1 Setup:

Let T_i be the true effectiveness of an intervention and assume your prior on the cost-effectiveness of the intervenion (before measurement) is $T_i \sim \text{lognormal}(\mu_\text{T}, \sigma_\text{T})$. You then make a noisy measurement $M_i = E_i \cdot T_i$ of the cost-effectiveness, where $E_i \sim \text{lognormal}(0, \sigma_\text{E})$ is the measurement noise and is assumed to have a mean of one (the measurement is unbiased). What is your final (posterior) estimate of the cost-effectiveness?

First, to work with normal distributions instead of lognormal ones, just take the log of all variables, $t_i = \log(T_i)$, $m_i = \log(M_i)$, $e_i = \log(E_i)$. Then $m_i = t_i + e_i$, and $t_i \sim N_{\mu_T,\sigma_T}$ and $e_i \sim N_{0,\sigma_E}$.

By Bayes' rule, the posterior distribution $P(t_i|m_i)$ is:

$$\begin{aligned} P(t_i|m_i) &= P(t_i)P(m_i|t_i)/P(m_i) \\ &= cons \cdot P(t_i)P(m_i|t_i) \\ &= cons \cdot P(t_i)P(e_i = m_i - t_i) \\ &= cons \cdot \text{N}_{\mu_T,\sigma_T}(t_i) \cdot N_{0,\sigma_E}(m_i - t_i) \\ &= cons \cdot \text{N}_{\mu_T,\sigma_T}(t_i) \cdot N_{m_i,\sigma_E}(t_i) \end{aligned}$$

This is the (pointwise) product of two normal distributions.

2 Math:

Product of two normal distributions:

$$\begin{aligned} \exp\Big[-(x-\mu_1)^2/2\sigma_1^2\Big]\exp\Big[-(x-\mu_2)^2/2\sigma_2^2\Big] &= cons \cdot \exp\Big[-x^2/2(\sigma_1^{-2}+\sigma_2^{-2})^{-1} + x\Big(\mu_1/\sigma_1^2 + \mu_2/\sigma_2^2\Big)\Big] \\ &= cons \cdot \exp\Big[-\Big(x - (\mu_1/\sigma_1^2 + \mu_2/\sigma_2^2)(\sigma_1^{-2}+\sigma_2^{-2})^{-1}\Big)^2/2(\sigma_1^{-2}+\sigma_2^{-2})^{-1}\Big] \\ &= cons \cdot \exp\Big[-(x-\mu)^2/2\sigma^2\Big] \end{aligned}$$

where

$$\begin{aligned} \mu &= (\mu_1\sigma_1^{-2} + \mu_2\sigma_2^{-2})/(\sigma_1^{-2}+\sigma_2^{-2}) \\ \sigma^{-2} &= (\sigma_1^{-2}+\sigma_2^{-2}) \end{aligned}$$

This is, when you multiply together (pointwise) two normal distributions, you get another normal distribution with mean equal to the average of the two means weighted by their inverse variances, and variance equal to the harmonic sum of the two variances. This makes sense: when your information comes from two sources with differing levels of confidence, your overall estimate should be an average of your two data points weighted by your confidence in each.

3 Conclusion

Thus, your posterior estimate of the log cost effectiveness after the measurement is

$$\mu = (\mu_T\sigma_T^{-2} + m_i\sigma_E^{-2})/(\sigma_T^{-2}+\sigma_E^{-2})$$

Thus, in an example where mens' height is normally distributed with a mean of 69 inches and a standard deviation of 3, and you make a noisy measurement of the height, with measurement error having a mean zero and a standard deviation 30 inches, and then measure the hieght of two men, Allan and Bob, to be 100 inches and 20 inches respectively, your posterior estimates of their heights should be:

$$\begin{aligned} \mu_{Allan} &= (69 \cdot 3^{-2} + 100 \cdot 30^{-2})/(3^{-2}+30^{-2}) = 69.3 \\ \mu_{Bob} &= (69 \cdot 3^{-2} + 20 \cdot 30^{-2})/(3^{-2}+30^{-2}) = 68.5 \end{aligned}$$

and the standard deviation for both estimates is 2.98 inches. Thus, it is likely that 50% of Allan's height is less than 100% of Bob's. In general if the uncertainty of your prior distribution is much lower than the measurement uncertainty, your posterior expected value will be very close to your prior expected value, and large differences in measurements will lead to small differences in posterior estimates.

Therefore, if your measurement of the cost effectiveness of an intervention (or class of intervention) is much more uncertain than your priors on the cost effectiveness, than large differences in measured cost-effectiveness will lead to small differences in your final Bayesian estimae of the cost-effectiveness. Thus, a factor of 2 can overcome an order of magnitude measured difference in cost-effectiveness, provided that the factor of 2 comes from a model which does NOT have similarly constrained priors.