Phase Transitions and Renormalization Group: from Theory to Numbers

Jean Zinn-Justin
Service de Physique Théorique*
CEA/Saclay
F-91191 Gif-sur-Yvette Cedex
France

Abstract. During the last century, in two apparently distinct domains of physics, the theory of fundamental interactions and the theory of phase transitions in condensed matter physics, one of the most basic ideas in physics, the decoupling of physics on different length scales, has been challenged. To deal with such a new situation, a new strategy was invented, known under the name of renormalization group. It has allowed not only explaining the survival of universal long distance properties in a situation of coupling between microscopic and macroscopic scales, but also calculating precisely universal quantities.

We here briefly recall the origin of renormalization group ideas; we describe the general renormalization group framework and its implementation in quantum field theory. It has been then possible to employ quantum field theory methods to determine many universal properties concerning the singular behaviour of thermodynamical quantities near a continuous phase transition. Results take the form of divergent perturbative series, to which summation methods have to be applied. The large order behaviour analysis and the Borel transformation have been especially useful in this respect.

As an illustration, we review here the calculation of the simplest quantities, critical exponents.

More details can be found in the work

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press 1989, (Oxford 4th ed. 2002).

1 Renormalization group: Motivation and basic ideas

During the last century, in two apparently distinct domains of physics, the theory of fundamental interactions and the theory of phase transitions in condensed matter physics, one of the most basic ideas in physics has been challenged:

We have all been taught that physical phenomena should be described in terms of degrees of freedom adapted to their typical scale. For instance, we conclude from dimensional considerations that the period of the pendulum scales like the square root of its length. This result implicitly assumes that other lengths in the problem, like the size of constituent atoms or the radius of the earth, are not relevant because they are much too small or much too large. In the same way, in newtonian mechanics the motion of planets around the sun can be determined, to a very good approximation, by considering planets and sun as point-like, because their sizes are very small compared with the size of the orbits.

It is clear that if this property also called *the decoupling of different scales of physics*, would not generally hold, progress in physics would have been very slow, maybe even impossible.

However, starting from about 1930, it was discovered that the quantum extension of Electrodynamics was plagued with infinities due to the point-like nature of the electron. The basic reason for this disease, the non-decoupling of scales, was understood only much later, but in the mean time physicists had discovered empirically a recipe to do finite calculations, called *renormalization*. Superficially, the renormalization idea is conventional: to describe physics, use parameters adapted to the scale of observation, like the observed strength of the electromagnetic interaction and the observed mass of the electron, rather than the initial parameters of the quantum lagrangian. However,
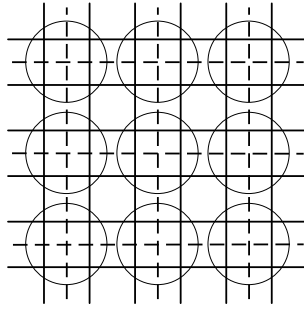

Figure 1: Initial lattice and lattice with double spacing.

there remained two peculiarities, the relation between initial parameters and so-called renormalized parameters involved infinities and the values of the renormalized parameters varied with the length or energy scale at which they were defined. This effect was eventually observed very directly in experiments; for example, the fine structure constant $\alpha = e^2/4\pi\hbar c$ is about $1/137$ at the scale given by the electron mass, but increases to $1/128$ at the scale of Z vector boson mass (one of the particles mediating weak interactions). The relation between the strength of interactions at different scales was called renormalization group (RG).

Later, similar difficulties were discovered in another branch of physics, in the study of continuous phase transitions (liquid–vapour, ferromagnetic, superfluid helium). Near a continuous phase transition a length, called the *correlation length*, becomes very large. This means that dynamically a length scale is generated, which is much larger than the scale characterizing the microscopic interactions. In such a situation, some non-trivial macroscopic physics is generated and it could have been expected that phenomena at the scale of the correlation length could be described by a small number of degrees of freedom adapted to this scale. Such an assumption leads to universal quasi-gaussian or mean field critical behaviour, but it became slowly apparent that critical phenomena could not be described by mean field theory. Again the deep reason for this failure is the non-decoupling of scales, that is the initial microscopic scale is never completely forgotten.

Both in the theory of fundamental interactions and in statistical physics, this coupling of very different scales is the sign that an *infinite* number of "stochastic" (i.e. subject to quantum or statistical fluctuations) degrees of freedom are involved.

One could then have feared that even at large scales physics remained completely dependent on the initial microscopic interactions, rendering a predictive theory impossible. However, this is not what empirically was discovered. Instead, phenomena could be gathered in *universality classes* that shared a number of universal properties, a situation that indicated that only a limited number of qualitative properties of the initial microscopic interactions were important.

Remark. We have already referred to the correlation length without defining it. In statistical systems, the correlation length ξ describes the exponential decay of correlation functions in the disordered phase. For instance, for a system where the degrees of freedom are spins $S(x)$ at space position x, the two-point correlation function $\langle S(x)S(y)\rangle$ decays exponentially at large distance like

$$\frac{\ln \langle S(x)S(y)\rangle}{|x-y|} \underset{|x-y|\to\infty}{\sim} -\frac{1}{\xi}.$$

The renormalization group idea. To explain this puzzling situation a new concept had to be invented, which was given again the name of RG. The idea that we will shortly describe, involved determining inductively the effective interactions at a given scale. The relation between effective interactions at neighbouring scales is called a RG transformation. A way to construct such a RG was proposed initially by Kadanoff. One considers a statistical model initially defined in terms of classical spin variables on some lattice of spacing a and configuration energy $\mathcal{H}_a(S)$. The partition function is obtained by summing over all spin configurations with a Boltzmann weight $e^{-\mathcal{H}_a(S)/T}$. The idea then is to sum over the initial spins, keeping their average on the coarser lattice of spacing $2a$ fixed (figure 1). After this summation, the partition function is given by summing over the

average spins on a lattice of spacing $2a$ with an *effective* configuration energy $\mathcal{H}_{2a}(S)$. It is clear that this transformation can be iterated as long as the lattice spacing remains much smaller than the correlation length ξ that describes the decay of correlation functions. This defines effective hamiltonians $\mathcal{H}_{2^n a}(S)$ on lattices of spacing $2^n a$. The recursion relation

$$\mathcal{H}_{2^n a}(S) = \mathcal{T}\left[\mathcal{H}_{2^{n-1} a}(S)\right],$$

is a renormalization group transformation. If the transformation $\mathcal{T}$ has fixed points:

$$\mathcal{H}_{2^n a}(S) \underset{n \to \infty}{\to} \mathcal{H}^*(S),$$

or fixed surfaces, then both the non-gaussian behaviour and universality can be understood. Wilson transformed this idea based on an iterative summation of short distance degree of freedom, whose initial formulation was somewhat vague, into a more precise framework, replacing, in particular, RG in space by integration over large momenta in the Fourier representation. Wegner, Wilson and others then discovered exact functional RG equations in the continuum with fixed points.

However, these general equations do not provide a very efficient framework for finding fixed points and calculating explicitly universal quantities. On the other hand, it can be argued that the simplest universality classes contain some standard quantum field theories. Moreover, the field theory RG that had been identified previously, appeared as an asymptotic RG in the more general framework. Therefore, previously developed quantum field theory (QFT) techniques could be used to prove universality and devise efficient methods of calculation, a domain in which the Saclay group has been especially active.

A strong limitation of this strategy is that the construction is possible only when fixed points are gaussian or, in the sense of some external parameter, close to a gaussian fixed point. This explains the role of Wilson–Fisher's ε-expansion, where ε is the deviation from the dimension 4: in dimension 4, non-trivial IR fixed points relevant for many simple phase transitions merge with the gaussian fixed point.

Note, however, that a combination of clever tricks has allowed doing calculations also at fixed dimensions, like the physical dimension 3.

Finally, let me notice that the understanding of non-decoupling of scales and universality resulting from RG fixed points, has also led to an understanding of the renormalization procedure in the theory of fundamental interactions. The quantum field theory that describes almost all known phenomena in particle physics except gravitation (the Standard Model) is now viewed as an *effective* low energy theory in the RG sense, and the cut-off as the remnant of some initial still unknown microscopic physics.

2   Renormalization Group: The General Idea

Even, if initially a statistical model is defined in terms of lattice variables taking a discrete set of values, asymptotically after RG transformations, the averaged variables will have a continuous distribution, and space will also be continuous. Therefore, RG fixed points belong to the class of statistical field theories in the continuum.

We thus consider a general statistical model defined in terms of some, translation invariant, hamiltonian $\mathcal{H}(\phi)$, function of a field $\phi(x)$ $(x \in \mathbb{R}^d)$, which is assumed to be expandable in powers of ϕ:

$$\mathcal{H}(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 d^d x_2 \ldots d^d x_n \, \mathcal{H}_n(x_1, x_2, \ldots, x_n) \phi(x_1) \ldots \phi(x_n), \qquad (2.1)$$

and has all the properties of the thermodynamic potential of Landau's theory. For example, the Fourier transforms of the functions $\mathcal{H}_n$, after factorization of a δ function of momentum conservation, are regular at low momenta (assumption of short-range forces or locality). In this framework, the space of all possible hamiltonians is infinite dimensional.

To a hamiltonian $\mathcal{H}(\phi)$ (really a configuration energy), corresponds a set of connected correlation functions $W^{(n)}(x_1, \ldots, x_n)$:

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \left[\int [\mathrm{d}\phi]\, \phi(x_1) \ldots \phi(x_n)\, \mathrm{e}^{-\beta \mathcal{H}(\phi)} \right]_{\text{connect.}}. \tag{2.2}$$

Connected correlation functions decay at large distance. One of the central problems is the determination of the long distance behaviour of correlation functions, that is the behaviour of $W^{(n)}(\lambda x_1, \ldots, \lambda x_n)$ when the dilatation parameter λ becomes large, near a continuous phase transition. In what follows we will only discuss critical correlation functions, that is correlation functions at the critical temperature where the correlation length is infinite ($T = T_c$, $\xi = \infty$), although universal behaviour extends to the neighbourhood of the critical temperature where the correlation length is large.

2.1   The renormalization group idea. Fixed points

The RG idea is to trade the initial problem, studying the behaviour of correlation functions as a function of dilatation parameter λ acting on space variables, for the study of the flow of a scale-dependent hamiltonian $\mathcal{H}_\lambda(\phi)$ which has essentially the same correlation functions at fixed space positions. More precisely, one wants to construct a hamiltonian $\mathcal{H}_\lambda(\phi)$ which has correlation functions $W_\lambda^{(n)}(x_i)$ satisfying

$$W_\lambda^{(n)}(x_1, \ldots, x_n) = Z^{-n/2}(\lambda) W^{(n)}(\lambda x_1, \ldots, \lambda x_n). \tag{2.3}$$

The mapping $\mathcal{H}(\phi) \mapsto \mathcal{H}_\lambda(\phi)$ is called a RG transformation. We define the transformation such that $\mathcal{H}_{\lambda=1}(\phi) \equiv \mathcal{H}(\phi)$. The choice of the function $Z(\lambda)$ depends on RG transformations.

In the case of models invariant under space translations, equation (2.3) after a Fourier transformation reads

$$\widetilde{W}_\lambda^{(n)}(p_1, \ldots, p_n) = Z^{-n/2}(\lambda) \lambda^{(1-n)d} \widetilde{W}^{(n)}(p_1/\lambda, \ldots, p_n/\lambda). \tag{2.4}$$

The simplest such RG transformation corresponds to rescalings of space and field. However, this transformation has a fixed point only in exceptional cases (gaussian models) and thus more general transformations have to be considered.

The fixed point hamiltonian. Let us assume that a RG transformation has been found such that, when λ becomes large, the hamiltonian $\mathcal{H}_\lambda(\phi)$ has a limit $\mathcal{H}^*(\phi)$, the fixed point hamiltonian. If such a fixed point exists in hamiltonian space, then the correlation functions $W_\lambda^{(n)}$ have corresponding limits $W_*^{(n)}$ and equation (2.3) becomes

$$W^{(n)}(\lambda x_1, \ldots, \lambda x_n) \underset{\lambda \to \infty}{\sim} Z^{n/2}(\lambda) W_*^{(n)}(x_1, \ldots, x_n). \tag{2.5}$$

We now introduce a second scale parameter μ and calculating $W^{(n)}(\lambda \mu x_i)$ from equation (2.5) in two different ways, we obtain a relation involving only $W_*^{(n)}$:

$$W_*^{(n)}(\mu x_1, \ldots, \mu x_n) = Z_*^{n/2}(\mu) W_*^{(n)}(x_1, \ldots, x_n) \tag{2.6}$$

with

$$Z_*(\mu) = \lim_{\lambda \to \infty} Z(\lambda \mu)/Z(\lambda). \tag{2.7}$$

Equation (2.6) being valid for arbitrary μ immediately implies that Z_* forms a representation of the dilatation semi-group. Thus, under reasonable assumptions,

$$Z_*(\lambda) = \lambda^{-2d_\phi}. \tag{2.8}$$

The fixed point correlation functions have a power law behaviour characterized by a positive number d_ϕ which is called the dimension of the field or order parameter $\phi(x)$.

Returning now to equation (2.7), we conclude that $Z(\lambda)$ also has asymptotically a power law behaviour. Equation (2.5) then shows that the correlation functions $W^{(n)}$ have a scaling behaviour at large distances:

$$W^{(n)}(\lambda x_1, \ldots, \lambda x_n) \underset{\lambda \to \infty}{\sim} \lambda^{-nd_\phi} W_*^{(n)}(x_1, \ldots, x_n) \tag{2.9}$$

with a power d_ϕ which is a property of the fixed point. The r.h.s. of the equation, which determines the critical behaviour of correlation functions, therefore, depends only on the fixed point hamiltonian. In other words, the correlation functions corresponding to all hamiltonians which flow after RG transformations into the same fixed point, have the same critical behaviour. This property is an example of *universality*. The space of hamiltonians is thus divided into *universality classes*. Universality, beyond the gaussian theory, relies upon the existence of IR fixed points in the space of hamiltonians.

2.2   Hamiltonian flows. Scaling operators

Let us consider an infinitesimal dilatation which leads from the scale λ to the scale $\lambda(1 + \mathrm{d}\lambda/\lambda)$. The variation of the hamiltonian $\mathcal{H}_\lambda$, consistent with equation (2.5), takes the form of a differential equation which involves a mapping $\mathcal{T}$ of the space of hamiltonians into itself and a real function η defined on the space of hamiltonians:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}_\lambda = \mathcal{T}[\mathcal{H}_\lambda], \tag{2.10}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = -2d_\phi[\mathcal{H}_\lambda]. \tag{2.11}$$

Equation (2.10) is a RG transformation in differential form. Moreover, we look only for markovian flows as a function of the "time" $\ln \lambda$, that is such that $\mathcal{T}$ does not depend on λ.

A fixed point hamiltonian $\mathcal{H}^*$ is then a solution of the fixed point equation

$$\mathcal{T}[\mathcal{H}^*] = 0. \tag{2.12}$$

The dimension d_ϕ of the field ϕ follows

$$d_\phi = d_\phi[\mathcal{H}^*]. \tag{2.13}$$

Linearized flow equations. To study the local stability of fixed points, we apply the RG transformation (2.10) to a hamiltonian $\mathcal{H}_\lambda = \mathcal{H}^* + \Delta\mathcal{H}_\lambda$ close to the fixed point $\mathcal{H}^*$. The linearized RG equation takes the form

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \Delta\mathcal{H}_\lambda = L^*(\Delta\mathcal{H}_\lambda), \tag{2.14}$$

where L^* is a linear operator, also independent of λ, acting on hamiltonian space. Let us assume that L^* has a discrete set of eigenvalues l_i corresponding to a set of eigenoperators $\mathcal{O}_i$. Then, $\Delta\mathcal{H}_\lambda$ can be expanded on the $\mathcal{O}_i$'s:

$$\Delta\mathcal{H}_\lambda = \sum h_i(\lambda)\mathcal{O}_i, \tag{2.15}$$

and the transformation (2.14) becomes

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} h_i(\lambda) = l_i h_i(\lambda) \ \Rightarrow \ h_i(\lambda) = \lambda^{l_i} h_i(1). \tag{2.16}$$

Classification of eigenvectors or scaling fields. The eigenvectors $\mathcal{O}_i$ can be classified into four families depending on the corresponding eigenvalues l_i:

(i) Eigenvalues with a positive real part. The corresponding eigenoperators are called *relevant*. If $\mathcal{H}_\lambda$ has a component on one of these operators, this component will grow with λ, and $\mathcal{H}_\lambda$ will move away from the neighbourhood of $\mathcal{H}^*$. Operators associated with a deviation from criticality are clearly relevant since a dilatation decreases the effective correlation length.

(ii) Eigenvalues with $\mathrm{Re}(l_i) = 0$. Then, two situations can arise: either $\mathrm{Im}(l_i)$ does not vanish, and the corresponding component has a periodic behaviour, or $l_i = 0$. Eigenoperators corresponding to a vanishing eigenvalue are called *marginal*. To determine the behaviour of the corresponding component h_i, it is necessary to expand beyond the linear approximation. Generically, one finds

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} h_i(\lambda) \sim B h_i^2 \,. \tag{2.17}$$

Depending on the sign of the constant B and the initial sign of h_i, the fixed point then is marginally unstable or stable. In the latter case, the solution takes for λ large the form

$$h_i(\lambda) \sim -1/(B \ln \lambda) \,. \tag{2.18}$$

A marginal operator generally leads to a logarithmic approach to a fixed point. In section 3.2, we show that in the ϕ^4 field theory, the operator $\phi^4(x)$ is marginally irrelevant in four dimensions.

An exceptional example is provided by the XY model in two dimensions ($O(2)$ symmetric non-linear σ-model) which instead of an isolated fixed point, has a line of fixed points. The operator which corresponds to a motion along the line is obviously marginal.

(iii) Eigenvalues with a negative real part. The corresponding operators are called *irrelevant*. The effective components on these operators go to zero for large dilatations.

(iv) Finally, some operators do not affect the physics. An example is provided by the operator realizing a constant multiplicative renormalization of the dynamical variables $\phi(x)$. These operators are called *redundant*. In QFT, quantum equation of motions correspond to redundant operators with vanishing eigenvalue.

Classification of fixed points. Fixed points can be classified according to their local stability properties, that is, to the number of relevant operators. This number is also the number of conditions a general hamiltonian must satisfy to belong to the surface which flows into the fixed point.

The critical domain. Universality is not limited to the critical theory. For temperatures close to T_c, and more generally for theories in which the hamiltonian is the sum of a critical hamiltonian and a linear combination of relevant operators with very small amplitudes, universal properties can be derived. Indeed, for small dilatations, the RG flow is hardly affected. After some large dilatation, the flow starts deviating substantially from the flow of the critical hamiltonian. But at this point the components of the hamiltonian on all irrelevant operators are already small.

This argument indicates that the behaviour of correlation functions as a function of amplitudes of relevant operators is universal in the limit of asymptotically small amplitudes. One calls *critical domain* the domain of parameters in which universality can be expected.

3   Critical behaviour: The effective ϕ^4 field theory

In the discussion, we restrict ourselves to Ising-like systems, the field ϕ having only one component. A generalization to the N-vector model with $O(N)$ symmetry is straightforward.

The main difficulty with the general RG approach is that it requires an explicit construction of RG transformations for hamiltonians, which have a chance to possess fixed points. The general idea is to integrate over the large momentum modes of the dynamical variables, but its practical implementation is far from being straightforward. In the continuum, RG equations, known under the name of Exact or Functional RG, have been discovered, which in simple examples have indeed fixed points. They can be written

$$
\begin{aligned}
\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}(\phi, \lambda) \;=\; & -\int \mathrm{d}^d x \, \frac{\delta \mathcal{H}(\phi, \lambda)}{\delta \phi(x)} \left[d_\phi(\mathcal{H}) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu} \right] \phi(x) \\
& -\frac{1}{2} \int \mathrm{d}^d x \mathrm{d}^d y \, D(x - y) \left[\frac{\delta^2 \mathcal{H}}{\delta \phi(x) \delta \phi(y)} - \frac{\delta \mathcal{H}}{\delta \phi(x)} \frac{\delta \mathcal{H}}{\delta \phi(y)} \right] \\
& -\int \mathrm{d}^d x \, \mathrm{d}^d y \, L(x - y) \frac{\delta \mathcal{H}}{\delta \phi(x)} \phi(y),
\end{aligned}
\tag{3.1}
$$

where the functions D and L are defined in terms of a propagator Δ, whose Fourier transform $\tilde{\Delta}(k)$ can be written

$$\tilde{\Delta}(k) = C(k^2)/k^2, \quad C(0) = 1,$$

the regular function $C(k^2)$ decreasing faster than any power for $|k| \to \infty$. Then, the Fourier transform of the function D is

$$\tilde{D}(k^2) = 2C'(k^2),$$

and

$$L(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \; \mathrm{e}^{ikx} \, \tilde{D}(k)\tilde{\Delta}^{-1}(k),$$

Various approximation schemes like derivative expansions reduce these equations to partial differential equations, which can be studied numerically. They are quite useful for exploring general properties, beyond perturbative expansions, but are somewhat complicated for precise calculations of universal quantities.

Another strategy is to start from the only fixed point that can be analyzed completely, the gaussian fixed point, the statistical analogue of free QFT. It corresponds to the hamiltonian

$$\mathcal{H}_{\mathrm{G}}(\phi) = \frac{1}{2} \int \mathrm{d}^d x \sum_{\mu} \left(\partial_\mu \phi(x)\right)^2.$$

An analysis of local perturbations, even functions of ϕ, shows that $\phi^2(x)$, which affects the correlation length, is always relevant. For $d > 4$, all other perturbations are irrelevant. For $d = 4$, $\phi^4(x)$ becomes marginal and relevant for $d < 4$. For lower dimensions eventually other terms become relevant too. The idea then is to work in dimension $d = 4$ or in the neighbourhood of dimension 4 (the famous $\varepsilon = 4 - d$ expansion) and try to write an asymptotic RG (in a sense that will be explained in next section) for the simplified effective local hamiltonian

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left\{ \frac{1}{2} \nabla \phi(x) K(-\nabla^2/\Lambda^2) \nabla \phi(x) + \frac{1}{2} r \phi^2(x) + \frac{1}{4!} g \Lambda^{4-d} \phi^4(x) \right\}, \tag{3.2}$$

where K is a positive differential operator, $K(z) = 1 + O(z)$, r and g are *regular* functions of the temperature for T close to T_c and Λ is a large momentum analogous to the cut-off used to regularize QFT, that is $1/\Lambda$ represents the scale of distance at which this effective hamiltonian is no longer generally valid. The parameter g is chosen here dimensionless.

The hamiltonian (3.2) generates a perturbative expansion of field theory type, which can be described in terms of Feynman, diagrams. The quadratic term in (3.2) contains additional higher order derivatives, corresponding to irrelevant operators, reflexion of the initial microscopic structure. They are needed to render perturbation theory finite and this is another manifestation of the non-decoupling of scales.

At g fixed, the correlation length ξ diverges at a value $r = r_c$, which thus corresponds to the critical temperature T_c. In terms of the scale Λ, the critical domain, where universality is expected, is then defined by $|r - r_c| \ll \Lambda^2$, distances large compared to $1/\Lambda$ or momenta much smaller than Λ, and magnetization $M \equiv \langle \phi(x) \rangle \ll \Lambda^{(d/2)-1}$. These conditions are met if Λ is identified with the cut-off of a usual QFT. However, an inspection of the action (3.2) also shows that, in contrast with conventional QFT, the ϕ^4 coupling constant has a dependence in Λ given *a priori*. This follows from the assumption that the effective hamiltonian is derived from some initial microscopic model, and, thus, all operators have coefficients proportional to powers of the cut-off given by their dimension at the gaussian fixed point. For $d < 4$, the ϕ^4 coupling is thus very large in terms of the scale relevant for the critical domain. In the usual formulation of QFT, by contrast, the coupling constant is also an adjustable parameter and the resulting QFT thus is less generic.

3.1   Renormalization group equations near dimension 4

The hamiltonian (3.2) can be studied by QFT methods. Rather than writing RG equations for the hamiltonian, it appears that it is simpler to first derive RG equations for correlation or vertex

functions directly, as we now explain. Using a power counting argument, one verifies that the critical theory does not exist in perturbation theory for any dimension smaller than 4. If one defines, by dimensional continuation, a critical theory in dimension $d = 4 - \varepsilon$, even for arbitrarily small ε there always exists an order in perturbation ($\sim 2/\varepsilon$) at which infrared (IR, i.e. zero momentum) divergences appear. Therefore, the idea, originally due to Wilson and Fisher, is to perform a double series expansion in powers of the coupling constant g and ε. Order by order in this expansion, the critical behaviour differs from the gaussian behaviour only by powers of logarithm, and one can construct a perturbative critical theory by adjusting r to its critical value $r_c (T = T_c)$.

In the critical theory, correlation functions have the form

$$W^{(n)}(x_i, g, \Lambda) = \Lambda^{n(d-2)/2} W^{(n)}(\Lambda x_i, g, 1).$$

Therefore, studying the large distance behaviour is equivalent to studying the large cut-off behaviour. One then can use methods developed for the construction of the renormalized massless ϕ^4 field theory. One considers correlation functions of the Fourier components of the field, after factorization of the δ function of momentum conservation due to translation invariance. Furthermore, it is more convenient to work with algebraic combinations of correlation functions called vertex functions, denoted below by $\Gamma^{(n)}$, and derived from the Legendre transform of the generating functional of connected correlation functions. For example,

$$\widetilde{W}^{(2)}(p; g, \Lambda) \Gamma^{(2)}(p; g, \Lambda) = 1.$$

One then introduces rescaled (or renormalized) vertex functions characterized by a new scale $\mu \ll \Lambda$ at which universal behaviour is expected,

$$\Gamma_r^{(n)}(p_i; g_r, \mu, \Lambda) = Z^{n/2}(g, \Lambda/\mu) \Gamma^{(n)}(p_i; g, \Lambda), \tag{3.3}$$

where $Z(g, \Lambda/\mu)$ is a field renormalization constant and g_r a renormalized coupling constant, which characterizes the strength of the ϕ^4 interaction at scale μ. At criticality

$$\Gamma^{(2)}(p = 0; g, \Lambda) = \Gamma_r^{(2)}(0; g_r, \mu, \Lambda) = 0.$$

The renormalization factor $Z(g, \Lambda/\mu)$ and the renormalized coupling constant g_r are then determined by additional conditions, for example, by renormalization conditions of the form

$$\begin{aligned} \frac{\partial}{\partial p^2} \Gamma_r^{(2)}(p; g_r, \mu, \Lambda)\big|_{p^2 = \mu^2} &= 1, \\ \Gamma_r^{(4)}(p_i = \mu\theta_i; g_r, \mu, \Lambda) &= \mu^\varepsilon g_r, \end{aligned} \tag{3.4}$$

in which θ_i is a numerical vector ($\theta_i \neq 0$).

From renormalization theory (more precisely a slightly extended version adapted to the ε-expansion), one then infers that the functions $\Gamma_r^{(n)}(p_i; g_r, \mu, \Lambda)$ of equation (3.3) have at p_i, g_r and μ fixed, large cut-off limits which are the renormalized vertex functions $\Gamma_r^{(n)}(p_i; g_r, \mu)$. Moreover, renormalized functions $\Gamma_r^{(n)}$ do not depend on the specific cut-off procedure and, given the normalization conditions (3.4), are universal. Since the renormalized functions $\Gamma_r^{(n)}$ and the initial ones $\Gamma^{(n)}$ are asymptotically proportional, both functions have the same small momentum or large distance behaviour. The renormalized functions thus contain the whole information about the asymptotic universal critical behaviour. One could, therefore, study only renormalized correlation functions, which indeed are the ones useful for many explicit calculations of universal quantities. However, universality is not limited to the asymptotic critical behaviour; leading corrections have also some interesting universal properties. Moreover, renormalized quantities are not directly obtained in non-perturbative calculations. For these various reasons, it is also useful to study the implications of equation (3.3) directly for the initial correlation functions.

RG equations. Differentiating equation (3.3) with respect to Λ at g_r and μ fixed, one obtains

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_r, \mu \text{ fixed}} Z^{n/2}(g, \Lambda/\mu) \Gamma^{(n)}(p_i; g, \Lambda) = O(\Lambda^{-2}(\ln \Lambda)^L). \tag{3.5}$$

We now neglect corrections subleading by powers of Λ order by order in the double series expansion of g and ε. We assume that these corrections, generated by operators irrelevant from the point of view of the gaussian fixed point, remain, after summation, corrections, that is that irrelevant operators are continuously deformed into irrelevant operators for the non-trivial fixed points.

Then, using chain rule, one infers from equation (3.5):

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g,\Lambda/\mu)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g,\Lambda/\mu)\right]\Gamma^{(n)}(p_i;g,\Lambda) = 0. \tag{3.6}$$

The functions β and η, which are dimensionless and may thus depend only on the dimensionless quantities g and Λ/μ, are defined by

$$\beta(g,\Lambda/\mu) = \Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_r,\mu} g, \tag{3.7}$$

$$\eta(g,\Lambda/\mu) = -\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_r,\mu}\ln Z(g,\Lambda/\mu). \tag{3.8}$$

However, the functions β and η can also be directly calculated from equation (3.6) in terms of functions $\Gamma^{(n)}$ which do not depend on μ. Therefore, the functions β and η cannot depend on the ratio Λ/μ and equation (3.6) simplifies as

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g)\right]\Gamma^{(n)}(p_i;g,\Lambda) = 0. \tag{3.9}$$

Equation (3.9), consequence of the existence of a renormalized theory, is satisfied, when the cut-off is large, by the physical vertex functions of statistical mechanics which are also the *bare* vertex functions of QFT. It follows implicitly from the solution of equation (3.9) (see section 3.2) that, conversely, the equation implies the existence of a renormalized theory.

This RG is only asymptotic because the r.s.h. of equation (3.5) and thus (3.6) have been neglected.

3.2 Solution of the RG equations: The ε-expansion

Equation (3.9) can be solved by the method of characteristics. One introduces a dilatation parameter λ and looks for functions $g(\lambda)$ and $Z(\lambda)$ such that

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[Z^{-n/2}(\lambda)\Gamma^{(n)}\left(p_i;g(\lambda),\lambda\Lambda\right)\right] = 0. \tag{3.10}$$

Consistency with equation '3.9) implies

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}g(\lambda) = \beta\big(g(\lambda)\big), \quad g(1) = g, \tag{3.11}$$

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln Z(\lambda) = \eta\big(g(\lambda)\big)), \quad Z(1) = 1. \tag{3.12}$$

The function $g(\lambda)$ is the effective coupling at the scale λ, and is governed by the flow equation (3.11). Equation (3.10) implies

$$\Gamma^{(n)}(p_i;g,\Lambda) = Z^{-n/2}(\lambda)\Gamma^{(n)}\left(p_i;g(\lambda),\lambda\Lambda\right).$$

It is actually convenient to rescale Λ by a factor $1/\lambda$ and write the equation

$$\Gamma^{(n)}(p_i;g,\Lambda/\lambda) = Z^{-n/2}(\lambda)\Gamma^{(n)}\left(p_i;g(\lambda),\Lambda\right). \tag{3.13}$$

Equations (3.11)-(3.12) and (3.13) implement approximately (because terms subleading by powers of Λ have been neglected) the RG ideas as presented in section 2: since the coupling constant $g(\lambda)$

characterizes the hamiltonian $\mathcal{H}_\lambda$, equation (3.11) is the equivalent of equation (2.10) (up to the change $\lambda \mapsto 1/\lambda$); equations (2.11) and (3.12) differ only by the definition of $Z(\lambda)$.

The solutions of equations (3.11)-(3.12) can be written as

$$\int_g^{g(\lambda)} \frac{\mathrm{d}g'}{\beta(g')} = \ln\lambda\,, \tag{3.14}$$

$$\int_1^\lambda \frac{\mathrm{d}\sigma}{\sigma}\eta\big(g(\sigma)\big) = \ln Z(\lambda)\,. \tag{3.15}$$

Equation (3.9) is the RG equation in differential form. Equations (3.13) and (3.14)-(3.15) are the integrated RG equations. In equation (3.13), we see that it is equivalent to increase Λ or to decrease λ. To investigate the large Λ limit we, therefore, study the behaviour of the effective coupling constant $g(\lambda)$ when λ goes to zero. Equation (3.14) shows that $g(\lambda)$ increases if the function β is negative, or decreases in the opposite case. Fixed points correspond to zeros of the β-function which, therefore, play an essential role in the analysis of critical behaviour. Those where the β-function has a negative slope are IR repulsive: the effective coupling moves away from such zeros, except if the initial coupling has exactly a fixed point value. Conversely, those where the slope is positive are IR attractive.

The RG functions have been calculated in perturbation theory and one finds

$$\beta(g,\varepsilon) = -\varepsilon g + \frac{3g^2}{16\pi^2} + O(g^3, g^2\varepsilon)\,. \tag{3.16}$$

The explicit expression (3.16) of the β-function shows that above dimension 4, that is, $\varepsilon < 0$, if initially g is small, $g(\lambda)$ decreases approaching the origin $g = 0$. We recover that the gaussian fixed point is IR stable.

By contrast, below four dimensions, if initially g is very small, $g(\lambda)$ first increases, a behaviour reflecting the instability of the gaussian fixed point.

However, the explicit expression (3.16) shows that, in the sense of an expansion in powers of ε, $\beta(g)$ has another zero

$$g^* = 16\pi^2\varepsilon/3 + O(\varepsilon^2) \;\Rightarrow\; \beta(g^*) = 0\,, \tag{3.17}$$

with a positive slope for ε infinitesimal:

$$\omega \equiv \beta'(g^*) = \varepsilon + O(\varepsilon^2) > 0\,. \tag{3.18}$$

Then, equation (3.14) shows that $g(\lambda)$ has g^* as an asymptotic limit. Below dimension 4, at least for ε infinitesimal, this non-gaussian fixed point is IR stable. In dimension 4, it merges with the gaussian fixed point and the eigenvalue ω vanishes indicating the appearance of the marginal operator.

From equation (3.15), we then derive the behaviour of $Z(\lambda)$ for λ small. The integral in the l.h.s. is dominated by small values of σ. It follows that

$$\ln Z(\lambda) \underset{\lambda\to 0}{\sim} \eta\ln\lambda\,, \tag{3.19}$$

where we have set

$$\eta = \eta(g^*).$$

Equation (3.13) then determines the behaviour of $\Gamma^{(n)}(p_i; g, \Lambda)$ for Λ large:

$$\Gamma^{(n)}(p_i; g, \Lambda/\lambda) \sim \lambda^{-n\eta/2}\Gamma^{(n)}(p_i; g^*, \Lambda)\,. \tag{3.20}$$

On the other hand, from simple dimensional considerations, we know that

$$\Gamma^{(n)}(p_i; g, \Lambda/\lambda) = \lambda^{-d+(n/2)(d-2)}\Gamma^{(n)}(\lambda p_i; g, \Lambda)\,. \tag{3.21}$$

Combining this equation with equation (3.20), we obtain

$$\Gamma^{(n)}(\lambda p_i; g, \Lambda) \underset{\lambda\to 0}{\sim} \lambda^{d-(n/2)(d-2+\eta)}\Gamma^{(n)}(p_i; g^*, \Lambda)\,. \tag{3.22}$$

This equation shows that critical vertex functions have a power law behaviour for small momenta, independent of the initial value of the ϕ^4 coupling constant g, at least if g initially is small enough for perturbation theory to be meaningful, or if the β-function has no other zero.

Equation (3.22) yields for $n = 2$ the small momentum behaviour of the vertex two-point function, and thus of the two-point correlation function

$$\widetilde{W}^{(2)}(p) = \left[\Gamma^{(2)}(p)\right]^{-1} \underset{|p| \to 0}{\sim} 1/p^{2-\eta}. \tag{3.23}$$

The spectral representation of the two-point function implies $\eta > 0$. A short calculation yields

$$\eta = \frac{\varepsilon^2}{54} + O(\varepsilon^3). \tag{3.24}$$

Finally, we note that equation (3.22) can be interpreted by saying that the field $\phi(x)$, which had at the gaussian fixed point a canonical dimension $(d-2)/2$, has now acquired an anomalous dimension (equation (2.13))

$$d_\phi = \tfrac{1}{2}(d - 2 + \eta).$$

Universality. Within the framework of the ε-expansion, all correlation functions have, for $d < 4$, a long distance behaviour different from the one predicted by a quasi-gaussian or mean field theory. In addition, the critical behaviour does not depend on the initial value of the ϕ^4 coupling constant g. Therefore, the critical behaviour is *universal*, although less universal than in the quasi-gaussian theory, in the sense that it depends only on some small number of qualitative properties of the physical system under study.

4 Calculation of universal quantities

We present here some explicit results obtained within the framework of the $O(N)$ symmetric $(\phi^2)^2$ statistical field theory. The results of the $(\phi^2)^2$ field theory, or N-vector model do not apply only to ferromagnetic systems. The superfluid helium transition corresponds to $N = 2$, the $N = 0$ limit is related to the statistical properties of polymers and the Ising-like $N = 1$ model also describes the physics of the liquid–vapour transition.

We discuss only critical exponents, although a number of other universal quantities have been calculated like the scaling equation of state or ratios of critical amplitudes.

4.1 The ε-expansion

Critical exponents in the N-vector model are known up to order ε^5. The higher order calculations have been done using dimensional regularization and a minimal subtraction scheme. The equation of state is known up to order ε^2 for arbitrary N and to order ε^3 for $N = 1$. A number of results have also been obtained for the two-point correlation function.

4.2 Critical exponents

Although the RG functions of the $(\phi^2)^2$ theory and, therefore, the critical exponents are known up to five-loop order, we give here only two successive terms in the expansion for illustration purpose, referring to the literature for higher order results. In terms of the variable

$$\tilde{g} = N_d\, g\,, \qquad N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\,, \tag{4.1}$$

the RG functions $\beta(\tilde{g})$ and $\eta_2(\tilde{g})$ at two-loop order, $\eta(\tilde{g})$ at three-loop order are

$$\beta(\tilde{g}) = -\varepsilon\tilde{g} + \frac{(N+8)}{6}\tilde{g}^2 - \frac{(3N+14)}{12}\tilde{g}^3 + O\left(\tilde{g}^4\right), \tag{4.2}$$

$$\eta(\tilde{g}) = \frac{(N+2)}{72}\tilde{g}^2\left[1 - \frac{(N+8)}{24}\tilde{g}\right] + O\left(\tilde{g}^4\right), \tag{4.3}$$

$$\eta_2(\tilde{g}) = -\frac{(N+2)}{6}\tilde{g}\left[1 - \frac{5}{12}\tilde{g}\right] + O\left(\tilde{g}^3\right), \tag{4.4}$$

k	0	1	2	3	4	5
γ	1.000	1.1667	1.2438	1.1948	1.3384	0.8918
η	$0.0\ldots$	$0.0\ldots$	0.0185	0.0372	0.0289	0.0545

Table 1: Sum of the successive terms of the ε-expansion of γ and η for $\varepsilon = 1$ and $N = 1$

The zero $\tilde{g}^*(\varepsilon)$ of the β-function then is $\tilde{g}^*(\varepsilon) = 6\varepsilon/(N+8) + O(\varepsilon^2)$. The values of the critical exponents η, γ and the correction exponent ω,

$$\eta = \eta(\tilde{g}^*), \quad \gamma = \frac{2-\eta}{2+\eta_2(\tilde{g}^*)}, \quad \omega = \beta'(\tilde{g}^*),$$

follow

$$\eta = \frac{\varepsilon^2(N+2)}{2(N+8)^2}\left[1 + \frac{(-N^2+56N+272)}{4(N+8)^2}\varepsilon\right] + O\left(\varepsilon^4\right), \tag{4.5}$$

$$\gamma = 1 + \frac{(N+2)}{2(N+8)}\varepsilon + \frac{(N+2)}{4(N+8)^3}\left(N^2+22N+52\right)\varepsilon^2 + O\left(\varepsilon^3\right), \tag{4.6}$$

$$\omega = \varepsilon - \frac{3(3N+14)}{(N+8)^2}\varepsilon^2 + O\left(\varepsilon^3\right). \tag{4.7}$$

Other exponents can be obtained from scaling relations. Note that the results at next order involve $\zeta(3)$. At higher orders $\zeta(5)$ and $\zeta(7)$ successively appear. In table 1, we give the values of the critical exponents γ and η obtained by simply adding the successive terms of the ε-expansion for $\varepsilon = 1$ and $N = 1$.

One immediately observes a striking phenomenon: the sums first seem to settle near some reasonable value and then begin to diverge with increasing oscillations. This is an indication that the ε-expansion is divergent for all values of ε. Divergent series can be used for small values of the argument. However, only a limited number of terms of the series can then be taken into account. The last term added gives an indication of the size of the irreducible error. For the exponents γ and η we roughly conclude from the series

$$\gamma = 1.244 \pm 0.050, \qquad \eta = 0.037 \pm 0.008,$$

where the errors are only indicative.

4.3   The perturbative expansion at fixed dimension

Critical exponents and various universal quantities have also been calculated within the framework of the massive $(\phi^2)^2$ field theory, as perturbative series at fixed dimension 3. The basic reason is that in dimension 3 one-loop diagrams have simple analytic expressions that can be used to simplify most higher order diagrams. It has, therefore, been possible to calculate the RG functions of the N-vector model up to six- and partially seven-loop order.

Note that this massive ϕ^4 field theory is a somewhat artificial construction: when the correlation length increases, simultaneously the coefficient of the relevant ϕ^4 operator is tuned to decrease like $g \propto (\Lambda\xi)^{d-4}$. Then, all correlation functions have a limit for $\Lambda \to \infty$, order by order in an expansion in powers of g at fixed dimension $d < 4$. However, the usual critical theory corresponds in this framework to an infinite coupling constant. Therefore, correlation functions renormalized at zero momentum are introduced, and correspondingly a renormalized coupling constant g_{r}, which is a universal function of g. Within the framework of the ε-expansion, one proves that g_{r} has a

finite limit g_r^* when $g \to \infty$. To the mapping $g \mapsto g_r$ is associated a function $\beta(g_r)$. For example, the RG β-function in three dimensions, for $N = 1$, has the expansion

$$\begin{aligned}
\beta(\tilde{g}) &= -\tilde{g} + \tilde{g}^2 - \tfrac{308}{729}\tilde{g}^3 + 0.3510695978\tilde{g}^4 - 0.3765268283\tilde{g}^5 \\
&\quad + 0.49554751\tilde{g}^6 - 0.749689\tilde{g}^7 + O\left(\tilde{g}^8\right)
\end{aligned} \tag{4.8}$$

with the normalization

$$\tilde{g} = 3g_r/(16\pi). \tag{4.9}$$

To calculate exponents or other universal quantities, one has first to find the IR stable zero g_r^* of the function $\beta(g_r)$, which is given by a few terms of a divergent expansion. An obvious problem is the absence of any small parameter: g^* is a number of order 1. Already at this stage a summation method is required. Estimates of critical exponents are displayed in table 3. In recent years universal ratios of critical amplitudes as well as the equation of state for Ising-like systems ($N = 1$) have also been calculated. Note, however, that in this framework, the calculation of physical quantities in the ordered phase leads to additional technical problems because the theory is parametrized in terms of the disordered phase correlation length m^{-1} which is singular at T_c. Also, the normalization of correlation functions is singular at T_c. This required developing a combination of techniques based on series summation, parametric representation and a method of order-dependent mapping.

4.4   Series summation

Because all series, ε-expansion or perturbative expansions at fixed dimension, are divergent, summation methods had to be developed. We describe here methods based on generalized Borel transformations. The necessary analytic continuation of the Borel transform outside its circle of convergence is then achieved by a conformal mapping.

The method. Several different variants based on the Borel–Leroy transformation have been implemented and tested. Let $R(z)$ be the function whose expansion has to be summed (z here stands for ε or the coupling constant $\tilde{g}$):

$$R(z) = \sum_{k=0} R_k z^k. \tag{4.10}$$

A plausible assumption is that the Borel transform is analytic in a cut-plane. One thus transforms the series into

$$R(z) = \sum_{k=0}^{\infty} B_k(\rho) \int_0^{\infty} t^\rho\, \mathrm{e}^{-t}\, [u(zt)]^k\, \mathrm{d}t, \tag{4.11}$$

$$u(z) = \frac{\sqrt{1 + az} - 1}{\sqrt{1 + za} + 1}. \tag{4.12}$$

The coefficients B_k are calculated by identifying the expansion of the r.h.s. of equation (4.11) in powers of z with the expansion (4.10). The constant a is known from the large order behaviour analysis in QFT based on instantons,

$$a(d = 3) = 0.147774232 \times \left(9/(N + 8)\right), \tag{4.13}$$

and ρ is a free parameter, adjusted empirically to improve the convergence of the transformed series by weakening the singularities of the Borel transform near $z = -a$. Eventually, the method has been refined, which involved also introducing two additional free parameters.

Needless to say, with three parameters and short initial series it becomes possible to find occasionally some transformed series whose apparent convergence is deceptively good. It is, therefore, essential to vary the parameters in some range around the optimal values to examine the sensitivity of the results upon their variations. Finally, it is useful to sum independently series for exponents related by scaling relations. An underestimation of the apparent errors leads to inconsistent results. It is clear from these remarks that the errors quoted in the final results are educated guesses based on a large number of consistency checks.

k	2	3	4	5	6	7
$\tilde{g}^*$	1.8774	1.5135	1.4149	1.4107	1.4103	1.4105
ν	0.6338	0.6328	0.62966	0.6302	0.6302	0.6302
γ	1.2257	1.2370	1.2386	1.2398	1.2398	1.2398

Table 2: Series summed by the method based on Borel transformation and mapping for the zero $\tilde{g}^*$ of the $\beta(g)$ function and the exponents γ and ν in the ϕ_3^4 field theory

A few examples of transformed series are displayed in table 2 to illustrate the convergence.
The $(\phi^2)^2$ field theory at fixed dimensions. The RG β-function has been determined up to six-loop order in three dimensions, while the series for the dimensions of the fields ϕ and ϕ^2 have recently been extended to seven loops. The series of the RG β-function has been first summed and its zero $\tilde{g}^*$ determined ($\tilde{g} = g_{\rm r}(N + 8)/(48\pi)$ for $d = 3$. The series of the other RG functions have then been summed for $\tilde{g} = \tilde{g}^*$. Examples of convergence are given in table 2.

The ε-expansion. The ε-expansion has one advantage: it allows connecting the results in three and two dimensions. In particular, in the cases $N = 1$ and $N = 0$, it is possible to compare the ϕ^4 results with exact results coming from lattice models and to test both universality and the reliability of the summation procedure. Moreover, it is possible to improve the three-dimensional results by imposing the exact two-dimensional values or the behaviour near two dimensions for $N > 1$. However, since the series in ε are shorter than the series at fixed dimension 3, the apparent errors are larger. Finally, as already emphasized, the comparison between the different results is a check of the consistency of QFT methods combined with the summation procedures.

4.5   Numerical estimates of critical exponents

Fixed dimension 3. Table 3 displays the results obtained from summed perturbation series at fixed dimension 3. The last exponent $\theta = \omega\nu$ characterizes corrections to scaling in the temperature variable.

Note that shorter series have been generated in dimension 2 (five loops). Because the series are short and the fixed coupling constant larger, the apparent errors are large, but the results are consistent with exact $N = 1$ results.

The ε-expansion. In table 4, we give the results coming from the summed ε-expansion for $\varepsilon = 2$ and compare them with exact results.

We see in this table that the agreement for $N = 0$ and $N = 1$ QFT and lattice models is satisfactory. We feel justified, therefore, in using a summation procedure of the ε-expansion which automatically incorporates the $d = 2$, $\varepsilon = 2$ values. Note, however, that in both cases, the identification of ω remains a problem.

Table 5 then displays the results for $\varepsilon = 1$, both for the ε series (free) and a modified ε series where the $d = 2$ results are imposed (bc).

Discussion. One can now compare the two sets of results coming from the perturbation series at fixed dimension, and the ε-expansion. First let us emphasize that the agreement is quite spectacular, although the apparent errors of the ε-expansion are in general larger because the series are shorter. Moreover, the agreement has improved with longer series.

N	0	1	2	3
$\tilde{g}^*$	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\theta = \omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Table 3: Estimates of critical exponents in the $O(N)$ symmetric $(\phi^2)_3^2$ field theory

	γ	ν	η	β	ω
$N = 0$	1.39 ± 0.04	0.76 ± 0.03	0.21 ± 0.05	0.065 ± 0.015	1.7 ± 0.2
Exact	1.34375	0.75	$0.2083\cdots$	$0.0781\cdots$	?
$N = 1$	1.73 ± 0.06	0.99 ± 0.04	0.26 ± 0.05	0.120 ± 0.015	1.6 ± 0.2
Ising	1.75	$1.$	0.25	0.125	$1.33\ldots?$

Table 4: Critical exponents in the ϕ_2^4 field theory from the ε-expansion

N	0	1	2	3
γ (free)	1.1575 ± 0.0060	1.2355 ± 0.0050	1.3110 ± 0.0070	1.3820 ± 0.0090
γ (bc)	1.1571 ± 0.0030	1.2380 ± 0.0050	1.317	1.392
ν (free)	0.5875 ± 0.0025	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
ν (bc)	0.5878 ± 0.0011	0.6305 ± 0.0025	0.671	0.708
η (free)	0.0300 ± 0.0050	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
η (bc)	0.0315 ± 0.0035	0.0365 ± 0.0050	0.0370	0.0355
β (free)	0.3025 ± 0.0025	0.3257 ± 0.0025	0.3465 ± 0.0035	0.3655 ± 0.0035
β (bc)	0.3032 ± 0.0014	0.3265 ± 0.0015		
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018
θ	0.486 ± 0.016	0.512 ± 0.013	0.536 ± 0.015	0.559 ± 0.017

Table 5: Critical exponents in the $(\phi^2)_3^2$ field theory from the ε-expansion

N	0	1	2	3
γ	1.1575 ± 0.0006	1.2385 ± 0.0025	1.322 ± 0.005	1.400 ± 0.006
ν	0.5877 ± 0.0006	0.631 ± 0.002	0.674 ± 0.003	0.710 ± 0.006
α	0.237 ± 0.002	0.103 ± 0.005	-0.022 ± 0.009	-0.133 ± 0.018
β	0.3028 ± 0.0012	0.329 ± 0.009	0.350 ± 0.007	0.365 ± 0.012
θ	0.56 ± 0.03	0.53 ± 0.04	0.60 ± 0.08	0.54 ± 0.10

Table 6: Critical exponents in the N-vector model on the lattice

The best agreement is found for the exponents ν and β. On the other hand, the values of η coming from the ε-expansion are systematically larger by about 3×10^{-3}, though the error bars always overlap. The corresponding effect is observed on γ.

Comparison with lattice model estimates. The N-vector with nearest-neighbour interactions has been studied on various lattices. Most of the results for critical exponents come from the analysis of high temperature (HT) series expansion by different types of ratio methods, Padé approximants or differential approximants. Some results have also been obtained from low temperature expansions, computer calculations using stochastic methods, and in low dimensions, transfer matrix methods. Table 6 tries to give an idea of the agreement between lattice and QFT results. A historical remark is here in order: the agreement between both types of theoretical results has improved as the HT series became longer which is of course encouraging. The main reason is that, in the analysis of longer HT series, it has become possible to take into account the influence of confluent singularities due to corrections to the leading power law behaviour, as predicted by the RG. The effect of this improvement has been specially spectacular for the exponents γ and ν of the 3D Ising model.

The obvious conclusion is that one observes no systematic differences. In particular, the agreement is extremely good in the case of the Ising model where the HT series are the most accurate. To the best of our knowledge, the N-vector lattice models and the $(\phi^2)^2$ field theory

	γ	ν	β	α	θ
(a)	1.236 ± 0.008	0.625 ± 0.010	0.325 ± 0.005	0.112 ± 0.005	0.50 ± 0.03
(b)	$1.23{-}1.25$	0.625 ± 0.006	$0.316{-}0.327$	0.107 ± 0.006	0.50 ± 0.03
(c)	1.25 ± 0.01	0.64 ± 0.01	0.328 ± 0.009	0.112 ± 0.007	

Table 7: Critical exponents in fluids and antiferromagnets

γ	ν	β	α	θ
1.40 ± 0.03	$0.700{-}0.725$	0.35 ± 0.03	$-0.09{-}-0.012$	0.54 ± 0.10

Table 8: Ferromagnetic systems

belong to the same universality class.

Critical exponents from experiments. We have discussed the N-vector model in the ferromagnetic language, even though most of our experimental knowledge comes from physical systems that are non-magnetic, but belong to the universality class of the N-vector model. The case $N = 0$ describes the statistical properties of long polymers, that is, long non-intersecting chains or self-avoiding walks. The case $N = 1$ (Ising-like systems) describes liquid–vapour transitions in classical fluids, critical binary fluids and uniaxial antiferromagnets. The helium superfluid transition corresponds to $N = 2$. Finally, for $N = 3$, the experimental information comes from ferromagnetic systems.

Critical exponents and polymers. In the case of polymers, only the exponent ν is easily accessible. The best results are

$$\nu = 0.586 \pm 0.004 \,,$$

in excellent agreement with the RG result.

Ising-like systems $N = 1$. Table 7 gives a survey of the experimental situation for critical binary fluids (*a*), liquid–vapour transition in classical fluids (*b*), and antiferromagnets (*c*). For binary mixtures, we quote a weighted world average. In the case of the liquid–vapour transition, we quote a range of experimental results rather than statistical errors for all exponents but ν, the reason being that the values depend much on the method of analysis of the experimental data. The agreement with RG results is clearly impressive.

Helium superfluid transition, $N = 2$. The helium transition allows measurements very close to T_c and this explains the remarkably precise determination of the critical exponents α and ν. The order parameter, however, is not directly accessible in helium. Most recent reported values are

$$\nu = 0.6705 \pm 0.0006 \,, \quad \nu = 0.6708 \pm 0.0004 \quad \text{and} \quad \alpha = -0.01285 \pm 0.00038 \,.$$

The agreement with RG values is quite remarkable but the precision of ν is now a challenge to field theory.

Ferromagnetic systems, $N = 3$. Finally, table 8 displays some results concerning magnetic systems.

Conclusion and prospects. If one takes into account all data (critical exponents, equation of state, amplitude ratios,...) one is forced to conclude that the RG predictions are remarkably consistent
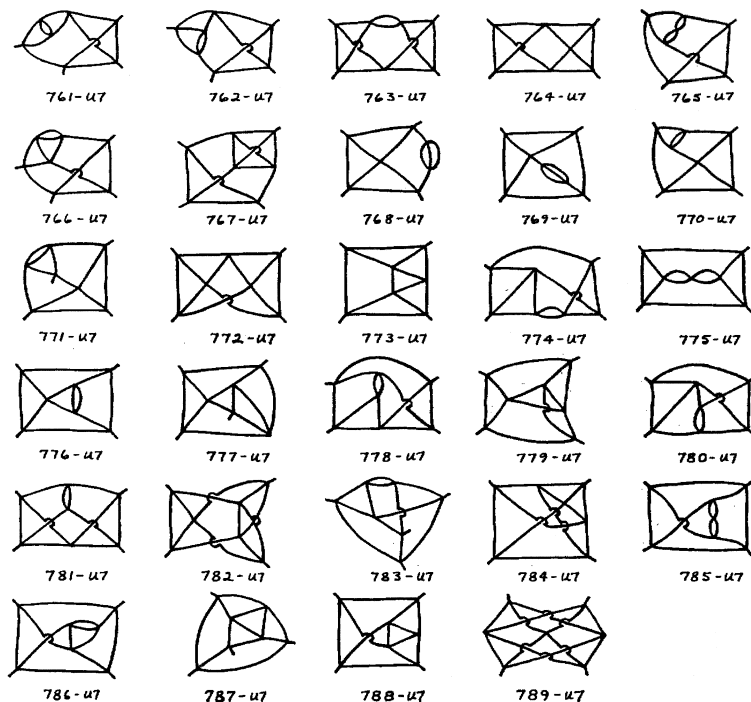
Figure 2: A few seven-loop diagrams.

with the whole experimental and lattice information available. Considering the variety of experimental situations, this is a spectacular confirmation of the RG ideas and the concept of universality.

The current effort goes in several directions. First, improve the precision of critical exponents, in particular trying to complete the seven loop calculation in three dimensions, which is a very demanding problem from the point of view of computer algebra and numerical integration: it involves calculating about 3500 Feynman diagrams, each of them being given *a priori* by a 21-dimensional integral (a few Feynman diagrams are displayed in figure 2). After a large number of tricks have been used the number of integrations can be reduced (figure 3).

Critical exponents are only the simplest universal quantities, but many other universal quantities are worth calculating, like the equation of state, in particular for $N > 1$ in 3 dimensions, $N = 1$ at higher orders in the ε expansion, or the two-point correlation function.

Then, other models with more than one coupling are also of interest, like the model with cubic anisotropy, which has been investigated.

These efforts are paralleled by similar efforts using HT series and simulations.

Much interesting work has been done in recent years using the functional RG equations expanded in the form of a derivative expansion. The main problem there is that it is difficult to go beyond the simplest approximation, and thus difficult to assess the reliability of the results which are obtained. In the future efforts to improve the approximation should be undertaken.

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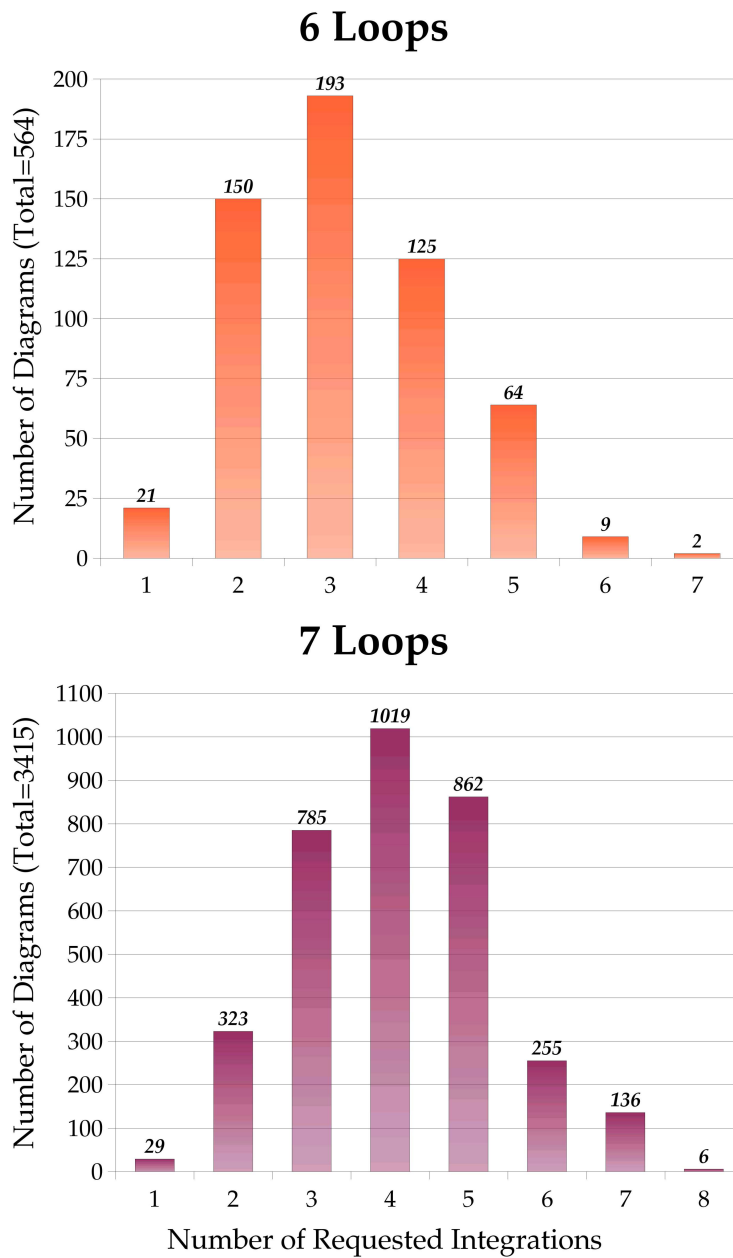
6 Loops

7 Loops

Figure 3: Number of remaining integrations after many tricks have been used (B.G. Nickel, R. Guida, P. Ribeca).