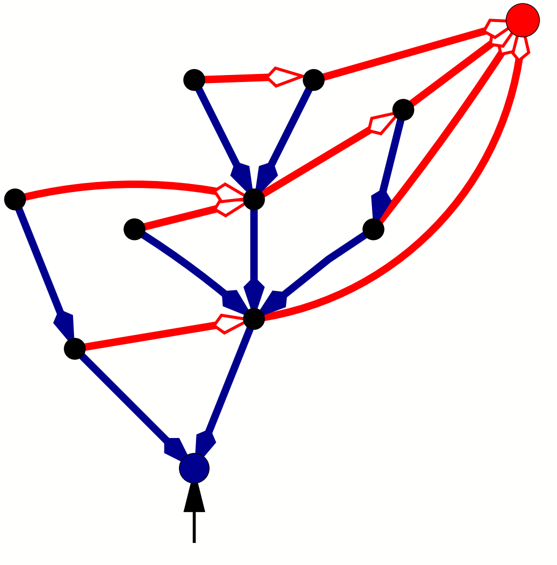


Inverse mapping

$(\color{blue}{P}, \color{red}{Q})$



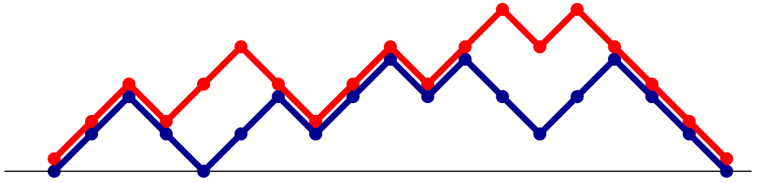
Step 3: Join tails and head.



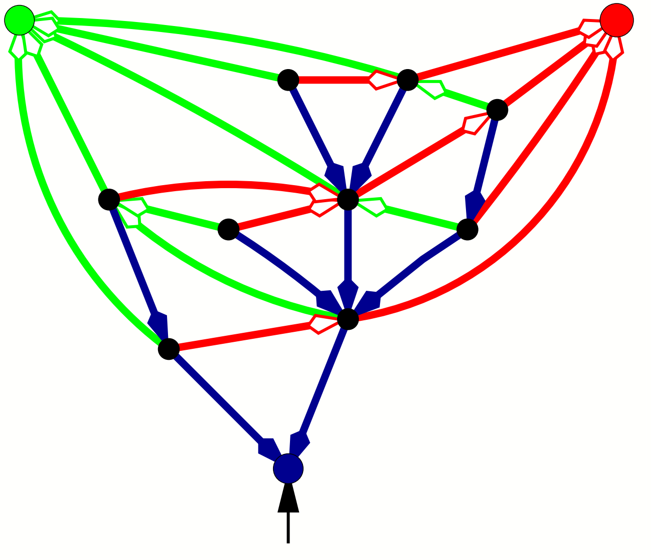
Claim : There is only one way of joining tails and heads.
This creates a tree.

Inverse mapping

(P, Q)



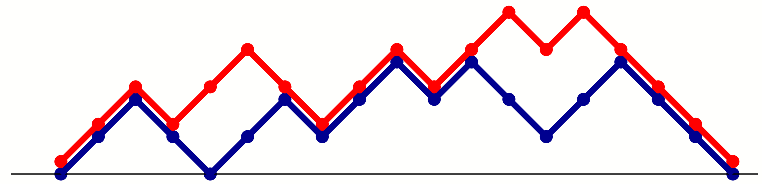
Step 4: Construct the green tree.



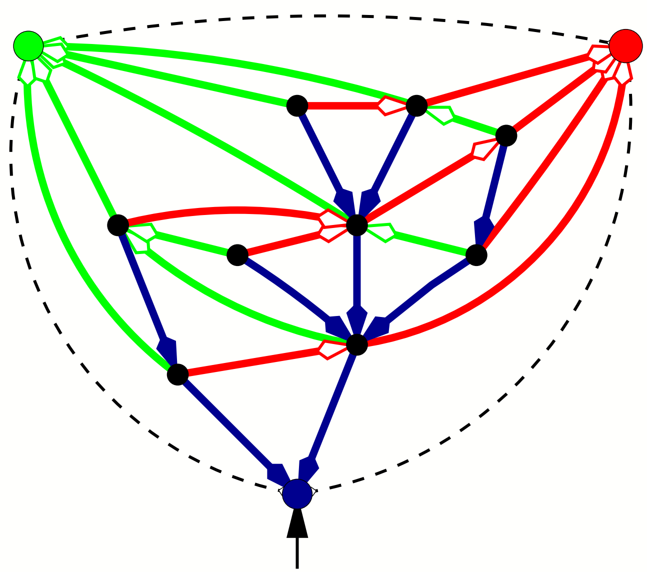
Claim : There exist a unique green tree.

Inverse mapping

(P, Q)

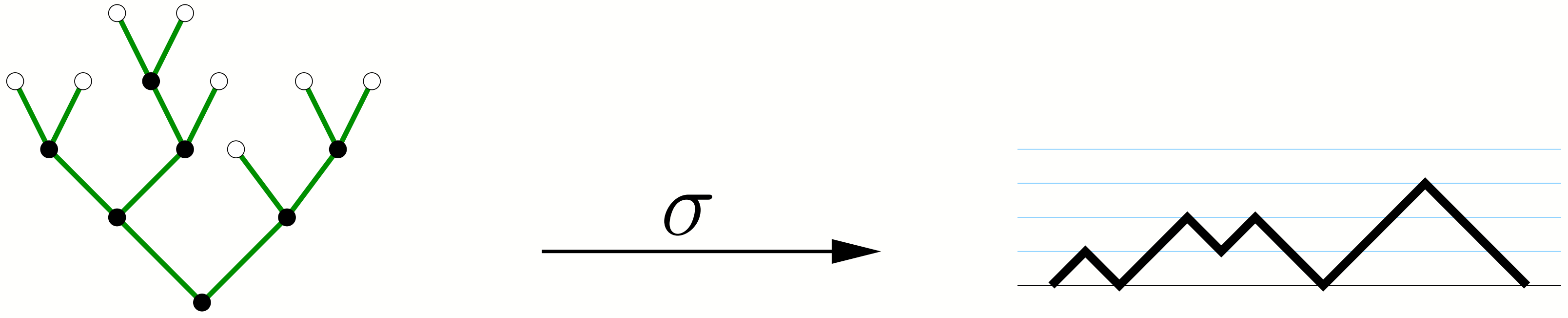


Step 5: Close the map.



Refinement Tamari

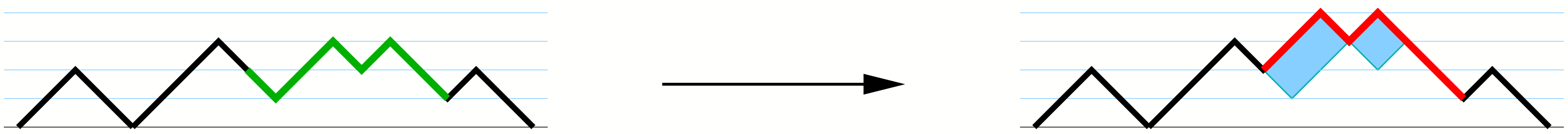
- Chose a good bijection binary-trees $\mapsto$ Dyck paths.



Refinement Tamari

• Chose a good bijection binary-trees $\mapsto$ Dyck paths.

• Characterize the covering relation of the Tamari lattice in terms of Dyck paths.

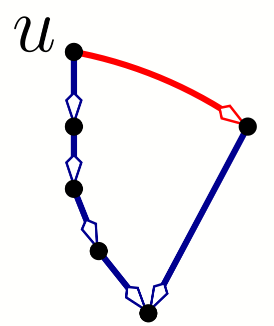


Refinement Tamari

• Chose a good bijection binary-trees $\mapsto$ Dyck paths.

• Characterize the covering relation of the Tamari lattice in terms of Dyck paths.

• Characterize the minimal realizers **[Bon, Gav, Han 02]**.



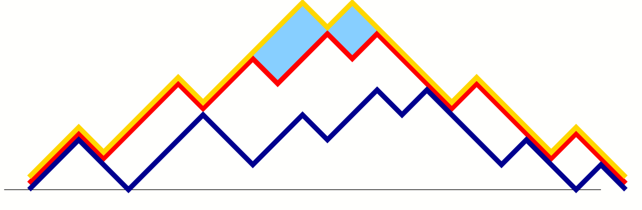
Refinement Tamari

• Chose a good bijection binary-trees $\mapsto$ Dyck paths.

• Characterize the covering relation of the Tamari lattice in terms of Dyck paths.

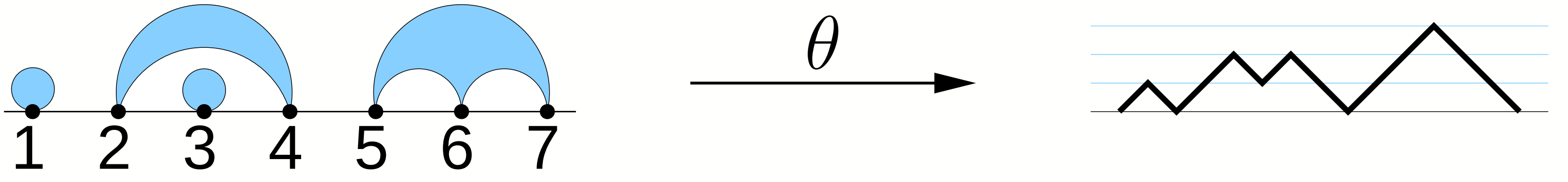
• Characterize the minimal realizers **[Bon, Gav, Han 02]**.

• Make an induction on $\Delta(P, Q)$ to prove that P and Q are comparable in the Tamari lattice if and only if the realizer $\Psi(P, Q)$ is minimal.



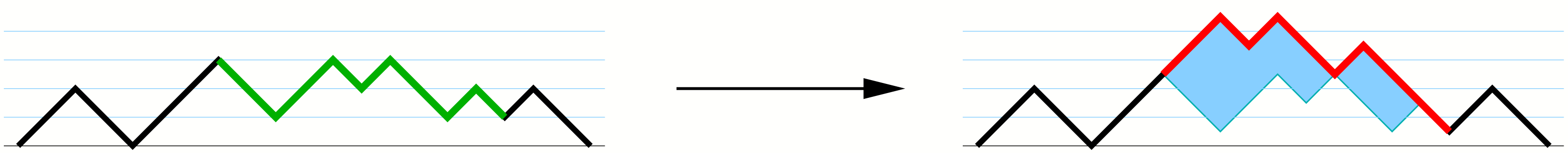
Refinement Kreweras

• Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.



Refinement Kreweras

- Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.
- Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.

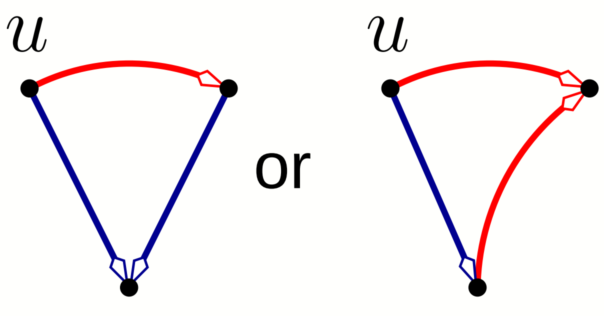


Refinement Kreweras

• Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.

• Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.

• Characterize the minimal and maximal realizers **[Bon, Gav, Han 02]**.



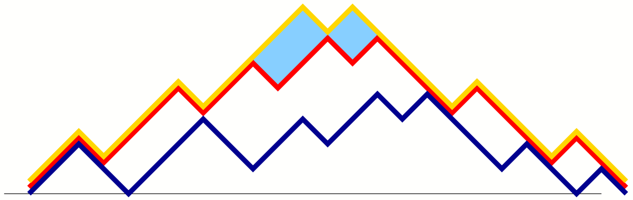
Refinement Kreweras

• Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.

• Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.

• Characterize the minimal and maximal realizers **[Bon, Gav, Han 02]**.

• Make an induction on $\Delta(P,Q)$ to prove that P and Q are comparable in the Kreweras lattice if and only if the realizer $\Psi(P,Q)$ is minimal and maximal.

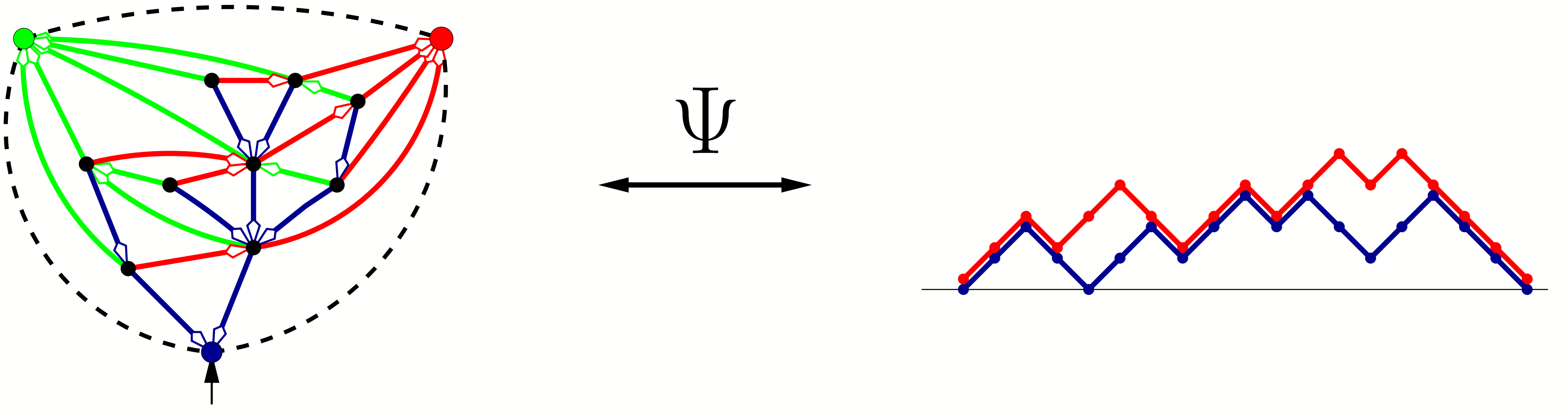


Refinement Kreweras

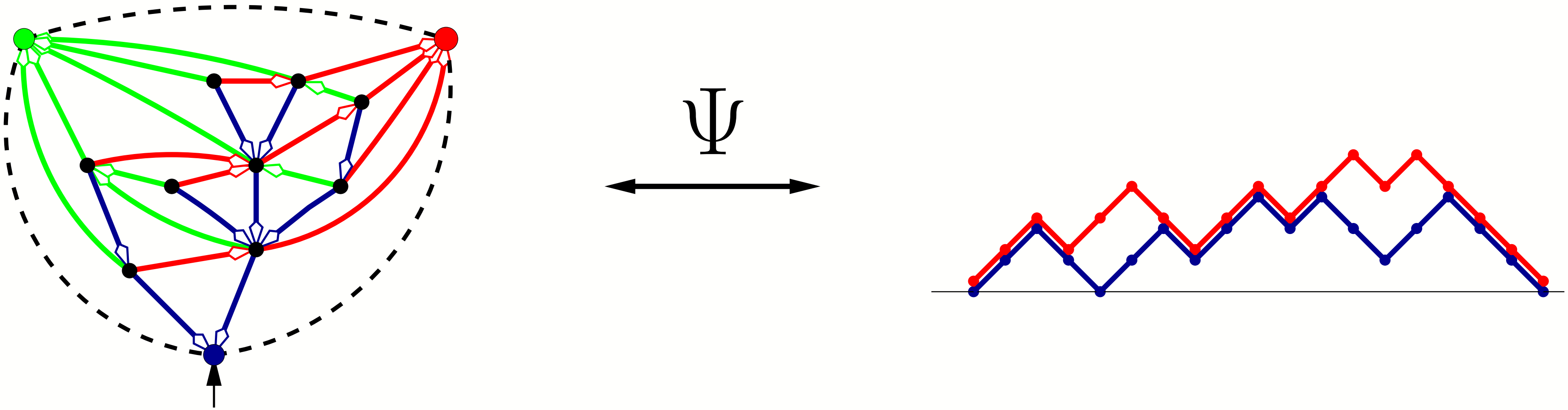
- Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.
- Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.
- Characterize the minimal and maximal realizers **[Bon, Gav, Han 02]**.
- Make an induction on $\Delta(P, Q)$ to prove that P and Q are comparable in the Kreweras lattice if and only if the realizer $\Psi(P, Q)$ is minimal and maximal.
- Prove that a triangulation has a unique realizer if and only if it is stack.

Summary

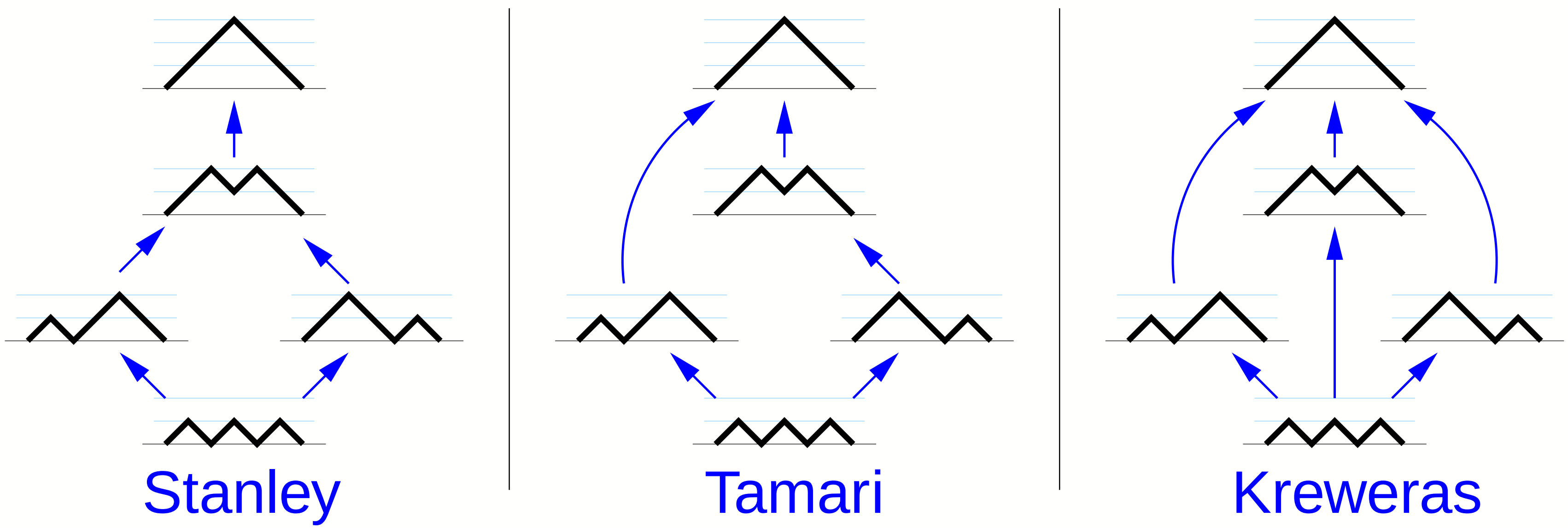
- **Bijection:** Realizers $\Longleftrightarrow$ Stanley intervals



- **Bijection:** Realizers $\Longleftrightarrow$ Stanley intervals

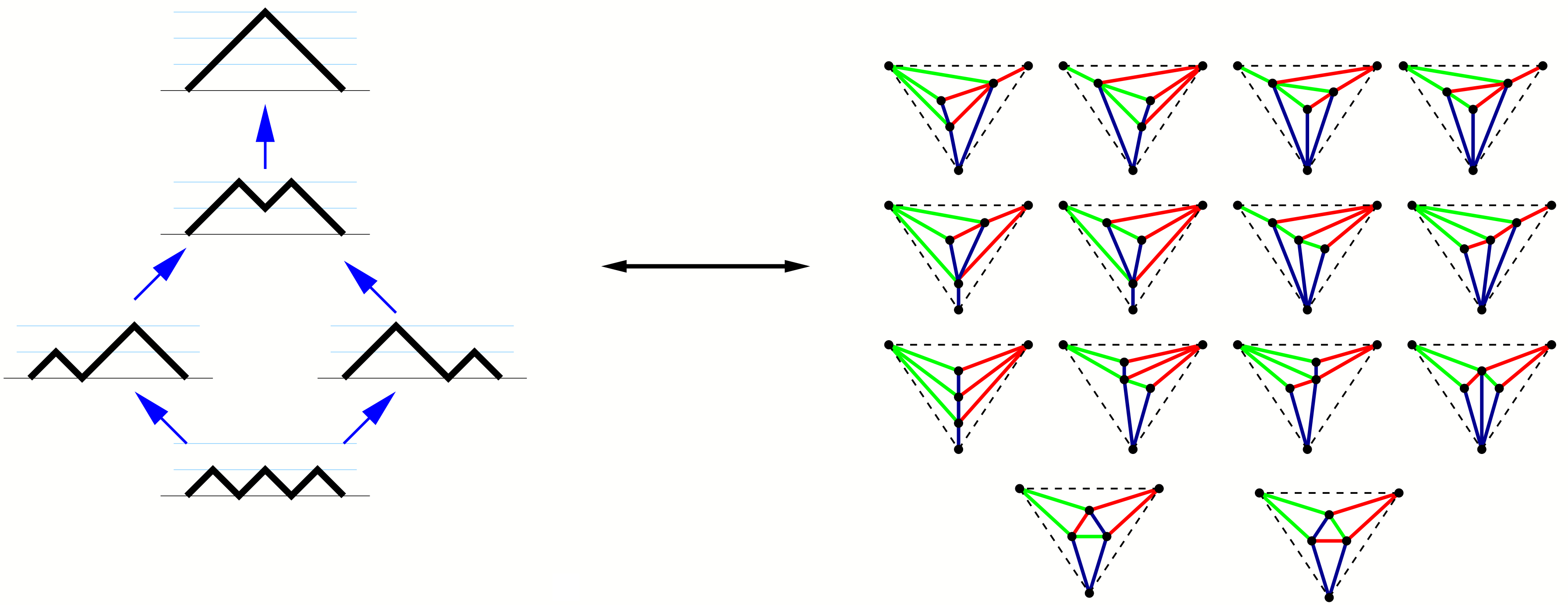


- **Refinements:**



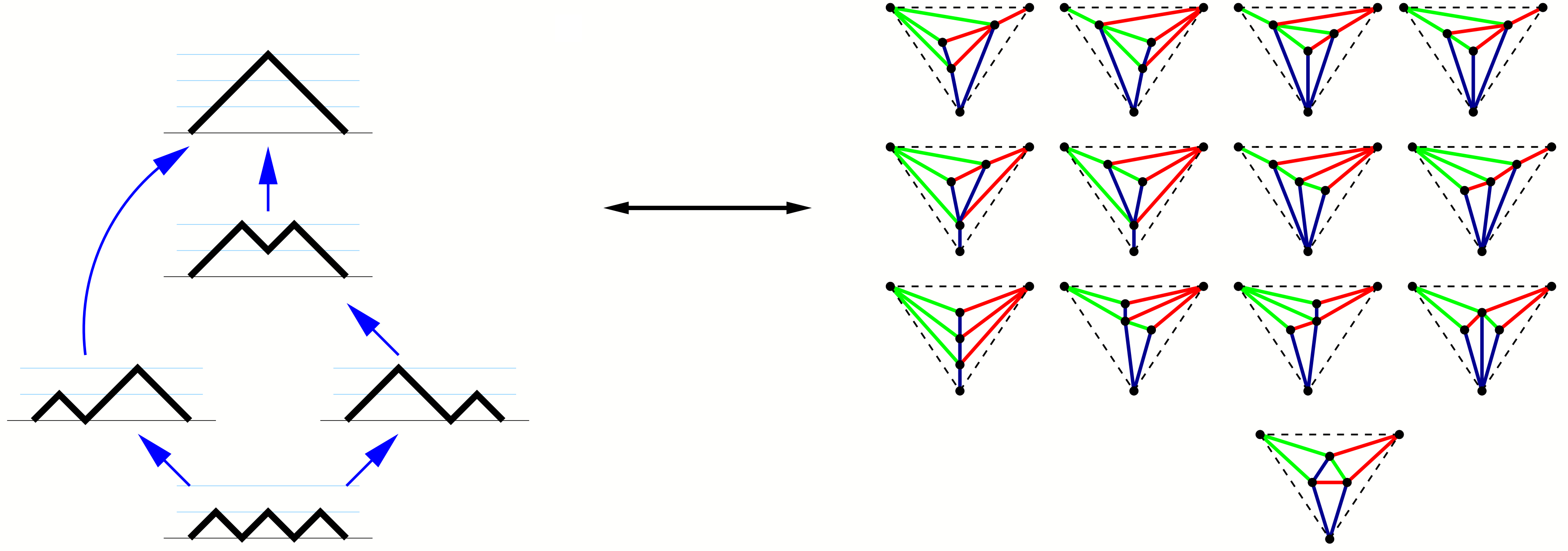
Bijection

Stanley intervals $\Longleftrightarrow$ Realizers



Refinement Tamari

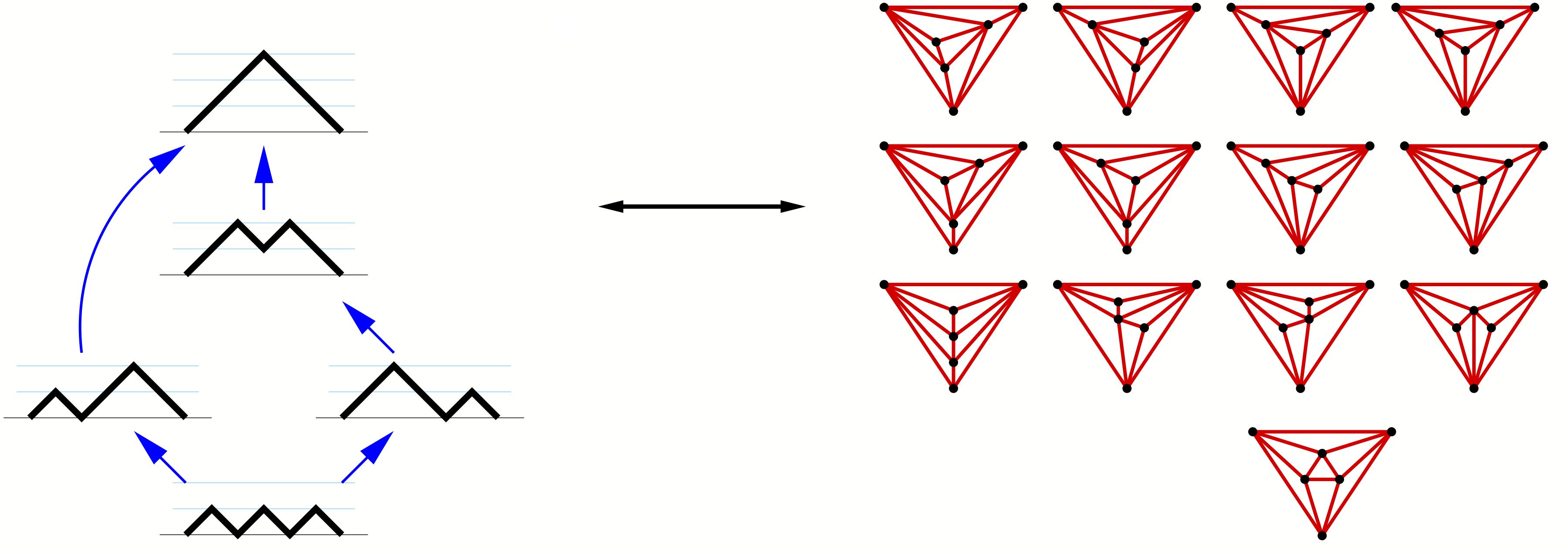
Tamari intervals $\Longleftrightarrow$ Minimal realizers



Refinement Tamari

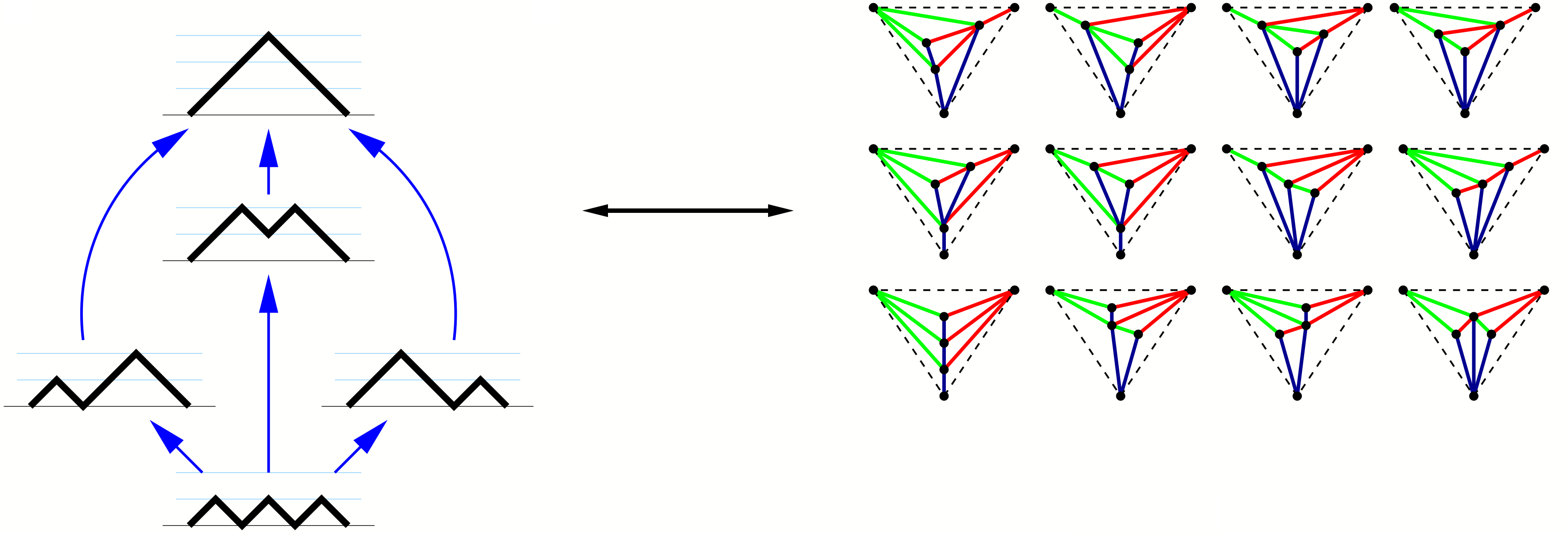
Tamari intervals $\iff$ Minimal realizers

$\iff$ Triangulations



Refinement Kreweras

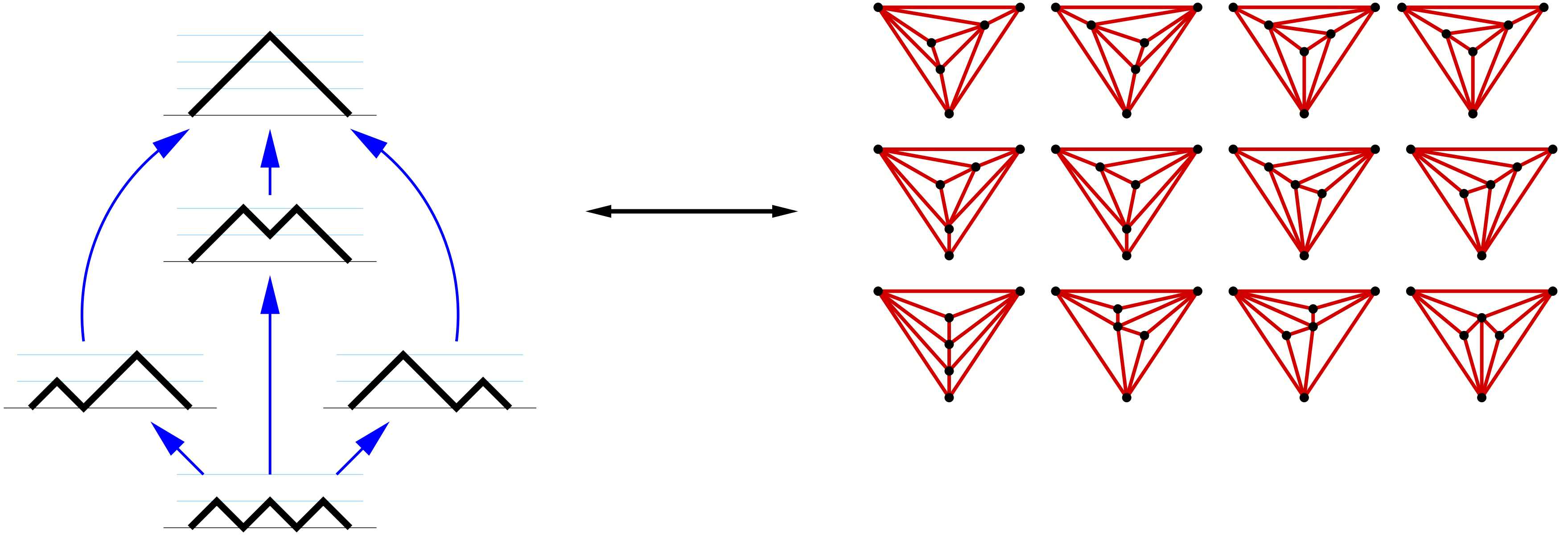
Kreweras intervals $\Longleftrightarrow$ Minimal and maximal realizers



Refinement Kreweras

Kreweras intervals $\Longleftrightarrow$ Minimal and maximal realizers

$\Longleftrightarrow$ Stack triangulations ($\Leftrightarrow$ Ternary trees)



Thanks.