THE CONTINUED FRACTION EXPANSION OF GAUSS' HYPERGEOMETRIC FUNCTION AND A NEW APPLICATION TO THE TANGENT FUNCTION

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Abstract. Starting from a formula for $\tan(nx)$ in the celebrated Hakmem report [1] we find a continued fraction expansion for $\tan(nx)$ in terms of $\tan(x)$.

1. Introduction

Gauss' hypergeometric function is defined by

$$F\Big(\begin{matrix}a,b\\c\end{matrix}\Big|z\Big)=\sum_{n\ge0}\frac{a^{\overline{n}}b^{\overline{n}}z^n}{c^{\overline{n}}n!},$$

with $x^{\overline{n}}=x(x+1)\dots(x+n-1)$, a rising factorial. (An older way to write this rising factorial is $(x)_n$; for the hypergeometric functions there exist different notations as well.) This function has two upper and one lower parameter and is thus often called the "two–eff–one". Hypergeometric functions are studied with any numbers of upper and lower parameters, but this one is the most prominent one; compare [2]. Hypergeometric functions are omnipresent in mathematics, and also well established in modern computer algebra systems such as Maple or Mathematica.

The following description is borrowed from [6].

Gauss' hypergeometric function satisfies the *contiguous relation*

$$F\Big(\begin{matrix}a,b\\c\end{matrix}\Big|z\Big)=F\Big(\begin{matrix}a,b+1\\c+1\end{matrix}\Big|z\Big)-\frac{a(c-b)}{c(c+1)}zF\Big(\begin{matrix}a+1,b+1\\c+2\end{matrix}\Big|z\Big).$$

To see this, we compare coefficients of z^n. For $n=0$, they match, so let us assume that $n\ge1$. We start with the right-hand side:

$$\frac{a^{\overline{n}}(b+1)^{\overline{n}}}{(c+1)^{\overline{n}}n!}-\frac{a(c-b)}{c(c+1)}\frac{(a+1)^{\overline{n-1}}(b+1)^{\overline{n-1}}}{(c+2)^{\overline{n-1}}(n-1)!}=\frac{(a+1)^{\overline{n-1}}(b+1)^{\overline{n-1}}}{(c+2)^{\overline{n-1}}n!}\bigg[\frac{a(b+n)}{c+1}-\frac{a(c-b)n}{c(c+1)}\bigg]$$

$$=\frac{(a+1)^{\overline{n-1}}(b+1)^{\overline{n-1}}}{(c+2)^{\overline{n-1}}n!}\frac{ab(c+n)}{c(c+1)}=\frac{a^{\overline{n}}b^{\overline{n}}}{c^{\overline{n}}n!},$$

as predicted. We also need a variation of this: First, we interchange a and b:

$$F\Big(\begin{matrix}a,b\\c\end{matrix}\Big|z\Big)=F\Big(\begin{matrix}a+1,b\\c+1\end{matrix}\Big|z\Big)-\frac{b(c-a)}{c(c+1)}zF\Big(\begin{matrix}a+1,b+1\\c+2\end{matrix}\Big|z\Big).$$

This author was supported by an incentive grant of the NRF of South Africa.

Now we increase both b and c by 1:

$$F\binom{a,b+1}{c+1}\Big|z\Big) = F\Big(\genfrac{}{}{0pt}{}{a+1,b+1}{c+2}\Big|z\Big) - \frac{(b+1)(c+1-a)}{(c+1)(c+2)}zF\Big(\genfrac{}{}{0pt}{}{a+1,b+2}{c+3}\Big|z\Big).$$

We rewrite the first contiguous relation as

$$\frac{F\Big(\genfrac{}{}{0pt}{}{a,b}{c}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a,b+1}{c+1}\Big|z\Big)} = 1 - \frac{a(c-b)}{c(c+1)}z\frac{F\Big(\genfrac{}{}{0pt}{}{a+1,b+1}{c+2}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a,b+1}{c+1}\Big|z\Big)}$$

and further as

$$\frac{F\Big(\genfrac{}{}{0pt}{}{a,b+1}{c+1}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a,b}{c}\Big|z\Big)} = \cfrac{1}{1 - \frac{a(c-b)}{c(c+1)}z\frac{F\Big(\genfrac{}{}{0pt}{}{a+1,b+1}{c+2}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a,b+1}{c+1}\Big|z\Big)}}.$$

A similar procedure applied to the variant leads to

$$\frac{F\Big(\genfrac{}{}{0pt}{}{a+1,b+1}{c+2}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a,b+1}{c+1}\Big|z\Big)} = \cfrac{1}{1 - \frac{(b+1)(c-a+1)}{(c+1)(c+2)}z\frac{F\Big(\genfrac{}{}{0pt}{}{a+1,b+2}{c+3}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a+1,b+1}{c+2}\Big|z\Big)}}.$$

This can be used in the first form:

$$\frac{F\Big(\genfrac{}{}{0pt}{}{a,b+1}{c+1}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a,b}{c}\Big|z\Big)} = \cfrac{1}{1 - \cfrac{\frac{a(c-b)}{c(c+1)}z}{1 - \frac{(b+1)(c-a+1)}{(c+1)(c+2)}z\frac{F\Big(\genfrac{}{}{0pt}{}{a+1,b+2}{c+3}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a+1,b+1}{c+2}\Big|z\Big)}}}.$$

Now the first form can be used again, with a, b, c replaced by $a+1, b+1, c+2$. In the resulting form, the variant can be used, and so on. The result is the *continued fraction of Gauss*:

$$\frac{F\Big(\genfrac{}{}{0pt}{}{a,b+1}{c+1}\Big|z\Big)}{F\Big(\genfrac{}{}{0pt}{}{a,b}{c}\Big|z\Big)} = \cfrac{1}{1 - \cfrac{\frac{a(c-b)}{c(c+1)}z}{1 - \cfrac{\frac{(b+1)(c-a+1)}{(c+1)(c+2)}z}{1 - \cfrac{\frac{(a+1)(c-b+1)}{(c+2)(c+3)}z}{1 - \cfrac{\frac{(b+2)(c-a+2)}{(c+3)(c+4)}z}{1 - \ddots}}}}}.$$

The expansion is purely formal and provides more and more correct coefficients of the powers of z. The book [6] discusses also the analytic validity of the expansion.

2. An application to tangents of multiple values

Almost everybody knows that

$$\frac{\sin(2x)}{\sin(x)} = 2\cos(x), \quad \frac{\sin(3x)}{\sin(x)} = 4\cos^2(x) - 1, \quad \frac{\sin(4x)}{\sin(x)} = 8\cos^3(x) - 4\cos(x), \quad \&c.,$$

and

$$\cos(2x) = 2\cos^2(x)-1, \quad \cos(3x) = 4\cos^3(x)-3\cos(x), \quad \cos(4x) = 8\cos^4(x)-8\cos^2(x)+1, \quad \&c.,$$

and that the polynomials that appear here are the *Chebyshev polynomials*.

Most people might have asked themselves whether there is something similar for the tangent function. Yes, there is, but it is not widely known.

In the celebrated HAKMEM report [1] we find entry 16:

$$\tan(n \arctan(t)) = \frac{1}{i} \frac{(1+it)^n - (1-it)^n}{(1+it)^n + (1-it)^n} \qquad \text{for an integer } n \geq 0,$$

which is equivalent to

$$\tan(nx) = \frac{\sum\limits_{0\le k\le n/2} (-1)^k \binom{n}{2k+1} \tan^{2k+1}(x)}{\sum\limits_{0\le k\le n/2} (-1)^k \binom{n}{2k} \tan^{2k}(x)}.$$

Compare with the sequences [5, A034839, A034867].

This is the formula of interest; it expresses $\tan(nx)$ as a *rational function* of $\tan(x)$ (not a polynomial, as in the simpler cases of $\sin(nx)$ and $\cos(nx)$). For computational (and aestethic!) reasons it is, however, beneficial to express this rational function as a *continued fraction.*

Let

$$f(z) = \sum_{k\ge 0} \binom{n}{2k+1} z^k \quad \text{and} \quad g(z) = \sum_{k\ge 0} \binom{n}{2k} z^k,$$

then we get

$$\frac{f(z)}{g(z)} = n \frac{F\left({\frac{-n+1}{2}, \frac{n+1}{2} \atop \frac{3}{2}} \middle| \frac{z}{z-1} \right)}{F\left({\frac{-n+1}{2}, \frac{n+1}{2} \atop \frac{1}{2}} \middle| \frac{z}{z-1} \right)};$$

this conversion into hypergeometric functions is best done nowadays by a computer. Using *Pfaff's reflection law* [2]

$$F\left({a, b \atop c} \middle| \frac{z}{z-1} \right) = (1-z)^a F\left({a, c-b \atop c} \middle| z \right),$$

this can be rewritten as

$$\frac{f(z)}{g(z)} = n \frac{F\left({\frac{n+1}{2}, \frac{n}{2}+1 \atop \frac{3}{2}} \middle| z \right)}{F\left({\frac{n+1}{2}, \frac{n}{2} \atop \frac{1}{2}} \middle| z \right)}.$$

Now it is in good shape to apply Gauss' continued fraction to it:

$$\frac{f(z)}{g(z)} = \cfrac{n}{1 + \cfrac{\frac{(n+1)(n-1)}{1\cdot 3}z}{1 + \cfrac{\frac{(n+2)(n-2)}{3\cdot 5}z}{1 + \cfrac{\frac{(n+3)(n-3)}{5\cdot 7}z}{\ddots}}}}$$

Observe that for natural numbers n this expansion is always finite, and one does not have to worry about convergence.

For our application, we must replace z by $-\tan^2(x)$, and multiply the whole expansion by $\tan(x)$. The result is

$$\tan(nx) = \cfrac{n\tan(x)}{1 - \cfrac{\frac{(n+1)(n-1)}{1\cdot 3}\tan^2(x)}{1 - \cfrac{\frac{(n+2)(n-2)}{3\cdot 5}\tan^2(x)}{1 - \cfrac{\frac{(n+3)(n-3)}{5\cdot 7}\tan^2(x)}{\ddots}}}}.$$

For example,

$$\tan(5x) = \cfrac{5\tan(x)}{1 - \cfrac{8\tan^2(x)}{1 - \cfrac{\frac{7}{5}\tan^2(x)}{1 - \cfrac{\frac{16}{35}\tan^2(x)}{1 - \frac{1}{7}\tan^2(x)}}}}.$$

3. An independent derivation of the continued fraction expansion

Not everybody is completely comfortable with *hypergeometric functions*, *hypergeometric transformations*, *contiguous relations*, etc. We demonstrate how such people can also derive the continued fraction expansion for $\tan(nx)$, by using a technique that has produced many other beautiful expansions [3, 4].

We will show that

$$\frac{zf(z)}{g(z)} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\cdots}}}},$$

with

$$a_{2k} = (4k+1)\frac{\prod_{j=-k+1}^{k}(n+1-2j)}{\prod_{j=-k}^{k}(n-2j)} \quad \text{and} \quad a_{2k+1} = (4k+3)\frac{\prod_{j=-k}^{k}(n-2j)}{\prod_{j=-k}^{k+1}(n+1-2j)}.$$

This translates then readily into

$$\tan(nx) = \cfrac{\tan(x)}{a_0 - \cfrac{\tan^2(x)}{a_1 - \cfrac{\tan^2(x)}{a_2 - \cfrac{\tan^2(x)}{\ddots}}}};$$

since the a_k's eventually become zero, this is a finite continued fraction expansion.

Here is an example, which is of course equivalent to our previously given example:

$$\tan(5x) = \cfrac{\tan(x)}{\frac{1}{5} - \cfrac{\tan^2(x)}{\frac{5}{8} - \cfrac{\tan^2(x)}{\frac{8}{7} - \cfrac{\tan^2(x)}{\frac{245}{128} - \cfrac{\tan^2(x)}{\frac{128}{35}}}}}}.$$

The technique that we use is to *guess* the form of the numbers a_k, by computing a sufficient number of them with a computer and detecting the pattern. Of course, one also has to give a proof, and for that, more guessing has to be done.

Define

$$s_{2k}(z) := \sum_{N\geq 0} \frac{z^N}{2^N N!} \frac{\prod_{j=-k}^{k+N}(n-2j) \prod_{j=1}^{N}(n+1-2j-2k)}{\prod_{j=0}^{2k+N}(2j+1)},$$

$$s_{2k+1}(z) := \sum_{N\geq 0} \frac{z^N}{2^N N!} \frac{\prod_{j=-k}^{k+N+1}(n+1-2j) \prod_{j=1}^{N}(n-2j-2k)}{\prod_{j=0}^{2k+N+1}(2j+1)}.$$

These formal power series are in fact just polynomials, since the coefficients become eventually zero. They were also guessed, using a computer. Further,

$$s_0(z) = \sum_{N\geq 0} \frac{z^N}{2^N N!} \frac{\prod_{j=0}^{N}(n-2j) \prod_{j=1}^{N}(n+1-2j)}{\prod_{j=0}^{N}(2j+1)} = \sum_{N\geq 0} \binom{n}{2N+1} z^N = f(z),$$

and

$$s_{-1}(z)=\sum_{N\ge0}\frac{z^N}{2^NN!}\frac{\prod\limits_{j=1}^{N}(n+1-2j)\prod\limits_{j=1}^{N}(n-2j+2)}{\prod\limits_{j=0}^{N-1}(2j+1)}=\sum_{N\ge0}\binom{n}{2N}z^N=g(z).$$

Now we will show that

$$s_{k+1}(z)=\frac{s_{k-1}(z)-a_ks_k(z)}{z},$$

by distinguishing two cases:

$$[z^N]\Big(s_{2k-1}(z)-a_{2k}s_{2k}(z)\Big)=\frac{1}{2^NN!}\frac{\prod\limits_{j=-k+1}^{k+N}(n+1-2j)\prod\limits_{j=1}^{N}(n-2j-2k+2)}{\prod\limits_{j=0}^{2k+N-1}(2j+1)}$$

$$-(4k+1)\frac{1}{2^NN!}\frac{\prod\limits_{j=1}^{N}(n-2j-2k)\prod\limits_{j=-k+1}^{k+N}(n+1-2j)}{\prod\limits_{j=0}^{2k+N}(2j+1)}$$

$$=\frac{\prod\limits_{j=-k+1}^{k+N}(n+1-2j)\prod\limits_{j=1}^{N-1}(n-2j-2k)}{2^NN!\prod\limits_{j=0}^{2k+N}(2j+1)}\times$$

$$\times\Big[(n-2k)(4k+2N+1)-(4k+1)(n-2N-2k)\Big]$$

$$=\frac{\prod\limits_{j=-k+1}^{k+N}(n+1-2j)\prod\limits_{j=1}^{N-1}(n-2j-2k)}{2^NN!\prod\limits_{j=0}^{2k+N}(2j+1)}2N(n+2k+1)$$

$$=\frac{\prod\limits_{j=-k}^{k+N}(n+1-2j)\prod\limits_{j=1}^{N-1}(n-2j-2k)}{2^{N-1}(N-1)!\prod\limits_{j=0}^{2k+N}(2j+1)}=[z^{N-1}]s_{2k+1}(z);$$

the proof that

$$[z^N]\Big(s_{2k}(z)-a_{2k+1}s_{2k+1}(z)\Big)=[z^{N-1}]s_{2k+2}(z)$$

is similar. Furthermore, for $N=0$, the differences are zero, so that we get the claimed recursions. (For the guessing, these recursions were *used* to compute a sufficient number of these polynomials.)

Consequently,

$$\frac{zf(z)}{g(z)} = \frac{zs_0(z)}{s_{-1}(z)} = \frac{zs_0(z)}{a_0 s_0(z) + z s_1(z)} = \cfrac{z}{a_0 + \cfrac{zs_1(z)}{s_0(z)}} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{zs_2(z)}{s_1(z)}}} = \dots,$$

which leads to the promised continued fraction. Notice again that the process stops since n is a non-negative integer.

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