8

Tournaments (n ≤ 6)

Tournaments are just what they look like. Team X beats team Y, indicated by an arrow directed from X to Y. All matches are one-on-one, every team plays every other team, and there are no draws.

The degree sequence for a tournament is an inventory of wins. For example the transitive 3-tournament has sequence 210, meaning the players have 2 wins, 1 win, and no wins. Degree sequences are far from unique.

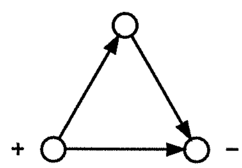
The proper (or Condorcet) winner of a tournament beats all other players (with degree $n - 1$), but the existence of a proper winner is unlikely. By examining all possible states of the arrows, we can assign a probability to a tournament class as a random outcome (listed up to **n** = 5). From these we can find the probability of a proper winner for **n** players (see Table 8.1).

Converse tournaments (connected by the sign ~) are related by reversing all of their arrows. Those tournaments not paired with a converse are self-converse.

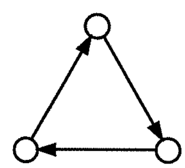
n = 2

10

n = 3

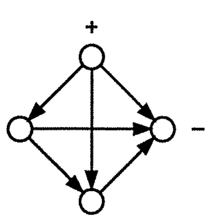


210 transitive P = 3/4

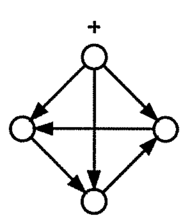


111 cyclic P = 1/4

n = 4

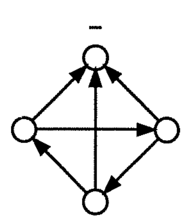


3210
P = 3/8
transitive

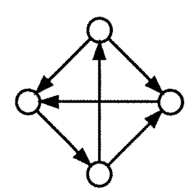


3111
P = 1/8

~



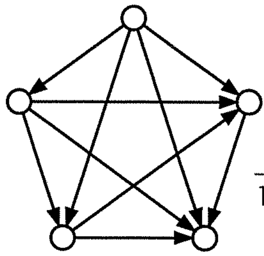
2220
P = 1/8



2211
P = 3/8
4-cycle

n = 5

+

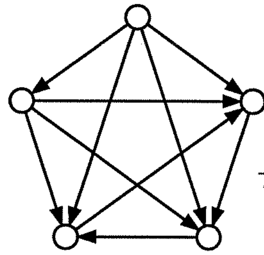


$\frac{15}{128}$

−

43210

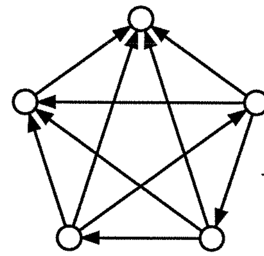
+



$\frac{5}{128}$

43111 ~

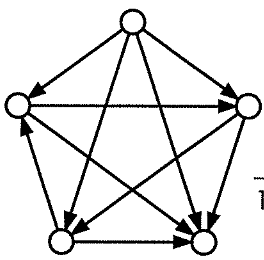
−



$\frac{5}{128}$

33310

+

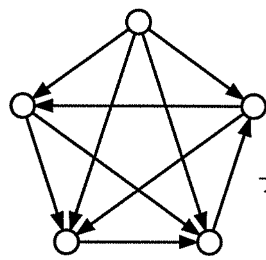


$\frac{5}{128}$

−

42220

+



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42211 ~

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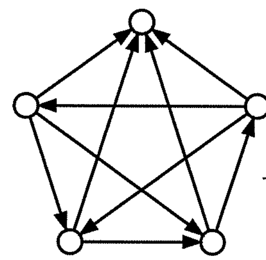

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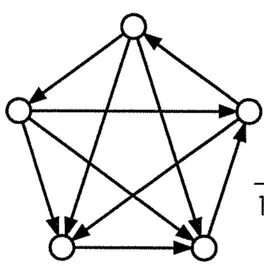

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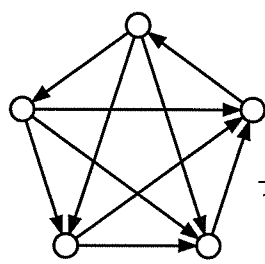

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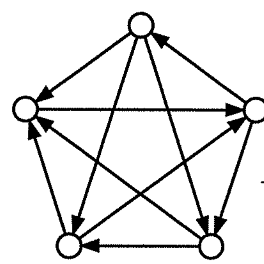

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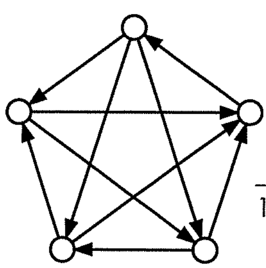

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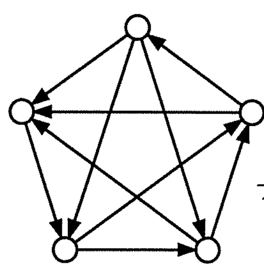

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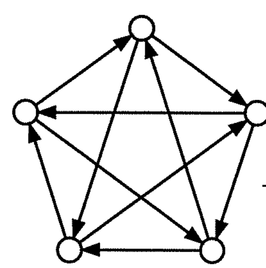

$\frac{3}{128}$

22222

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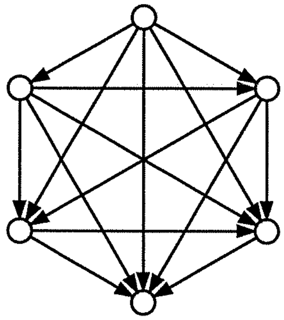
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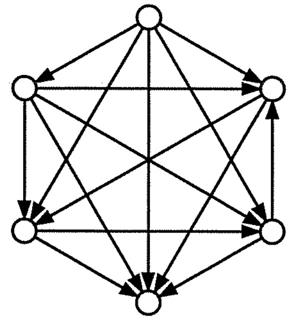
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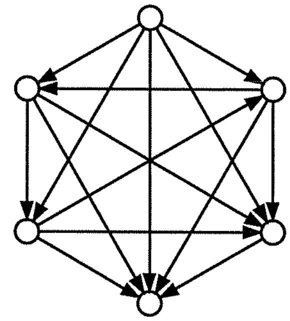
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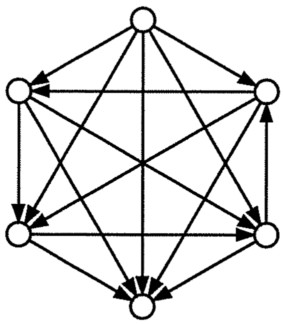
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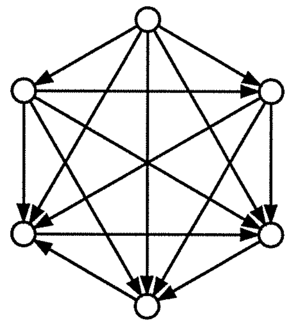
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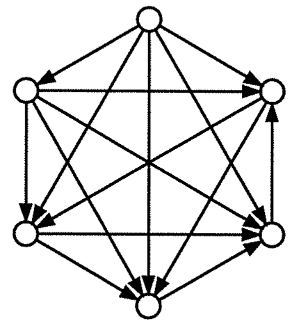
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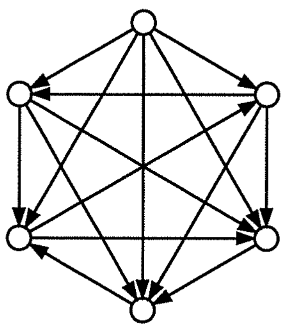
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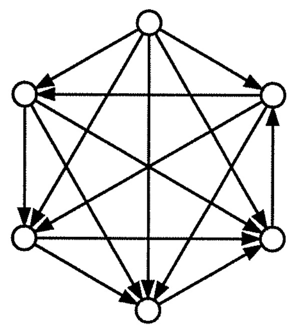
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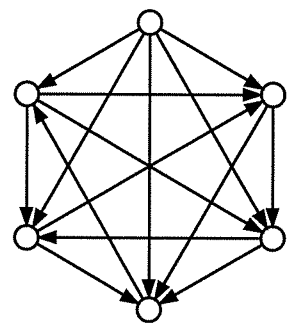
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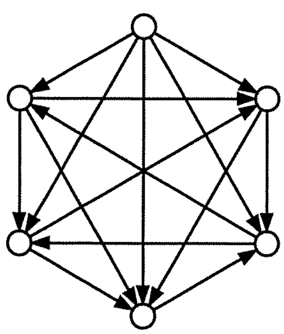
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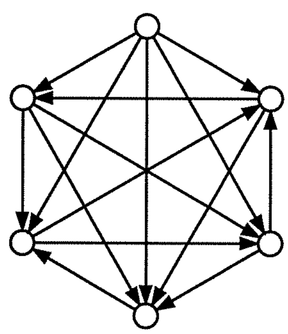
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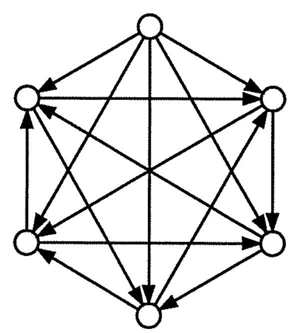
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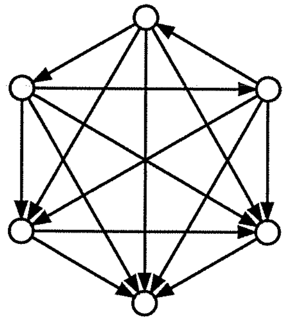

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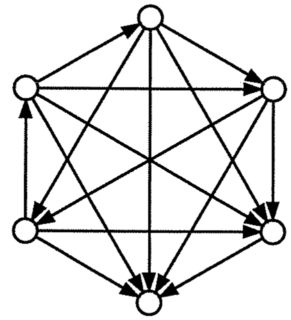

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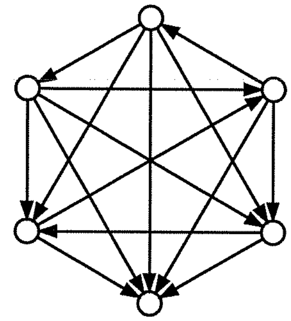

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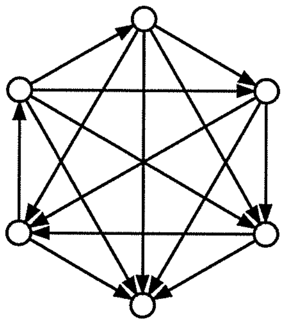

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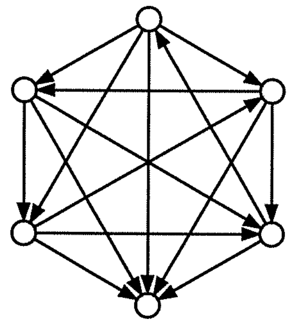

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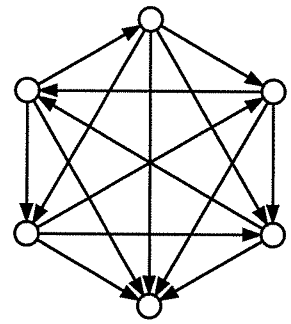

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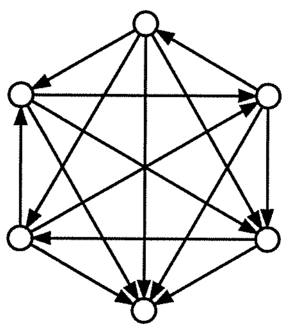

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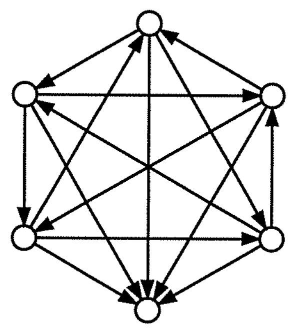

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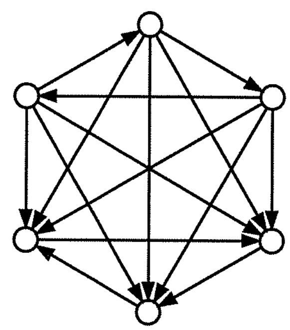

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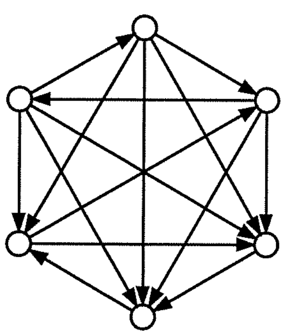

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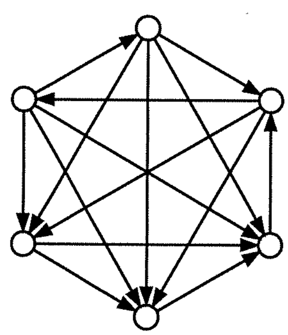

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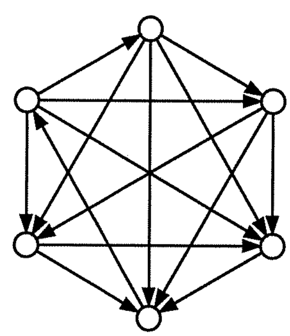

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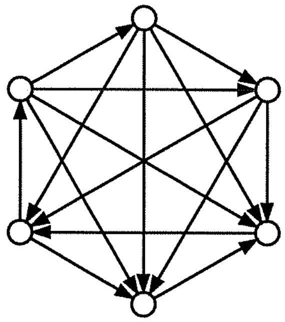

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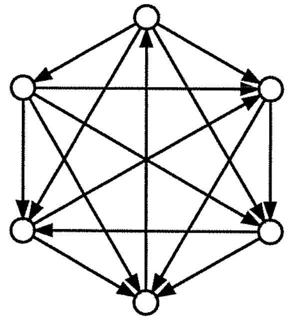

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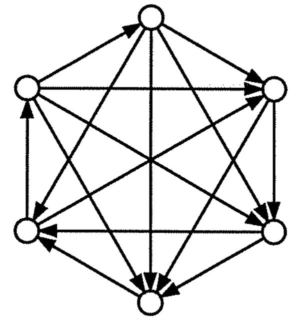

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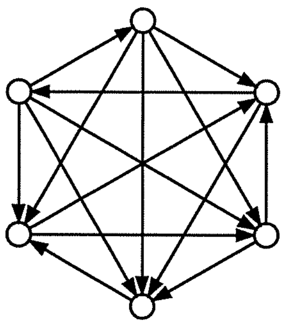

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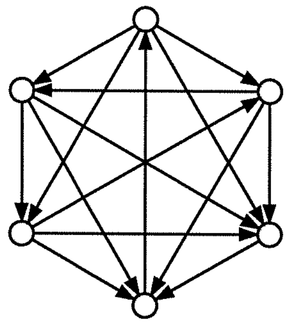

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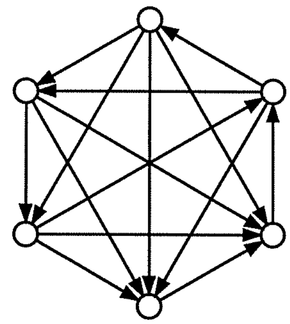

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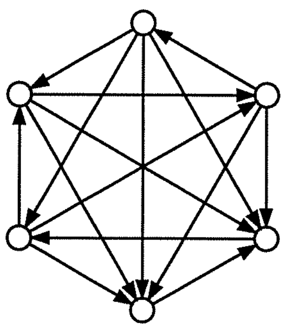

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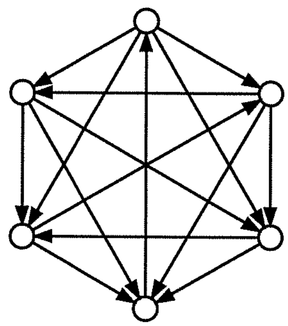

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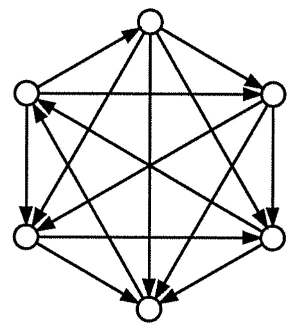

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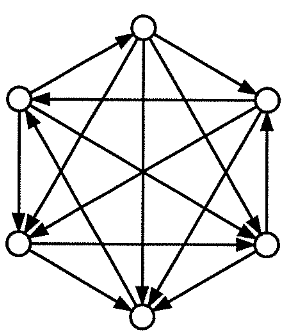

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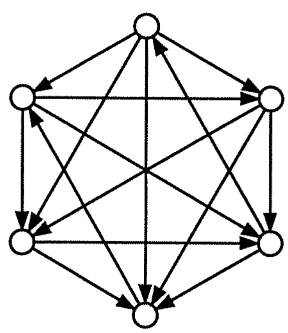

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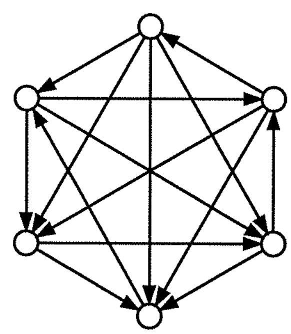

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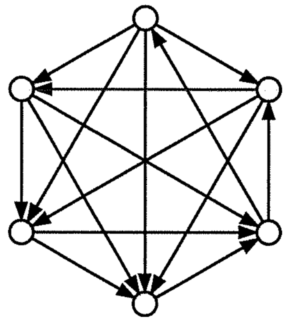

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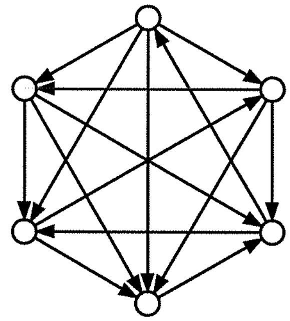

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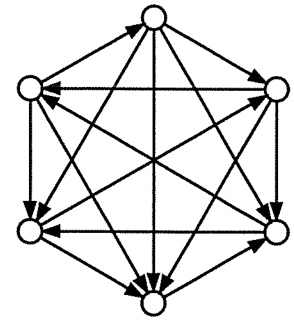

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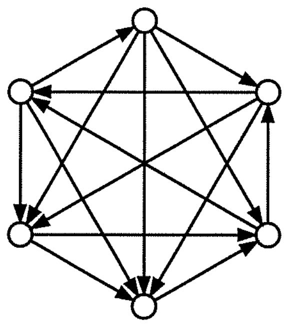

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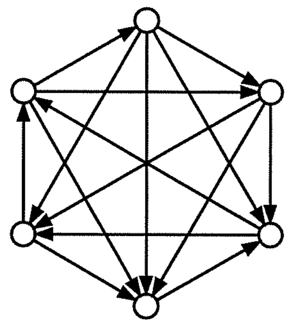

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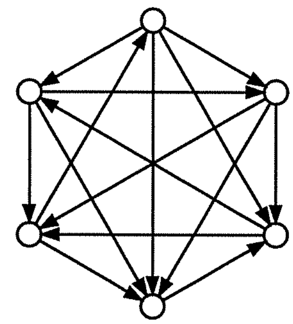

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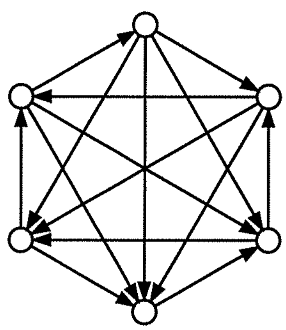

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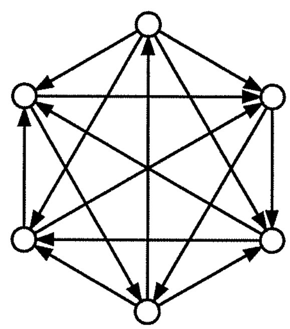

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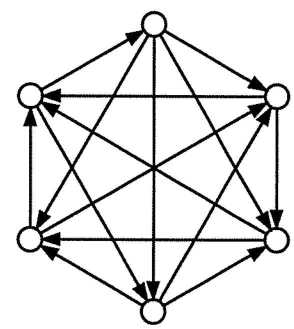

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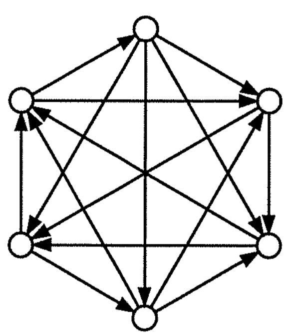

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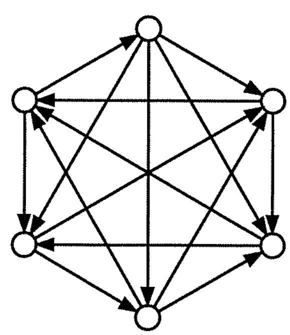

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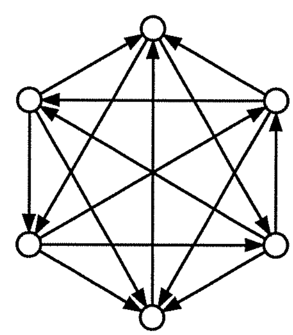

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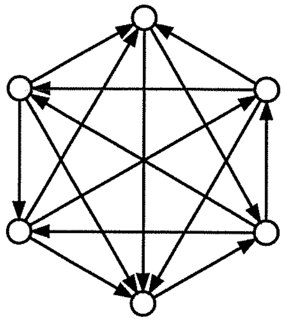
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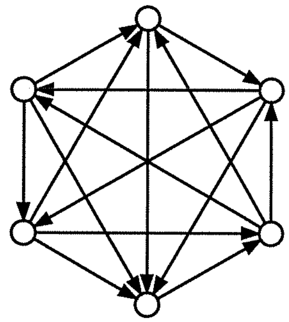
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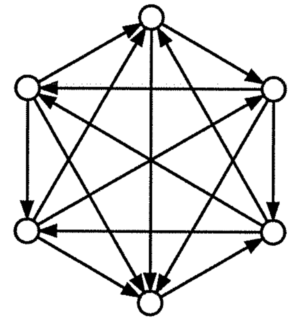
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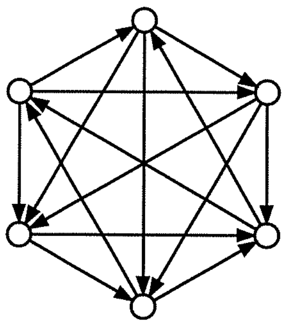
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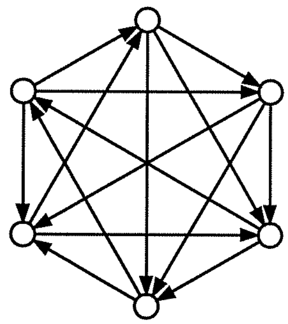
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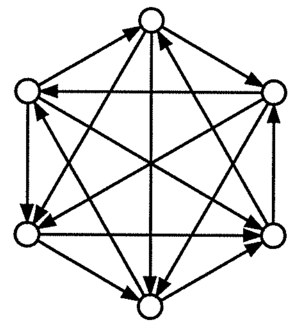
54

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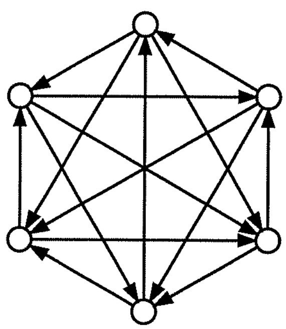
55

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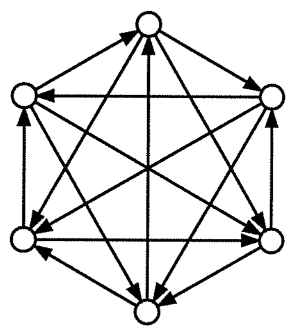
56

333222

Table 8.1 Numbers of Tournaments

n	T(n)	*self-converse	#Hamil-tonian	Condorcet probability $n/2^{n-1}$
1	1	1	1	1
2	1	1	0	1
3	2	2	1	3/4
4	4	2	1	1/2
5	12	8	6	5/16
6	56	12	35	3/16
7	456	88	353	7/64
8	6 880	176	6 008	1/16
9	191 536	2752	178 133	9/256
10	9 733 056	8784	9 355 949	5/256

*Alistair Farrugia, Univ of Malta
#A Hamiltonian tournament has at least one n-cycle. Equivalently, every node is reachable from every node.

Table 8.2 Converse pairs for n = 6

1 sc	21 sc	39 sc
2 ~ 3	22 sc	40 ~ 41
3 ~ 2	23 ~ 25	43 sc
4 sc	24 sc	44 ~ 48
5 ~ 13	26 ~ 29	45 ~ 49
6 ~ 14	27 ~ 30	46 ~ 50
7 ~ 15	28 ~ 31	47 ~ 51
8 ~ 16	32 sc	52 sc
9 ~ 17	33 ~ 37	53 ~ 54
10 ~ 18	34 ~ 38	55 sc
11 ~ 19	35 sc	56 sc
12 ~ 20	36 ~ 42	