Open questions within the Newtonian N-body problem .

The first two questions are well-known, and generally agreed to be quite important.

Stability: the oldest question in dynamical systems.

Call a solution to the N-body problem "bounded" if all the interparticle distances r_{ij} are bounded functions of time.

Q1. Is the subset of phase space swept out by bounded solutions nowhere dense?

Begin with the zero angular momentum planar three body problem.

For the two-body problem the answer is a resounding 'No': all negative energy orbits are periodic, hence bounded. For the planar restricted three-body problem the answer is again no, this time by the KAM theorem combined with dimensionality. The planar restricted three-body problem has two degrees of freedom, and the KAM tori bound within an energy surface for 2 degree of freedom systems. For the spatial restricted three-body problem the problem is also open.

M. Herman called this problem the "oldest question in dynamical systems" in his ICM lecture of 1998.

Notes on degrees of freedom. After dividing by translations and rotations, the planar N body problem has $2N-3$ degrees of freedom, so its phase space has dimension $2(2N-3)$.

Notes on KAM. The KAM theorem asserts that the subset swept out by the bounded orbits has positive measure provided we can find a single KAM stable orbit, meaning an orbit satisfying the hypothesis of the KAM theorem. KAM guarantees a set of stable solutions arbitrarily close to the stable orbit – the union of the KAM tori. This stable set of tori is a kind of fat Cantor set: nowhere dense, but of positive measure. In the interstices between the KAM tori are the remnants of 'destroyed tori'. General opinion is that most solutions starting in these interstices leak out way from a neighborhood of the KAM-stable orbit. (The time it takes to 'leak away' is extremely long if one starts very close.)

On the other side, we know of open sets of orbits for negative energy which are unbounded. In these two planets form a "tight binary" moving away from the third mass.

Central configurations.

Q2. Is the number of central configurations finite for all N and all positive masses?

Smale included Q2 in his list of problems for the 21st century.

Central configurations play the role of singular points for the vector field which defines the N-body equations. More precisely, there is a way to quotient Newton's equations by the action of all similarities. The central configurations are the singular points (zeros) of this quotient vector field. (See Chenciner: " A l'infini en temps fini".) Surely, if we want to understand the flow of a vector field, we would like to know if its zeros are finite in number. The set of central configurations is invariant under isometries and indeed similarities, so we count them modulo similarities. For $N = 3$ there are precisely five central configurations modulo oriented similarity, corresponding to the three collinear solutions of Euler plus the two equilateral ones of Lagrange. For $N = 4$ and general masses it is not even known if the number of central configurations is finite!

See Pollard or Hall and Meyer for more on central configurations.

Alain Albouy has proved the answer 'yes' for 4 equal masses. Moeckel has proved 'yes' for an open dense set of mass ratios when $N = 4$. Moeckel also has written a beautiful computer program to generate central configurations and it seems you get millions (billions?) when N is in the teens.

Variational Problems.

The next bunch of problems are variational in nature. There are two basic action principles in mechanics, the standard one, and the Jacobi action. The standard action is $\int_c (K-V)dt$ where c is a curve, K is the kinetic energy and V is the potential energy. One fixes the time interval of a solution, and some endpoint conditions and searches for extremals. In the Jacobi action principle the time is not fixed, but instead the energy $E = K + V$ is fixed. The Jacobi action is $\int_c ds_J$ where the Jacobi metric ds_J is given by $ds_J = (E-V)ds_K^2$ where d^2s_K is the metric associated to the kinetic energy K. We must restrict to the set of points for which $E-V \geq 0$, since $E-V = K$ for curves of energy E, and since K is nonnegative. Geodesics in the Jacobi metric are reparameterizations of solutions to Newton's equations. (The reparameterization factor is K.) One must be especially careful at the Hill boundary $\{E-V=0\}$ since the metric degenerates here. There one must insist curves hit the boundary orthogonally to get the correspondence between geodesics and Newton.

It is well-known that a collision-free local minimizer of the action is a solution, provided we put appropriate endpoint conditions on the competing class of curves. The collisions are the central difficulty in using either action principle. The Hill boundary is an additional difficulty with the Jacobi action.

A priori, all we know about a minimizer for the standard action is that it consists of a countable collection of solution arcs concatenated continuously at collisions. The set of collision times could be any closed set of measure zero, eg. the Cantor set.

In this regard the following theorem is fundamental.

Theorem. *(Chenciner, 2002 ICM) For the fixed endpoint problem with collision-free endpoints, and for the standard action for the N-body problem, all action minimizers are collision-free.*

Q3. Is an analogous theorem true for Jacobi minimizers, provided the two points are sufficiently far (how far??) from the Hill boundary?

Note: if both points are on the Hill boundary then the infimum of the Jacobi action is zero, and is realized by any curve on the Hill boundary connecting these endpoints.

Stability of the eight.

Q4. Give an analytic proof, preferably variatonal, which explains why the eight is KAM stable.

Dan Offin has been working on this problem.

Braid type of solutions.

This question was raised by Wu-Yi Hsiang in around 1996. It is inspired by similar ideas regarding geodesic flow on hyperbolic surfaces, where every free homotopy class, and hence every symbol sequence, is realized by a geodesic.

Consider a motion of the planar three-body problem. We say it has an eclipse of type 2 at instant t if at that instant the masses are collinear with 2 lying between 1 and 3. Similarly we have eclipses of type 1 and 3. Any collision-free planar solution yields an eclipse sequence. A "tight binary" in which 1 and 2 rotate around each other while 3 stays far away has eclipse sequence 121212.. , or simply 12 if we use the convention that we extend the word periodically. The eight has sequence 123123. Lagrange's solution has eclipse sequence the empty word $\emptyset$.

Q5. Is every sequence of eclipses realized by some solution? If not, describe the set of eclipse sequences which are realized by solutions.

We have not insisted on periodicity, either of the solution, or of the eclipse sequence. If we insist on periodic solutions, then the eclipse sequence is periodic, and the energy of the solution is negative. The configuration space for the three-body problem has topology when we forbid collisions. The free homotopy type of such a collision-free curve, taken modulo rotations is encoded by its eclipse sequence. To make sure we have just one word for each free homotopy type we impose two rules of grammar on the words. These rules are (1) "no stuttering" and (2) the word must be thought of as a cylic word. (1) means that we cannot have two successive eclipses of the same type. Thus, for example: 1223 is not allowed. (2) means that we must view the word as a cyclic word. Thus for for example, 123123 = 23123 = 312312. We call these "the rules of grammar".

Q3B. Subject to the above rules of grammar, is every periodic sequence of eclipses realized by a reduced periodic solution to the planar Newtonian three body problem? To the zero angular momentum planar three-body problem? If not describe those classes that can be realized.

Here "reduced periodic" means "periodic modulo rotation".

No 'spinning' for total collapse.

A solution which tends to total collapse (N-tuple collision) does so like $t^{2/3}$ when $t = 0$ is the time of total collision. Saari has claimed to have proved that any such total collapse solution $x(t)$ has the property that $t^{-2/3}x(t)$ tends to a fixed N-gon as $t \to 0$. His proof is incomplete, in that it relies on two unproven hypotheses. These hypothesis are (1) that the central configurations are finite in number (see previous problem), and (2) that when viewed on the collision manifold (a la McGehee) the central configurations define hyperbolic rest points.

Q6. Prove Saari's "theorem" .

Q6 may be easier than Q2.

Existence of choreographies. After Chenciner-Montgomery established the eight's existence, Joseph Gerver found by a combination of geometric and numerical arguments an $N = 4$ solution in which all 4 equal masses chase each other around the same "supereight" curve, while forming a parallelogram at every instant of time. Soon after, and Simo found , numerically, hordes of new solutions called "choreographies" in which all N masses are equal and chase each other around the same planar curve. None of these new planar choreographies ones have been proved to exist, although I have no doubt that they do exist.

Q7. Prove that Gerver's orbit exists, or prove the existence of any one of Simo's new choreographies.

NOTE: Venturelli, using the ICM result of Chenciner, the existence of SPATIAL CHOREOGRAPHIES of "hip-hop" type have been shown to exist for any N.

On Shape curves

Q8.

[via Julian Barbour.] Can you "read off" the basic invariants of a solution, eg. energy, angular momentum, dimension of space in which the bodies are moving, from its "shape curve", meaning the projection of this solution to the shape sphere? More precisely, since the shape curve is invariant under homothethy, as is the "Dziobek constant" $D = EC^2$, can we read off the sign of EC^2? If $EC^2 = 0$ can we tell if it is $E = 0$, or $C = 0$ or both? If not, can we read these off from a family of curves?

Remark: The Lagrange and Euler homographic solutions have shape curves which are fixed points, and they can have any D, so naively the answer is no, and we need to allow "generic" curves, whatever that means.

Q9. [**Levi's problem.**] Does every bounded zero angular momentum solution to the planar three-body problem suffer infinitely many eclipes.

ANSWERED! See "Infinitely Many Syzygies" on my web page.

Constancy of moment of inertia.

Numerically we know that the moment of inertia of the eight is not constant, although it is very nearly so.

Q10. Prove that the moment of inertia of the eight is not constant.

Q10 was formulated by Chenciner. It is the simplest nontrivial instance of

Q11. Saari's conjecture. Is it true that if I is constant along a solution then that solution corresponds to a rotating central configuration.