Math Olympiad Hardness Scale (MOHS)

because arguing about problem difficulty is fun :P

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In this document I provide my personal ratings of difficulties of problems from selected recent contests. This involves defining (rather carefully) a rubric by which I evaluate difficulty. I call this the MOHS hardness scale (pronounced "moez"); I also go sometimes use the unit "M" (for "Mohs").

The scale proceeds in increments of 5M, with a lowest possible rating of 0M and a highest possible rating of 60M; but in practice few problems are rated higher than 50M, so it is better thought of as a scale from 0M to 50M, with a few "off-the-chart" ratings.

1 Warning

§1.1 These ratings are subjective

Despite everything that's written here, at the end of the day, these ratings are ultimately my personal opinion. I make no claim that these ratings are objective or that they represent some sort of absolute truth.

For comedic value:

Remark (Warranty statement). The ratings are provided "as is", without warranty of any kind, express or implied, including but not limited to the warranties of merchantability, fitness for a particular purpose, and noninfringement. In no event shall Evan be liable for any claim, damages or other liability, whether in an action of contract, tort or otherwise, arising from, out of, or in connection to, these ratings.

§1.2 Suggested usage

More important warning: excessive use of these ratings can hinder you.

For example, if you end up choosing to not seriously attempt certain problems because their rating is 40M or higher, then you may hurt yourself in your confusion by depriving yourself of occasional exposure to difficult problems.^{[1](#page-2-0)} If you don't occasionally try IMO3 level problems with real conviction, then you will never get to a point of actually being able to solve them.^{[2](#page-2-1)} For these purposes, paradoxically, it's often better to *not* know the problem is hard, so you do not automatically adopt a defeatist attitude.

These ratings are instead meant as a reference. In particular you could choose to usually not look at the rating for a problem until after you've done it; this simulates competition conditions the best, when you have no idea how hard a problem is until you either solve it or time runs out and you see who else solved it.

You have been warned. Good luck!

¹This will also be my excuse for declining "why don't you also rate X contest?"; to ensure that there is an ample supply of great problems that don't have a rating by me. For example, I will not publish ratings for IMO shortlist; it is so important of a training resource that I don't want it to be affected by MOHS. The PSC ordering is already enough. I also want to avoid publishing ratings for junior olympiads since I feel younger students are more likely to be discouraged or intimidated than older students.

 2 Fun story: in Taiwan, during "team selection quizzes" (which were only 110 minutes $/$ 2 problems and don't count too much), one often encountered some difficult problems, in fact sometimes harder than what appeared on the actual TST. My guess is the intention was for training purposes, to get some experience points with a super-hard problem for at least a little time, even if almost no one could actually solve it in the time limit.

2 Specification

Here is what each of the possible ratings means.^{[1](#page-3-0)}

- Rating OM: Sub-IMO. Problems rated 0 are too easy to use at IMO. I can often imagine such a problem could be solved by a strong student in an honors math class, even without olympiad training.
- **Rating 5M: Very easy.** This is the easiest rating which could actually appear while upholding the standards of IMO. They may still be very quick.

Recent examples:

- IMO 2019/1 on $f(2a) + 2f(b) = f(f(a+b))$
- IMO 2017/1 on $\sqrt{a_n}$ or $a_n + 3$
- Rating 10M: Easy. This is the rating assigned to an IMO 1/4 which would cause no issue to most students. Nevertheless, there is still some work to do here. For example, the second problem of each shortlist often falls into this category. These problems would still be too easy to use as IMO 2/5.

Recent examples:

- IMO 2019/4 on $k! = (2^n 1) \dots$
- IMO 2018/1 on \overline{DE} || \overline{FG}
- **Rating 15M: Somewhat easy.** This is the easiest rating of problems that could appear as IMO 2/5 (and sometimes do), though they often would be more appropriate as IMO 1/4. A defining characteristic of these problems is that they should be solved comfortably by students from the top 10 countries at the IMO even when placed in the 2/5 slot (as this is not always the case for 2/5 problems).

Recent examples:

- IMO 2019/5 on Bank of Bath
- IMO 2018/4 with sites and stones on a 20×20 grid, ft. Amy/Ben
- IMO 2017/4 with KT tangent to Γ
- **Rating 20M: Medium-easy.** This is the first rating of problem which would probably be too difficult to use as an IMO $1/4$, though still not up to the difficulty of an average IMO 2/5. Nevertheless, top countries often find such problems routine anyways.

Recent examples:

- IMO 2018/5 on $\frac{a_1}{a_2} + \cdots + \frac{a_n}{a_1}$ $\frac{a_n}{a_1} \in \mathbb{Z}$.
- Rating 25M: Medium. Placed at the center of the scale, problems in this rating fit comfortably as IMO 2/5 problems. This is the lowest rating for which team members in top countries could conceivably face difficulty.

Recent examples:

¹I deliberately chose to use multiples of 5 in this scale to avoid accidentally confusing problem numbers (e.g. "6") with difficulty ratings (e.g. "30M"). Originally used multiples of 10 until I clashed with a different scale for some other contest which used multiples of 10. This led to a lot of headache for me, so I switched to 5. Anyways, 50 felt like a nice effective maximum.

- IMO 2019/2 on P_1 , Q_1 , P , Q cyclic.
- Rating 30M: Medium-hard. These are problems which are just slightly tougher than the average IMO 2/5, but which I would be unhappy with as IMO 3/6 (although this can still happen). Problems rated 30M or higher often cause issues for top-10 countries at the IMO.

Recent examples:

- IMO 2018/2 on $a_i a_{i+1} + 1 = a_{i+2}$
- Rating 35M: Tough. This is the highest rating that should appear as an IMO 2/5; I think IMO5 has a reputation for sometimes being unexpectedly tricky, and this category grabs a lot of them. The most accessible IMO 3/6's also fall into the same rating, and these are often described as "not that bad for a 3/6" in this case.

Recent examples:

- IMO 2019/6 on $DI \cap PQ$ meeting on external ∠A-bisector
- IMO 2017/5 on Sir Alex and soccer players
- Rating 40M: Hard. This is the lowest rating of problems which are too tough to appear in the IMO 2/5 slot. Experienced countries may still do well on problems like this, but no country should have full marks on this problem.

Recent examples:

- IMO 2019/3 on social network and triangle xor
- IMO 2017/2 on $f(f(x) f(y)) + f(x + y) = f(xy)$
- \bullet IMO 2017/3 on hunter and rabbit
- IMO 2017/6 on homogeneous polynomial interpolation
- **Rating 45M: Super hard.** Problems in this category are usually solved only by a handful of students. It comprises most of the "harder end of IMO 3/6".

Recent examples:

- IMO 2018/3 on anti-Pascal triangle
- IMO 2018/6 on $\angle BXA + \angle DXC = 180^\circ$.
- Rating 50M: Brutal. This is the highest rating a problem can receive while still being usable for a high-stakes timed exam, although one would have to do so with severe caution. Relative to IMO, these are the hardest problems to ever appear (say, solved by fewer than five or so students). They also may appear on top-country team selection tests.
- Rating 55M: Not suitable for exam. Problems with this rating are so tedious as to be unsuitable for a timed exam (for example, too long to be carried out by hand). This means that maybe such a problem *could* be solved by a high-school student in 4.5 hours, but in practice the chance of this occurring is low enough that this problem should not be used. Some problems of this caliber could nonetheless be published, for example, on the IMO Shortlist.
- **Rating 60M: Completely unsuitable for exam.** This rating is usually given to problems which simply could not be solved by a high-school student in 4.5 hours, but might still be eventually solvable by a high-school student. For example, a result from a

combinatorics REU whose proof is a 15-page paper could fit in this category. (In contrast, a deep result like Fermat's last theorem would simply be considered not rate-able, rather than 60M.)

3 The fine print

Of course, difficulties are subjective in many ways; even the definition of the word "difficult" might vary from person to person. To help provide further calibration, problems rated with the MOHS difficulty scale will use the following conventions.

§3.1 Assumed background knowledge

One of the subtle parts of rating the difficulty of problems is the knowledge that a student knowns. To quote Arthur Engel:

"Too much depends on the previous training by an ever-changing set of hundreds of trainers. A problem changes from impossible to trivial if a related problem was solved in training".

We will try to at least calibrate as follows. First, we consider the following table, which lists several standard results.

These indicate rough guidelines for the difficulty tiers in which several standard theorems or techniques should be taken into account.

Here are some notes on what this table means.

- The table refers to minimal exposure, rather than mastery. For example, when we write "FE's" as 0M, we just mean that a student has seen a functional equation before and knows what such a problem is asking, and maybe has seen enough examples to see the words "injective" and "surjective". It does not assert that a student is "well-trained" at FE's (so for this reason, IMO 2019 is rated 5M, not 0M).
- Here is an example of interpretation. Projective geometry is rated at 15M. This means that, if a problem has an extremely straightforward solution to students exposed to projective geometry, but does not have a simple solution without requiring such knowledge, then an appropriate rating is 15M. (One could not rate it lower without being unfair to students who do not know the result, and vice-versa.)
- This table is not exhaustive and meant to serve only as a guideline. For other results which are considered standard, for example lemmas in geometry, a judgment call should be made using this table as reference.
- This table is quite skewed to be knowledge-favoring, reflecting a decision that MOHS is aimed at rating difficulties for well-trained students. For example, many students arrive at the IMO without knowledge of what AM-GM is. Despite this, AM-GM is rated as 0M, meaning if a problem is completely routine for a student who knows the AM-GM theorem, then it could be rated 0M (even though, if actually given at the IMO, necessarily students not knowing AM-GM might not solve it).
- In cases where results are sufficiently specialized (say, few students from top countries know them), then we will generally make the assumption that a student has not seen a problem or result which could be considered as "trivializing the problem". For example, when rating IMO 2007/6 we assume the student has not seen combinatorial nullstellensatz before and when rating USAMO 2016/2 we assume the student has not seen hook-length formula before.

§3.2 Details count towards difficulty

I believe that in Olympiads, we should try to encourage students to produce complete solutions without flaws, rather than only emphasizing finding the main idea and then allowing some hand-waving with details (I understand not everyone agrees with this philosophy.) Consequently, in the MOHS hardness scale, the difficulty of a problem takes into account the complexity of the details as well. Therefore, a problem which can be solved by a long but routine calculation may still have a high difficulty rating; and a problem which has a "pitfall" or common mistake is likely to be rated higher than people might expect, too.

A good example is IMO 2012/4, on the functional equation $f(a)^2 + f(b)^2 + f(c)^2 =$ $2[f(a)f(b) + f(b)f(c) + f(c)f(a)]$. Even though the problem does not really have any deep idea or "trick" to it, the rating is nonetheless set at 15M. The reason is that the pathological casework is notoriously slippery, and is rather time-consuming. Therefore I do not regard this as especially easy.

This is also the same reason why IMO 2017/1 is rated 5M instead of 0M. When I first solved it, I scoffed at the problem, thinking it was too easy for the IMO. But as I wrote up the solution later on, I found the details ended up being longer and more nuanced than I remembered, and I made mistakes multiple times. Therefore I no longer think this problem is too easy for IMO (and even considered rating it 10M).

§3.3 Length of solution

I should say at once that it is a common mistake to judge the difficulty of a problem by the length of the solution.

Nonetheless, I believe the length of the solution cannot be ignored entirely when judging the difficulty of problems. The reason is that I often witness what I like to jokingly call "the **infinite monkey theorem**": if the solution to a TST problem is sufficiently short then, no matter how tricky it is to find, *somebody* out there will get it (often quickly), and that will decrease the difficulty rating of the problem a bit.

See USA TSTST 2019/3 about cars for a hilarious example.

For the same reason, problems which are rated as 55M or 60M (while still being rate-able) are most commonly rated this way for the solution being too long to work out during a 4.5-hour exam.

§3.4 Multiple approaches

When a problem has multiple correct (essentially different) approaches, in general, this seems to suggest that the problem is easier than the difficulty of any particular approach.

This is most useful to keep in mind in cases where a problem has a lot of correct approaches; even if each individual approach is not easy to find, the overall problem might end up being quite accessible anyways.

§3.5 How to use statistics

I think that problem statistics (e.g. those on imo-official.org) are quite useful for calibration. They are completely objective with no room for human bias, so they can help with avoiding the "PSC effect" in which problems appear much easier than they are after thinking about the shortlist for days or even weeks (while the students will only have 4.5 hours).

Despite this, I think statistics should not supersede the experience of having done the problem yourself; and therefore there are a few examples of situations in which I rated a problem much lower than the statistics in the problem might suggest.

The biggest confounding factor seems to be the fact that problems are not given to students in isolation, but in sets of three. This means that if $#2$ is unusually hard, then the scores for $#3$ will be unusually low, for example. Even the position of a problem can intimidate students into not trying it. Other confounding factors include the strength of students taking the exam (which is not fixed across years of the IMO, say) and the way that partial credit is given.

Here are a few illustrative examples.

A story of IMO 2017/2, on $f(f(x)f(y)) + f(x + y) = f(xy)$

The problem IMO 2017/2 is rated as 40M, despite an average score of 2.304. In fact, if one looks at the rubric for that year, one will find that it is unreasonably generous in many ways, starting with the first line:

(1 point) State that $x \mapsto 0$, and that at least one of $x \mapsto x-1$ or $x \mapsto 1-x$ are solutions to the functional equation.

And it got worse from there. What happened in practice (in what I saw as an observer) was that many students were getting 4-5 points for 0^+ solutions.

Naturally, this inflates the scores by an obscene amount, and this leads to a misleading historical average. In truth, even among top countries most teams were getting something like 2 of 6 solves. Even worse, the problem was an enormous time-sink; even students who did solve the problem ended up with very little time to think about the final problem.

A story of IMO 2017/3, on hunter and rabbit

On the other end, the IMO 2017/3 is rated as 40M, the same difficulty as IMO $\#2$ that year, despite having an average score of 0.042. (In fact, several of my students have told me they think it should be rated 35M.)

In my head, the reason for this is very clear — the reason so few people solved the problem is because they ran out of time due to IMO 2; under time pressure, few students would rather spend time on this scary-looking problem than an innocent-looking (but ultimately pernicious) IMO2, where one can simply continue futile substitutions.

My evidence in this belief is alas anecdotal:

- The problem was C5 in the shortlist packet, and the PSC regarded it as mediumhard. Some leaders even voted for it as IMO5, during the jury meeting.
- I solved the problem with no paper while at a shopping mall (during the phase while jury works on problems), and so had no reason to expect to become such a historically difficult problem.
- I have given this problem to students in isolation before, and many of them solve it outright. So it is certainly not impossible.
- After finding the main idea, there aren't many details to stop you. The "calculation" part of the problem is pretty short.

Moreover, because this problem was essentially binary grading, there is almost no partial credit awarded, leading to such an intimidatingly low average.

A story of IMO 2005/1, on hexagon geometry

I would like to give one example in the other direction, where the statistics of the problem were a big part of my rating.

When I tried IMO 2005/1 myself, I found the solution immediately (having seen the solution to USAMO 2011/3 helped a lot!). But the solution felt unusual to me, and I sensed that others may run into difficulty. So I looked up the statistics for the problem, and found that many top countries had students who did not solve the problem, confirming my suspicion.

This is why I decided to assign a rating of 20M even though I solved the problem quickly. Since the problem was slotted as $#1$, there was really no plausible explanation to me why top students would miss the problem other than it being harder than expected (contrary to the previous example of IMO 2017/3 where I did have an explanation).

§3.6 Bond, James Bond

Even when it seems impossible, someone will often manage to score 007 on some day of the contest.

Which just goes to say: problem difficulty is actually a personal thing. On every exam, someone finds the second problem easier than the first one, and someone finds the third problem easier than the second one. The personal aspect is what makes deciding the difficulty of problems so difficult.

These ratings are a whole lot of nonsense. Don't take them seriously.

4 Ratings of contests

As stated in [Chapter 1,](#page-2-2) these rating are ultimately my personal opinion. Included are:

- IMO (International Math Olympiad) from 2000 to present
- USAMO (USA Math Olympiad) from 2000 to present
- USA TST (USA IMO Team Selection Test) from 2014 to present
- USA TSTST (USA TST Selection Test) from 2014 to present
- USEMO (US Ersatz Math Olympiad), all years

§4.1 IMO ratings, colored by difficulty

§4.2 USAMO ratings, colored by difficulty

§4.3 USA TSTST ratings, colored by difficulty

§4.4 USA TST ratings, colored by difficulty

§4.5 USEMO ratings, colored by difficulty

§4.6 IMO ratings, colored by subject

§4.7 USAMO ratings, colored by subject

§4.8 USA TSTST ratings, colored by subject

§4.9 USA TST ratings, colored by subject

§4.10 USEMO ratings, colored by subject

