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# DETERMINANTS WHOSE ELEMENTS ARE EULERIAN, PREPARED BERNOULLIAN, AND OTHER NUMBERS

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[The object of this paper is to find the values of determinants of the form

$$\begin{vmatrix} a & b & c & \dots \\ b & c & d & \dots \\ c & d & e & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

known as the 'per symmetra,' the elements $a, b, c, \dots$ being Eulerian, prepared Bernoullian, and other specially defined numbers. It is believed that the values of these determinants have not been obtained by any previous writer.]

§ 1. *Notation*: We use the $\Sigma$ notation throughout this paper in a particular sense as explained below.

We have $\Sigma n^2 = 1^2 + 2^2 + \dots + n^2$.

Consider the series $u_{n-1} = (n-1)^2 \Sigma n^2$ which has $(n-1)$ terms.

We write

$$\Sigma u_{n-1} = \Sigma \{(n-1)^2 \Sigma n^2\} = \Sigma (n-1)^2 \Sigma (n^2),$$

omitting flower brackets.

Since $\Sigma u_{n-1} = \Sigma u_{n-2} + u_{n-1}$, we have

$$\Sigma (n-1)^2 \Sigma n^2 = (n-1)^2 \Sigma n^2 + \Sigma (n-2)^2 \Sigma (n-1)^2,$$

which is the reduction formula for our notation.

We can generalise this further in the form

$$\Sigma(a+1)^2 \Sigma(a+2)^2 \dots \Sigma(a+n)^2 = (a+1)^2 \Sigma(a+2)^2 \dots \Sigma(a+n)^2 + \Sigma a^2 \Sigma(a+1)^2 \dots \Sigma(a+\overline{n-1})^2.$$

With this notation, it is easily proved that the $n^{th}$ Eulerian number

$$E_n = \Sigma 1^2 \Sigma 2^2 \dots \Sigma n^2.\ *$$

In the same way, if we write

$$1.n + 2(n+1) + \dots + r(n+r-1) = \Sigma(r, n+r-1)$$

and if $w_{r-1} = (r-1)(n+r-2)\Sigma(r, n+r-1)$.

then $\Sigma w_{r-1} = \Sigma(r-1, n+r-2)\Sigma(r, n+r-1)$

$$= (r-1)(n+r-2)\Sigma(r, n+r-1) + \Sigma(r-2, n+r-3)\Sigma(r-1, n+r-2)$$

* This formula and some interesting properties of these numbers are discussed by the authors in a paper on "Some Properties of Eulerian and Prepared Bernoullian Numbers" communicated to the *Third Conference of the Indian Mathematical Society*, March 1921.

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Of course $\Sigma w_{r-1}$ contains $(r-1)$ terms, and the second term above is the sum of $(r-2)$ terms, and the first is $w_{r-1}$. This can obviously be generalized in the form

$$\Sigma(a, b)\,\Sigma(a+1, b+1)\ \dots\ \Sigma(a+r, b+r)$$
$$= a\,.\,b\,\Sigma(a+1, b+1)\ \dots\ \Sigma(a+r, b+r)$$
$$+\ \Sigma(a-1, b-1)\,\Sigma(a, b)\ \dots\ \Sigma(a+r-1, b+r-1),$$

which is the general formula of reduction.

With this notation, it is seen that

$$b_n = 2^{2n}\,(2^{2n}-1)\,\frac{B_n}{2_n}.$$
$$= \Sigma(1, 2)\,\Sigma(2, 3)\ \dots\ \Sigma(n-1, n)$$

where $B_n$ = the $n^{th}$ Bernoullian number, that is, $b_n$ is the prepared Bernoullian number.*

In the above reduction formulæ the integers increase as we proceed to the right.

Similar formulæ hold in the case of *descending* integers. Thus

$$\Sigma(n-2)^2 = 1^2 + 2^2 + \dots + (n-2)^2.$$

and if $\qquad v_{n-2} = n^2\,\Sigma(n-2)^2,$

then $\qquad \Sigma v_{n-2} = \Sigma\{n^2\,\Sigma(n-2)^2\} = \Sigma n^2\,\Sigma(n-2)^2$, say.

We have

$$\Sigma(n)^2\,\Sigma(n-2)^2 = \Sigma v_{n-2} = n^2\,\Sigma(n-2)^2 + \Sigma(n-1)^2\,\Sigma(n-3)^2;$$

and so on.

The following particular cases are noteworthy :—

(i) $\Sigma(2n+1)^2\,\Sigma(2n-1)^2 \dots \Sigma 3^2\,\Sigma 1^2 = (2n+1)^2\,(2n-1)^2 \dots 3^2.1^2$ ;

(ii) $\Sigma(2n)^2\,\Sigma(2n-2)^2 \dots \Sigma(2^2) = (2n)^2\,\Sigma(2n-2)^2\,\Sigma(2n-4)^2 \dots \Sigma(2^2)$
$\qquad + (2n-1)^2\,(2n-3)^2 \dots$ $(1^2)$.

§ 2. It is well-known that

$$\sec x = \Sigma\left\{E_r\,\frac{x^{2r}}{(2r)!}\right\},$$

where $E_r$ is the $r^{th}$ Eulerian number.

Differentiating

$$\sec x.\tan x = \Sigma\left\{E_r\,\frac{x^{2r-1}}{(2r-1)!}\right\}.$$

* This formula and some interesting properties of these numbers are discussed by the authors in a paper on "Some Properties of Eulerian and Prepared Bernoullian Numbers" communicated to the *Third Conference of the Indian Mathematical Society*, March 1921.



Generally, it can be proved by *mathematical induction*,* that

(i) $(2n)!\ \sec x.\tan^{2n} x = \Sigma\{E_{r+n} - \Sigma(2n-1)^2.\,E_{(r+n-1)}$
$$+\ \Sigma(2n-1)^2\,\Sigma(2n-3)^2\,E_{r+n-2} - + \dots\Big\}\frac{x^{2r}}{(2r)!};\ (1)$$

(ii) $(2n+1)!\ \sec x.\tan^{2n+1} x = \Sigma\{E_{r+n} - \Sigma(2n)^2.\,E_{r+n-1} +$
$$\Sigma(2n)^2\,\Sigma(2n-2)^2\,E_{r+n-2} - + \dots\Big\}\frac{x^{2r-1}}{(2r-1)!}.\ (2)$$

§ 3. We have proved in a paper on "A Table of Values of Thirty Eulerian Numbers" that the expansion of $(\sec x \tan^m x)$ in ascending powers of $x$ may be written in the form

$$\sec x\ \tan^m x = (m)!\left\{\frac{x^m}{m!} + \frac{x^{m+2}}{m+2!}\,\Sigma(m+1)^2 + \right.$$
$$\frac{x^{m+4}}{m+4!}\,\Sigma(m+1)^2\,\Sigma(m+2)^2 + \dots, \qquad (3)$$
$$\left. + \frac{x^{m+2r}}{(m+2r)!}\,\Sigma(n+1)^2\,\Sigma(n+r)^2 \dots \Sigma(n+r)^2 + \dots\right\}$$

when $m$ is an integer.

§ 4. Now comparing the expansion (1) with (3), we find that the co-efficients of powers of $x$ lower than $x^{2n}$ must vanish in (1). If we denote by $S_r$ the quantity $\Sigma(2n-1)^2\,\Sigma(2n-3)^2 \dots \Sigma(2n-2r+1)^2$, we thus obtain the following equations :—

$$\left.\begin{array}{l}
E_n - S_1 E_{n-1} + S_2 E_{n-2} - \dots + (-1)^n S_n E_0 = 0 \\
E_{n+1} - S_1 E_n + S_2 E_{n-1} - \dots + (-1)^n S_n E_1 = 0 \\
E_{n+2} - S_1 E_{n+1} + S\ E_n - \dots + (-1)^n S_n E_2 = 0 \\
\dots\quad \dots\quad \dots\quad \dots\quad \dots\quad \dots \\
E_{2n-1} - S_1 E_{2n-2} + S_2 E_{2n-3} - \dots + (-1)^n S_n E_{n-1} = 0.
\end{array}\right\}(4).$$

* For, assuming the results for values up to $2n$ and differentiating (i), we have

$$(2n+1)!\ \sec x.\tan^{2n+1} x + 2n.\,(2n)!\sec x.\tan^{2n-1} x$$
$$= \Sigma\{E_{r+n} - \Sigma(2n-1)^2.\ E_{r+n-1} +$$
$$\Sigma(2n-1)^2\,\Sigma(2n-3)^2.\,E_{r+n-2} - + \dots\Big\}\frac{x^{2r-1}}{(2r-1)!}$$

which gives (ii) for the next higher value, if we bear in mind that

$$2n.\,(2n)!\sec x.\tan^{2n-1} x = (2n)^2.\,\Sigma\{E_{r+n-1} - \Sigma(2n-2)^2\,E_{r+n-2}$$
$$+\ \Sigma(2n-2)^2\,\Sigma(2n-4)^2\,E_{r+n-3} - + \dots\Big\}\frac{x^{2r-1}}{(2r-1)!}$$

and $\quad \Sigma(2n)^2\,\Sigma(2n-2)^2 = (2n)^2\,\Sigma(2n-2)^2 + \Sigma(2n-2)^2\,\Sigma(2n-4)^2$
&c. &c.

Similarly for the other case.

Equating the co-efficients of $x^{2n}$ in (1) and (3),

$$E_{2n} - S_1 E_{2n-1} + S_2 E_{2n-2} - \dots + (-1)^n S_n E_n = (2n!)^2.\quad (5)$$

Now write

$$\Delta_{2n} = \begin{vmatrix} E_0 & E_1 & \dots & E_n \\ E_1 & E_2 & \dots & E_{n+1} \\ E_2 & E_3 & \dots & E_{n+2} \\ \dots & \dots & \dots & \dots \\ E_n & E_{n+1} & \dots & E_{2n} \end{vmatrix}$$

The above equations easily give (after a little reduction and re-arrangement), the relation

$$\Delta_{2n} = (2n\,!)^2 \,.\, \Delta_{2n-2},$$

whence

$$\Delta_{2n} = (2n\,!)^2 (2n-2\,!)^2 \,.\, (2n-4\,!)^2 \,..\, (2\,!)^2$$

$$= \{ (1.2)^n (3.4)^{n-1} (5.6)^{n-2} \,..\, (2n-1.2n) \}^2, \qquad (6)$$

Example.

$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_2 & E_3 & E_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 5 & 61 \\ 5 & 61 & 1385 \end{vmatrix} = 2304. = (4\,!)^2 (2\,!)^2.$$

Also, solving for $S_1$ from (4) and (5),

$$-S_1 \Delta_{2n} = \begin{vmatrix} E_n & E_{n-2} & E_{n-3} & \dots & E_0 & 0 \\ E_{n+1} & E_{n-1} & E_{n-2} & \dots & E_1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ E_{2n} & E_{2n-2} & E_{2n-3} & \dots & E_n & -(2n\,!)^2 \end{vmatrix}$$

$$\therefore\ S_1 \Delta_{2n} = (2n\,!)^2. \begin{vmatrix} E_0 & E_1 & \dots & E_{n-1} \\ E_1 & E_2 & \dots & E_n \\ \dots & \dots & \dots & \dots \\ E_{n-2} & E_{n-1} & \dots & E_{2n-3} \\ E_n & E_{n+1} & \dots & E_{2n-1} \end{vmatrix} \qquad (7)$$

[Notice that the row $| E_{n-1}, E_n \dots E_{2n-2} |$ is absent in the above.]

We thus obtain the result

$$\begin{vmatrix} E_0 & E_1 & E_2 & \dots & E_{n-1} \\ E_1 & E_2 & E_3 & \dots & E_n \\ \dots & \dots & \dots & \dots & \dots \\ E_{n-2} & E_{n-1} & E_n & \dots & E_{2n-3} \\ E_n & E_{n+1} & E_{n+2} & \dots & E_{2n-1} \end{vmatrix} = \frac{2n\,(2n-1)\,(4n-1)}{6} \times (2n-2\,!)^2 (2n-4\,!)^2 \dots\dots (2\,!)^2. \qquad (8)$$

Ex. If $m=3$, we get

$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_3 & E_4 & E_5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 5 & 61 \\ 61 & 1385. & 50521 \end{vmatrix} = 126720.$$

Solving generally for $S_r$, we have

$$\begin{vmatrix} E_0 & E_1 & \dots & E_{n-r-1} & E_{n-r+1} & \dots & E_n \\ E_1 & E_2 & \dots & E_{n-r} & E_{n-r+2} & \dots & E_{n+1} \\ \dots & \dots & & \dots & & \dots & \\ E_{n-1} & E_n & \dots & E_{2n-r-2} & E_{2n-r} & \dots & E_{2n-1} \end{vmatrix} = (2n-2\,!)^2 (2n-4\,!)^2 \dots (2\,!)^2\, S_r. \qquad (9)$$



Since $S_r$ can be readily calculated when $r$ is given, we thus obtain the values of such determinants for different $r$. Thus writing $n=3$, $r=2$, we have $S_r = 439$,

and

$$\begin{vmatrix} E_0 & E_2 & E_3 \\ E_1 & E_3 & E_4 \\ E_2 & E_4 & E_5 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 61 \\ 1 & 61 & 1385 \\ 5 & 1385 & 50521 \end{vmatrix}$$

$$= 1{,}011{,}456$$

$$= (24)^2 \,.\, (2)^2 \,.\, 439.$$

One case of great interest is that obtained by putting $r = n$; then

$$S_n = \Sigma(2n-1)^2\, \Sigma(2n-3)^2 \dots \Sigma 1^2$$

$$= (2n-1)^2 (2n-3^2) (2n-5)^2 \,..\, 1^2;$$

and we obtain the equation

$$\begin{vmatrix} E_1 & E_2 & \dots & E_n \\ E_2 & E_3 & \dots & E_{n+1} \\ \dots & \dots & \dots & \dots \\ E_n & E_{n+1} & \dots & E_{2n-1} \end{vmatrix} = (2\,!)^2 (4\,!)^2 \,..\, (2n-2\,!)^2 \dots 1^2.\, 3^2.\, 5^2 \dots (2n-1)^2. \qquad (10)$$

$$= (2n-1\,!)^2 (2n-3\,!)^2 \dots (3\,!)^2 (1\,!)^2$$

$$= [1^n \,.\, (2\,.\,3)^{n-1} (4\,.\,5)^{n-2} \dots (2n-2\,.\,2n-1)^1]^2.$$

Ex. If $n = 3$, we have

$$\begin{vmatrix} 1 & 5 & 61 \\ 5 & 61 & 1385 \\ 61 & 1385 & 50521 \end{vmatrix} = 518,\, 400 = 2^2.\, 24^2.\, 3^2.\, 5^2.$$

§ 5. Equating the co-efficients of $x^{2n+2s}$, we obtain from (1) and (3)

$$E_{2n+s} - \Sigma(2n-1)^2 E_{2n+s-1} + \dots$$

$$+ (-1)^n\, \Sigma(2n-1)^2\, \Sigma(2n-3)^2 \,..\, \Sigma(1^2) . E_{n+s}$$

$$= (2n\,!)^2\, \Sigma(2n+1)^2\, \Sigma(2n+2)^2 \dots \Sigma(2n+s)^2. \qquad (11)$$

Hence from (4) and (11), we obtain,

$$\begin{vmatrix} E_0 & E_1 & \dots & E_n \\ E_1 & E_2 & \dots & E_{n+1} \\ \dots & \dots & \dots & \dots \\ E_{n-1} & E_n & \dots & E_{2n-1} \\ E_{n+s} & E_{n+s+1} & \dots & E_{2n+s} \end{vmatrix} = (2n\,!)^2\, \Sigma(2n+1)^2\, \Sigma(2n+2)^2 \dots \Sigma(2n+s)^2.\ \ \Delta_{2n-2}$$

$$= (2n\,!)^2 (2n-2\,!)^2 \,..\, (2\,!)^2\, \Sigma(2n+1)^2\, \Sigma(2n+2)^2 \,..\, \Sigma(2n+s)^2. \qquad (12)$$

Ex. Writing $n=s=2$, we have

$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_4 & E_5 & E_6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 5 & 61 \\ 1385 & 50521 & 2702765 \end{vmatrix}$$

$$= 8,\, 031,\, 744.$$

$$= 2^2.\, (24)^2.\, 3486.$$

$$= 2^2,\, (24)^2.\, (\Sigma 5^2\, \Sigma 6^2).$$

Writing $n=2$, $5=3$, we get

$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_5 & E_6 & E_7 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 1 & 61 \\ 50{,}521 & 2{,}702{,}765 & 199{,}360{,}981 \end{vmatrix}$$

$= 647,\ 907,\ 840.$

$= 2^2.\ (24)^2\ (281,\ 210)$

$= 2^2.\ (24)^2\ (\Sigma\, 5^2\ \Sigma\, 6^2\ \Sigma\, 7^2)$

§ 6. Now considering (2) and comparing it with (3), for $m$ odd, we obtain the following equations:—

$$\left.\begin{aligned} E_{n+1} - k_1 . E_n + k_2\, E_{n-1} - \dots + (-1)^n k_n\, E_1 &= 0 \\ E_{n+2} - k_1 . E_{n+1} + k_2\, E_n - \dots + (-1)^n k_n\, E_2 &= 0 \\ E_{2n} - k_1 . E_{2n-1} + k_2\, E_{2n-2} - \dots + (-1)^n k_n\, E_n &= 0 \end{aligned}\right\} \quad (4 . a)$$

$$E_{2n+1} - k_1\, E_{2n} + k_2\, E_{2n-1} - \dots + (-1)^n k_n\, E_{n+1} = (2n+1\,!)^2. \quad (5 . a)$$

where $k_r = \Sigma(2n)^2\ \Sigma(2n-2)^2 . \Sigma(2n-2r+2)^2.$

Without going into details, we may state in particular that we obtain (10) and also the persymmetric,

$$\begin{vmatrix} E_2 & E_3 & \dots & E_{n+1} \\ E_3 & E_4 & \dots & E_{n+2} \\ \dots & \dots & \dots & \dots \\ E_{n+1} & E_{n+2} & \dots & E_{2n} \end{vmatrix} = [1^n\,(2.3)^{n-1}\ \dots\ (2n-2\,.\,2n-1)^1]^2.$$

$$\times\ \Sigma\,(2n)^2\ \Sigma\,(2n-2)^2\ \dots\ \Sigma\, 2^2. \quad (10 . a)$$

*Ex.* $\begin{vmatrix} E_2 & E_3 \\ E_3 & E_4 \end{vmatrix} = \begin{vmatrix} 5 & 61 \\ 61 & 1385 \end{vmatrix} = 3204 = 6^2 . (89)$

§ 7. We next proceed to evaluate in a similar manner the values of determinants involving the prepared Bernoullian numbers.

We shall write

$$b_n = 2^{2n}\,(2^{2n}-1)\frac{B_n}{2n}.$$

It is well-known that

$$\tan x = \Sigma\, b_n \frac{x^{2n-1}}{2n-1\,!}.$$

Differentiating and transposing

$$\tan^2 x = \sum_{1}^{\infty} b_n \frac{x^{2n-2}}{2n-2\,!}.$$

Differentiating again and using the above

$$\therefore\quad 2\,!\,\tan^3 x = \Sigma\,(b_{n+1} - 1.2 b_n)\frac{x^{2n-1}}{2n-1\,!}.$$

Proceeding in this manner, we can write down, by induction, that,

$$(2n-2)!\tan^{2n-1} x = \Sigma \frac{x^{2r-1}}{2r-1\,!}\Big\{ b_{r+n-1} - L_1\, b_{r+n-2} + L_2\, b_{r+n-3}$$

$$- \dots + (-1)^{n-1} L_{n-1}\, b_r \Big\} \dots \quad (13)$$



where $L_r = \Sigma\,(2n-3,\ 2n-2)\ \Sigma\,(2n-5,\ 2n-4) .. \Sigma\,(2n-2r-1,\ 2n-2r).$

$$(2n-1)\,!\ \tan^{2n} x = \Sigma\,\frac{x^{2r}}{2r!}\Big\{ b_{r+n} - T_1\, b_{r+n-1} + T_2 b_{r+n-2} \dots$$

$$+ (-1)^{n-1}\ T_{n-1}\ b_{r+1} \Big\} \quad (14)$$

where $T_r = \Sigma\,(2n-2,\ 2n-1)\ \Sigma\,(2n-4,\ 2n-3) \dots \Sigma\,(2n-2r,\ 2n-2r+1)$

§ 8. From (13), we obtain the following equations:—

$$\left.\begin{aligned} & b_n - L_1 b_{n-1} + L_2 b_{n-2} - \dots + (-1)^{n-1} L_{n-1} b_1 = 0 \\ & b_{n+1} - L_1 b_n + L_2 b_{n-1} - \dots + (-1)^{n-1} L_{n-1} b_2 = 0 \\ & b_{n+2} - L_1 b_{n+1} + L_2 b_n - \dots + (-1)^{n-1} L_{n-1} b_3 = 0 \\ & \dots \quad \dots \quad \dots \quad \dots \\ & b_{2n-2} - L_1 b_{2n-3} + L_2 b_{2n-4} - \dots + (-1)^{n-1} L_{n-1} b_{n-1} = 0 \\ & b_{2n-1} - L_1 b_{2n-2} + L_2 b_{2n-3} - \dots + (-1)^{n-1} L_{n-1} b_n \\ & \qquad = (2n-1)\,!\,(2n-2)\,! \end{aligned}\right\} \quad (15)$$

Hence, we obtain

$$\begin{vmatrix} b_1 & b_2 & b_3 & \dots & b_n \\ b_2 & b_3 & b_4 & \dots & b_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ b_n & b_{n+1} & \dots & \dots & b_{2n-1} \end{vmatrix} = 2n-1\,!\ 2n-2\,! \begin{vmatrix} b_1 & b_2 & \dots & b_{n-1} \\ b_2 & b_3 & \dots & b_n \\ \dots & \dots & \dots & \dots \\ b_{n-1} & b_n & \dots & b_{2n-3} \end{vmatrix}$$

$$= 2n-1\,!\ 2n-2\,!\ 2n-3\,!\ 2n-4\,! \dots 2\,!\ 1\,!. \quad (16)$$

Ex. $\begin{vmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \\ b_3 & b_4 & b_5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 16 \\ 2 & 16 & 272 \\ 16 & 272 & 7936 \end{vmatrix}$

$= 34,\ 560$

$= 1.\ 2.\ 6.\ 24.\ 120.$

§ 9. Solving for $L_r$ in general, we have,

$$\begin{vmatrix} b_1 & b_2 & \dots & b_{n-r-1} & b_{n-r+1} & \dots & b_n \\ b_2 & b_3 & \dots & b_{n-r} & b_{n-r+2} & \dots & b_{n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1} & b_n & \dots & b_{2n-r-3} & b_{2n-r-1} & \dots & b_{2n-2} \end{vmatrix} 2n-1\,!\ 2n-2$$

$$= L_r.\ 2n-1\,!\ 2n-2\,! \dots\ 2\,!\ 1\,!.$$

Thus

$$\begin{vmatrix} b_1 & b_2 & \dots & b_{n-r-1} & b_{n-r+1} & \dots & b_n \\ b_2 & b_3 & \dots & b_{n-r} & b_{n-r+2} & \dots & b_{n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1} & b_n & \dots & b_{2n-r-3} & b_{2n-r-1} & \dots & b_{2n-2} \end{vmatrix}$$

$$= L_r\ (2n-3)\,!\ (2n-4)\,!\ \dots\ 2\,!\ 1\,!. \quad (17)$$

In particular, writing $r = 1$,

$$\begin{vmatrix} b_1 & b_2 & \cdots & b_{n-2} & b_n \\ b_2 & b_3 & \cdots & b_{n-1} & b_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_n & \cdots & b_{2n-4} & b_{2n-2} \end{vmatrix}$$

$$= 2n-3\,!\; 2n-4\,!\; 2n-5\,! \;\ldots\; 1\,! \times L_r.$$

and $$L_r = 1\,.\,2 + 2\,.\,3 + \ldots + (2n-3)(2n-2)$$
$$= (2n-3)(2n-2)(2n-1)/3.$$

$$\therefore \quad D = \frac{(2n-3)(2n-2)(2n-1)}{3} (2n-3)\,!\,(2n-4)\,!\ldots 1\,!. \quad (18)$$

§ 10. From (14), we obtain the following equations—

$$\left.\begin{array}{l} b_{n+1} - T_1 b_n + T_2 b_{n-1} - \ldots + (-1)^{n-1} T_{n-1} b_2 = 0 \\ b_{n+2} - T_1 b_{n+1} + T_2 b_n - \ldots + (-1)^{n-1} T_{n-1} b_3 = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ b_{2n-1} - T_1 b_{2n-2} + T_2 b_{2n-3} - \ldots + (-1)^{n-1} T_{n-1} b_n = 0 \end{array}\right\} \quad (19)$$

$$b_{2n} - T_1 b_{2n-1} + T_2 b_{2n-2} - \ldots + (-1)^{n-1} T_{n-1} b_{n+1}$$
$$= (2n)\,!\,(2n-1)\,!\,. \quad \ldots (20)$$

Hence as before,

$$\begin{vmatrix} b_2 & b_3 & \cdots & b_{n+1} \\ b_3 & b_4 & \cdots & b_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{vmatrix} = 2n\,!\,(2n-1)\,!\,(2n-2)\,! \ldots 2\,!\,1\,! \quad (21)$$

and

$$\begin{vmatrix} b_2 & b_3 & \cdots & b_{n-r} & b_{n-r+2} & \cdots & b_{n+1} \\ b_3 & b_4 & \cdots & b_{n-r+1} & b_{n-r+3} & \cdots & b_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n+1} & b_{n+2} & \cdots & b_{2n-r-1} & b_{2n-r+1} & \cdots & b_{2n-1} \end{vmatrix}$$
$$= T_r.\ (2n-2)\,!\,(2n-3)\,! \ldots 1\,! \quad \ldots (22)$$

§ 11. Similarly we obtain, from (19), by solving for $T_{n-1}$, that,

$$\begin{vmatrix} b_3 & b_4 & \cdots & b_{n+1} \\ b_4 & b_5 & \cdots & b_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n+1} & b_{n+2} & \cdots & b_{2n-1} \end{vmatrix} = T_{n-1} \begin{vmatrix} b_2 & b_3 & \cdots & b_n \\ b_3 & b_4 & \cdots & b_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ b_n & b_{n+1} & \cdots & b_{2n-2} \end{vmatrix}$$

$$= T_{n-1}\,.\, 2n-2\,!\; 2n-3\,! \;\ldots\; 2\,!\,.\,1\,!.$$

[*To be continued.*]



# MATHEMATICS IN INDIA—THEN AND NOW.

By V. Sankaran.

(*Continued from page 224, Vol. XIII*).

Let us next consider the Tamil work *Kanakkadikaram*. Its contents are roughly :—(1) notation, (2) tables of weights, (3) relative weights and sizes, (4) measurement of time, (5) alloys, (6) lengths, (7) table of capacity, (8) tables of time and life of different animals, (9) areas, (10) military units, (11) mensuration, (12) time and work, (13) partnership, (14) mixtures, and (15) miscellaneous examples. Only such problems are dealt with as are likely to be useful in the practical application to other sciences and to life. The style is popular, and there are few technical phrases employed in the exposition of the subject-matter.

We shall examine these contents at some length : The fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{20}$, $\frac{1}{80}$, $\frac{1}{320}$ have special names—*arai*, *kâl*, *araikâl*, *mû*, *kâni*, and *mundirai*. Any given fraction is usually expressed in terms of these and the approximation is taken to the nearest mundirai. $\frac{1}{320^2}$ was termed the *kīlmundirai*; there was the fraction *immi* equal to $\frac{2}{21}$ of a *kīlmundirai*. This shows that, where necessary, people used to calculate $\frac{2}{21} \times \frac{1}{320} \times \frac{1}{320}$ of a unit. However, they generally stopped with the nearest *mundirai*. Sub-multiples of *immi* are also mentioned.

In connection with the tables of weights and measures, we find the following table for gold :—

| | | |
|---|---|---|
| 2 grains | = 1 pilavu. | |
| 2 pilavus | = 1 kundri. | |
| 2 kundris | = 1 manjadi. | |
| 20 manjadis | = 1 kalanji | (= $\frac{2}{5}$ sovereign). |
| 2 kalanjis | = 1 kaisa. | |
| 4 kaisas | = 1 palam. | |
| 100 palams | = 1 nirai. | |
| 2 nirais | = 1 tulam. | |
| 32 tulams | = 1 baram. | |

This system is still in vogue in parts of the Tinnevelly District. Particulars about the weights of well-known substances are added and are followed by other interesting approximations. Thus we are told that

| | | | | |
|---|---|---|---|---|
| one measure of | clay | weighs | 17 | palams |
| " | sand | " | 20 | " |
| " | paddy | " | 6 | " |
| " | rice | " | 10 | " |
| " | salt | " | 16 | " |

# DETERMINANTS INVOLVING SPECIFIED NUMBERS.

(*Continued from page* 62, *Vol. XIV, No.* 2, *J.I.M.S.*)

By C. Krishnamachary & M. Bhimasena Rao.

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§ 1. Let $a_n$ be any function of the integral variable $n$. Consider the following table :—

| | | | | | |
|---|---|---|---|---|---|
| $0$ | $\Sigma a_1$ | $0$ | $\Sigma a_1 \Sigma a_2$ | $0$ | $\Sigma a_1 \Sigma a_2 \Sigma a_3$ |
| $a_1$ | $\Sigma a_2$ | $a_1 \Sigma a_2$ | $\Sigma a_2 \Sigma a_3$ | $a_1 \Sigma a_2 \Sigma a_3$ | $\Sigma a_2 \Sigma a_3 \Sigma a_4$ |
| $a_2$ | $\Sigma a_3$ | $a_2 \Sigma a_3$ | $\Sigma a_3 \Sigma a_4$ | $a_2 \Sigma a_3 \Sigma a_4$ | $\Sigma a_3 \Sigma a_4 \Sigma a_5$ |
| $a_3$ | $\Sigma a_4$ | $a_3 \Sigma a_4$ | $\Sigma a_4 \Sigma a_5$ | $a_3 \Sigma a_4 \Sigma a_5$ | |
| $a_4$ | $\Sigma a_5$ | $a_4 \Sigma a_5$ | | | |

The process of constructing the table is obvious. The second column is obtained from the first by a process of addition. Thus

$$\Sigma a_1 = a_1,\ \Sigma a_2 = a_1 + a_2,\ \ldots$$

The third is obtained from the second by multiplying by the corresponding numbers of the first column. The fourth is obtained from the third by addition, and so on. Thus

$$\Sigma a_1 \Sigma a_2 = a_1 \Sigma a_2 \,;$$

$$\Sigma a_2 \Sigma a_3 = a_2 \Sigma a_3 + a_1 \Sigma a_2 \,;$$

Calling $a_1, a_2, a_3$, the *elements*, and the various *numbers in the even columns, the elemental numbers* of the table, we may denote

$$\Sigma a_n \Sigma a_{n+1} \ldots \Sigma a_{n+r} \quad \text{by } {}_nA_{n+r}\,; \qquad (1)$$

so that the left suffix indicates the element which commences the $\Sigma$ and the right suffix the one which ends it. A is employed to denote the numbers defined from the $a$ elements. Similarly, we may use B to denote the elemental numbers defined by the $b$ elements.

The numbers in the *even* columns of the $r$th row are

$${}_rA_r,\ {}_rA_{r+1},\ {}_rA_{r+2},\ \ldots\ldots$$

The left suffix indicates the row, and the right suffix, the number of the *even* column to which the number belongs ; *e.g.*

$${}_rA_{r+k-1} = \Sigma a_r \Sigma a_{r+1} \ldots \Sigma a_{r+k-1}$$

is the number in the $k^{th}$ even column of the $r^{th}$ row.



From the way in which the table is formed, we have

$$\Sigma a_r \Sigma a_{r+1} \ldots \Sigma a_m = a_r \Sigma a_{r+1} \Sigma a_{r+2} \ldots \Sigma a_n + \Sigma a_{r-1} \Sigma a_r \ldots \Sigma a_{n-1}\text{*}$$

Hence the general formula of reduction, *viz.*

$${}_rA_n = a_r \cdot {}_{r+1}A_n + {}_{r-1}A_{n-1} \qquad \ldots \qquad (2)$$

1·1. Similarly we may denote

$$\Sigma a_n \Sigma a_{n-2} \ldots \Sigma a_{n-2r} \quad \text{by } {}_n\alpha_{n-2r} \qquad \ldots \qquad (3)$$

Here again the left suffix indicates the element which begins the $\Sigma$ and the right, the one which ends it. Also, we have the general formula of reduction, *viz.*

$${}_n\alpha_{n-2r} = a^n \cdot {}_{n-2}\alpha_{n-2r} + {}_{n-1}\alpha_{n-2r-1}. \qquad (3.1)$$

Thus

$${}_{2n-1}\alpha_1 = a_{2n-1}\, a_{2n-3} \ldots a_3\, a_1.$$

$${}_{2n-2}\alpha_2 = a_{2n-2} \cdot {}_{2n-4}\alpha_2 + a_{2n-3}\, a_{2n-5} \ldots a_3\, a_1. \qquad (3.2)$$

1·2. Let us reduce (2) further for the elemental numbers of the first row as follows.—

$${}_1A_n = a_1 \cdot {}_2A_n$$

$$= a_1 \cdot (a_2 \cdot {}_3A_n + {}_1A_{n-1})$$

*i.e.* $${}_1A_n - a_1 \cdot {}^1A_{n-1} = a_1\, a_2 \cdot {}^3A_n \qquad \ldots \qquad (4\cdot1)$$

$$= a_1\, a_2\, (a_3 \cdot {}_4A_n + {}_2A_{n-1}).$$

$$= a_1\, a_2\, a_3 \cdot {}_4A_n + a_2 \cdot {}_1A_{n-1} \qquad \ldots \qquad (4.2)$$

*i.e.* $${}_1A_n - (a_1 + a_2)\, {}_1A_{n-1} = a_1\, a_2\, a_3 \cdot {}_4A_n.$$

$$= a_1\, a_2\, a_3 \cdot (a_4 \cdot {}_5A_n + {}_3A_{n-1})$$

$$= a_1\, a_2\, a_3\, a_4 \cdot {}_5A_n + a_3\, ({}_1A^{n-1} - a_1 \cdot {}_1A_{n-2})$$

from (4.1).

*i.e.* $${}_1A_n - (a_1 + a_2 + a_3)\, {}_1A_{n-1} + a_3\, a_1 \cdot {}_1A_{n-2}$$

$$= a_1\, a_2\, a_3\, a_4 \cdot {}_5A_n. \qquad \ldots \qquad \ldots \qquad (4.3)$$

$$= a_1\, a_2\, a_3\, a_4\, (a_5 \cdot {}_6A_n + {}_4A_{n-1})$$

$$= a_1\, a_2\, a_3\, a_4\, a_5 \cdot {}_6A_n$$

$$+ a_4\, \{ {}_1A_{n-1} - (a_1 + a_2)\, {}_1A_{n-2} \} \text{ from (4.2)}$$

Rearranging, this may be written

$${}_1A_n - {}_4\alpha_4 \cdot {}_1A_{n-1} + {}_4\alpha_2 \cdot {}_1A_{n-2} = a_1\, a_2\, a_3\, a_4\, a_5 \cdot {}_6A_n \ldots \qquad (4.4)$$

---

* The use of the $\Sigma$ is clearly explained in the first section of the last paper. (See Page 55, Vol. *XIV*, No. 2, *J. I. M. S.*) The repetition of the $\Sigma$ merely stands for repeated summation, the brackets being omitted for convenience.

After $r$ reductions as above, we obtain the following general relation between the $\alpha$'s and the A's.

$$ {}_1A^n - {}^{r-1}\alpha_{r-1}\ {}_1A_{n-1} + {}_{r-1}\alpha_{r-3}\ .\ {}_1A_{n-2} - \dots $$
$$ + (-1)^{k-1}\ {}_{r-1}\alpha_{r-2k-1}\ .\ {}_1A_{n-k-1} $$
$$ = a_1\ a_2\ a_3\ \dots\ a_r\ .\ {}_{r+1}A_n, \qquad \dots \qquad \dots \quad (4) $$

where $r-2k-1=2$, if $(r-1)$ is even ; and 1 if $(r-1)$ is odd.

*The right suffix for $\alpha$ in the last term is always 2 or 1.* The series on the left is to be continued till then. Remembering the reduction formulæ for $\alpha$'s expressed in (3·1) and (3·2), (4) can be easily proved by induction. (4) is fundamental in the theory of the functions we deal with. It is true whatever be the value of $r$ provided it is less than $n$. If $r = n$, it is easily seen that (4) reduces to

$$ {}_1A_n - {}_{n-1}\alpha_{n-1}\ .\ {}_1A_{n-1} + {}^{n-1}\alpha_{n-3}\ .\ {}_1A_{n-2} - . = a_1\ a_2\ a_3\ \dots\ a_n. \quad (4') $$

A question naturally arises, can we find relations so that the last elemental number occurring in (4) is ${}_1A_1$? Or, what is the same thing: what is the value of the left-hand side expression in (4) for values of $r > n$? We can, consistently with the original table, give to $r$ any values which make the right hand suffix of the last A in (4) equal to any positive integer down to unity. We proceed to prove that *if $r > n$, the value of the expression on the left is zero.*

1·3. The following purely arithmetical method of establishing the fundamental equations (13) and (14) below is given on account of its directness. The method of § 4 is important in the theory, and is therefore added.

It is obvious from (4) that after $n-1$ reductions (*i.e.* when $r=n-1$), we have

$$ {}_1A_n - {}_{n-2}\alpha_{n-2}\ .\ {}_1A_{n-1} + {}_{n-2}\alpha_{n-4}\ .\ {}_1A_{n-2} - \dots $$
$$ = a_1 a_2\ \dots\ a_{n-1}\ .\ {}_nA_n \qquad \dots \qquad \dots \qquad \dots \quad (4.5) $$

*the series on the left being continued till the right suffix of $\alpha$ is 1 or 2.*

For the sake of clearness, let $n = 2m$ and omit the left suffix of A's. The above equation is

$$ A_{2m} - {}_{2m-2}\alpha_{2m-2}\cdot A_{2m-1} + {}_{2m-2}\alpha_{2m-4}\cdot A_{2m-2} - \dots $$
$$ + (-1)^r\ {}_{2m-2}\alpha_{2m-2r}\cdot A_{2m-r} + \dots + (-1)^{m-1}\ {}_{2m-2}\alpha_2\ .\ A_{m+1} $$
$$ = a_1\ a_2\ \dots\ a_{2m-1}\cdot {}_{2m}A_{2m} \qquad \dots \qquad \dots \quad (4.6) $$



Let $n = 2m-1$. Then,

$$ A_{2m-1} - {}_{2m-3}\cdot\alpha_{2m-3}\ A_{2m-2} + {}_{2m-3}\alpha_{2m-5}\cdot A_{2m-3} - \dots $$
$$ + (-1)^r\ {}_{2m-3}\alpha_{2m-2r-1}\cdot A_{2m-2r-1} + \dots + (-1)^{m-1}\ {}_{2m-3}\alpha_1 . A_m $$
$$ = a_1\ a_2\ \dots\ a_{2m-2}\cdot {}_{2m-1}A_{2m-1}\cdot \dots \qquad (4.7) $$

Now from (4.6),

$$ A_{2m} - {}_{2m-2}\alpha_{2m-2}\cdot A_{2m-1} + {}_{2m-2}\alpha_{2m-4}\cdot A_{2m-2} - \dots $$
$$ + (-1)^r\ {}_{2m-2}\alpha_{2m-2r}\cdot A_{2m-r} + \dots + (-1)^{m-1}\ {}_{2m-2}\alpha_2 . A_{m+1} $$
$$ = a_1\, a_2 \dots a_{2m-1}\,(a_{2m} + {}_{2m-1}A_{2m-1}) $$
$$ = a_1\, a_2 \dots a_{2m} + a_{2m-1}\,\{ A_{2m-1} - {}_{2m-3}\alpha_{2m-3}\cdot A_{2m-2} + \dots $$
$$ + (-1)^r\ {}_{2m-3}\alpha_{2m-2r-1}\cdot A_{2m-r-1} + \dots + (-1)^{m-1}\ {}_{2m-3}\alpha_1\cdot A_m \} $$

by substituting from (4·7). Transposing the terms within the flower brackets on the right, to the left and remembering (3·1), we obtain,

$$ A_{2m} - {}_{2m-1}\alpha_{2m-1}\cdot A_{2m-1} + {}_{2m-1}\alpha_{2m-3}\cdot A_{2m-2} - \dots + (-1)^r $$
$$ {}_{2m-1}\alpha_{2m-2r+1}\cdot A_{2m-r} + \dots + (-1)^m {}_{2m-1}\alpha_1\cdot A_m = a_1\, a_2\, \dots a_{2m}. \quad (4.8) $$

which is the formula (4′) for $n=2m$. Similarly we can obtain (4′) when $n=2m-1$, *viz.*

$$ A_{2m-1} - {}_{2m-2}\alpha_{2m-2}\cdot A_{2m-2} + \dots + (-1)^r\ {}_{2m-2}\alpha_{2m-2r}\cdot A_{2m-r-1} $$
$$ + (-1)_{m-1}\ {}_{2m-2}\alpha_2\cdot A_m = a_1 a_2 .\ \alpha_{2m-1}. \quad (4.9) $$

Substituting from (4·9) in (4·8) for $a_1\ a_2 \dots a_{2m-1}$, we have,

$$ A_{2m} - {}_{2m-1}\alpha_{2m-1}\cdot A_{2m-1} + {}_{2m-1}\alpha_{2m-3}\cdot A_{2m-2} - \dots $$
$$ + (-1)^r {}_{2m-1}\alpha_{2m-2r+1}\cdot A_{2m-r} + \dots + (-1)^m {}_{2m-1}\alpha_1\cdot A_m $$
$$ = a_{2m}\ \{ A_{2m-1} - {}_{2m-2}\alpha_{2m-2}\cdot A_{2m-2} + \dots $$
$$ + (-1)^r {}_{2m-2}\alpha_{2m-2r}\ A_{2m-r-1} + \dots + (-1)^{m-1}\ {}_{2m-2}\alpha_2\cdot A_m \}. $$

*i.e.* $$ A_{2m} - {}_{2m}\alpha_{2m}\cdot A_{2m-1} + {}_{2m}\alpha_{2m-2}\cdot A_{2m-2} - \dots $$
$$ (-1)^r {}_{2m}\alpha_{2m-2r+2}\cdot A_{2m-r} + \dots + (-1)^m\ {}_{2m}\alpha_2\ A_m \} = 0. \quad (4.10) $$

Similarly from (4·9), by substituting for $a_1 a_2 \dots a_{2m-2}$,

$$ A_{2m-1} - {}_{2m-1}\alpha_{2m-1}\cdot A_{2m-2} + {}_{2m-1}\alpha_{2m-3}\cdot A_{2m-3} - \dots $$
$$ + (-1)^r\ {}_{2m-1}\alpha_{2m-2r+1}\cdot A_{2m-r-1} + \dots + (-1)^m {}_{2m-1}\alpha_1\cdot A_{m-1} = 0. $$
$$ (4.11) $$

From (4·10), writing $m-1$ for $m$, we have

$$A_{2m-2} - {}_{2m-2}a_{2m-2}\, A_{2m-3} + \ldots + (-1)^r\, {}_{2m-2}a_{2m-2r} \cdot A_{2m-r-2}$$
$$+ \ldots + (-1)^{m-1}\, {}_{2m-2}a_2 \cdot A_{m-1} = 0. \qquad (4{\cdot}12)$$

Multiply (4·12) by $-a_{2m}$, and add to (4·11). We get

$$A_{2m-1} - {}_{2m}a_{2m} \cdot A_{2m-2} + {}_{2m}a_{2m-2} \cdot A_{2m-3} - \ldots$$
$$+ (-1)^m\, {}_{2m}a_2 \cdot \; A_{m-1} = 0. \qquad (4{\cdot}13)$$

Similarly to (4·12) add $-a_{2m-1}$ times the expression on the left in (4·11) after writing $m-1$ for $m$. We obtain,

$$A_{2m-2} - {}_{2m-1}a_{2m-1}\, A_{2m-3} + \ldots$$
$$+ (-1)^m\, {}_{2m-1}a_1\, A_{m-2} = 0 \qquad (4{\cdot}14)$$

The equations (4·10) and (4·13) are the first two of the equations in (14·3) § 4, below ; and (4·11) and (4·14) are the last two of the equations in (13·3). It is now obvious that the other equations in (13·3) and (14·3) can be similarly obtained *in a purely arithmetical manner* without any reference to the method in § 4.

§ 2. Let

$$M(x) = \frac{x^r}{r!} - \Sigma a_{r+1} \frac{x^{r+2}}{r+2} + \Sigma a_{r+1} \Sigma a_{r+2} \frac{x^{r+4}}{r+4!} - \ldots$$
$$= \frac{x^r}{r!} - {}_{r+1}A_{r+1} \frac{x^{r+2}}{r+2} + {}_{r+1}A_{r+2} \frac{x^{r+4}}{r+4!} - \ldots \qquad (5)$$

the coefficients A being the elemental numbers in the even columns of the $(r+1)^{th}$ row. Thus,

$$M_0(x) = 1 - \Sigma a_1 \cdot \frac{x^2}{2!} + \Sigma a_1 \Sigma a_2 \cdot \frac{x^4}{4!} - \ldots$$
$$= 1 - {}_1A_1 \frac{x^2}{2!} + {}_1A_2 \cdot \frac{x^4}{4!} - \ldots$$
$$M_1(x) = \frac{x}{1!} - {}_2A_2 \frac{x^3}{3!} + {}_2A_3 \cdot \frac{x^5}{5!} - \ldots$$

The functions $M^r(x)$ are very interesting and general, and because of their wide generality, we propose to deal with their properties at some length in a future paper. In earlier papers *, we have identified the functions $M_r(x)$ with well known functions as follows.—

(1) $a_n = n^2$. $\quad {}_iA = {}_nE_n$, $n^{th}$ Eulerian number.

$$M_0(x) = \text{sech}\, x,\ M_1(x) = \text{sech}\, x \tanh x$$
$$M_r(x) = \frac{\text{sech}\, x\ (\tanh x)^r}{r!}. \qquad (6)$$

* "Some properties of Eulerian and prepared Bernoullian numbers" presented to the Third Conference of the Indian Mathematical Society.



(2) $a = r\,(n+r-1)$.

$$M_0(x) = (\text{sech}\, x)^n, \qquad M_1(x) = (\text{sech}\, x)^n \tanh x.$$
$$M_r(x) = (\text{sech}\, x)^n \frac{(\tanh x)^r}{r!}. \qquad (7)$$

(3) $a_n = n^*$. $\quad {}_1A_n = 1{\cdot}3{\cdot}5\ldots(2n-1)$.

$$M_0(x) = e^{-x^2/2}, \qquad M_r(x) = \frac{x_r}{r!} \cdot e^{-x^2/2}. \qquad (8)$$

$$M_0(x) = (\text{sech}\, x\sqrt{a})^{\frac{b}{a}+1}.$$

(4) $\quad a_n = an + bn. \qquad M^r(x) = (\text{sech}\, x\sqrt{a})^{\frac{a}{b}+1} \dfrac{(\tanh x\sqrt{a})^r}{r!\ \ a^{\frac{1}{2}r}}.$

(5) $\quad a_n = 1.\ M_0(x) = \dfrac{1}{x} J_1(2x)$

$$M_r(x) = \frac{r+1}{x} J_{r+1}(2x) = J_r(2x) + J_{r+2}(2x).$$

In fact

$$M_r(x) = \frac{x^r}{r!} - \frac{r+1}{1!} \cdot \frac{x^{r+2}}{r+2!} + \frac{(r+1)(r+4)}{2!} \frac{x^{r+4}}{r+4!} - \frac{(r+1)(r+5)(r+6)}{3!} \frac{x^{r+6}}{r+6!} + \ldots$$

[A table of values is found in Appendix I.]

The following algebraical method of proof may be found interesting :

$$\Sigma a_{r+1} = 1 + 1 + \ldots \text{ to } r+1 \text{ terms } = \frac{r+1}{1}.$$

$\Sigma a_{r+1} \Sigma a_{r+2}$ = sum of $r+1$ terms of the series $\Sigma a_{r+2}$

$$= \frac{r+2}{1} + \frac{r+1}{1} + \ldots + \frac{2}{1} = \frac{(r+1)(r+4)}{1{\cdot}2}$$

* The cas of $a_n = n$ presents remarkable simplicity in the evaluation of $a_n \Sigma a_{n+1} \ldots \Sigma a_{n+r}$. Thus

$$\Sigma a_{n+1} = \frac{(n+1)(n+2)}{2}$$
$$\Sigma a_n \Sigma a_{n+1} = \sum_1^n \frac{n(n+1)(n+2)}{2}.$$
$$= \frac{n(n+1)(n+2)(n+3)}{2{\cdot}4}.$$

$\Sigma a_{r+1} \Sigma a_{r+2} \Sigma a_{r+3}$ = sum of $(r+1)$ terms of the series whose last term is $\Sigma a_{r+2} \Sigma a_{r+}$

= sum of $(r+1)$ terms of the series whose $r$th term is

$$\frac{(r+1)\ (r+4}{1\cdot 2}$$

$$= \frac{(r+1)\ (r+5)\ (r+6)}{1\cdot 2\cdot 3} \quad \text{and so on.}$$

§ 3. Since ${}_1A_n = a_1 \cdot {}_2A_n$, we have

$$\frac{d}{dx} M_0 (x) = -a_1 M_1 (x). \tag{11·1}$$

In virtue of the recurrence formula (2), we have the general and fundamental relation between any three consecutive series,

$$\frac{d}{dx} M_r (x) = M_{r-1} (x) - a_{r+1} M_{r+1} (x). \tag{11}$$

This relation is true for all positive integral values of $r$ and for $r = 0$, if we consider $M_{-1} (x) = 0$ in virtue of (11·1).

From (11), it follows that $M_r (x)$ can be expressed in terms of $M_0 (x)$ and its differential coefficients.

And we easily obtain

$$(D^r + {}_{r-1}\alpha_{r-1} D^{r-2} + {}_{r-1}\alpha_{r-3} D^{r-4} + \ldots) M_0 (x)$$
$$= (-1)^r a_1 a_2 \ldots a_r M_r (x). \tag{12}$$

This is directly proved by induction. The last term on the left is

$${}_{r-1}\alpha_1 = a_{r-1} a_{r-3} \ldots a_3 a_1, \text{ if } r \text{ is even;}$$

and

$${}_{r-1}\alpha_2 \cdot D = \Sigma a_{r-1} \Sigma a_{r-3} \ldots \Sigma a_2 D, \text{ if } r \text{ is odd.} \tag{12·3}$$

Following Brioschi (Muir, *Theory of Determinants*, Vol. II, page 344), we can write the equation (12) in the form

$$\begin{vmatrix} D & a_1 & \ldots & \ldots & \\ -1 & D & a_2 & \ldots & \\ \cdot & -1 & D & a_3 & \ldots \end{vmatrix} M_0 (x) = (-1)^r a_1 a_2 \ldots a_r M_r (x). \tag{12·4}$$

there being $r$ rows and columns, D standing for $\frac{d}{dx}$.



It will be proved in a continuation of this paper that the denominators of the convergents of the continued fraction

$$\frac{1}{D-}\ \frac{a_1}{D-}\ \frac{a_2}{D-} \ldots \tag{12·5}$$

is the expression on the left hand side in (12).

3·1. One interesting point about (12) may be noticed in passing, viz. whenever one of the elements $a_r$ vanishes, the right hand side in (12) vanishes, and hence we can obtain the function $M_0 (x)$ by solving a linear differential equation with constant co-efficients. Since such an equation can always be solved (at least theoretically), the function $M_0 (x)$ can in such a case be determined completely, but for constants.

*Ex.* 1. Write $a_n = n(n-3)$. $a_3 = 0$.

The equation (12) is

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0.$$

$$\therefore \quad y = A + B \cosh 2x + c \sinh 2x.$$

Here $a_n = n^2 - [3n$, the case in § 2, (4) where $a = 1, b = -3$,] so that $M_0 (x) = \cosh^2 x = \frac{1}{2} (1 + \cosh 2x)$.

Since $\frac{d}{dx} M_0 (x) = -a_1 M_1 (x) = 2 M_1 (x)$,

the above equation can be written

$$\left(\frac{d^2}{dx^2} - 4\right) M_1 (x) = 0.$$

*i.e.* $M_1 (x) = A \cosh 2x + B \sinh 2x$.

From § 2, (4), $M_1 (x) = \cosh^2 x \tanh x$.
$= \frac{1}{2} \sinh 2x$.

*Ex.* 2. Write $a_n = (n-2)(n-3)$.

The differential equation for $M_0 (x)$ is

$$\frac{d^2y}{dx^2} + 2y = 0.$$

$$\therefore \quad M_0 (x) = A \cos(\sqrt{2}x) + B \sin(\sqrt{2}x).$$

By forming the table, it will be seen that the first row gives the co-efficients in cos $(\sqrt{2}x)$ viz. 2, $2^2$, $2^3$, ...... Similarly but for a factor $\sqrt{2}$, the co-efficients in the second row will be found to be the co-efficients of sin $\sqrt{2}x$.

3·2. Another advantage of (12) is that $M_r(x)$ can be found in all cases provided $M_0(x)$ is known.

*Ex.* $a_n = n^2$. Here (12) gives, for $r = 3$,

$$\frac{d^3y}{dx^3} + 5\frac{dy}{dx} = (D^3 + 5D) \text{ sech } x.$$

$$= -1^2 \cdot 2^2 \cdot 3^2 \cdot \left(\frac{\text{sech } x \tanh^3 x}{3!}\right). \text{ [Cf. (6) above.]}$$

§ 4. In (12), write $r = 2n$, and equate the co-efficients of $x^{2s}$ on both sides. We obtain the following equations.

If $s > m$,

$$A_{m+s} - {}_{2m-1}\alpha_{2m-1} \cdot A_{m+s-1} + {}_{2m-1}\alpha_{2m-3} \cdot A_{m+s-2} - \dots + (-1)^m\, {}_{2m-1}\alpha_1 \cdot A_s = a_1 a_2 \dots a_{2m} \cdot {}_{2m+1}A_{m+s}. \quad (13\cdot1)$$

If $s = m$,

$$A_{2m} - {}_{2m-1}\alpha_{2m-1} A_{2m-1} + {}_{2m-1}\alpha_{2m-3} A_{2m-2} - \dots + (-1)^m\, {}_{2m-1}\alpha_1 \cdot A_m = a_1 a_2 \dots a_{2m}. \quad (13\cdot2)$$

If $s < m$,

$$\left.\begin{aligned} &A_m - {}_{2m-1}\alpha_{2m-1} \cdot A_{m-1} + {}_{2m-1}\alpha_{2m-3} \cdot A_{m-2} - \dots + (-1)^m\, {}_{2m-1}\alpha_1 \cdot A_0 = 0. \\ &A_{m+1} - {}_{2m-1}\alpha_{2m-1} A_m + {}_{2m-1}\alpha_{2m-3} A_{m-1} - \dots + (-1)^m\, {}_{2m-1}\alpha_1 \cdot A_1 = 0. \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &A_{2m-1} - {}_{2m-1}\alpha_{2m-1} \cdot A_{2m-2} + {}_{2m-1}\alpha_{2m-3} \cdot A_{2m-3} - \dots + (-1)^m\, {}_{2m-1}\alpha_1 \cdot A_{m-1} = 0 \end{aligned}\right\} \dots (13\cdot3)$$

where $A_0 = 1$.

4·1. Similarly in the differential equation, write $r = 2m + 1$, and equate the co-efficients of $x^{2s+1}$ on both sides. We obtain the following equations:



If $s > m$,

$$A_{m+s+1} - {}_{2m}\alpha_{2m} \cdot A_{m+s} + {}_{2m}\alpha_{2m-2} \cdot A_{m+s-1} - \dots + (-1)^m\, {}_{2m}\alpha_2\, A_{s+1} = a_1 a_2 \dots a_{2m+1} \cdot {}_{2m+2}A_{s+m+1}. \quad (14\cdot1)$$

If $s = m$,

$$A_{2m+1} - {}_{2m}\alpha_{2m} A_{2m} + {}_{2m}\alpha_{2m-2} A_{2m-1} - \dots + (-)^m\, {}_{2m}\alpha_2\, A_{m+1} = a_1 a_2 \dots a_{2m+1}. \quad (14\cdot2)$$

If $s < m$,

$$\left.\begin{aligned} &A_{2m} - {}_{2m}\alpha_{2m} A_{2m-1} + {}_{2m}\alpha_{2m-2} A_{2m-2} - \dots + (-1)^m\, {}_{2m}\alpha_2\, A_m = 0. \\ &A_{2m-1} - {}_{2m}\alpha_{2m} \cdot A_{2m-2} + {}_{2m}\alpha_{2m-2} \cdot A_{2m-3} - \dots + (-1)^m\, {}_{2m}\alpha_2 \cdot A_{m-1} = 0. \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &A_{m+1} - {}_{2m}\alpha_{2m} \cdot A_m + {}_{2m}\alpha_{2m-2}\, A_{m-1} - \dots + (-1)^m\, {}_{2m}\alpha_2 \cdot A_1 = 0. \end{aligned}\right\} \dots (14\cdot3)$$

§ 5. From the equations (13...) and (14...), we may evaluate determinants whose elements are the first row elemental numbers of the table. The method of procedure is exactly similar to the one we adopted in our last paper. We content ourselves with stating the main results. We add some examples of cases in which the elements are different from those already considered.

$$\begin{vmatrix} A_0 & A_1 & A_2 & \dots & A_m \\ A_1 & A_2 & A_3 & \dots & A_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_m & A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = (a_1 a_2)^m (a_3 a_4)^{m-1} (a_5 a_6)^{m-2} \dots (a_{2m-1} a_{2m})^1 \quad (15)$$

$$\begin{vmatrix} A_1 & A_2 & A_3 & \dots & A_m \\ A_2 & A_3 & A_4 & \dots & A_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_m & A_{m+1} & A_{m+2} & \dots & A_{2m-1} \end{vmatrix} = a_1^m (a_2 a_3)^{m-1} (a_4 a_5)^{m-2} \dots (a_{2m-2} a_{2m-1})^1 \quad (16)$$

$$\begin{vmatrix} A_2 & A_3 & \dots & A_{m+1} \\ A_3 & A_4 & \dots & A_{m+2} \\ \dots & \dots & \dots & \dots \\ A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = a_1^m (a_2 a_3)^{m-1} (a_4 a_5)^{m-2} \dots (a_{2m-2} a_{2m-1}) \times {}_{2m}\alpha_2. \quad (17)$$

$$\begin{vmatrix} A_0 & A_1 & A_2 & \dots & A_m \\ A_1 & A_2 & A_3 & \dots & A_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{m-2} & A_{m-1} & \dots & \dots & A_{2m-2} \\ A_{m-1} & A_m & \dots & \dots & A_{2m-1} \\ A_s & A_{s+1} & \dots & \dots & A_{m+s} \end{vmatrix} = (a_1a_2)^m (a_3 a_4)^{m-1} \dots (a_{2m-1} a_{2m})' \times {}_{2m+1}A_{s+m}. \qquad (18)$$

where $s$ is any integer equal to, or, greater than $m$, ${}_{2m+1}A_{2m}$ being equal to unity.

$$\begin{vmatrix} A_1 & A_2 & A_3 & \dots & A_{m+1} \\ A_2 & A_3 & A_4 & \dots & A_{m+2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{m-1} & A_m & \dots & \dots & A_{2m-1} \\ A_m & A_{m+1} & \dots & \dots & A_{2m} \\ A_{s+1} & A_{s+2} & \dots & \dots & A_{m+s+1} \end{vmatrix} = a_1^{m+1} (a_2 a_3)^m \dots (a_{2m-2} \; a_{2m-1})^2 (a_{2m} \; a_{2m+1})^1 \times {}_{2m+2}A_{m+s+1}. \qquad (19)$$

where $s$ is any integer equal to, or greater than $m$.

5.1. Particular cases of the above determinants were obtained independently in the last paper, in view of the fact that their elements are Euler's and Bernoulli's numbers.

We give here other examples.

(a) $a_n = n$. $\qquad {}_1A_n = 1.3.5 \dots (2n-1)$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 15 \\ 3 & 15 & 105 \end{vmatrix} = 48 = (1.2)^2 (3.4)^1 = (1.2)^2 (3.4)^1 \qquad \begin{vmatrix} 1 & 3 & 15 \\ 3 & 15 & 105 \\ 15 & 105 & 945 \end{vmatrix} = 720 = 1(2.3)(4.5)^1 = 1^3 (2.3)^2 (4.5).$$

(b) $a_1 = a_2 = \dots = a_n = 1$.

The values of the A's are calculated in Appendix I with the help of which we write down some examples.

$$\begin{vmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{vmatrix} = 1, \qquad \begin{vmatrix} 2 & 5 & 14 \\ 5 & 14 & 42 \\ 14 & 42 & 132 \end{vmatrix} = 4 \text{ and } {}_6\alpha_2 = 4.$$

Again $\qquad {}_n\alpha_n = \Sigma a_n = n.$

$${}_n\alpha_{n-2} = (n-2) + (n-3) + \dots + 1 = \frac{(n-2)(n-1)}{2!}.$$



$${}_n\alpha_{n-4} = \text{sum of terms } (\Sigma a_{n-2} \; \Sigma a_{n-4}), \; (n-4) \text{ in number}$$

$$= \frac{1}{2!}\{1{\cdot}2 + 2{\cdot}3 + \dots + (n-4)(n-3)\}$$

$$= \frac{1}{3!}(n-4)(n-3)(n-2),$$

$${}_n\alpha_{n-6} = \text{sum of terms } (\Sigma a_{n-2} \; \Sigma a_{n-4} \; \Sigma a_{n-6}), \; (n-6) \text{ in number.}$$

$$= \frac{1}{3!}\{1.2.3 + 2.3.4 + 3.4.5 + \dots\}$$

$$= \frac{1}{4!}(n-6)(n-5)(n-4)(n-3);$$

and generally $${}_n\alpha_{n-2r} = \frac{1}{r+1!}(n-2r)(n-2r+1)\dots(n-r-1)(n-r).$$

§ 6. We give further examples of important coefficients being obtained from the table. Their proof depends upon a fundamental result relating to the representation by a continued fraction of the integral

$$\int_0^\infty M^\circ(x)\, e^{-zt}\, dx.$$

That result being established, the examples given here follow easily from the continued fractions given by Prof. L. J. Rogers in his paper, "*Asymptotic Series as Convergent Continued Fractions*" in the Proceedings of the London Mathematical Society, Series II, Vol. IV.

(a) Let $a_{2n-1} = (2n-1)^2 k^2, \; a_{2n} = (2n)^2$.

The first row numbers of the table give the coefficients in the expansion in an ascending series of powers of $x$ of the function dn $(x, k)$.

i.e. $\qquad M_0(x) = \text{dn}(x, k).$

$$M_{2r}(x) = \frac{\text{sn}^{2r}(x,k)\,\text{dn}(x,k)}{2r!}, \quad M_{2r+1}(x) = \frac{\text{sn}^{2r+1}(x,k)\,\text{cn}(x,k)}{2r+1!}.$$

(b) Let $a_{2n-1} = (2n-1)^2, \; a_{2n} = (2n)^2 k^2$.

$$M_0(x) = \text{cn}(x, k).$$

$$M_{2r}(x) = \frac{\text{sn}^{2r}(x,k)\,\text{cn}(x,k)}{2r!}. \quad M_{2r+1}(x) = \frac{\text{sn}^{2r+1}(x,k)\,\text{dn}(x,k)}{2r+1!}$$

(c) Let $a_{2n-1} = a_{2n} = n$.

${}_1A_n = n!, \; {}_2A_n = n+1!, \; {}_3A_n = (n+1)(n+1)!$ etc.

(*d*) Let $a_1 = m,\ a_2 = 1,\ a_3 = m + 1,\ a_4 = 2, \ldots\ldots$

$$a_{2n-1} = m + n_{m+n-1},\ a_{2n} = n, \ldots$$

$$_1A_n = m\,(m+1)\,(m+2)\ldots(m+n-1)$$

(*e*) Let $a_n = \dfrac{n^4}{(2n-1,(2n+1)}$.

$$_1A_n = 2\,(2^{2n-1} - 1)\,B_n,$$

where $B_n$ is the $n^{th}$ Bernoullian number. We may write down the values of some determinants here. If $c_n = (2^{2n-1} - 1)\,B_n$, we have

$$\begin{vmatrix} 1 & c_1 & \ldots & \ldots & c_n \\ c_1 & c_2 & \ldots & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & \ldots & c_{2n} \end{vmatrix} = \frac{1}{2^{n+1}} \frac{[(1.2)^n\,(3.4)^{n-1}\ldots(2n-1,2n)^1]^4}{3^{2n}\,5^{2n-1}\,7^{2n-2}\ldots(4n-1)^2\,(4n+1)^1}$$

$$\begin{vmatrix} c_1 & c_2 & \ldots & c_n \\ c_2 & c_3 & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & c_{2n-1} \end{vmatrix} = \frac{1}{2^n}. \frac{[1_n(2\cdot3)^{n-1}(4\cdot5)^{n-2}\ldots(2n-2.2n-1)^1]^4}{3^{2n-1}\,5^{2n-2}\ldots(4n-3)^2\,(4n-1)^1}$$

(*f*) Let $a_1 = \dfrac{1.2^2.3}{2^2.3.5},\ a_2 = \dfrac{2.3^2\cdot4}{2^2.5.7},\ a_n = \dfrac{n(n+1)^2\,(n+2)}{2^2\,(2n+1)\,(2n+3)}$.

$_1A_n = 6\,B_{n+1}$, where $B_n$ is the $n^{th}$ Bernoullian number. The construction of a table in (*e*) or (*f*) is out of the question owing to the obvious tediousness of the work. But it is given here in connection with the evaluation of determinants whose elements are Bernoullian numbers, or involve them. Hence,

$$\begin{vmatrix} B_1 & B_2 & \ldots & B_{n+1} \\ B_2 & B_3 & \ldots & B_{n+2} \\ \ldots & \ldots & \ldots & \ldots \\ B_{n+1} & & \ldots & B_{2n+1} \end{vmatrix} = \frac{1}{2^{n+1}}.\,1^{2n}\cdot 2^{2n-1}\,2^{2n-2}\cdot 3^{2n-3}.\ 3^{2n-4}\ldots n^8\cdot n^3\,(n+1)\times$$

$$\frac{3^{2n}\cdot 3^{2n-1}\cdot 5^{2n-2}\,5^{2n-3}\ldots(2n+1)^3(2n+1)}{3^{2n+1}\,5^{2n}\,7^{2n-1}\ldots(4n+1)^2\,(4n+3)}.$$

$$\begin{vmatrix} B_2 & B_3 & \ldots & B_{n+1} \\ B_2 & B_4 & \ldots & B_{n+2} \\ \ldots & \ldots & \ldots & \ldots \\ B_{n+1} & B_{n+2} & \ldots & \ldots B_{2n} \end{vmatrix} = \frac{1}{6^n}\,(1\cdot2^3)^{n-1}\,(2\cdot3^3)^{n-2}\,(3\cdot4^3)^{n-3}.\ \ldots(\overline{n-1}\,n^3)^1.\ \times$$

$$\frac{(3^3\cdot5)^{n-1}\,(5^3\cdot7)^{n-2}\,(7^3\cdot9)^{n-3}\ldots\times(\overline{2n-1}^3\cdot2n+1)}{5^{2n-1}\,7^{2n-2}\,9^{2n-3}\ldots(4n-1)^2\,(4n+1)}.$$



(*g*) Let $a_1 = \dfrac{1^3}{1\cdot3},\ a_2 = \dfrac{2^3}{3\cdot5},\ \ldots\ a_n = \dfrac{n^3}{(2n-1)\,(2n+1)}$.

Then $$_1A_n = \frac{1}{2n+1}.$$

We therefore evaluate the following two determinants given by Rouche (1858) in another connection. It does not appear however that the determinants in question were evaluated by him. (See Muir: *Theory of Determinants*, Vol. II, Page 354). Rouche's expressions for Legendre's polynomials are proved by us in another paper.

$$\begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \ldots & \frac{1}{2n+1} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \ldots & \frac{1}{2n+3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \ldots & \ldots & \frac{1}{4n+1} \end{vmatrix} = \left(1\cdot\frac{1}{5}\cdot\frac{1}{9}\cdot\frac{1}{13}\ldots\frac{1}{4^{n+1}}\right)\times \left\{\left(\frac{1\cdot2}{1\cdot3}\right)^n\left(\frac{3\cdot4}{5\cdot7}\right)^{n-1}\ldots\times\left(\frac{2n-1\cdot2n}{4n-3\cdot4n-1}\right)^1\right\}^2$$

$$\begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \ldots & \frac{1}{2n+1} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \ldots & \frac{1}{2n+3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \ldots & \ldots & \frac{1}{4n-1} \end{vmatrix} = \left(\frac{1}{3}\cdot\frac{1}{7}\cdot\frac{1}{11}\ldots\frac{1}{4n-1}\right)\times \left\{\left(\frac{2\cdot3}{3\cdot5}\right)^{n-1}\left(\frac{4\cdot5}{7\cdot9}\right)^{n-2}\ldots\left(\frac{2n-2\cdot2n-1}{4n-5\cdot4n-3}\right)^1\right\}^2$$

§. 7. Hitherto the quantities $A_n$ have been numbers, but they also occur in the theory as functions of a variable. As examples, we give below without proof the following interesting results relating to Legendre's, Euler's and Bernoulli's polynomials.

(*a*) Write $a_1 = 1,\ a_2 = x,\ a_3 = 0,\ a_4 = 1,\ a_5 = x,\ a_6 = 0;\ \ldots\ldots$ we find that $_1A_n = (1+x)^{n-1}$.

(*b*) Write $a_1 = \dfrac{x+1}{2},\ a_2 = \dfrac{x-1}{2}$,

$$a_3 = \frac{x+1}{2} = a_{2n-1},\ a_4 = \frac{x-1}{2} = a_{2n}.$$

Then ${}_1A_n = \frac{P_{n+1}(x) - P_{n-1}(x)}{(2n+1)(x-1)} = \frac{1}{x-1}\int_1^x P_n(x)\,dx$,

where $P_n(x)$ is Legendre's ploynomial of the $n^{\text{th}}$ degree.

From (15), (16) and (17) above, we obtain the following interesting results relating to determinants.

Let $A_n$ stand for $\int_1^x P_n(x)\,dx, = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$

Then $A_0 = \frac{P_1(x) - P_{-1}(x)}{1} = (x-1)$.

$$\begin{vmatrix} A_0 & A_1 & A_2 & \cdots & A_n \\ A_1 & A_2 & \cdots & \cdots & A_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n+1} & & \cdots & A_{2n} \end{vmatrix} = \frac{(x+1)^{\frac{n(n+1)}{2}} \cdot (x-1)^{\frac{(n+1)(n+2)}{2}}}{2^{n(n+1)}}.$$

$$\begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ A_2 & A_3 & \cdots & A_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n+1} & \cdots & A_{2n-1} \end{vmatrix} = \frac{(x^2-1)^{\frac{n(n+1)}{2}}}{2^{n^2}}.$$

$$\begin{vmatrix} A_2 & A_3 & \cdots & A_{n+1} \\ A_3 & A_4 & \cdots & A_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+1} & A_{n+2} & \cdots & A_{2n} \end{vmatrix} = \frac{(x^2-1)^{\frac{n(n+1)}{2}}}{2^{n^2+n+1}} \cdot \left[(x+1)^{n+1} - (x-1)^{n+1}\right]$$

(c) Form a table with $a_1 = x(1-x)$, $a_2 = 1^2$, $a_3 = (1+x)(2-x)$, $a_4 = 2^2$; $a_{2n-1} = (n-1+x)(n-x)$; $a_{2n} = n^2$.

Then ${}_1A_1 = -2\psi_2(x)$, ${}_1A_2 = 2\psi_4(x)$, ... ${}_1A_n = (-1)^n\, 2\psi_{2n}(x)$,...

where $\psi_n(x)$ is the co-efficient of $\frac{t^n}{n!}$ in the expansion of $\frac{e^{xt}}{e^t+1}$.

$$2^{n+1}\begin{vmatrix} \frac{1}{2} & \psi_2 & \psi_4 & \cdots & \psi_{2n} \\ \psi_2 & \psi_4 & \psi_6 & \cdots & \psi_{2n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{2n} & \psi_{2n+2} & \cdots & \cdots & \psi_{4n} \end{vmatrix} = (1^n\, 2^{n-1}\, 3^{n-2} \ldots n^1)^2\, x^n \times (1-x)(2-x)\ldots(n-x) \times (1^2-x^2)^{n-1} (2^2-x^2)^{n-2} \ldots (\overline{n-1}^2 - x^2)^1.$$



$$(-)^n\, 2^n \begin{vmatrix} \psi_2 & \psi_4 & \cdots & \psi_{2n} \\ \psi_4 & \psi_6 & \cdots & \psi_{2n+2} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{2n} & \psi_{2n+2} & \cdots & \psi_{4n-2} \end{vmatrix} = [1^{n-1}\, 2^{n-2}\, 3^{n-3} \ldots (n-1)^1]^2 \times x^n (1-x)(2-x)\ldots(n-x)\, (1^2-x^2)^{n-1} (2^2-x^2)^{n-2} \ldots (\overline{n-1}^2 - x^2)^1.$$

(d) Write $a_1 = 1^2\,\frac{2x(2-2x)}{1 \cdot 3}$, $a^2 = \frac{2^2(1+2x)(3-2x)}{3 \cdot 5}$,...

$a_3 = \frac{3^2(2+2x)(4-2x)}{5 \cdot 7}$ ... $a_n = \frac{n^2(n-1+2x)(n+1-2x)}{(2n-1)(2n+1)}$, ...

Then ${}_1A_1 = \frac{-2^3\,\phi_3(x)}{3(2x-1)}$, ${}_1A_2 = \frac{2^5\phi_5(x)}{5(2x-1)}$, ...

$${}_1A_n = \frac{(-1)^n\, 2^{2n+1}\, \phi_{2n+1}(x)}{(2n+1)(2x-1)}$$

where $\phi_n(x)$ is Bernoulli's polynomial of the $n^{\text{th}}$ degree, and is the coefficient of $\frac{t^n}{n!}$ in the expansion of $t\,\frac{e^{xt}-1}{e^t-1}$.

$$2^{n(n+1)}\begin{vmatrix} x-\frac{1}{2} & \frac{\phi_3}{3} & \frac{\phi_5}{5} & \cdots & \frac{\phi_{2n+1}}{2n+1} \\ \frac{\phi_3}{3} & \frac{\phi_5}{5} & \frac{\phi_7}{7} & \cdots & \frac{\phi_{2n+3}}{2n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\phi_{2n+1}}{2n+1} & \frac{\phi_{2n+3}}{2n+3} & \cdots & \cdots & \frac{\phi_{4n+1}}{4n+1} \end{vmatrix}$$

$$= (x-\tfrac{1}{2})^{n+1} \cdot [(1\cdot 2)^n \times (3\cdot 4)^{n-1} \ldots (2n-1.2n)^1]^2 \times \frac{1}{[(1\cdot 3)^n (5\cdot 7)^{n-1} \ldots (4n-3\cdot 4n-1)^1]^2} \times \frac{1}{1\cdot 5\cdot 9 \ldots (4n+1)}$$

$$\times \{x(1-x)\}^n \{(1+x)(2-x)\}^{n-1} \ldots \{(n-1+x)(n-x)\}^1$$

$$\times \{(1+2x)(3-2x)\}^n \{(3+2x)(5-2x)\}^{n-1} \ldots \{(2n-1+2x)(2n+1-2x)\}^1.$$

$$(-)^n 2^{n(n-1)} \begin{vmatrix} \frac{\phi_3}{3} & \frac{\phi_5}{5} & \cdots & \frac{\phi_{2n+1}}{2n+1} \\ \frac{\phi_5}{5} & \frac{\phi_7}{7} & \cdots & \frac{\phi_{2n+3}}{2n+3} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\phi_{2n+1}}{2n+1} & \cdots & \cdots & \frac{\phi_{4n-1}}{4n-1} \end{vmatrix}$$

$$= (x-\tfrac{1}{2})^n \frac{[1^n \cdot (2\cdot 3)^{n-1} \ldots (2n-2 \cdot 2n-1)]^2}{3 \cdot 7 \cdot 11 \ldots (4n-1)} \times \frac{1}{[1^n (3 \cdot 5)^{n-1} \ldots (4n-5\ 4n-3)^1]^2}$$

$$\times \{x(1-x)\}^n \{(1+x)(2-x)\}_{n-1} \ldots \{(n-1+x)(n-x)\}^1 \times \{(1+2x)$$

$$(3-2x)\}^{n-1} \{(3+2x)(5-2x)\}^{n-2} \ldots \{(2n-3+2x)(2n-1-2x)\}^1.$$

[*To be concluded.*]

## APPENDIX I.

Table for $a_n = 1$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 14 | 42 | 132 | 439 | 1450 | 4912 | 16936 |
| 1 | 3 | 9 | 28 | 90 | 297 | 1011 | 3462 | 12024 | 42270 |
| 1 | 4 | 14 | 48 | 165 | 572 | 2012 | 7112 | 25334 | 90920 |
| 1 | 5 | 20 | 75 | 275 | 1001 | 3650 | 13310 | 48650 | 178400 |
| 1 | 6 | 27 | 110 | 429 | 1638 | 6198 | 23316 | 87480 | 327976 |
| 1 | 7 | 35 | 154 | 637 | 2548 | 10006 | 38830 | 149576 | |
| 1 | 8 | 44 | 208 | 910 | 3808 | 15514 | 62096 | | |
| 1 | 9 | 54 | 273 | 1260 | 5508 | 23266 | | | |
| 1 | 10 | 65 | 350 | 1700 | 7752 | | | | |
| 1 | 11 | 77 | 440 | 2244 | | | | | |
| 1 | 12 | 90 | 544 | | | | | | |
| 1 | 13 | 104 | | | | | | | |
| 1 | 14 | | | | | | | | |
| 1 | | | | | | | | | |

*N. B.*—The process of multiplying by the numbers of the first column, *viz.*, by unity is omitted.

2, 5, 14, ... are the coefficients in the expansion of $\frac{2}{x} J_2(2x)$.

3, 9, 28. ... „ „ $\frac{3}{x} J_3(2x)$.

4, 14, 48, ... „ „ $\frac{4}{x} J_4(2x)$.

1, 2, 5, 14, „ „ $\frac{1}{x} J_1(2x)$.



## SHORT NOTES

### On the Product of all Numbers less than $N$ and Prime to it.

Let $\pi d(N)$ denote the product of all the positive integers less than $N$ and prime to it.

I. If $N$ is prime, $\pi d(N)$ is evidently $(N-1)!$

II. If $N$ is composite, we shall first find the product of all the integers less than $N$ and not prime to $N$.

Let $p, q, r, s, t, \ldots\ldots$ be the different primes which divide $N$; *i.e.* let

$$N = p^\alpha \ q^\beta \ r^\gamma \ s^\delta \ \ldots\ldots$$

Consider the series of integers $1, 2, 3 \ldots\ldots\ldots N-1$. Of these the following are multiples of $p$: $1.p, 2.p, \ldots\ldots \left(\frac{N}{p}-1\right).p$.

The product of these $= \left(\frac{N}{p}-1\right)!\ p^{\frac{N}{p}-1}$.

Similarly the product of all the multiples of $q = \left(\frac{N}{q}-1\right)!\ q^{\frac{N}{q}-1}$.

Hence the product of all the multiples of $p$, all the multiples of $q$, &c., in the series is given by

$$P_1 = \pi\left\{\left(\frac{N}{p}-1\right)!\ p^{\frac{N}{p}-1}\right\}.$$

In the same series there are $\left(\frac{N}{pq}-1\right)$ multiples of $pq$ and their product is

$$\left\{\left(\frac{N}{pq}-1\right)!\ (pq)^{\frac{N}{pq}-1}\right\}.$$

Similarly for multiples of $pr, ps, \ldots qr, \ldots$ (taking all the binary products of $p, q, r, \ldots\ldots$)

Hence the product of all such multiples is given by

$$P_2 = \pi\left\{\left(\frac{N}{pq}-1\right)!\ (pq)^{\frac{N}{pq}-1}\right\}.$$

Similarly the product of all the multiples of the ternary products $pqr, pqs, prs, qrs, \ldots\ldots$ is

$$P_3 = \pi\left\{\left(\frac{N}{pqr}-1\right)!\ (pqr)^{\frac{N}{pqr}-1}\right\}.$$

And so on.

The correct answer in the case considered is given by (A). Thus, here $p = 2, q = 3, r = 5, \alpha = 2, \beta = 1, \gamma = 1$, so that by (A),

$$\pi\, d\,(60) = \frac{59\,!\; 9\,!\; 5\,!\, 3\,!}{29\,!\; 19\,!\; 11\,!\; 2^{16}\, 3^{8}\, 5^{4}}.$$

Now the numerator of this fraction is $= (2^{54}\, 3^{27}\, 5^{13}\, 7^{9}\, 11^{5}\, 13^{4}\, 17^{3}\, 19^{3}\, 23^{2}\, 29^{2}.\, 31.\, 37.\, 41.\, 43.\, 47.\, 53.\, 59)\, (2^{7}\, 3^{4}\, 5.\, 7)\, (2^{3}\, 3.\, 5)\, (2.3)$ and the denominator $= (2^{25} 3^{13} 5^{6} 7^{4}\, 11^{2}\, 13^{2}\, 17.\, 19.\, 23.\, 29)\, (2^{16} 3^{8} 5^{3} 7^{2} 11.\, 13.\, 17.\, 19)\, (2^{8}\, 3^{4}\, 5^{2}\, 7.\, 11)\, 2^{16} 3^{8} 5^{4}$ so that $\pi\, d\,(60) = 7.^{3}11.\, 13.\, 17.\, 19.\, 23.\, 29.\, 31.\, 37.\, 41.\, 43.\, 47.\, 53.\, 59.$

That this is the correct answer can be easily verified; for the integers less than 60, and prime to it are 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 52, 59.

Question 1013 of the J. I.M.S. (Vol. X, Dec. 1918, p. 492) is similarly wrongly stated.

N. B. MITRA.

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## Three Fundamental Formulæ.

[The formulæ here proved have been assumed in our Paper on Determinants. (Vide : J.I.M.S., Vol. XIV, p. 55, §§ 1 and 3.)]

1. Let $y = \sec x = \Sigma\, E_n \frac{x}{2n!}$, where $E_n$ is the $n^{th}$ Eulerian number, so that

$$\left.\frac{d^{2n}y}{dx^{2}}\right]_{x=0} = E_n.$$

Now let us obtain the expansion of the $n^{th}$ differential coefficient of $\sec x$ as the product of $s = \sec x$ and a power series in $t = \tan x$.

It is easily seen that

$$\frac{d^4y}{dx^4} = s\ \{\, 4\,!\, t^4 + 2\,!\ (1^2 + 2^2 + 3^2)\, t^2 + (1^2 + 2^2)\,\}$$

$$\therefore \qquad E_2 = 1^2 + 2^2.$$

$$\frac{d^6y}{dx^6} = s\ \{\, 6\,!\, t^6 + 4\,!\, t^4 \times (1^2 + 2^2 + 3^2 + 4^2 + 5^2)$$

$$+\, 2\,!\, t^2\ \{\, 1^2\,(1^2 + 2^2) + 2^2\,(1^2 + 2^2 + 3^2)$$

$$+\, 3^2\,(1^2 + 2^2 + 3^2 + 4^2)\,\}$$

$$+\ \{\, 1^2\,(1^2 + 2^2) + 2^2\,(1^2 + 2^2 + 3^2)\,\}$$

$$\therefore\ E_3 = 1^2\,(1^2 + 2^2) + 2^2\,(1^2 + 2^2 + 3^2) = \Sigma 1^2\, \Sigma 2^2\, \Sigma 3^2.$$

Similarly it is easily seen that

$$\frac{d^8y}{dx^8} = s\Big\{\, 8\,!\, t^8 + 6\,!\, \Sigma 7^2\, t^6 + 4\,!\, t^4\, \Sigma 5^2\, \Sigma 6^2$$

$$+\, 2\,!\, t^2\, \Sigma 3^2\, \Sigma 4^2\, \Sigma 5^2 + \Sigma 1^2\, \Sigma 2^2\, \Sigma 3^2\, \Sigma 4^2\,\Big\} \ldots$$

$$\therefore\ E_4 = \Sigma 1^2\, \Sigma 2^2\, \Sigma 3^2\, \Sigma 4^2.$$

Now assume that

$$\frac{d^ny}{dx^n} = s\ \Big\{\, n\,!\, t^n + \overline{n-2}\,!\ t^{n-2}\, \Sigma(n-1)^2$$

$$+\, \overline{n-4}\,!\, t^{n-4}\, \Sigma(n-3)^2\, \Sigma(n-2)^2 + \ldots$$

$$+\, \overline{n-2r}\,!\, t^{n-2r}\, \Sigma(n-2r+1)^2 \ldots\ldots$$

$$\Sigma(n-r+1)^2\, \Sigma(n-r)^2 + \ldots\,\Big\} \qquad (1{\cdot}1)$$

the last term being $\Sigma 2^2\ \Sigma 3^2 \ldots \Sigma \frac{(n+1)^2}{2}\ t$, if $n$ is odd,

and $$\Sigma 1^2\ \Sigma 2^2 \ldots \Sigma \left(\frac{n}{2}\right)^2 = E_{\frac{n}{2}}, \text{ if } n \text{ is even.} \qquad (1)$$

Differentiating, writing $\frac{dt}{dx} = 1 + t^2$, and collecting the coefficients of $t^{n-2r+1}$, we find this coefficient is

$$\begin{aligned}
&n-2r!\ \Sigma(n-2r+1)^2\ \Sigma(n-2r+2)^2 \ldots \Sigma(n-r)^2\\
&+ (n-2r+2)\ n-2r+2!\ \Sigma(n-2r+3)^2\\
&\quad \Sigma(n-2r+4)^2 \ldots \Sigma(n-r+1)^2 + n-2r!\ (n-2r)\\
&\quad \Sigma(n-2r+1)^2\ \Sigma(n-2r+2)^2 \ldots \Sigma(n-r)^2.\\
&= n-2r+1!\ \{(n-2r+{}^2\Sigma 2)(n-2r+3)^2 \ldots\\
&\quad \Sigma(n-r+1)^2 + \Sigma(n-2r+1)^2 \ldots \Sigma(n-r)^2\}.\\
&= n-2r+1!\ \Sigma(n-2r+2)^2\ \Sigma(n-2r+3)^2 \ldots \Sigma(n-r+1)^2 \qquad (1{\cdot}4)
\end{aligned}$$

Hence by induction, the formulæ (1.) and (1.1) follow immediately.

[NOTE :—The formula (1.1) is elegantly expressed by the table in Table I for Euler's numbers, which was kindly suggested to us by Mr. K. B. Madhava.

The first column contains the squares of the natural numbers, *viz.* 1, 4, 9, 16,... The second is obtained from the first by an obvious method of addition, *e.g.* $5 = 1^2 + 2^2$, $14 = 1^2 + 2^2 + 3^2$,... The third is obtained by multiplying the numbers in the second column by the corresponding numbers in the first. *e.g.* 56 = 4.14, 270 = 9·30,... The fourth is obtained from the third by addition as before and the process is repeated. We obtain Euler's numbers in pairs at the top.]

2. In a similar manner, since

$$\tan x = \Sigma b_n\, \frac{x^{2n-1}}{2n-1!}$$

where $b_n = 2_{2n}(2_{2n}-1)\ \frac{B_n}{2n} = n^{\text{th}}$ prepared Bernoullian number (an integer), we can prove that

$$\begin{aligned}
\frac{d^n t}{dx^n} &= n!\ t^{n+1} + n-2!\ t^{n-1}\ \Sigma(n-1.n)\\
&+ n-4!\ t^{n-3}\ \Sigma(n-3.\,n-2)\ \Sigma(n-2.\,n-1) + \ldots
\end{aligned}$$



$$\begin{aligned}
&+ (n-2r)!\ t^{n-2r+1}\ \Sigma(n-2r+1.\,n-2r) +\\
&\Sigma(n-2r.\,n-2r-1) \ldots \Sigma(n-r.\,n+r+1) + \ldots \quad (2{\cdot}1)
\end{aligned}$$

the last term being

$$t\ \Sigma(1.2)\ \Sigma(2.3) \ldots \Sigma\left(\frac{n}{2}.\ \frac{n}{2}+1\right), \text{ if } n \text{ is even,}$$

and $$\Sigma(1.2)\ \Sigma(2.3) \ldots \Sigma\left(\frac{n-1}{2}.\ \frac{n+1}{2}\right) = b_{\frac{n+1}{2}}, \text{ if } n \text{ is odd.} \quad (2)$$

Hence, if a table is formed exactly as in 2 above with 1.2, 2.3, 3.4, ... in the first column, we obtain the prepared Bernoullians beginning with the second ($b_2$) in the odd columns of the top row. (See Table II.)

3. Again $$\frac{1}{\cos x - a \sin x} = \sec x \left\{1 + \Sigma a^n \tan^n x\right\}.$$

Also if we write

$$\frac{1}{\cos x - a \sin x} = 1 + \Sigma A_n(a).\ \frac{x^n}{n!}, \qquad (3.1)$$

then $A_n$ is obviously a function of $a$ of degree $n$, odd or even according as $n$ is odd or even, and

$$A_n(a) = \frac{d^n}{dx^n}\left(\frac{1}{\cos x - a \sin x}\right)\Bigg]_{x=0}.$$

$$= \cos\theta\ \frac{d^n}{dr_n}(\sec\theta), \text{ where } \tan\theta = a,$$

so that

$$\begin{aligned}
A_n(a) &= n!\,a^n + (n-2)!\,a^{n-2}\,\Sigma(n-1)^2 + \ldots\\
&+ (n-2r)!\,a^{n-2r}\,\Sigma(n-2r+1)^2 \ldots \Sigma(n-r)^2 + \ldots\ldots \quad (3{\cdot}2)
\end{aligned}$$

Hence by rearranging (3.1) in powers of $a$ with the help of (3), we have

$$\left.\begin{aligned}
\sec x \tan^n x = n!\ \Big\{&\frac{x^n}{n!} + \frac{x^{n+2}}{n+2!}\ \Sigma(n+1)^2\\
&+ \frac{x^{n+4}}{n+4!}\ \Sigma(n+1)^2\ \Sigma(n+2)^2 + \ldots\ldots\\
&+ \frac{x^{n+2r}}{n+2r!}\ \Sigma(n+1)^2\ \Sigma(n+2)^2 \ldots \Sigma(n+r)^2 + \ldots\ldots
\end{aligned}\right\} \quad (3)$$

All the coefficients in (3) are to be found in the even columns of Table I.

## TABLE I.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 5 | 61 | 61 | 1385 | 1385 | 50521 | 50521 |
| 4 | 14 | 56 | 331 | 1324 | 12284 | 49136 | | |
| 9 | 30 | 270 | 1211 | 10899 | | | | |
| 16 | 55 | 880 | | | | | | |
| 25 | | | | | | | | |

## TABLE II.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 8 | 16 | 136 | 272 | 3968 | 7936 | 176896 | 353792 |
| 6 | 20 | 120 | 616 | 3696 | 28160 | 168960 | | |
| 12 | 40 | 480 | 2016 | 24192 | | | | |
| 20 | 70 | 1400 | | | | | | |
| 30 | | | | | | | | |

C. KRISHNAMACHARI.

M. BHIMASENA RAO.

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## SOLUTIONS.

### Question 1127.

(K. J. SANJANA, M.A.):—Prove that there are two and only two Tucker circles of a triangle which touch a given straight line. These circles coalesce when the given line is one of the sides of the triangle.

If O and K be the circumcentre and symmedian point of a triangle ABC, $T_1$ the centre and $R_1$ the length of the radius of the Tucker circle touching BC, prove that

$$KT_1 : T_1O = b^2 + c^2 - a^2 : b^2 + c^2 + a^2 \text{ and } R_1 : R = bc : (b^2 + c^2).$$

*Additional Solution by the Proposer.*

An elegant geometrical solution of this question is given by Mr. M. M. Thomas in the February (1922) number of our Journal. The following analytical solution may prove of interest.

As proved in my paper on Tucker Circles printed in the Journal for December 1917, the trilinear equation of a Tucker circle of *anti-parallel intercept* $\mu$ is

$$abc\,(\Sigma\alpha\beta\gamma) - \mu\,(\Sigma a\alpha)\,.\,\Sigma\{(bc - a\mu)\,\alpha\} = 0.$$

This may be written in the form

$$\Sigma(\mu^2a^2 - \mu abc)\,\alpha^2 + \Sigma\{a^2bc - \mu a\,(b^2+c^2) + 2\mu^2bc\}\,\beta\gamma = 0.$$

The condition that the straight line $l\alpha + m\beta + n\gamma = 0$ should touch the circle (the tangential equation of the Tucker circle) being written in the form $\Sigma Al^2 + \Sigma 2Fmn = 0$, it will be found that

$$A = -\tfrac{1}{4}a^2\,\{\mu\,(b^2 + c^2) - abc\}^2,$$

with similar values for B and C, and that

$$F = \tfrac{1}{4}\,\mu^2bc(-a^4 + a^2b^2 + a^2c^2 + b^2c^2) - \tfrac{1}{4}\mu ab^2c^2\,(b^2+c^2) + \tfrac{1}{4}\,a^2b^3c^3,$$

with similar values for G and H.

Since these values of A, B, C and F, G, H involve $\mu$ only to the second power, it follows that when $l, m, n$ are given we get a quadratic equation to determine $\mu$. Hence there cannot be more than two Tucker circles of a triangle touching a given straight line in the plane of the triangle.

When the given line is a side of the triangle, say BC, we have $m = n = 0$, and the condition of tangency reduces to

$$Al^2 = 0, \text{ or } \{\mu(b^2+c^2) - abc\}^2 = 0.$$

Thus the two circles coalesce, the anti-parallel intercept becoming $abc/(b^2 + c^2)$.