Bi-interpretation in set theory

Joel David Hamkins
Professor of Logic
Sir Peter Strawson Fellow

University of Oxford
University College, Oxford

University of Bristol
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Joint work with:

Alfredo Roque Freire
Professor of Logic
University of Brasília, Brasília
alfrfreire@gmail.com
https://www.alfredoroquefreire.com/

[FH20] Alfredo Roque Freire and Joel David Hamkins. Bi-interpretation in weak set theories. 2020, Mathematics ArXiv: 2001.05262.

Introduction

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Let's begin by reviewing what it means to interpret one model in another or one theory in another.

This is a very general model-theoretic concept, which makes sense with any kind of model or theory.

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$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc)\end{aligned}$$

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Conversely, $\mathbb{R}$ is not actually interpretable in $\mathbb{C}$, as fields.

But $\mathbb{R}$ is interpretable in $\langle \mathbb{C}, +, \cdot, \bar{z} \rangle$, with conjugation $z \mapsto \bar{z}$, or in the complex plane $\langle \mathbb{C}, +, \cdot, \mathrm{Re}, \mathrm{Im} \rangle$, which is bi-interpretable with $\langle \mathbb{R}, +, \cdot \rangle$.

Integer ring $\langle \mathbb{Z}, +, \cdot \rangle$ is interpretable in natural numbers $\langle \mathbb{N}, +, \cdot \rangle$

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$$(n,m)\equiv(s,t)\quad\iff\quad n-m=s-t\quad\iff\quad n+t=s+m.$$

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Rational field $\langle \mathbb{Q}, +, \cdot \rangle$ is interpretable in integer ring $\langle \mathbb{Z}, +, \cdot \rangle$

Rational number are represented as fractions p/q, essentially integer pairs (p, q), with $q \neq 0$ under the *same ratio* relation.

$$\frac{p}{q} \simeq \frac{r}{s} \quad \iff \quad ps = rq.$$

The familiar fractional arithmetic is well-defined.

Finite set theory

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Use the Ackermann relation

$$n\ E\ m \iff n^{th} \text{ binary digit of } m \text{ is } 1.$$

This relation is definable in arithmetic and it is easily verified that $\langle \mathrm{HF}, \in \rangle \cong \langle \mathbb{N}, E \rangle$.

General definition

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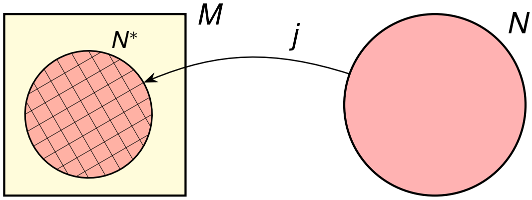
More specifically, $\langle N, R, f, c, \ldots\rangle \cong \langle N^*, R^{N^*}, f^{N^*}, c^{N^*}, \ldots\rangle/\simeq$

- where $N^* \subseteq M^k$ is a definable set of k-tuples in M;
- R^{N^*}, f^{N^*}, c^{N^*} are M-definable relations/functions;
- $\simeq$ is an M-definable equivalence relation, a congruence.



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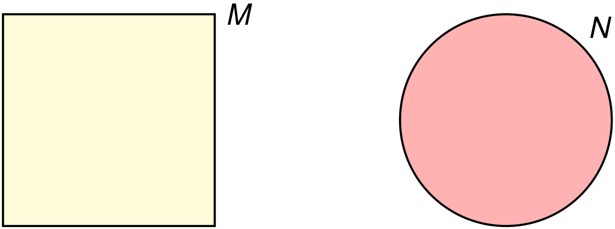
- In ZF, even without global choice, can eliminate need for $\simeq$ via Scott's trick with minimal rank representatives.
- (foreshadowing: can't generally eliminate $\simeq$ in ZFC^-)

Mutual interpretation of models

Models M and N are *mutually interpretable*, if each of them is interpreted in the other.



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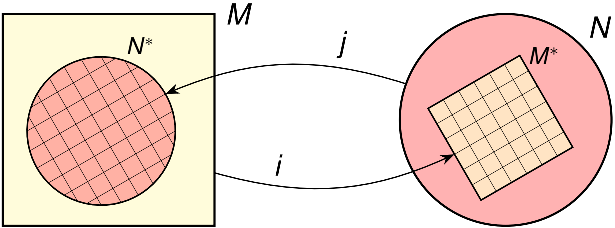

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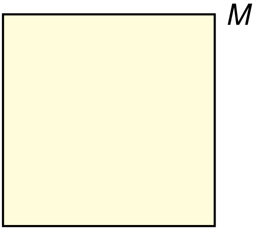

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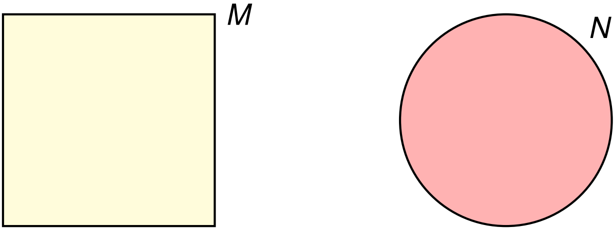

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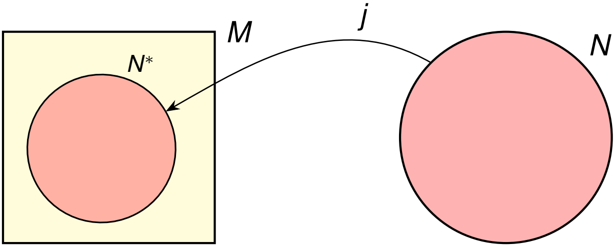

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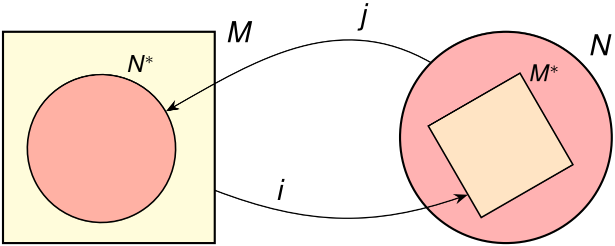

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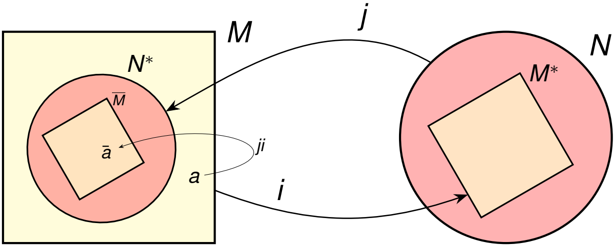

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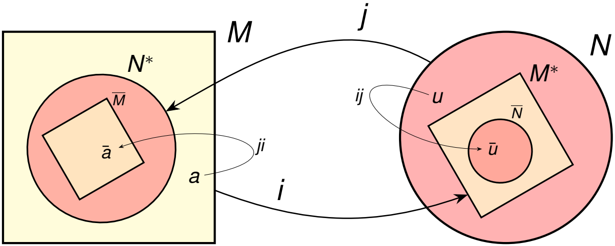

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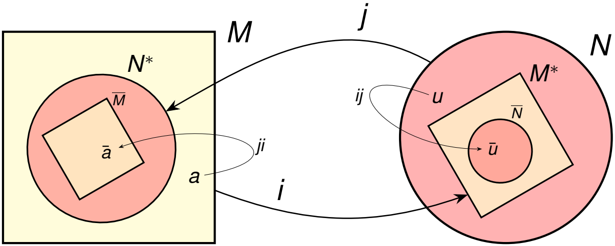
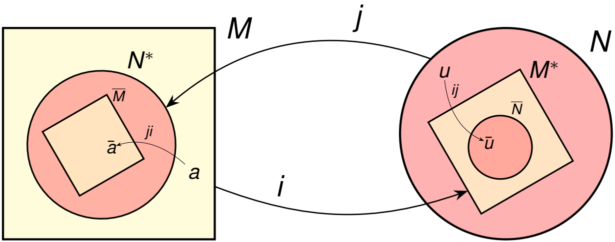


Each model is isomorphic to its iterated interpreted copy

$$ji : M \cong \overline{M} \qquad ij : N \cong \overline{N}.$$

Bi-interpretation

Models M and N are *bi-interpretable*, if they are mutually interpretable in such a way that the isomorphisms $ji : M \cong \overline{M}$ and $ij : N \cong \overline{N}$ arising in the interpretation are each definable in the original models.



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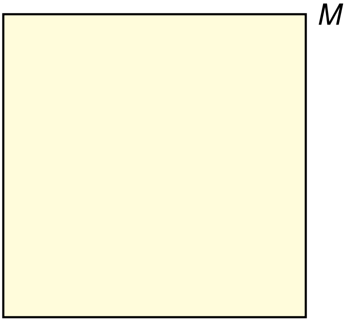

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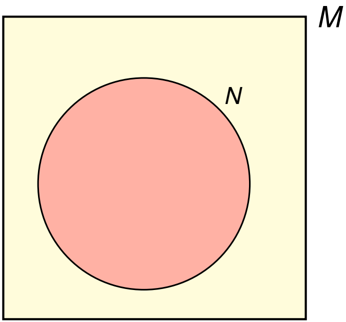

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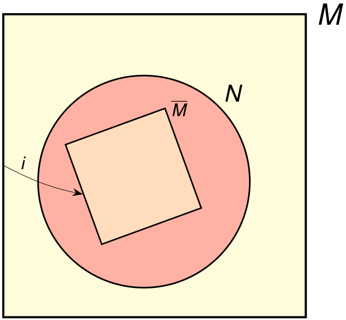

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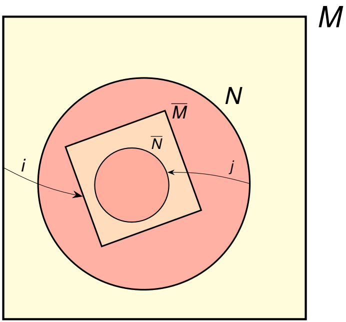

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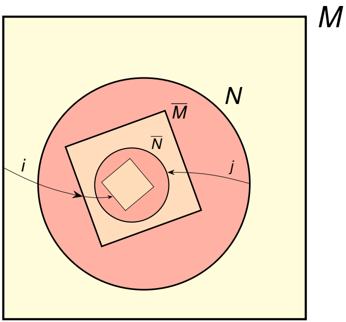

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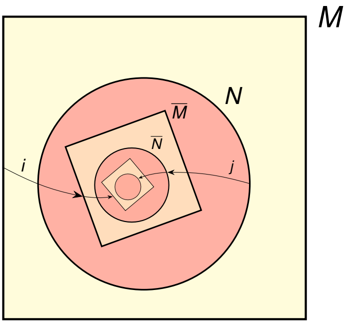

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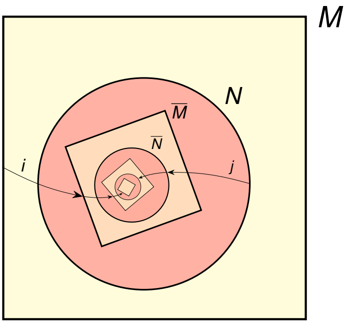

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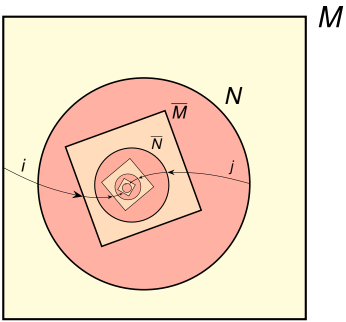

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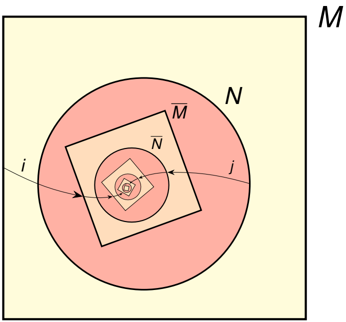

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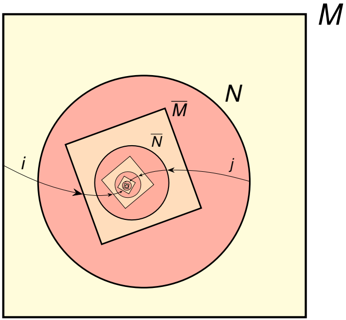

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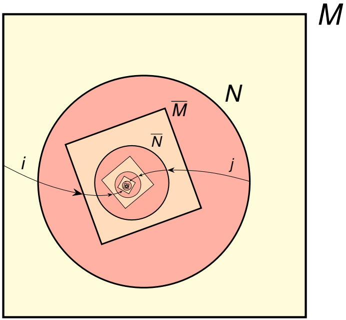

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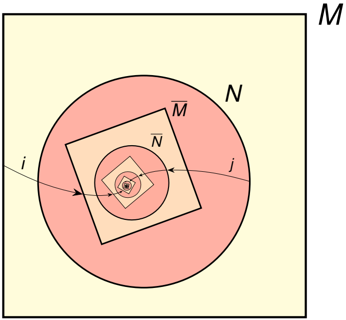

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Synonymy

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Every instance of bi-interpretation between models of ZF can be transformed to an instance of bi-interpretation synonymy.

- Don't need k-tuples, since can encode sequences internally.
- Don't need equivalence relations, by Scott's trick.
- Can use whole domain, by Cantor-Schröder–Bernstein theorem for classes.

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There should be $\mathcal{L}_2$-formulas defining a domain of k-tuples, defining interpretations of the $\mathcal{L}_1$ structure and defining an equivalence relation, which provide recursively a translation of the $\mathcal{L}_1$ assertions into the language of $\mathcal{L}_2$,

$$\varphi \mapsto \varphi^*$$

in such a way that

$$T_1 \vdash \varphi \qquad \Longrightarrow \qquad T_2 \vdash \varphi^*.$$

So theory T_2 proves that the interpretation is a model of T_1.

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For bi-interpretation, the theory T_1 proves that the universe is isomorphic, by a definable isomorphism map, to the model resulting by first interpreting to the defined model of T_2 and then interpreting to the model of T_1 inside that model; and similarly T_2 proves that its universe is definably isomorphic to the iterated interpreted model.

Interpretation in ZF set theory

There is an extremely robust mutual interpretability phenomenon in set theory.

Theorem

The following theories are pairwise mutually interpretable.

1. ZF
2. ZFC
3. ZFC + GCH
4. ZFC + $V = L$
5. ZF + ¬AC
6. ZFC + ¬CH
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And many corresponding theorems for theories of higher consistency strength.

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- $\mathrm{ZFC}+\mathrm{CH}$ is interpretable in ZF.
- $\mathrm{ZFC}+V=L_\mu$ is interpretable in $\mathrm{ZFC}+\exists$ measurable cardinal.

In each case, we can go to a definable inner model where the interpreted theory holds.

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Nevertheless, one can use forcing to define interpreted models by means of the Boolean ultrapower.

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Define Boolean ultrapower model $M^{\mathbb{B}}/U$, using

$$\begin{aligned} \sigma =_U \tau &\iff [\![\sigma = \tau]\!] \in U; \\ \sigma \in_U \tau &\iff [\![\sigma \in \tau]\!] \in U. \end{aligned}$$

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The Łoś theorem shows

$$M^{\mathbb{B}}/U \models \varphi \iff [\![\varphi]\!] \in U.$$

So this is a model of everything forced by $\mathbb{B}$.

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For example, in any model of ZFC, can define L and the forcing $\mathrm{Add}(\omega, \omega_2)^L$ and the L-least ultrafilter U on Boolean completion $\mathbb{B}$.

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Therefore, can define $L^{\mathbb{B}}/U$, which is a model of $\mathrm{ZFC} + \neg\mathrm{CH}$.

From mutual interpretation to bi-interpretation?

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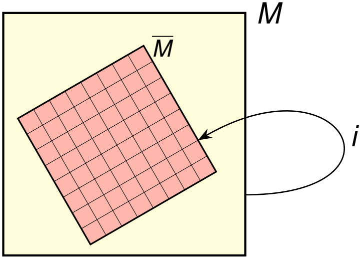
If not, does following an interpretation in set theory necessarily involve the loss of information?

Automatic bi-interpretability

Theorem

If a well-founded model M of ZF^- is interpreted in itself via $i : M \to \overline{M}/\simeq$, then i is unique and definable.



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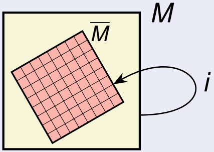

Proof.

Assume $\langle M, \in\rangle \models \mathrm{ZF}^-$ is interpreted in itself $i : \langle M, \in\rangle \cong \langle \overline{M}, \bar{\in}\rangle / \simeq$.

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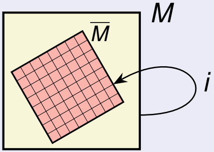

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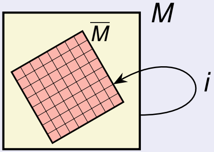
The relation $\bar{\in}$ is well-founded and extensional (modulo $\simeq$).

Theorem

If a well-founded model M of ZF^- *is interpreted in itself via* $i : M \to \overline{M}/\simeq$*, then* i *is unique and definable.*



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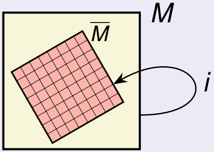

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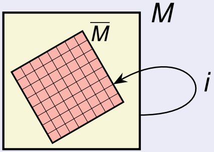

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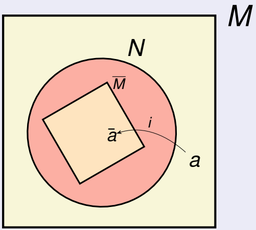
So the map is definable. □

Automatic bi-interpretability

Corollary

Every instance of mutual interpretation amongst well-founded models of ZF^- *is a bi-interpretation. Indeed, if M is a well-founded model of* ZF^- *and mutually interpreted with any structure N of any theory, as in the figure below, then the isomorphism $i : M \to \overline{M}$ is definable in M.*



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Theorem (Enayat [Ena16])

1. *Distinct non-isomorphic models of* ZF *are never bi-interpretable.* ZF *is solid.*
2. *Distinct theories extending* ZF *are never bi-interpretable.* ZF *is tight.*

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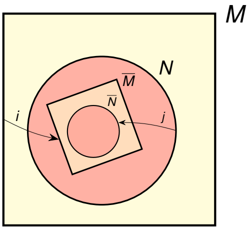

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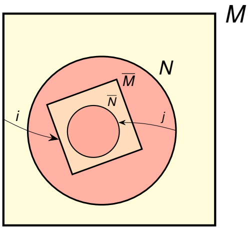

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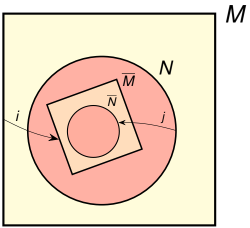

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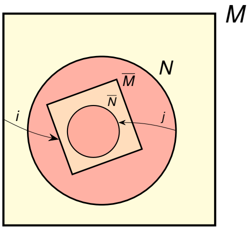

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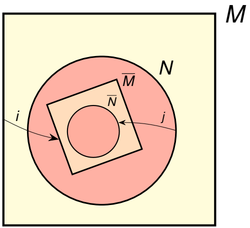

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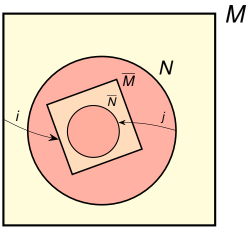

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And similarly if $\mathrm{Ord}^M < \mathrm{Ord}^N$. So $\langle M, \in^M\rangle \cong \langle N, \in^N\rangle$, as desired. $\square$

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ZF is tight. That is, distinct theories extending ZF are never bi-interpretable.

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There is no nontrivial bi-interpretation phenomenon in set theory amongst the models or theories strengthening ZF.

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- Result also follows from internal categoricity result of Väänänen [Vä19].

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Every instance of mutual interpretation amongst the well-founded models of ZF is a bi-interpretation, but bi-interpretation occurs only between isomorphic models. □

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One cannot get by interpretation back to the original model, even if one gets back to a model of the original theory.

Internal categoricity

Theorem (Väänänen [Vä19])

If $\langle V, \in, \bar{\in}\rangle$ is a model of $\mathrm{ZF}(\in, \bar{\in})$, then

$$\langle V, \in\rangle \cong \langle V, \bar{\in}\rangle.$$

Furthermore, there is a unique definable isomorphism in $\langle V, \in, \bar{\in}\rangle$.

The hypothesis asserts, more precisely:

- $\mathrm{ZF}_{\in}(\bar{\in})$, using $\in$ as membership and $\bar{\in}$ as predicate; and
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Zermelo's quasi-categoricity theorem

The internal categoricity argument is similar in important respects to Zermelo's 1930 quasi-categoricity argument, showing that for any two models of ZF_2, one of them is isomorphic to a rank-initial segment of the other.

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For theories where the synonymy methods work, therefore, one can view internal categoricity as a strengthening of solidity/tightness, dropping the definability requirements.

Fundamental use of the V_α hierarchy

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Do the results hold for ZFC^-, without power set?

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Can one prove tightness and internal categoricity for weak set theories?

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We shall provide the counterexample model.

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There is a transitive model $\langle M, \in, \bar{\in} \rangle \models \mathrm{ZFC}^-(\in, \bar{\in})$, where $\langle M, \in \rangle$ is not isomorphic to $\langle M, \bar{\in} \rangle$, both well-founded.

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For outright existence, omit Luzin via Shoenfield absoluteness. □

Nonsolidity of ZFC^-

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Both H_{ω_1} and H_{ω_2} can see how the coding works, and from this one can show it is a bi-interpretation. □

Achieving synonymy for H_{ω_1} and H_{ω_2}

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It is relatively consistent with ZFC *that there is relation* $\bar{\in}$ *definable in* $\langle H_{\omega_1}, \in\rangle$ *for which*

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This allows one to pick representatives, and avoid the quotient.

Meanwhile

In stronger large cardinal settings, however, we cannot expect to interpret H_{ω_2} inside H_{ω_1}.

Theorem

If there is no projectively definable ω_1-sequence of distinct reals, then $\langle H_{\omega_2}, \in\rangle$ cannot be interpreted in $\langle H_{\omega_1}, \in\rangle$. In particular, in this case the structures are not bi-interpretable nor even mutually interpretable.

The hypothesis is a consequence of sufficient large cardinals, since it is a consequence of $\mathrm{AD}^{L(\mathbb{R})}$.

ZFC^- is not solid

Can appeal to absoluteness to get the outright result, instead of mere consistency.

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The theory ZFC^- is not solid, not even for well-founded models. Indeed, there are transitive models $\langle M, \in\rangle$, $\langle N, \in\rangle$ of ZFC^- that form a bi-interpretation synonymy, but are not isomorphic.

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Proof.

There are such transitive sets in $L[G]$. Can find countable such sets. Apply Shoenfield absoluteness to get them in V. □

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Let T_1 and T_2 be theories describing the situation of $\langle H_{\omega_1}, \in\rangle$ and $\langle H_{\omega_2}, \in\rangle$ in the previous theorem.

So T_2 asserts ZFC$^-$ plus ω_1 exists but not ω_2, that $\omega_1 = \omega_1^L$, that $\omega_2 = \omega_2^L$, and that every subset of ω_1 is coded by a real using the almost-disjoint coding with respect to the L-least almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1\rangle$.

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These two theories are bi-interpretable, but incompatible. □

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Let's now consider Zermelo set theory Z.

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Theorem (Freire, Hamkins)

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1. *Z is not solid, not even for well-founded models. There are bi-interpretable well-founded models of Zermelo set theory that are not isomorphic.*
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2. *Z is not tight. There are distinct bi-interpretable strengthenings of Z.*
3. *Every model of ZF is bi-interpretable with a transitive inner model of Zermelo set theory, with prescribed failures of replacement.*

Mathias slim model technique

We use Mathias's slim model construction [Mat01].

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A class C is *fruitful*, if

1. every $x \in C$ is transitive;
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If C is fruitful, then $M = \bigcup C$ is a supertransitive model of Zermelo set theory with the foundation axiom.

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Key idea: construct fruitful classes by specifying allowed rate-of-growth $|x \cap V_n|$.

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One such slim model M has sets x obeying rate of growth

$$\exists k\ \forall n \qquad |\,\mathrm{TC}(x)\cap V_n| \leq 2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}\Big\}k.$$

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This slim model M is a transitive model of Zermelo with foundation, containing all ordinals, in which V_ω does not exist.

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We claim the original ZF model $\langle V, \in \rangle$ is bi-interpretable with the slim model M.

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$$\varnothing^{(a)} = a.$$
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We replace all hereditary copies of $\varnothing$ in x with a.

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Can define $V^{(a)}$ inside M: all $\in$-descents pass through a.

So this is a bi-interpretation of $\langle V, \in\rangle$ with $\langle M, \in\rangle$.

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Zermelo set theory is neither solid nor tight

We've proved that every ZF model $\langle V, \in\rangle$ is bi-interpretable with a model M of Zermelo set theory.

So Z is not solid.

Consider theories describing the situation. Let ZM assert Z plus the assertion that the Zermelo tower $V^{(\omega)}$ is a model of ZF, and that the universe M is isomorphic to $M^{(\omega)}$ by our map.

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These theories are different, but bi-interpretable, so Z is not tight.

Flexibility about which V_λ is excluded

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We can include V_ω and V_α for all α up to some desired limit ordinal λ, but V_λ is excluded.

Model-by-model bi-interpretation

Consider bi-interpretation in models vs. theories.

Definition

Theories T_1, T_2 are *model-by-model* bi-interpretable if every model of one is bi-interpretable with a model of the other.

In effect we drop the uniformity requirement on the interpretation.

It could be different interpretations that work in some models than in others, with perhaps no uniform interpretation.

Theorem

There are theories T_1 and T_2 that are model-by-model bi-interpretable, but not bi-interpretable.

Proof.

Consider the theories

1. $T_1 = \mathrm{ZF}$.
2. $T_2 = \mathrm{ZF} \vee \mathrm{ZM} = \{\alpha \vee \beta \mid \alpha \in \mathrm{ZF}, \beta \in \mathrm{ZM}\}$.

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Conversely, every model of T_2 is either a model of ZF or of ZM, which is bi-interpretable with a model of ZF.

But not bi-interpretable: let $M \models \mathrm{ZM} + \neg\mathrm{ZF}$, interpret $N \models \mathrm{ZF}$, hence T_2, so intepret further $N^* \models \mathrm{ZF}$. N and N^* bi-interpretable, hence isomorphic. But interpreting back to T_1 from N or N^* produces M and N, not isomorphic. Contradiction. □

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- But there is no nontrivial bi-interpretation for ZF and stronger.
- The moral: by following the mutual interpretations of set theory, you can never go back home.
- Meanwhile, bi-interpretation occurs in weak set theories, such as ZFC^- and Z.
- Even H_{ω_1} and H_{ω_2} can be bi-interpretable.
- Every ZF model is bi-interpretable with a slim Zermelo inner model.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
Oxford University

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