The Inverse Mellin Transform, Bell Polynomials, a Generalized Dobinski Relation, and the Confluent Hypergeometric Functions

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1 Inverse Transform Representations of Delta Fct.

1.1 Inverse Fourier Transform

$$\delta(y) \;=\; \lim_{l \to \infty} \frac{\sin(l\ \pi\ y)}{\pi\ y} = \lim_{l \to \infty} \int_{-\ l/2}^{l/2} e^{2\pi\ i\ x\ y}\ dx \;=\; \int_{-\ \infty}^{\infty} e^{2\pi\ i\ x\ y}\ dx$$

(The limits, as usual, are to be taken outside any integral containing the delta fct.)

1.2 Inverse Laplace Transform

Letting $p = 2\pi\ i\ x$ in the inverse Fourier Tranform gives

$$\delta(y) \;=\; \frac{1}{2\pi\ i} \int_{-\ i\infty}^{i\infty} e^{p\ y}\ dp\ .$$

1.3 Inverse Mellin Transform

In $\delta(z-w) \;=\; \frac{1}{2\pi\ i} \int_{-\ i\infty}^{i\infty} e^{\ p\ (\ z-w)}\ dp$, let

$z = \ln(y)$, $w = \ln(x)$, and $p = s$ with $x, y > 0$, giving

$$\delta[\ln(y) - \ln(x)] \;=\; \frac{1}{2\pi\ i} \int_{-\ i\infty}^{i\infty} y^{s}\ x^{-s}\ ds\ .$$ But,

$$\delta[f(u)] = \sum_{u_i=\text{zeros of } f} \delta[\tfrac{df(u_i)}{du}\,(u-u_i)] = \sum_{\text{zeros of } f} \delta[\,(u-u_i)]\,/\,|\tfrac{df(u_i)}{du}|$$, so

$\delta[\ln(y)-\ln(x)] = \delta[\ln(x)-\ln(y)] = y\,\delta[x-y]$. Therefore,

$$\delta(y-x) = \delta(x-y) = \frac{1}{2\pi\, i} \int_{-\, i\infty}^{i\infty} y^{s-1}\, x^{-s}\, ds$$, the inverse Mellin Transform rep.

Then formally, with $H(x)$ the Heaviside step function,

$$H(x)\, f(x) = \int_0^{\infty} f(y)\,\delta(y-x)\, dy = \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} \int_0^{\infty} f(y)\, y^{s-1}\; dy\, x^{-s}\, ds$$

$$= \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} \bar{f}(s)\, x^{-s}\, ds$$, i.e., the inverse Mellin Transform of $\bar{f}(s)$,

identifying the Mellin transform of $f(x)$ as

$\bar{f}(s) = \int_0^{\infty} f(x)\, x^{s-1}\; dx$. (An appropriate contour must be chosen to obtain the desired $f(x)$ using the inverse Mellin transform.)

Example I: Window Function

$$H(y)\, H(1-y) = \int_0^1 \delta(y-x)\; dx = H(y) \int_0^1 \{\frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} y^{s-1} x^{-s}\; ds\}\, dx$$

$$= H(y)\; \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} y^{s-1}\{ \int_0^1 x^{-s}\;\; dx\}\, ds$$

$$= H(y)\; \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} y^{s-1}\{\frac{1}{1-s}\}\, ds \quad \text{for } \sigma < 1$$

$$= H(y)\, \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} y^{-s}\{\frac{1}{s}\}\, ds \quad \text{for } \sigma > 0$$

where $H(x)$ is the Heaviside step function that vanishes for $x < 0$, equals $\frac{1}{2}$ for $x = 0$, and is 1 for $x > 0$.

Conversely,

$$\bar{f}(s) = \int_0^\infty H(1-x)\, x^{s-1}\ dx\ = \int_0^1 x^{s-1}\ dx = \tfrac{1}{s} \text{ for } Re(s) > 0\,.$$

Example II: Dirichlet Series

Let $f(x) = \sum_{n=1}^{\infty} a_n\, \delta(n\,x-1)$

then $\bar{f}(s)\ = \sum_{n=1}^{\infty} a_n\ \int_0^\infty \delta(n\,x-1)\, x^{s-1}\ dx$

$= \sum_{n=1}^{\infty} a_n\ \int_0^\infty \frac{\delta(\,x-\frac{1}{n})}{n} x^{s-1}\ dx\ = \sum_{n=1}^{\infty} a_n\, n^{-s}$, a Dirichlet series.

Conversely,

$$f(x) = \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} \bar{f}(s)\, x^{-s}\ ds\ = \sum_{n=1}^{\infty} a_n\ \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} n^{-s}\, x^{-s}\ ds$$

$$= \sum_{n=1}^{\infty} a_n\ \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+\, i\infty} \frac{(1/n)^{s-1}}{n} x^{-s}\ ds = \sum_{n=1}^{\infty} a_n \frac{\delta(\,x-\frac{1}{n})}{n} = \sum_{n=1}^{\infty} a_n\, \delta(n\,x-1)\ .$$

2 Interpolation and Ramanujan's Master Formula

With the finite difference operator $\nabla_n^{s-1} a_n = \sum_{n=0}^{\infty} (-1)^n \binom{s-1}{n}\ a_n$,

$\nabla_n^{s-1} \nabla_j^{n} \frac{x^j}{j!}\ = \frac{x^{s-1}}{(s-1)!}$, the Newton interpolation of $\frac{x^j}{j!}$ for $Re(s) > 0$.

Defining the modifed Mellin Transform and its inverse as

$$\tilde{f}(s) = \int_0^\infty f(x)\, \frac{x^{s-1}}{(s-1)!}\ dx \ \text{ and}$$

$$H(x)\, f(x) = \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} (s-1)!\ \ \tilde{f}(s)\, x^{-s}\ ds\ = \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)}\, \tilde{f}(s)\, \frac{x^{-s}}{(-s)!} ds\ ,$$

then formally

$$\tilde{f}(s) \;=\; \int_0^\infty f(x)\,\frac{x^{s-1}}{(s-1)!}\;dx \;=\; \int_0^\infty f(x)\,\nabla_n^{s-1}\nabla_j^n\frac{x^j}{j!}\;dx$$

$= \nabla_n^{s-1}\nabla_j^n \int_0^\infty f(x)\,\frac{x^j}{j!}\;dx = \nabla_n^{s-1}\nabla_j^n\;\tilde{f}(j+1)$, a Newton interpolation, and

$H(x)\,f(x) = H(x)\sum_{n=0}^{\infty}\,(-1)^n\,\tilde{f}(-n)\,\frac{x^n}{n!}$ <RMF>

by closing the inversion contour to the left with $\mathrm{Re}(\sigma) > 0$ and picking up the singularities of $\frac{\pi}{\sin(\pi s)}$ if $f(x)$ is such that $\tilde{f}(s)$ has no singularities and decays sufficiently fast in the neighborhood of infinity.

Example III: Exponential Fct.

$\tilde{f}(s) = \text{Finite Part}\int_0^\infty e^{-x}\,\frac{x^{s-1}}{(s-1)!}\;dx$

$= \int_0^\infty \{e^{-x} - [\sum_{j=0}^{n}\frac{(-x)^j}{j!}]\}\,\frac{x^{s-1}}{(s-1)!}\,dx = 1$ for $n-1 < Re(s) < n$

$= \int_0^\infty e^{-x}\,\frac{x^{s-1}}{(s-1)!}\,dx = 1$ for $Re(s) > 0$.

$$H(x)\,\{e^{-x} - [\sum_{j=0}^{n}\frac{(-x)^j}{j!}]\} = \frac{1}{2\pi\, i}\int_{\sigma-\,i\infty}^{\sigma+i\infty}(s-1)!\;x^{-s}\,ds = \frac{1}{2\pi\, i}\int_{\sigma-\,i\infty}^{\sigma+i\infty}\frac{\pi}{\sin(\pi s)}\,\frac{x^{-s}}{(-s)!}ds$$

for $n-1 < \sigma < n$.

$$H(x)\,e^{-x} = \frac{1}{2\pi\, i}\int_{\sigma-\,i\infty}^{\sigma+i\infty}(s-1)!\;x^{-s}\,ds = \frac{1}{2\pi\, i}\int_{\sigma-\,i\infty}^{\sigma+i\infty}\frac{\pi}{\sin(\pi s)}\,\frac{x^{-s}}{(-s)!}ds$$

for $\sigma > 0$.

$\tilde{f}(s) = \nabla_n^{s-1}\nabla_j^n\;\tilde{f}(j+1) = \nabla_n^{s-1}\nabla_j^n\;1 = 1$ for all s .

3 Bell / Exponential / Touchard Polynomials

For the number operator $x\,\frac{d}{dx}$, $\frac{x^{-s}}{(-s)!}$ is an eigenfunction with eigenvalue $-s$, i.e.,

$x\,\frac{d}{dx}\,\frac{x^{-s}}{(-s)!} = -s\frac{x^{-s}}{(-s)!}$.

For $\sigma > 0$ and n , a natural number,

$$H(x)\,e^{x}\,[x\,\tfrac{d}{dx}]^{n}\exp(-x) = H(x)\,e^{x}\,[x\,\tfrac{d}{dx}]^{n}\tfrac{1}{2\pi\, i}\int_{\sigma-\,i\infty}^{\sigma+i\infty}\tfrac{\pi}{\sin(\pi s)}\,\tfrac{x^{-s}}{(-s)!}\,ds$$

$$= H(x)\,e^{x}\tfrac{1}{2\pi\, i}\int_{\sigma-\,i\infty}^{\sigma+i\infty}\tfrac{\pi}{\sin(\pi s)}(-s)^{n}\,\tfrac{x^{-s}}{(-s)!}\,ds$$

$$= H(x)\,e^{x}\sum_{j=0}^{\infty}(-1)^{j}j^{n}\tfrac{x^{j}}{j!} = H(x)\sum_{j=0}^{n}\nabla_{k}^{j}\;k^{n}\tfrac{x^{j}}{j!} = H(x)\phi_{n}(-x)$$

$= H(x)\,e^{x}\phi_{n}(:x\frac{d}{dx}:)\;e^{-x}$, where $\phi_n(x)$ is the n'th *Bell polynomial* ,

$(:x\frac{d}{dx}:)^{k} = x^{k}\frac{d^{k}}{dx^{k}}$.

(The binomial transform has been used above:

$e^{x}e^{-a.\;x} = e^{(1-a.\;)x}$ or $e^{x}\sum_{j=0}^{\infty}(-1)^{j}a_{j}\frac{x^{j}}{j!} = \sum_{j=0}^{\infty}\nabla_{k}^{j}\;a_{k}\frac{x^{j}}{j!}$).

Conversely,

$$MMT[e^{-x}\phi_{n}(-x)] = \int_{0}^{\infty}e^{-x}\phi_{n}(-x)\,\tfrac{x^{s-1}}{(s-1)!}\,dx = (-s)^{n} \text{ for } Re(s) > -1$$

$= \nabla_{k}^{s-1}\nabla_{j}^{k}\;[-(j+1)]^{n}$ for all s , agreeing with the Newton interpolator

$\tilde{f}(s) = \nabla_{k}^{s-1}\nabla_{j}^{k}\;\tilde{f}(j+1)$.

To determine the exponential generating fct. for the polynomials consider

$$H(x)\exp[t\,\phi_{.}(-x)] = H(x)\,e^{x}\exp[t\,\phi_{.}(:x\,\tfrac{d}{\mathrm{dx}}:)]\,e^{-x}$$

$$= H(x)\,e^{x}\exp(t\,x\,\tfrac{d}{dx})\,e^{-x} = e^{x}\exp(t\,x\,\tfrac{d}{dx})\,\tfrac{1}{2\pi\, i}\int_{\sigma-\,i\infty}^{\sigma+i\infty}\tfrac{\pi}{\sin(\pi s)}\,\tfrac{x^{-s}}{(-s)!}\,ds$$

$$= H(x)\, e^{x} \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} \exp(-t\, s) \frac{x^{-s}}{(-s)!}\, ds \text{ for } Re(s) > 0$$

$$= H(x)\, e^{x} \,\, [\sum_{j=0}^{\infty} (-1)^{j}\, e^{t\,j} \frac{x^{j}}{j!}] = H(x) \exp[-x\,(e^{t}-1)]$$

$$= H(x)\, e^{x} \exp[\,(e^{t}-1)(:x\, \frac{d}{\mathrm{dx}}:)]\, e^{-x}\ .$$

So also

$$\exp(t\, x\, \tfrac{d}{dx})\, G(x) = \exp[t\, \phi_{.}(:x\, \tfrac{d}{dx}:)]\, G(x) = \exp[\,(e^{t}-1)(:x\, \tfrac{d}{\mathrm{dx}}:)]\, G(x)$$

$$= G(x\, e^{t})\ .$$

4 Generalized Dobinski Relations: Confluent Hypergeometric Functions / Generalized Laguerre Functions

The results of the previous sections can formally be generalized operationally as

$$H(x)\ e^{x}\ f(x\, \tfrac{d}{dx})\ e^{-x} \ = H(x)\ e^{x}\ \ f(x\, \tfrac{d}{dx})\, \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} \frac{x^{-s}}{(-s)!}\, ds$$

$$= H(x)\ e^{x} \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} f(-s) \frac{x^{-s}}{(-s)!}\, ds$$

$$= H(x)\ e^{x}\ f[\,\phi_{.}(:x\, \tfrac{d}{\mathrm{dx}}:)]\, e^{-x}$$

$$= H(x)\ f[\,\phi_{.}(-x)]\ \ ,$$

giving a *generalized Dobinski relation*,

$$H(x)\ f[\,\phi_{.}(-x)] = H(x)\, e^{x} \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} f(-s) \frac{x^{-s}}{(-s)!}\, ds \quad ,$$

with, for $Re(s) > \sigma$, the Mellin transform giving the *umbral hybrid Laplace-Mellin transform*

$$f(-s) = \int_{0}^{\infty} e^{-x}\, f[\,\phi_{.}(-x)]\, \frac{x^{s-1}}{(s-1)!}\ dx\ .$$

Checking for consistency, substitute $-\phi_.(-y)$ for s , then formally

$$H(y)\, f(\phi_.(-y)) = \int_0^\infty e^{-x}\, f[\phi_.(-x)] \,:\, \frac{x^{-\phi_.(-y)-1}}{[-\phi_.(-y)-1-1]!} \,:\, dx$$

$$= \int_0^\infty e^{-x}\, f[\phi_.(-x)]\, e^y \sum_{n=0}^\infty (-1)^n \frac{x^{-n-1}}{(-n-1)!} \frac{y^n}{n!}\, dx$$

$$= \int_0^\infty e^{-x}\, f[\phi_.(-x)]\, e^y \sum_{n=0}^\infty (-1)^n \delta^n(x) \frac{y^n}{n!}\, dx$$

$$= \int_0^\infty e^{-x}\, f[\phi_.(-x)]\, e^y \exp(-y\tfrac{d}{\mathrm{dx}})\, \delta(x)\, dx$$

$$= \int_0^\infty e^{-x}\, f[\phi_.(-x)]\, e^y\, \delta(x-y)\, dx$$

$= H(y)\; f[\phi_.(-y)]$, and also

$$: \frac{x^{-\phi_.(-y)-1}}{[-\phi_.(-y)-1-1]!} := H(y)\, e^y \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} \frac{x^{s-1}}{(s-1)!} \frac{y^{-s}}{(-s)!} ds$$

$$= H(y)\, e^y \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{1}{x}\left(\frac{y}{x}\right)^{-s} ds = H(y)\, e^y\, \delta(x-y) \;.$$

Letting $f(x) = \binom{x+\alpha+\beta}{\beta}$, then

$$f(\phi_.(-x)) = \binom{\phi_.(-x)+\alpha+\beta}{\beta} = \binom{\alpha+\beta}{\beta} K(-\beta, \alpha+1, x) = L_\beta^\alpha(x)$$

where $K(a,b,x)$ is *Kummer's confluent hypergeometric function* and $L_\beta^\alpha(x)$, a *generalized Laguerre function*. Then

$$H(x)\, L_\beta^\alpha(x) = H(x) \binom{\phi_.(-x)+\alpha+\beta}{\beta} = H(x) e^x \sum_{n=0}^\infty (-1)^n \binom{n+\alpha+\beta}{\beta} \frac{x^n}{n!}$$

$$= H(x) \sum_{n=0}^\infty \nabla_j^n \binom{j+\alpha+\beta}{\beta} \frac{x^n}{n!} = H(x) \sum_{n=0}^\infty (-1)^n \binom{\alpha+\beta}{\alpha+n} \frac{x^n}{n!}$$

$$= H(x)\, e^x \frac{1}{2\pi\, i} \int_{\sigma-\, i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} \binom{-s+\alpha+\beta}{\beta} \frac{x^{-s}}{(-s)!} ds \;.$$

For the *associated Newton interpolation* consider the Mellin transform again.

$$f(-s) = \int_0^\infty e^{-x}\, f[\,\phi_.(-x)]\, \frac{x^{s-1}}{(s-1)!}\;dx\;,$$

$$= \int_0^\infty e^{-x}\, e^{x} \sum_{n=0}^\infty (-1)^n f(n) \frac{x^n}{n!}\;\frac{x^{s-1}}{(s-1)!}\;dx$$

$$= \int_0^\infty e^{-x} \sum_{n=0}^\infty \nabla_j^n\; f(j) \frac{x^n}{n!}\;\frac{x^{s-1}}{(s-1)!}\;dx$$

$$= \sum_{n=0}^\infty \nabla_j^n\; f(j)\; \int_0^\infty e^{-x}\;\frac{x^n}{n!}\;\frac{x^{s-1}}{(s-1)!}\;dx$$

$$= \sum_{n=0}^\infty \nabla_j^n\; f(j)\; \binom{s-1+n}{n} = \sum_{n=0}^\infty (-1)^n \binom{-s}{n}\; \nabla_j^n\, f(j)$$

$$= \nabla_n^{-s} \nabla_j^n\, f(j)\;.$$

Summarizing, the associated Newton interpolation is

$$f(-s) = \nabla_n^{-s} \nabla_j^n\, f(j)\;,$$

or changing variables,

$$f(s-1) = \nabla_n^{s-1} \nabla_j^n\, f(j),$$

or translating the function,

$$f(s) = \nabla_n^{s-1} \nabla_j^n\, f(j+1)\quad.$$

Specifically for $f(x) = \binom{x+\alpha+\beta}{\beta}$, we walk through the steps again to see that the Chu-Vandermonde identity appears:

$$\binom{-s+\alpha+\beta}{\beta} = \int_0^\infty e^{-x}\, \binom{\phi_.(-x)+\alpha+\beta}{\beta}\, \frac{x^{s-1}}{(s-1)!}\;dx$$

$$= \int_0^\infty e^{-x} \sum_{n=0}^\infty (-1)^n \binom{\alpha+\beta}{\alpha+n}\; \frac{x^n}{n!}\; \frac{x^{s-1}}{(s-1)!}\;dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+\beta}{\alpha+n} \binom{n+s-1}{n}$$

$$= \sum_{n=0}^{\infty} \binom{\alpha+\beta}{\beta-n} \binom{-s}{n} = \sum_{n=0}^{\infty} (-\,1)^n \binom{-s}{n} \nabla_j^n \binom{j+\alpha+\beta}{\beta}$$

$$= \nabla_n^{-s} \nabla_j^n \binom{j+\alpha+\beta}{\beta}.$$

Substituting $1-s$ for s and $\alpha+1$ for α (or translating the fct.) gives

$$\binom{s+\alpha+\beta}{\beta} = \nabla_n^{s-1} \nabla_j^n \binom{j+1+\alpha+\beta}{\beta} ,$$

or for $f(x) = \binom{x+\alpha+\beta}{\beta}$,

$$f(s) = \nabla_n^{s-1} \nabla_j^n f(j+1) .$$

Also,

$$\binom{-s+\alpha+\beta}{\beta} = \int_0^{\infty} e^{-x} \binom{\phi_.(-x)+\alpha+\beta}{\beta} \frac{x^{s-1}}{(s-1)!} \, dx$$

$$= \int_0^{\infty} e^{-x} e^{x} \sum_{n=0}^{\infty} (-1)^n \binom{n+\alpha+\beta}{\beta} \frac{x^n}{n!} \frac{x^{s-1}}{(s-1)!} \, dx$$

$$= \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n \binom{n+\alpha+\beta}{\beta} \frac{x^n}{n!} \frac{x^{s-1}}{(s-1)!} \, dx ,$$

and consistently

$$\binom{j+\alpha+\beta}{\beta} = \int_0^{\infty} e^{-x} \binom{\phi_.(-x)+\alpha+\beta}{\beta} \frac{x^{-j-1}}{(-j-1)!} \, dx$$

$$= \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n \binom{n+\alpha+\beta}{\beta} \frac{x^n}{n!} \frac{x^{-j-1}}{(-j-1)!} \, dx$$

$$= \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n \binom{n+\alpha+\beta}{\beta} \frac{x^n}{n!} \, \delta^j(x) \, dx$$

$$= \int_0^{\infty} e^{-x} \binom{\phi_.(-x)+\alpha+\beta}{\beta} \delta^j(x) \, dx$$

$$= (-1)^j \frac{d^j}{dx^j} e^{-x} \binom{\phi_.(-x)+\alpha+\beta}{\beta} \Big|_{x=0}$$

Now apply the umbral compositional inverse of $\phi_n(-x)$ to these equations.

For $f(x) = x^n$, from *the generalized Dobinski relation*,

$$H(x)\ f(\phi_.(-x)) = H(x)\,\phi_n(-x) = H(x)\,e^x \frac{1}{2\pi\, i} \int_{\sigma-\,i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} (-s)^n \frac{x^{-s}}{(-s)!}\, ds$$

$$= H(x)e^x \sum_{j=0}^{\infty} (-1)^j j^n \frac{x^j}{j!} = H(x) \sum_{k=0}^{n} \nabla_j^k\ j^n \frac{x^k}{k!}\ .$$

Replace x umbrally by the the negative of *the falling factorial* $(x)_.$, then

$$\phi_n[(x)_.] = \sum_{k=0}^{n} \nabla_j^k\ j^n \frac{(-1)^k (x)_k}{k!} = \nabla_n^x \nabla_j^n\ j^n = x^n\ .$$

The last equality follows from the Mellin transform results above with $f(x) = x^n$:

$$f(-s) = \int_0^\infty e^{-y}\, f[\,\phi_.(-y)]\, \frac{y^{s-1}}{(s-1)!}\ dy\ ,$$

giving with $x = -s$,

$$x^n = \int_0^\infty e^{-y}\ \phi_n(-y)\ \frac{y^{-x-1}}{(-x-1)!}\ dy$$

$$= \int_0^\infty e^{-y} \sum_{k=0}^{n} \nabla_j^k\ j^n\ \frac{y^k}{k!}\ \frac{y^{-x-1}}{(-x-1)!}\ dy$$

$$= \nabla_n^x \nabla_j^n\ j^n\ .$$

So, the *falling factorial* is the *umbral compostional inverse* of the *Bell polynomial*.

Recalling the formulas from above

$$H(x)\ L_\beta^\alpha(x) = H(x) \binom{\phi_.(-x)+\alpha+\beta}{\beta} = H(x)e^x \sum_{n=0}^{\infty} (-1)^n \binom{n+\alpha+\beta}{\beta} \frac{x^n}{n!}$$

$$= H(x) \sum_{n=0}^{\infty} \nabla_j^n \binom{j+\alpha+\beta}{\beta} \frac{x^n}{n!} = H(x) \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+\beta}{\alpha+n} \frac{x^n}{n!}$$

$$= H(x)\ e^x \frac{1}{2\pi\, i} \int_{\sigma-\,i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi s)} \binom{-s+\alpha+\beta}{\beta} \frac{x^{-s}}{(-s)!}\, ds\ ,$$

and substituting the negative of the falling factorial $-(s)_.$ for x gives

$$\binom{s+\alpha+\beta}{\beta} = \nabla_n^s \nabla_j^n \binom{j+\alpha+\beta}{\beta} = \sum_{n=0}^{\infty} \binom{s}{n}\binom{\alpha+\beta}{\alpha+n}$$

and from $e^{-x} L_\beta^\alpha(x)$

$$\sum_{n=0}^{\infty} \binom{n+\alpha+\beta}{\beta}\binom{s}{n} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\pi}{\sin(\pi\omega)} \binom{-\omega+\alpha+\beta}{\beta}\binom{s}{-\omega} (-1)^{-\omega} d\omega \quad \text{?diverges?}$$

$$= \sum_{n=0}^{\infty} \binom{s}{n} \sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+\beta}{\alpha+k} = \nabla_n^s (-1)^n \nabla_k^n (-1)^k \binom{\alpha+\beta}{\alpha+k}$$

$$= \frac{\sin(\pi\beta)}{\sin(\pi(\alpha+\beta))} \sum_{n=0}^{\infty} (-1)^n \binom{s}{n}\binom{-(\beta+1)}{\alpha+n}.$$

Note: $\sum_{n=0}^{\infty} \binom{n+\alpha+\beta}{\beta}\binom{s}{n} = \frac{\sin(\pi\beta)}{\sin(\pi(\alpha+\beta))} \sum_{n=0}^{\infty} (-1)^n \binom{s}{n}\binom{-\beta-1}{\alpha+n}$ can be obtained

by repeated use of $\frac{\sin(\pi\lambda)}{(\pi\lambda)} = \frac{1}{\lambda!(-\lambda)!}$ or by using the Kummer transformation (pg. 505, Abramowitz and Stegun, Handbook of Mathematical Functions) to obtain

$$e^{-x} L_\beta^\alpha(x) = \frac{\sin(\pi\beta)}{\sin(\pi(\alpha+\beta))} L^\alpha_{-(\alpha+\beta+1)}(-x).$$

4.0.1 Exercises

A) Confirm that $\int_0^\infty e^{-t} \frac{t^{\phi.(-x)+\alpha}}{(\phi.(-x)+\alpha)!} \frac{t^\beta}{\beta!} \, dt = \binom{\phi.(-x)+\alpha+\beta}{\beta} = L_\beta^\alpha(x).$

B) Show that $\frac{t^{-\phi.(-x)-1}}{(-\phi.(-x)-1)!} = e^t \ \delta(x-t) = e^x \ \delta(x-t).$

C) Show that $\binom{-\phi.(-x)-1+\beta}{\beta} = \frac{x^\beta}{\beta!}, \quad (\mathrm{Re}(\beta) > -1, x > 0).$

D) Show that $\binom{-\phi.(-x)-\alpha-1}{n} = (-1)^n L_n^\alpha(x) = (-1)^n \frac{U(-n,\alpha+1,x)}{n!}$
where $U(\alpha,\beta,x)$ is the *Tricomi confluent hypergeometric function*.

E) Show that $\psi(\alpha,\beta,x) = \binom{-\phi.(-x)-\beta}{-\alpha} = \frac{U(\alpha,\beta,x)}{(-\alpha)!}$, so that

$$\psi(-n,\alpha+1,x) = (-1)^n L_n^\alpha(x).$$

F) Using $\sum_{n=-\infty}^{\infty} \frac{\sin(\pi(\lambda-n)}{\pi(\lambda-n)} \binom{\nu}{\mu+n} = \binom{\nu}{\mu+\lambda}$ $\quad (\mathrm{Re}(\nu) > -1)$,

show that $\sum_{n=0}^{\infty} (-1)^n \frac{\sin(\pi(\alpha-n)}{\pi(\alpha-n)} L_n^{-\alpha}(x) = \frac{x^\alpha}{\alpha!}$ $\quad (\alpha > -1, x > 0)$.

G) Interpret $\quad \frac{1}{(\beta)!} D_t^\beta t^{\phi.(-x)+\beta+\alpha}\Big|_{t=1}$

$$= \frac{1}{(\beta)!}[: D_t t :]^\beta t^{\phi.(-x)+\alpha}\Big|_{t=1}$$

where $D_t^\beta \frac{t^\lambda}{\lambda!} = \frac{t^{\lambda-\beta}}{(\lambda-\beta)!} = FP \int_0^t \frac{(t-z)^{-\beta-1}}{(-\beta-1)!} \frac{z^\lambda}{\lambda!}\, dz$ $\quad$ (*FP=Finite Part*)

$$= FP \frac{1}{2\pi\, i} \oint_{|z-t|=t} \frac{\beta!}{(z-t)^{\beta+1}} \frac{z^\lambda}{\lambda!}\, dz \quad \text{and } [: D_t t :]^\beta = D_t^\beta t^\beta.$$

H) Interpret $\quad \frac{1}{(-\alpha)!} W_t^{-\alpha} t^{\phi.(-x)+\beta-\alpha-1}\Big|_{t=1}$

$$= \frac{1}{(-\alpha)!}[: W_t t :]^{-\alpha} t^{\phi.(-x)+\beta-1}\Big|_{t=1}$$

where $W_t^{-\alpha} \cdot = FP \int_t^\infty \frac{(z-t)^{\alpha-1}}{(\alpha-1)!} \cdot \; dz$ $\quad$ and $[: W_t t :]^{-\alpha} = W_t^{-\alpha} t^{-\alpha}$.

I) Interpret $\quad \frac{1}{(-\alpha)!} D_t^{-\alpha} t^{-\phi.(-x)-\beta}\Big|_{t=1}$

$$= \frac{1}{(-\alpha)!}[: D_t t :]^{-\alpha} t^{-\phi.(-x)-\beta+\alpha}\Big|_{t=1}.$$

J) Prove the relation

$$L_\beta^\alpha(x+y) = e^y \sum_{k=0}^{\infty} (-1)^k \; L_\beta^{\alpha+k}(x) \frac{y^k}{k!} = \; L_\beta^{\alpha+\phi.(-y)}(x)$$

$$= \binom{\phi.(-x-y)+\alpha+\beta}{\beta} = \binom{\phi.(-x) \; + \phi.(-y)+\alpha+\beta}{\beta}$$

starting with

$$\binom{s+\alpha+\beta}{\beta} = \nabla_n^s \nabla_j^n \; \binom{j+\alpha+\beta}{\beta}.$$

K) Explore $\; n! \binom{\phi.(-x)-1+n}{n} = n!\, L_n^{-1}(x)$ $\;$ and $\; n! \binom{\phi.(-x)+1+n}{n} = n! L_n^1(x)$.
Which of the above expressions are more fundamental, .i.e., are valid as they stand, for these two cases?