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THERE ARE 718 6-POINT TOPOLOGIES,

QUASIORDERINGS AND TRANSGRAPHS

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Abstract

The number of topologies, quasiorderings or transgraphs on n distinct points is the same. They may be denoted by certain (0,1)-matrices. They fall into equivalence classes under permutations of the underlying set. We express the numbers of n-point topologies or classes in terms of the numbers of connected ones for $m \leq n$, define an iterative computing procedure for counting the latter, and find for $n = 6$ that there are 718 classes, of which 512 are connected. In Table 3 we classify the latter as to T_0, duality (defined below), and class size.

1. Introduction

It is easy to show that for a topology $\mathcal{T}$ on a finite set $\{a_1, \dots, a_n\}$, the least basis is $\{\emptyset, A_1, \dots, A_n\}$, where $A_i = \cap \{U: a_i \in U \text{ open}\}$. Furthermore, the relation $<$ defined by:

$$a_i < a_j \quad \text{iff} \quad A_i \subseteq A_j \quad \text{iff} \quad a_i \in A_j \tag{1}$$

is a reflexive, transitive relation, or quasiordering, and any such relation conversely defines a basis for a topology [4]. We thus have a 1-to-1 correspondence between topologies and quasiorderings on n distinct points.

Quasiorderings correspond to transitive directed graphs, or <u>transgraphs</u> [2], and have a natural representation by the matrices $m(<) = m(\mathcal{T}) = M = (m_{ij})$, defined by:

$$m_{ij} = 1 \text{ if } a_i < a_j, \quad m_{ij} = 0 \text{ otherwise.} \tag{2}$$

Note that column j then describes the basic set A_j of the corresponding topology. Henceforth let n-<u>matrix</u> mean "n by n (0,1)-matrix". The conditions for an n-matrix to represent a quasiordering are:

$$m_{ii} = 1 \text{ ; and for all } i, j, k, \; m_{ij} = 1 = m_{jk} \Rightarrow m_{ik} = 1 \text{ .} \tag{3}$$

We will call such matrices <u>valid</u> . They fall into natural equivalence classes under permutations p of $\{1, \ldots, n\}$, in which $\mathcal{J}_2$ is homeomorphic to $\mathcal{J}_1$ iff there is a p with

$$U \in \mathcal{J}_1 \quad \text{iff} \quad \{a_{p(i)} : a_i \in U\} \in \mathcal{J}_2 ,$$

in which case, where $M = m(\mathcal{J}_1)$ and $N = m(\mathcal{J}_2)$,

$$n_{p(i),p(j)} = m_{ij} . \tag{4}$$

We express this by: $n = p(M)$, or $N \approx M$, or $\mathcal{J}_2 = p(\mathcal{J}_1)$, etc.

2. Definition of variables to be enumerated

Let t_n be the number of valid matrices, quasiorderings, transgraphs and topologies on n distinct points. Let h_n be the number of classes, which may be viewed as the number of topologies, transgraphs, etc. on n unlabeled points. For a class H of n-point topologies, let $|H|$ be the number of members of H. Note that $|H| < n!$ when there are symmetries in its members; if the unlabeled transgraph of H (drawn with arcs directed upward) has a set of points in lateral symmetry, such as K_2 in Figure 1, then a permutation that permutes only the labels of these points does not change the ordering of the set, hence does not change the topology.

Values of t_n and h_n are not known in general, but t_n has been published for $n \leq 7$ [2], and h_n for $n \leq 5$ [4]. See Table 1. In this paper we give h_6, and certain other enumerations defined below.

It has been noted by [4] and others that the following are equivalent:

$\mathfrak{J}$ is T_0

$<$ is antisymmetric (a partial ordering)

if $i \neq j$ then $A_i \neq A_j$

if $i \neq j$ then $m_{ij} = 0$ or $m_{ji} = 0$ (5)

if $i \neq j$ then columns i, j of M differ. (6)

Let t_n^o, h_n^o be the numbers of T_0 topologies and classes thereof. It is known [1, 2] that

$$t_n = \sum_{m=1}^{n} \lambda(n,m)\, t_m^o \tag{7}$$

where $\lambda(n,m) = \frac{1}{m!} \sum_{k=1}^{m} \binom{m}{k} k^n (-1)^{m-k}$, a Stirling number of the second kind. But the same relation does not hold for h and h^o, and no analogous one has been found.

$\mathcal{T}$ is connected iff its transgraph is connected; for the following are equivalent:

$a_i < a_j$

$a_i \in A_j$

a_j is a closure point of $\{a_i\}$

a_i , a_j lie in the same component.

The equivalent matrix condition is [4]:

$m(\mathcal{T})$ and $m(\mathcal{T})'$ have no common r by $(n-r)$ zero submatrix, for $0 < r < n$. (8)

Let t_n^c , h_n^c , t_n^{co} , h_n^{co} be the variables above with "connected" specified.

The family $\mathcal{T}'$ of closed sets of a finite topology is also a topology. Furthermore, it is easy to show that

$$m(\mathcal{T}') = m(\mathcal{T})^T , \text{ where } {}^T \text{ denotes "transpose".}$$

3. Some ways to reduce computation time

We can use the information above to write a computer program for enumeration of t_n , h_n , etc. However, the time required is expensively long. We can cut it down by using the following propositions. (The <u>minor</u> of m_{ij} is the submatrix of M obtained by striking out row i and column j.)

3.1. Proposition. In a valid matrix, the minor of each diagonal element m_{rr} is valid.

Proof. In (3), restrict i, j, k to be $\neq r$.

3.1.1. Corollary. The valid n-matrices may be constructed by taking the valid (n-1)-matrices and forming valid augmentations of them: that is, attaching a row and a column, of equal index, under restriction (3).

Note: This principle was used in [2] in enumerating t_n and t_n^o.

3.1.2. Corollary. If only one member of each (n-1)-point class is given, and the row and column are attached as the n'th, then every class of n-matrices will be represented among the augmentations so obtainable.

Proof. Suppose M belongs to an unrepresented class. But the minor M' of m_{nn} belongs to a class represented by some given (n-1)-matrix K. We can construct an augmentation K^* of K and a permutation p such that $p(K^*) = M$. In fact, if $K = q(M')$, let $k^*_{nn} = 1$, and for $i < n$ let $k^*_{in} = m_{q(i),n}$ and $k^*_{ni} = m_{n,q(i)}$. Let $p(i) = q^{-1}(i)$ for $i < n$, and $p(n) = n$.

Henceforth let augmentation of an (n-1)-matrix mean "attachment of an n'th row and column, not necessarily valid".

Note: Unfortunately, augmentations of (n-1)-matrices in distinct classes may be equivalent. See Figure 1, where $K_1 \not\sim K_2$ but their respective augmentations M_1 and M_3 are equivalent.

3.2. Proposition. Transposes of equivalent matrices are equivalent, and valid augmentations of M and M^T are transposition pairs. Proof: Evident from relation (4).

Note: M may be equivalent to M^T, even though not a symmetric matrix. (M_1 in Figure 1 is an example.) If so, we call the class $H(M)$ of M self-dual. The corresponding unlabeled trans-graph is isomorphic to its ~~inverse~~ transpose. If not, $H(M)$ and $H(M^T)$ are a dual pair of classes; and their union is the duality class of M, whether or not M is self-dual.

3.2.1. Corollary. To represent each n-point duality class it suffices to take one member of each (n-1)-point duality class and construct valid augmentations.

Note: If M is an augmentation of K, self-duality of $H(M)$ is independent of that of $H(K)$. See Figure 1, where $H(K_1)$ is self-dual and $H(K_2)$ is not; and each has both a self-dual and a non- self-dual augmentation, as shown.

3.3. Proposition. *If* K *is connected and* M *is an augmentation of* K *such that* $m_{in} \neq 0$ *or* $m_{ni} \neq 0$ *for some* $i < n$, *then* M *is connected.* Proof: Apply relation (8).

3.4. Proposition. (a) *If* K *is not* T_0 *then* M *is not.* Proof: T_0 is preserved in taking subspaces. Or, apply (5). Note: K may be T_0 and M not so. Example: K_2 and M_5 in Figure 1. However,

(b) *Every class of non-*T_0 *n-matrices, for* $n \geq 3$, *is represented by an augmentation of a* non-T_0 (n-1)-*matrix*.

Proof: Suppose M belongs to an unrepresented class. But by (6), M has a pair of like columns i, j. Let $r \neq i, j$. Then the minor of m_{rr} is not T_0, by (6). Its class is represented by some non-T_0 matrix K. By a construction like that of 3.1.2, we can find an augmentation of K equivalent to M.

3.5. Proposition. (a) *The components od a finite topological space are open, and it id the free union of its components.*

(b) *It is* T_0 *iff all the components are* T_0.

Proof is elementary.

Note: $\mathcal{T}$, as a set of subsets, can be considered as the cross-~~product of its restrictions to the components.~~

product of the topologies on the components. If there are r components, each open set is the union of an r-tuple of open sets (some of which may be empty).

4. Summation formulas

Let a partition of n be represented by

$$X = ((n_1, x_1), \ldots, (n_k, x_k))$$

meaning that there are $x_1 + \ldots + x_k$ cells, $n_1 < \ldots < n_k$, and n_i occurs x_i times; so $n = \Sigma\, n_i x_i$.

4.1. Theorem. <u>In terms of the numbers of classes of connected topologies, the number of classes of</u> n-<u>point topologies is</u>

$$h_n = \sum_X \prod_{i=1}^{k} \binom{h^c_{n_i} + x_i - 1}{x_i} . \qquad (10)$$

Proof. For each partition X of n undistinguished points, and for each (n_i, x_i), we choose from $h^c_{n_i}$ kinds of component, x_i kinds, allowing repetition and without regard to order. The appropriate combinatorial is as given above. [3, pg. 7]

4.1.1. Corollary. $$h^o_n = \sum_X \prod_{i=1}^{k} \binom{h^{co}_{n_i} + x_i - 1}{x_i} . \qquad (11)$$

Proof: As for the theorem, in view of 3.5 (b).

Let dist(X) be the number of ways of distributing n distinct things into distinct cells (unordered within a cell) according to partition X. It is well-known [3, pg. 3] that

$$\text{dist}(X) = \frac{n!}{(n_1!)^{x_1}(n_2!)^{x_2}\cdot\ldots\cdot(n_k!)^{x_k}} . \qquad (12)$$

4.2. Theorem. $$t_n = \sum_X \text{dist}(X) \prod_{i=1}^{k} (t^c_{n_i})^{x_i} \Big/ \prod_{i=1}^{k} (x_i!) . \qquad (13)$$

Proof. For each X, n distinct points can be put in Σx_i distinct components in dist(X) ways. Each x_i components introduce a "symmetry factor" $x_i!$, for, if one distribution puts the sets $S_1, \ldots, S_{x_i}$ of points in the components $K_1, \ldots, K_{x_i}$ of size n_i , then every topology having those sets in those components taken in some other order is identical with one having them in that order, by 3.5 (a), because free product is commutative. Finally, the n_i points assigned to a component can be arranged into a connected space in $t^c_{n_i}$ ways, by definition of the latter.

4.2.1. Corollary. $$t^o_n = \sum_X \text{dist}(X) \prod_{i=1}^{k} (t^{co}_{n_i})^{x_i} \Big/ \prod_{i=1}^{k} (x_i!) . \qquad (14)$$

Proof: As for theorem, in view of 3.5 (b).

4.3. Theorem. If H is a class of n-point topologies having Σy_i components, where for each i, y_i of them belong to one class H_i^c of connected n_i-point topologies, with $n_1 \leq \ldots \leq n_k$ (Note: we allow equality, in case components of equal size are taken from distinct classes), then, where

$Y = ((n_1,y_1) ,\ldots, (n_k,y_k))$,

$$|H| = \mathrm{dist}(Y) \prod_{i=1}^{k} |H_i^c|^{y_i} \Big/ \prod_{i=1}^{k} (y_i!) \quad . \qquad (15)$$

Proof. For each Y, n distinct points can be put in the components in dist(Y) ways. If y_i components are to be members of class H_i^c , there is a symmetry factor $y_i!$, as in 4.2. Note that if $n_i = n_j$ but $H_i^c \neq H_j^c$, then we must distinguish between distributions putting a given set of points into cells of different kinds. After a distribution of points to components, we can, in each one independently, "label" the unlabeled transgraph of H_i^c in $|H_i^c|$ ways; the product of these numbers, over all components, is therefore a factor of $|H|$.

Note: For T_0 classes, (15) applies unchanged, by 3.5 (b).

4.3.1. Corollary. We may write $t_n^c = \Sigma\{|H_j^c| : j = 1, \ldots, h_n^c\}$ (summing over all classes of n-point connected topologies), and similarly

$$t_n = \sum_{j=1}^{h_n} |H_j| = \sum_{j=1}^{h_n} \operatorname{dist}(Y_j) \prod_{i=1}^{k} |H_i^c|^{\gamma_i} \Big/ \prod_{i=1}^{k} (\gamma_i!) \quad , \qquad (16)$$

where Y_j and $\{H_i^c\}$ depend on H_j as described in the theorem.

Note: This also applies, mutatis mutandis, in the T_0 case.

4.4. Some open problems remaining are: To express h_n in terms of $\{h_m^o : m \leq n\}$ in some way analogous to (7); or to give complete algorithms for t_n, h_n, etcetera in terms of n.

5. Conclusion

We have shown that the enumeration of the variables defined in Section 2 can be accomplished with the closed-form computations in (10) to (16), together with a computer program having the following specifications:

Input: One member of each duality class of (n-1)-matrices, with an indication of whether it is T_0.

Procedures:
1. Construct connected augmemtations (using 3.3).
2. Test each for validity (using (3)).
3. If input was T_0, check whether new one is T_0. If not, discard it. If so, go on. (Use (5) and (6).)
4. Test whether new matrix or its transpose is equivalent to one already stored. (Use (4).) If so,

discard it; if not, go on.

5. Count its distinct permutations (using (4)).
6. Check whether its transpose is one of them, i.e. its class is self-dual. (Use (9).)
7. Store the matrix, a T_0 indicator, and the results of steps 5 and 6, for use in step 4, until all the input has been used.

Output: All the stored information, with matrices and T_0 indicators in a format suitable for use as input for the next value of n.

Such a program (not using 3.4, which was noticed later) has been written in PL-1 by William Arcuri and run on an IBM 360, model 65, at the University of Rochester, for $n \leq 6$. The totals t_n and t_n^o agree with the unclassified enumerations reported in [2]. A summary of values obtained is given in the tables. Interested parties may obtain from the author: (a) The program; (b) a list of the matrices found for $n = 5$ or 6, with hand-drawn diagrams. (Equivalent matrices for $n = 5$ have been published in [4].) The machine times were: 3 minutes for 5; 144 minutes for 6; estimated for 7, 120 hours.

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class of T_0-topologies　T_0-topologies　equiv classes of topologies　topologies

Table 1

n	h_n^{co}	t_n^{co}	h_n^{c}	t_n^{c}	h_n^{o}	t_n^{o}	h_n	t_n
1	1	1	1	1	1	1	1	1
2	1	2	2	3	2	3	3	4
3	3	12	6	19	5	19	9	29
4	10	146	21	233	16	219	33	355
5	44	3060	94	4851	63	4231	139	6942
6	238	101642	512	158175	318	130023	718	209527
7		5106612*		7724373*		6129859		9535241

*Obtained by (13) and (14), using t_7^o and t_7 as given in [2].

Table 2

Numbers of 5-point connected classes, grouped by class size, T_0, and duality.

Class size	1	5	10	15	20	30	60	120	Totals
T_0, self-dual	0	0	0	0	1	1	0	4	6
T_0, not s-d.	0	2	2	0	2	2	20	10	38
not T_0, s-d.	1	0	0	0	1	1	1	0	4
not T_0 or s-d	0	2	6	2	4	14	18	0	46
Totals	1	4	8	2	8	18	39	14	94

Table 3

Numbers of 6-point connected classes, grouped by class size, T_0, and duality.

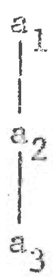
Class size	1	6	15	20	30	45	60	90	120	180	360	720	Totals
T_0, self-dual	0	0	0	1	1	0	0	1	1	4	7	13	28
T_0, not s-d.	0	2	2	0	2	0	6	0	16	24	90	68	210
not T_0, s-d.	1	0	0	1	1	0	0	4	1	6	4	0	18
not T_0 or s-d.	0	2	6	2	4	4	32	12	18	86	90	0	256
Totals	1	4	8	4	8	4	38	17	36	120	191	81	512

Figure 1

(All directions not marked are upward.)

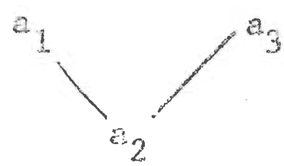
K_1

1 0 0
1 1 0
1 1 1



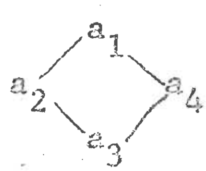
K_2

1 0 0
1 1 1
0 0 1



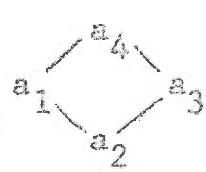
M_1

1 0 0 0
1 1 0 0
1 1 1 1
1 0 0 1



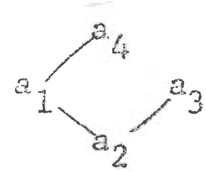
M_3

1 0 0 1
1 1 1 1
0 0 1 1
0 0 0 1



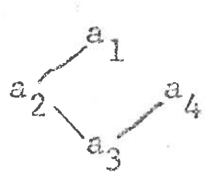
M_4

1 0 0 1
1 1 1 1
0 0 1 0
0 0 0 1



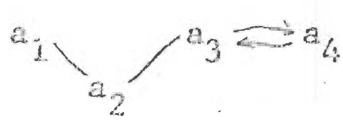
M_2

1 0 0 0
1 1 0 0
1 1 1 1
0 0 0 1



M_5

1 0 0 0
1 1 1 1
0 0 1 1
0 0 1 1



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