Dynamics of dissipative gravitational collapse

by

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Abstract

In this study we generate the matching conditions for a spherically symmetric radiating star in the presence of shear. Two new exact solutions to the Einstein field equations are presented which model a relativistic radiating sphere. We examine the role of anisotropy in the thermal evolution of a radiating star undergoing continued dissipative gravitational collapse in the presence of shear. Our model was the first study to incorporate both shear and pressure anisotropy, and these results were published in 2006. The physical viability of a recently proposed model of a shear-free spherically symmetric star undergoing gravitational collapse without the formation of a horizon is investigated. These original results were published in 2007. The temperature profiles of both models are studied within the framework of extended irreversible thermodynamics.

To

My parents,

for their love and encouragement.

Preface and Declaration

The study described in this dissertation was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban. This dissertation was completed under the supervision of Professor S D Maharaj and Dr M Govender.

The research contained in this dissertation represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

N F Naidu

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DECLARATION 1 - PLAGIARISM

I, Nolene Ferrari Naidu, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.
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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication)

Publication 1

Naidu N F, Govender M, and Govinder K S, Thermal evolution of a radiating anisotropic star with shear, *Int. J. Mod. Phys. D* **15**, 1053 (2006).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written my myself with some input from my supervisor.)

Publication 2

Naidu N F, and Govender M, Causal temperature profiles in horizon-free collapse, *J. Astrophys. Astr* **28**, 167 (2007).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written my myself with some input from my supervisor.)

Signed:

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Contents

Chapter 1

Introduction

Gravitational collapse of massive stars, and the possibilities for the end state of such a continued collapse, are outstanding problems in relativistic astrophysics and gravitation theory today. Once a massive star has exhausted its thermonuclear source of energy, it enters the state of endless gravitational collapse. It is thus crucial to study the possible conclusions of this collapse, dominated entirely by the force of gravity. Interest in this area started in 1939, when Oppenheimer and Snyder studied the problem of gravitational collapse in which a spherically symmetric dust cloud underwent continued collapse. Particularly interesting is the fact that general relativity generically admits the existence of spacetime singularities. These are extreme regions in the spacetime where densities and spacetime curvatures diverge and the theory must break down.

The Cosmic Censorship Conjecture proposed by Penrose in 1969, which states that any reasonable matter configuration undergoing continued gravitational collapse will form a black hole – and never a naked singularity, has attracted much attention in relativistic astrophysics. Joshi in 2002 assessed the situation due to several models that proposed naked singularities or black holes as the final outcome of collapse. The non-existence of trapped surfaces till the formation of the singularity in collapse was thought to be the signature of naked singularities. Penrose (1998) and Wald (1997) showed that this need not be the case. A singularity being naked means that there exist families of future directed non-spacelike curves, which in the past terminate at the singularity. No such families exist originating from the singularity when the product of collapse is a black hole. For black hole formation, the resultant spacetime singularity is hidden inside an event horizon of gravity, remaining unseen by external observers. For naked singularity formation, however, there is a causal connection between the region of singularity and distant observers, thus enabling communication from the superdense regions close to the singularity to distant observers. Joshi (2002) listed the physical conditions that could support the cosmic censorship conjecture and prevent the formation of a naked singularity as the end state of gravitational collapse. The conditions are: (i) A suitable energy condition must be obeyed, (ii) the collapse must develop from regular initial data, (iii) singularities from realistic collapse must be gravitationally strong (divergence of all important physical quantities such as pressure, density, curvature, etc.), (iv) the matter fields must be sufficiently general, (v) a realistic equation of state must be obeyed, (vi) all radiations from naked singularity must be infinitely red-shifted. Hence the final outcome of stellar gravitational collapse is still very much open to debate, primarily due to models that admit naked singularities (Harada *et al* 1998, Kudoh *et al* 2000). For the interior of the star forms of the energy momentum tensor, ranging from a perfect fluid to an imperfect fluid with heat flux and anisotropic pressure have been investigated (Herrera and Santos 1997a, Naidu *et al* 2006). It is well known that when a reasonable matter distribution undergoes gravitational collapse, in the absence of shear or with homogeneous density, the end result is a black hole. Shear has been identified as the factor that delays the formation of the apparent horizon, by making the final stages of collapse incoherent thereby leading to the generation of naked singularities (Joshi *et al* 2002, Goncalves 2008). Wagh and Govinder (2001) showed that all known naked singularites in spherically symmetric self-similar spacetimes arise as a result of singular initial matter distribution. This is a result of the peculiarity of the coordinate transformation that takes these spacetimes into a separable form. Such examples of naked singularites are therefore of no apparent consequence to astrophysics. Banerjee and Chatterjee (2004) showed that the non-occurence of a horizon is due to the fact that the rate of mass loss is exactly counterbalanced by the fall of boundary radius. This is a counter example to the cosmic censorship conjecture. Also observed was that the rate of collapse is delayed with the introduction of extra dimensions.

The first exact solution to the Einstein field equations was presented by Karl Schwarzschild in 1916. He considered the matter of a static spherically symmetric star to be a perfect fluid with constant density, thus simplifying the problem significantly. An important assumption made was that the pressure vanishes on the boundary of the star. In the Oppenheimer and Snyder model of 1939, the exterior spacetime was described by the exterior Schwarzschild solution. Vaidya derived the first exact solution for the exterior gravitational field of a spherically symmetric radiating object (Vaidya 1951). Thereafter, the junction conditions for the interior of a shear-free radiating star were derived by Santos in 1985; and various models of radiating stars could then be studied. Hundreds of solutions incorporating shear, pressure anisotropy, charge, and several matter distributions have been found since then (Herrera and Santos 1997a, Naidu *et al* 2006).

This dissertation is organized as follows:

- Chapter 1: Introduction.

- Chapter 2: In this chapter we provide an overview of the mathematics and differential geometry of general relativity essential to construct stellar models of gravitational collapse. The Einstein field equations are presented for both the spherically symmetric shearing metric, and the shear-free metric.

- Chapter 3: The junction conditions for the smooth matching of two spherically symmetric spacetimes on a timelike hypersurface are presented in this chapter. We derive the junction conditions for both the shearing and the shear-free line elements. These results are essential for subsequent chapters.

- Chapter 4: To obtain a model of a radiating star undergoing gravitational collapse and dissipating energy in the form of heat flux, the thermodynamics must be studied in detail. In this chapter we present an overview of Eckart's theory of thermodynamics, and discuss the problems with this theory. These problems can be avoided by taking a causal approach (Maartens 1996b). Certain aspects of causal thermodynamics relevant to this study are presented and discussed. We also include the Maxwell-Cattaneo equation for heat transport and motivate for the particular form for the mean collision time.

- Chapter 5: We present an exact model for a radiating anisotropic star undergoing gravitational collapse in the presence of shear. The Einstein field equations are solved exactly, and a detailed analysis of the thermodynamics is performed.

- Chapter 6: In this chapter we investigate the physical viability of a recently proposed model of a radiating star undergoing dissipative gravitational collapse without the formation of a horizon. The thermodynamics are studied in detail.

- Chapter 7: Conclusion

Chapter 2

Preliminaries

2.1 Introduction

Einstein's theory of general relativity describes the interaction between matter and the geometry of spacetime. This theory has been successful in its predictions and has been shown to be consistent with observations. In this chapter, we provide the basic framework of this theory in order to develop the mathematical background necessary to generate a stellar model. An outline of the essential differential geometry and the field equations of general relativity are presented in §2.2. In §2.3 we consider spherically symmetric metrics which describe the interior of a radiating star. The outgoing Vaidya line element, which describes the exterior spacetime of a radiating spherically symmetric star, is introduced in §2.4.

2.2 Field equations

The general theory of relativity was proposed by Albert Einstein in 1914 to describe gravity. The fundamental idea in this theory is that matter and the geometry of spacetime are intimately related, i.e., matter curves spacetime. In this metric theory of gravitation, spacetime is represented by a four–dimensional, differentiable manifold endowed with an indefinite metric tensor **g** with signature $(-+++)$. Local coordinates in the manifold are chosen so that $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike, and x^1, x^2, x^3 are spacelike. The invariant distance between two infinitesimally separated points in spacetime is given by the line element

$$ds^2 = g_{ab}dx^a dx^b$$

The metric connection Γ is defined in terms of the metric tensor field and its derivatives by

$$\Gamma^a{}_{bc} = \tfrac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2.1}$$

where commas denote partial differentiation. The fundamental theorem of Riemannnian geometry ensures the existence of a unique metric connection which preserves inner products under parallel transport. The Riemann tensor is constructed from the connection coefficients (2.2.1) as follows

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.2.2}$$

which provides a measure of the curvature of spacetime. The Ricci tensor is obtained from contracting the Riemann tensor (2.2.2):

$$R_{ab} = \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^e{}_{ab}\Gamma^d{}_{ed} - \Gamma^e{}_{ad}\Gamma^d{}_{eb} \tag{2.2.3}$$

A further contraction of the Ricci tensor (2.2.3) yields the Ricci scalar

$$R = R^a{}_a \tag{2.2.4}$$

The Einstein tensor is given by

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} \tag{2.2.5}$$

which obeys

$$G^{ab}{}_{;b} = 0 \tag{2.2.6}$$

by the contracted Bianchi identity. Semicolons denote covariant differentiation.

The energy–momentum tensor **T** is decomposed as follows

$$T_{ab} = T^{\mathrm{h}}{}_{ab} + T^{\mathrm{i}}{}_{ab} + T^{\mathrm{e}}{}_{ab} + T^{\mathrm{n}}{}_{ab} \tag{2.2.7}$$

which is based on the treatment of Krasiński (1997). We consider each term in (2.2.7) separately:

- The component

  $$T^{\mathrm{h}}{}_{ab} = (\rho + p)u_a u_b + p g_{ab}$$

  represents the dynamically isotropic perfect fluid where $\mathbf{u}$ is a unit timelike four–velocity vector, ρ is the energy density and p is the isotropic pressure.

- The component

  $$T^{\mathrm{i}}{}_{ab} = q_a u_b + q_b u_a + \pi_{ab}$$

  represents the dynamically anisotropic stress energy tensor where q_a is the heat flow vector and π_{ab} is the trace-free anisotropic stress tensor. The heat flow vector and stress tensor satisfy the conditions

  $$\begin{aligned} q^a u_a &= 0 \\ \pi^{ab} u_b &= 0 \end{aligned}$$

  relative to the fluid four–velocity $\mathbf{u}$.

- The component

  $$T^{\mathrm{e}}{}_{ab} = F_a{}^c F_{bc} - \tfrac{1}{4} g_{ab} F^{cd} F_{cd}$$

  represents the electromagnetic contribution to $\mathbf{T}$ and F_{ab} is the electromagnetic field tensor. We can express the electromagnetic field tensor as

  $$F_{ab} = \phi_{b;a} - \phi_{a;b}$$

where ϕ_a is the four–potential. Maxwell's equations, governing the behaviour of the electromagnetic field, are

$$F^{ab}{}_{;b} = J^a \tag{2.2.8a}$$

$$F_{[ab;c]} = 0 \tag{2.2.8b}$$

In the above

$$J^a = \mu u^a$$

is the four–current density and μ is the proper charge density.

- The component

  $$T^{\mathrm{n}}{}_{ab} = \epsilon w_a w_b$$

  is the energy–momentum tensor for null radiation where ϵ is the radiation energy density and w_a is the null four-vector.

Particular forms of the energy–momentum tensor (2.2.7) are used in later chapters.

The Einstein field equations can be written as

$$G_{ab} = T_{ab} \tag{2.2.9}$$

in appropriate units. Throughout this thesis we utilise standard geometrical units in which the coupling constant $8\pi G/c^4$ and the velocity of light c are taken to be unity. The Einstein field equations (2.2.9) couples the curvature of spacetime (2.2.5) to the matter content (2.2.7). A solution of this system of ten highly nonlinear partial differential equations is necessary to study the gravitational behaviour of a gravitating system. For a more extensive and detailed discussion of differential geometry and manifold structure applicable to general relativity see de Felice and Clarke (1990), Hawking and Ellis (1973) and Joshi (1993).

2.3 Spherical spacetimes

2.3.1 Shearing spacetimes

The most general spherically symmetric line element, in spherical coordinates $(x^a) = (t, r, \theta, \phi)$, can be written as

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3.1}$$

where A, B and Y are functions of the coordinates t and r. The fluid four–velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0$$

Note that the coordinates utilised in (2.3.1) are not isotropic. The kinematical quantities for the line element (2.3.1) are given by

$$\omega_{ab} = 0 \tag{2.3.2a}$$

$$\dot{u}^a = \left(0, \frac{A'}{A}, 0, 0\right) \tag{2.3.2b}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right) \tag{2.3.2c}$$

$$\sigma^1_1 = \sigma^2_2 = -\tfrac{1}{2}\sigma^3_3 = \frac{1}{3A}\left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B}\right) \tag{2.3.2d}$$

where ω_{ab} is the vorticity tensor, $\dot{u}^a$ is the four–acceleration vector, Θ is the expansion scalar and σ represents the magnitude of the shear (or the rate of shear) of the fluid. For a relativistic fluid the kinematical quantities are important for studying the evolution of the system. Dots and primes represent differentiation with respect to t and r respectively.

The nonzero Ricci tensor components (2.2.3) assume the following form

$$R_{00} = -\frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y} - 2\frac{\ddot{Y}}{Y}$$

$$+\frac{A^2}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + 2\frac{A'}{A}\frac{Y'}{Y}\right) \tag{2.3.3a}$$

$$R_{01} = 2\left(\frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y}\right) \tag{2.3.3b}$$

$$R_{11} = -\frac{A''}{A} + \frac{A'}{A}\frac{B'}{B} + 2\frac{B'}{B}\frac{Y'}{Y} - 2\frac{Y''}{Y}$$

$$+\frac{B^2}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y}\right) \tag{2.3.3c}$$

$$R_{22} = \frac{Y\dot{Y}}{A^2}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{\dot{Y}}\right)$$

$$+\frac{YY'}{B^2}\left(\frac{B'}{B} - \frac{A'}{A} - \frac{Y'}{Y} - \frac{Y''}{Y'}\right) + 1 \tag{2.3.3d}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.3.3e}$$

for the line element (2.3.1). The Ricci scalar is given by

$$R = \frac{2}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - 2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + 2\frac{\ddot{Y}}{Y}\right)$$

$$- \frac{2}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} - 2\frac{B'}{B}\frac{Y'}{Y} + 2\frac{A'}{A}\frac{Y'}{Y} + \frac{Y'^2}{Y} + 2\frac{Y''}{Y}\right) + \frac{2}{Y^2} \tag{2.3.4}$$

for the line element (2.3.1). By making use of (2.3.3) and (2.3.4) we obtain the nonvanishing Einstein tensor components

$$G_{00} = 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - \frac{A^2}{B^2}\left(-2\frac{B'}{B}\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{A^2}{Y^2} \tag{2.3.5a}$$

$$G_{11} = \frac{B^2}{A^2}\left(2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - 2\frac{\ddot{Y}}{Y}\right) + 2\frac{A'}{A}\frac{Y'}{Y} + \frac{Y'^2}{Y^2} - \frac{B^2}{Y^2} \tag{2.3.5b}$$

$$G_{01} = 2\left(\frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y}\right) \tag{2.3.5c}$$

$$G_{22} = -\frac{Y^2}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - \frac{\dot{A}}{A}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) + \frac{Y^2}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{Y'}{Y} - \frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right) \tag{2.3.5d}$$

$$G_{33} = \sin^2\theta G_{22} \tag{2.3.5e}$$

for spherical symmetry. The Einstein field equations are

$$\rho = \frac{2}{A^2}\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{1}{A^2}\frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right) \tag{2.3.6a}$$

$$p_R = \frac{1}{A^2}\left(-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y'^2}{Y^2} + 2\frac{A'}{A}\frac{Y'}{Y}\right) - \frac{1}{Y^2} \tag{2.3.6b}$$

$$\begin{aligned} p_\perp = & -\frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - \frac{\dot{A}}{A}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \\ & + \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{Y'}{Y} - \frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right) \end{aligned} \tag{2.3.6c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{\dot{Y}}{Y}\right) \tag{2.3.6d}$$

where we have used (2.3.5) and (2.2.7). In the above we have defined

$$q = q^1 \tag{2.3.7}$$

where $q^a = (0, q^1, 0, 0)$ has only a radial component.

The last equation (2.3.6d) can be written as

$$qB^2 = \frac{2}{3}(\Theta - \sigma)' - 2\sigma\frac{Y'}{Y} \tag{2.3.8}$$

using the formalism of Herrera *et al* (2008). We introduce a useful parameter Δ which is a measure of the pressure anisotropy:

$$\Delta = p_R - p_\perp \tag{2.3.9}$$

The field equations (2.3.6) describe the interaction of a shearing matter distribution which is expanding and accelerating with heat flux. In order to describe a physically reasonable model, the metric functions A, B and Y must be determined. The 'unknowns' include ρ, p, q, A, B and Y. In order to close the system, we have to either employ a suitable equation of state, or make a simplifying assumption such as: (i) the Weyl tensor vanishes, (ii) the model is acceleration-free, or (iii) the collapse is expansion-free.

2.3.2 Shear-free spacetimes

In the case of vanishing shear we have

$$\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B} = 0$$

from (2.3.2d) so that the line element (2.3.1) may be rewritten as

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.3.10}$$

which is simultaneously comoving and isotropic with $Y = rB$. Here A and B are functions of the coordinates t and r. The fluid four–velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0$$

For the line element (2.3.10) the kinematical quantities are

$$\omega_{ab} = 0 \tag{2.3.11a}$$

$$\dot{u}^a = \left(0, \frac{A'}{A}, 0, 0\right) \tag{2.3.11b}$$

$$\Theta = \frac{3\dot{B}}{AB} \tag{2.3.11c}$$

$$\sigma = 0 \tag{2.3.11d}$$

where ω_{ab} is the vorticity tensor, $\dot{u}^a$ is the four–acceleration vector, Θ is the expansion scalar and σ represents the magnitude of the shear (or the rate of shear) of the fluid.

The Einstein field equations for the line element (2.3.10) reduce to

$$\rho = \frac{3}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) \tag{2.3.12a}$$

$$p = \frac{1}{A^2}\left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right) + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right) \tag{2.3.12b}$$

$$p = -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} + \frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3} \tag{2.3.12c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) \tag{2.3.12d}$$

where we have assumed the pressure is isotropic.

We can rewrite (2.3.12d) as

$$qB^2 = \frac{2}{3}\Theta' \tag{2.3.13}$$

which is consistent with (2.3.8) when $\sigma = 0$. The field equations (2.3.12) describe the gravitational interaction of a shear–free matter distribution with heat flux and vanishing electromagnetic field. The condition for pressure isotropy (2.3.9) becomes

$$\frac{A''}{A} + \frac{B''}{B} = \left(2\frac{B'}{B} + \frac{1}{r}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) \tag{2.3.14}$$

from (2.3.12b) and (2.3.12c). We note that in this case, the anisotropic parameter $\Delta = 0$ since $p_R = p_\perp = p$. The above system of equations will be utilised in subsequent chapters to generate simple analytic models of radiative gravitational collapse.

2.4 Vaidya spacetime

The exterior of a radiating star is described by the Vaidya solution (Vaidya 1951, 1953). This solution may be generated from the exterior Schwarzchild solution by utilizing the Eddington–Finkelstein coordinate transformation, although this was not the original approach taken by Vaidya. For details of the coordinate transformation see Govender (1994).

The Vaidya line element is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.4.1}$$

where $m(v)$ is the Newtonian mass of the gravitating body as measured by an observer at infinity. The Vaidya solution is the unique spherically symmetric solution of the Einstein field equations (2.2.9) for radiation in the form of a null fluid, and is often used to describe the exterior gravitational field of a radiating star in applications. Some authors who have employed the Vaidya solution in applications include Kolassis *et al* (1988), de Oliveira *et al* (1985), and Kramer (1992). It is interesting to note that the Vaidya solution (3.3.4) is completely determined by the mass function $m(v)$. To ensure the exterior spacetime of the radiating star is physically reasonable, $m(v)$ must be a nonincreasing function, i.e., $\frac{dm}{dv} \leq 0$. Physically, we may conclude that the mass of the star is decreasing. This is because energy is being carried away in the form of radiation.

Chapter 3

Junction conditions

3.1 Introduction

In this chapter we review the junction conditions that match two spherically symmetric spacetimes across a timelike hypersurface Σ (Bonnor *et al* 1989, Santos 1985). We only provide an outline which is relevant for later chapters. The results obtained in this chapter are valid for the Einstein field equations (2.3.6) with a vanishing cosmological constant.

3.2 Matching hypersurfaces

We consider a spherical surface described by a timelike three–space Σ. The surface Σ divides the manifold into the two distinct regions $\mathcal{M}^-$ and $\mathcal{M}^+$. We take g_{ij} to be the intrinsic metric to Σ which enables us to write

$$ds^2_\Sigma = g_{ij} d\xi^i d\xi^j \tag{3.2.1}$$

The intrinsic coordinates to Σ are given by ξ^i where $i = 1, 2, 3$. The line elements in the regions $\mathcal{M}^\pm$ are given by

$$ds^2{}_\pm = g_{ab} d\mathcal{X}^a_\pm d\mathcal{X}^b_\pm \tag{3.2.2}$$

The coordinates in $\mathcal{M}^\pm$ are $\mathcal{X}^a_\pm$ where $a = 0, 1, 2, 3$. It is necessary that the metrics (3.2.1) and (3.2.2) match smoothly across Σ. This requirement generates the first junction condition in the form

$$(ds^2_-)_\Sigma = (ds^2_+)_\Sigma = ds^2_\Sigma \tag{3.2.3}$$

We are using the notation $(\ )_\Sigma$ to represent the value of $(\ )$ on the surface Σ. Consequently the coordinates of Σ in $\mathcal{M}^\pm$ are given by $\mathcal{X}^a_\pm = \mathcal{X}^a_\pm(\xi^i)$. The second junction condition is generated by requiring that the extrinsic curvature of Σ is continuous across the boundary. This requirement generates the second junction condition

$$K^+_{ij} = K^-_{ij} \tag{3.2.4}$$

where

$$K^\pm_{ij} \equiv -n^\pm_a \frac{\partial^2 \mathcal{X}^a_\pm}{\partial \xi^i \partial \xi^j} - n^\pm_a \Gamma^a{}_{cd} \frac{\partial \mathcal{X}^c_\pm}{\partial \xi^i} \frac{\partial \mathcal{X}^d_\pm}{\partial \xi^j} \tag{3.2.5}$$

and $n^\pm_a(\mathcal{X}^b_\pm)$ are the components of the vector normal to the surface Σ. Different forms for the first and second junction conditions have been obtained by other researchers. We should point out that the junction conditions (3.2.3) and (3.2.4) given above are equivalent to the junction conditions generated earlier by Lichnerowicz (1955) and O' Brien and Synge (1952). Lake (1987) provides a comprehensive review of the junction conditions for boundary surfaces and surface layers with several applications to general relativity and cosmology.

3.3 Shearing junction conditions

This section provides a summary of crucial results obtained by Govender (1998). The intrinsic metric to Σ is given by

$$ds^2_\Sigma = -d\tau^2 + \mathcal{R}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3.1}$$

with coordinates $\xi^i = (\tau, \theta, \phi)$ and $\mathcal{R} = \mathcal{R}(\tau)$. Note that the time coordinate τ is defined only on the surface Σ. In comoving coordinates we take the interior spacetime $\mathcal{M}^-$ to be given by the shearing line element (2.3.1):

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3.2}$$

For $\mathcal{M}^-$ the first junction condition (3.2.3), for the metrics (3.3.1) and (3.3.2), yields the restrictions

$$A(r_\Sigma, t)t_\tau = 1 \tag{3.3.3a}$$

$$Y(r_\Sigma, t) = \mathcal{R}(\tau) \tag{3.3.3b}$$

where differentiation with respect to τ is represented by subscripts.

We take the exterior spacetime $\mathcal{M}^+$ to be described by the Vaidya line element

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.3.4}$$

For $\mathcal{M}^+$ the first junction condition (3.2.3) for the line elements (3.3.1) and (3.3.4), generates the equations

$$\mathsf{r}_\Sigma(v) = \mathcal{R}(\tau) \tag{3.3.5a}$$

$$\left(1 - \frac{2m}{\mathsf{r}} + 2\frac{d\mathsf{r}}{dv}\right)_\Sigma = \left(\frac{1}{{v_\tau}^2}\right)_\Sigma \tag{3.3.5b}$$

The intermediary variable τ may be eliminated from these equations. Thus the necessary and sufficient conditions on the spacetimes which ensure the validity of the first junction condition (3.2.3) are that

$$A(r_\Sigma, t)dt = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{\frac{1}{2}} dv \tag{3.3.6a}$$

$$Y(r_\Sigma, t) = \mathsf{r}_\Sigma(v) \tag{3.3.6b}$$

The second junction condition (3.2.4) is obtained by equating the appropriate extrinsic curvature components. This gives

$$\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma = \left(\frac{v_{\tau\tau}}{v_\tau} - v_\tau\frac{m}{\mathsf{r}^2}\right)_\Sigma \tag{3.3.7a}$$

$$\left(\frac{YY'}{B}\right)_\Sigma = (v_\tau(\mathsf{r}-2m)+\mathsf{r}\mathsf{r}_\tau)_\Sigma \tag{3.3.7b}$$

The junction conditions (3.3.7) may be expressed in a form which is equivalent but more convenient for applications. We can generate an equation for the mass function $m(v)$ in terms of the metric functions only, from (3.3.7b) by eliminating r, r_τ and v_τ. In addition, we may rewrite relation (3.3.7b), with the help of (3.3.3) and (3.3.5), as

$$m(v) = \left[\frac{Y}{2}\left(1+\frac{\dot{Y}^2}{A^2}-\frac{Y'^2}{B^2}\right)\right]_\Sigma \tag{3.3.8}$$

We may interpret $m(v)$ as representing the total gravitational mass within the surface Σ. The expression (3.3.8) corresponds to the mass function of Cahill and McVittie (1970) (also see Hernandez and Misner 1966) for spheres of radius r inside Σ.

From (3.3.3) and (3.3.5a) we can write

$$(\mathsf{r}_\tau)_\Sigma = \left(\frac{\dot{Y}}{A}\right)_\Sigma$$

Using this expression for $(\mathsf{r}_\tau)_\Sigma$ and on substituting (3.3.8) in (3.3.7b) we have that

$$(v_\tau)_\Sigma = \left(\frac{\dot{Y}}{A}+\frac{Y'}{B}\right)^{-1}_\Sigma \tag{3.3.9}$$

If we now differentiate (3.3.9) with respect to τ and make use of (3.3.3a) we can write

$$(v_{\tau\tau})_\Sigma = \left[-\frac{1}{A}\left(\frac{\dot{Y}}{A}+\frac{Y'}{B}\right)^{-2}\left(\frac{\dot{Y}'}{B}-\frac{\dot{B}Y'}{B^2}-\frac{\dot{A}\dot{Y}}{A^2}+\frac{\ddot{Y}}{A}\right)\right]_\Sigma \tag{3.3.10}$$

Substituting (3.3.3b), (3.3.5a), (3.3.8), (3.3.9) and (3.3.10) into (3.3.7a) we obtain

$$\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma = \left[\left(-\frac{\dot{Y}'}{B}+\frac{\dot{B}Y'}{B^2}+\frac{\dot{A}\dot{Y}}{A^2}-\frac{\ddot{Y}}{A}-\frac{\dot{Y}^2}{2AY}+\frac{A}{2Y}\left(\frac{Y'^2}{B^2}-1\right)\right)\times \left(\frac{\dot{Y}}{A}+\frac{Y'}{B}\right)^{-1}\right]_\Sigma$$

On multiplying this equation by $\left(\frac{\dot{Y}}{A} + \frac{Y'}{B}\right)$ and simplifying we obtain the following result

$$(p_R)_\Sigma = (Q)_\Sigma$$

where we have utilised the field equations (2.3.6b) and (2.3.6d). This is an important result which relates the radial pressure p_R to the magnitude of the heat flow $Q = qB$. It was first established by Santos in 1985 for shear-free matter. The necessary and sufficient conditions on the spacetimes for the second junction condition (3.2.4) to be valid are that

$$m(v) = \left[\frac{Y}{2}\left(1 + \frac{\dot{Y}^2}{A^2} - \frac{Y'^2}{B^2}\right)\right]_\Sigma \qquad (3.3.11a)$$

$$(p_R)_\Sigma = (Q)_\Sigma \qquad (3.3.11b)$$

across the boundary.

The equations (3.3.6) and (3.3.11) are the general matching conditions for the spherically symmetric spacetimes $\mathcal{M}^+$ and $\mathcal{M}^-$. Relation (3.3.11b) implies that the isotropic pressure p_R is proportional to the magnitude of the heat flow q which is nonvanishing in general. The pressure p_Σ on the boundary can only be zero when q_Σ becomes zero. In this case there is no radial heat flow and the exterior spacetime consequently is not the Vaidya spacetime but is the exterior Schwarzschild spacetime.

A physical interpretation of (3.3.11b) is obtained by considering the radial momentum flux across the boundary. As the expression (3.3.8) also gives the total energy for a sphere of radius r within Σ we can write $m(v) = m(t, r)$. On differentiating partially with respect to t we obtain

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = \left[\dot{Y}\left(\frac{\ddot{Y}Y}{A^2} + \frac{\dot{Y}^2}{2A^2} - \frac{Y'^2}{2B^2} - \frac{\dot{A}\dot{Y}Y}{A^3} + \frac{1}{2}\right)\right.$$

$$\left. -\frac{Y'\dot{Y}'Y}{B^2} + \frac{\dot{B}Y'^2Y}{B^3}\right]_\Sigma \tag{3.3.12}$$

On using the field equations (2.3.6b) and (2.3.6d) we can rewrite $\left(\frac{\partial m}{\partial t}\right)_\Sigma$ as

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = \left[-\frac{AY^2}{2}\left(\frac{\dot{Y}}{A} + \frac{Y'}{B}\right)p_R\right]_\Sigma \tag{3.3.13}$$

The radial coordinate is comoving with respect to Σ so we can write

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = \left(\frac{dm}{dt}\right)_\Sigma = \left(\frac{v_\tau}{t_\tau}\frac{dm}{dv}\right)_\Sigma \tag{3.3.14}$$

Utilizing (3.3.3a), (3.3.5a), (3.3.13) and (3.3.14) we obtain

$$\left(-\frac{2}{\mathsf{r}^2}\frac{dm}{dv}{v_\tau}^2\right)_\Sigma = (p_R)_\Sigma \tag{3.3.15}$$

The radial flux of momentum of the radiation on both sides of Σ is given by

$$F^\pm = e_0^{\pm a} n^{\pm b} T^\pm_{ab}$$

The unit tangent vectors in the τ–direction of Σ are given by

$$e_0^{+a} = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{-\frac{1}{2}}\left(\delta^a_0 + \frac{d\mathsf{r}_\Sigma}{dv}\delta^a_1\right)$$

$$e_0^{-a} = {A_\Sigma}^{-1}\delta^a_0$$

For details of this result see Lindquist *et al* (1965). It is easy to show that

$$F^+ = \left[\frac{2}{\mathsf{r}^2}\frac{dm}{dv}{v_\tau}^2\right]_\Sigma$$

$$F^- = [-Q]_\Sigma$$

so that $F^+ = F^-$ which is equivalent to the junction condition (3.3.11b). Note that Q represents the magnitude of the heat flow. Therefore the result (3.3.11b) corresponds to the continuity of the radial flux of momentum of the radiation across the surface Σ, i.e., it expresses the local conservation of momentum.

We will now discuss the luminosity of the star. Lindquist *et al* (1965) define the total luminosity for an observer at rest at infinity by

$$L_\infty(v) = -\frac{dm}{dv} = \lim_{\mathsf{r}\to\infty} 4\pi \mathsf{r}^2\Phi^2 \tag{3.3.16}$$

An observer with four–velocity $v^a = (v_\tau, \mathsf{r}_\tau, 0, 0)$ located on Σ has proper time τ related to the time t by $d\tau = Adt$. The energy density that this observer measures on Σ is

$$\Phi^2{}_\Sigma = \frac{1}{4\pi}\left(-\frac{{v_\tau}^2}{\mathsf{r}^2}\frac{dm}{dv}\right)_\Sigma$$

and the luminosity observed on Σ is

$$L_\Sigma = 4\pi\mathsf{r}^2\Phi^2{}_\Sigma$$

The boundary redshift z_Σ of the radiation emitted by the star is given by

$$1 + z_\Sigma = \frac{dv}{d\tau}$$

which can be used to determine the time of formation of the horizon. The above expressions allow us to write

$$1 + z_\Sigma = \left(\frac{L_\Sigma}{L_\infty}\right)^{\frac{1}{2}} \tag{3.3.17}$$

which relates the luminosities L_Σ to L_∞ via the surface redshift.

Note that the result (3.3.11b) has been established in general for spherically symmetric, shearing spacetimes without assuming any particular forms for the metric functions. In addition, our expressions with nonzero shear have the same form as the expressions in Lindquist *et al.* However the contribution of the shear (σ_{ab}) is introduced via the metric functions in the definition of the mass function m in (3.3.8). The junction conditions for the smooth matching of a spherically symmetric line element to the Vaidya exterior have been extensively studied by various authors (Herrera 2006, Govender and Maharaj 1999). The junction conditions have also been extended to include nonspherical collapse and rotation in the slow approximation (Nath *et al* 2008, Herrera *et al* 1998b). In a recent investigation, the matching conditions applicable to spherically symmetric gravitational collapse were generated by Di Prisco *et al* (2007) to include nonadiabatic charged fluids.

3.4 Shear-free junction conditions

In comoving coordinates we take the interior spacetime to be given by the shear–free line element (2.3.10):

$$ds^2 = -A^2dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.4.1}$$

which is obtained by setting $Y = rB$ in (2.3.1). The results of the first junction condition (3.2.3) can be collectively written as

$$A(r_\Sigma, t)t_\tau = 1 \tag{3.4.2a}$$

$$r_\Sigma B(r_\Sigma, t) = \mathcal{R}(\tau) \tag{3.4.2b}$$

$$\mathsf{r}_\Sigma(v) = \mathcal{R}(\tau) \tag{3.4.2c}$$

$$\left(1 - \frac{2m}{\mathsf{r}} + 2\frac{d\mathsf{r}}{dv}\right)_\Sigma = \left(\frac{1}{{v_\tau}^2}\right)_\Sigma \tag{3.4.2d}$$

The intermediary variable τ may be eliminated from these equations, to generate the necessary and sufficient conditions for the first junction condition (3.2.3) to be valid on the spacetimes. We obtain

$$A(r_\Sigma, t)dt = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{\frac{1}{2}} dv \tag{3.4.3a}$$

$$r_\Sigma B(r_\Sigma, t) = \mathsf{r}_\Sigma(v) \tag{3.4.3b}$$

The second junction condition (3.2.4) is obtained by equating the appropriate extrinsic curvature components which yield

$$\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma = \left(\frac{v_{\tau\tau}}{v_\tau} - v_\tau\frac{m}{\mathsf{r}^2}\right)_\Sigma \tag{3.4.4a}$$

$$(r(rB)_r)_\Sigma = (v_\tau(\mathsf{r} - 2m) + \mathsf{r}\mathsf{r}_\tau)_\Sigma \tag{3.4.4b}$$

An expression for the mass function $m(v)$ in terms of the metric functions can be obtained from (3.4.4b) after eliminating r, r_τ and v_τ. This leads to

$$m(v) = \left(\frac{r^3 B}{2A^2}\dot{B}^2 - r^2 B' - \frac{r^3}{2B}B'^2\right)_\Sigma \tag{3.4.5}$$

We may interpret $m(v)$ as representing the total gravitational mass within the surface Σ.

From (3.3.3) and (3.3.5a) we can write

$$(\mathsf{r}_\tau)_\Sigma = \left(\frac{r}{A}\dot{B}\right)_\Sigma$$

Using this expression for $(\mathsf{r}_\tau)_\Sigma$ and on substituting (3.4.5) in (3.4.4b) we have that

$$(v_\tau)_\Sigma = \left(1 + r\frac{B'}{B} + r\frac{\dot{B}}{A}\right)^{-1}_\Sigma \tag{3.4.6}$$

If we now differentiate (3.4.6) with respect to τ and make use of (3.3.3a) we can write

$$(v_{\tau\tau})_\Sigma = \left[\frac{1}{A}\left(1 + r\frac{B'}{B} + r\frac{\dot{B}}{A}\right)^{-2} \times \right.$$

$$\left. \times \left(r\frac{B'\dot{B}}{B^2} - r\frac{\dot{B}'}{B} + r\frac{\dot{A}\dot{B}}{A^2} - r\frac{\ddot{B}}{A}\right)\right]_\Sigma \tag{3.4.7}$$

Substituting (3.3.3b), (3.3.5a), (3.4.5), (3.4.6) and (3.4.7) into (3.4.4a) we obtain

$$\begin{aligned}\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma &= \left[\left(1+r\frac{B'}{B}+r\frac{\dot{B}}{A}\right)^{-1}\times\right.\\ &\quad\times\left(\frac{r}{A}\frac{\dot{B}B'}{B^2}-\frac{r}{A}\frac{\dot{B}'}{B}+r\frac{\dot{A}}{A^3}\dot{B}-\frac{r}{A^2}\ddot{B}\right.\\ &\quad\left.\left.+\frac{B'}{B^2}+\frac{r}{2}\frac{B'^2}{B^3}-\frac{r}{2A^2}\frac{\dot{B}^2}{B}\right)\right]_\Sigma\end{aligned}$$

On multiplying this equation by $\left(1+r\left(\frac{B'}{B}\right)+r\left(\frac{\dot{B}}{A}\right)\right)$ and simplifying we obtain the following result

$$\begin{aligned}&-\frac{2}{A^2}\frac{\ddot{B}}{B}+2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B}-\frac{1}{A^2}\frac{\dot{B}^2}{B^2}+\frac{2}{r}\frac{A'}{A}\frac{1}{B^2}+\frac{2}{r}\frac{B'}{B^3}+\frac{B'^2}{B^4}+2\frac{A'}{A}\frac{B'}{B^3}\\ &= -\frac{2}{AB}\left(-\frac{\dot{B}'}{B}+\frac{B'\dot{B}}{B^2}+\frac{A'}{A}\frac{\dot{B}}{B}\right)\end{aligned}$$

which is equivalent to

$$p_\Sigma = (qB)_\Sigma$$

where we have utilised the field equations (2.3.12b) and (2.3.12d). Therefore the necessary and sufficient conditions on the spacetimes for the second junction condition (3.2.4) to be valid are that

$$m(v) = \left(\frac{r^3B}{2A^2}\dot{B}^2-r^2B'-\frac{r^3}{2B}B'^2\right)_\Sigma \qquad (3.4.8a)$$

$$p_\Sigma = (qB)_\Sigma \qquad (3.4.8b)$$

across the boundary.

The first attempt to generalise the above junction conditions to include shear for neutral matter was carried out by Glass (1989). To obtain a complete solution of radiative gravitational collapse we must solve the pressure isotropy condition ($\Delta = 0$) together with the junction condition (3.4.8b). It is crucial to check these equations and all the associated quantities for consistency in order to obtain a physically reasonable model. The junction conditions for shearing spacetimes for the special case with geodesic motion have been generated by Tomimura and Nunes (1993). de Oliveira and Santos (1987) investigated the shear–free case with a nonvanishing electromagnetic field. This work was extended by Maharaj and Govender (2000) to include the electromagnetic field. Several solutions to the junction conditions have been found when the shear is vanishing; recent models generated include the investigations of Herrera *et al* (2004b), Maharaj and Govender (2005) and Misthry *et al* (2008) where the Weyl tensor components vanish.

Chapter 4

Thermodynamics

4.1 Introduction

In this chapter we review the Eckart theory and point out its shortcomings. The essential features of causal thermodynamics are discussed and the truncated transport equations are presented in relativistic Maxwell-Cattaneo form. Under particular assumptions we obtain an explicit form for the causal temperature in spherically symmetric spacetimes. We employ the formalism of Maartens (1996a).

4.2 Aspects of irreversible thermodynamics

We take the particle four-current of a dissipative fluid to be of the form

$$na^3 = \text{comoving constant} \tag{4.2.1}$$

where a is the Hubble scale factor. This is analogous to choosing an average four-velocity in which there is no particle flux. We call this the particle frame. We also propose that at any event in spacetime, the fluid is thermodynamically close to some equilibrium state at the event. The local equilibrium scalars are represented by $\bar{n},\ \bar{\rho},\ \bar{p},\ \bar{S},\ \bar{T}$. The four-velocity of the local equilibrium is denoted $\bar{u}^\mu$. In the particle frame, we are able to choose $\bar{u}^\mu$ so that: (i) the number and energy densities coincide with the local equilibrium values, while (ii) the pressure does not, i.e.

$$n = \bar{n}, \quad \rho = \bar{\rho}, \quad p = \bar{p} + \Pi \tag{4.2.2}$$

where $\Pi = p - \bar{p}$ is the bulk viscous pressure. Hereafter, we will replace $\bar{p}$ by p. The effective non-equilibrium pressure becomes

$$p_{eff} = p + \Pi \quad (p \longrightarrow p_{eff},\ \bar{p} \longrightarrow p) \tag{4.2.3}$$

The conservation of particle number and energy-momentum are crucial to irreversible thermodynamics, whether we employ the standard or extended theory. These are represented by

$$n^{\alpha}{}_{;\alpha} = 0, \quad T^{\alpha\beta}{}_{;\beta} = 0 \tag{4.2.4}$$

If we consider the conservation of particle number, we obtain the same equation that holds in equilibrium (4.2.1).

The equilibrium energy and momentum conservation equations are

$$\begin{aligned} \dot{\rho} + 3H(\rho + p) &= 0 \quad (4.2.5)\\ (\rho + p)\dot{u_\alpha} + D_\alpha p &= 0 \quad (4.2.6) \end{aligned}$$

The above equations are changed by the dissipative terms in the energy-momentum tensor:

$$\begin{aligned} \dot{\rho} + 3H(\rho + p + \Pi) + D^\alpha q_\alpha + 2\dot{u}_\alpha q^\alpha + \sigma_{\alpha\beta}\pi^{\alpha\beta} &= 0 \quad (4.2.7)\\ (\rho + p + \Pi)\dot{u}_\alpha + D_\alpha(p + \Pi) - D^\beta \pi_{\alpha\beta} + \dot{u}^\beta \pi_{\alpha\beta} + h_\alpha{}^\beta \dot{q}_\beta & \\ + (4Hh_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta})q^\beta &= 0 \quad (4.2.8) \end{aligned}$$

The striking feature of irreversible thermodynamics is that entropy is no longer conserved. In this theory the entropy grows, in accordance with the second law of thermodynamics. The rate at which entropy is produced is given by the divergence of the entropy four-current. We may now give the covariant form of the second law of thermodynamics

$$S^{\alpha}{}_{;\alpha} \geq 0 \tag{4.2.9}$$

This gives a new form for S^α which contains a dissipative term

$$S^\alpha = Snu^\alpha + \frac{R^\alpha}{T} \tag{4.2.10}$$

where $S = \bar{S}$ and $T = \bar{T}$ are related by

$$TdS = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) \tag{4.2.11}$$

which is the Gibbs equation.

We assume that the dissipative part R^α of S^α is an algebraic function of n^α and $T^{\alpha\beta}$. Furthermore, we will assume that R^α vanishes in equilibrium. These assumptions are part of the hydrodynamical description. The assumption is that non-equilibrium states can be completely specified by the hydrodynamical tensors $n^\alpha, T^{\alpha\beta}$ alone. In irreversible thermodynamics, the form of R^α in the standard theory is different from that in the extended theory.

In order to satisfy (4.2.9) we follow the approach of Maartens (1996) and impose the following linear relationships between the thermodynamic 'fluxes' $\Pi, q_\alpha, \pi_{\alpha\beta}$ and the corresponding thermodynamic 'forces' $H,\ \dot{u}_\alpha + D_\alpha \ln T,\ \sigma_{\alpha\beta}$:

$$\Pi = -3\varsigma H \tag{4.2.12}$$

$$q_\alpha = -\lambda(D_\alpha T + T\dot{u}_\alpha) \tag{4.2.13}$$

$$\pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta} \tag{4.2.14}$$

In the standard Eckart theory of relativistic irreversible thermodynamics, these are the most important equations for dissipative quantities. They form the relativistic generalisations of the corresponding Newtonian laws:

$$\Pi = -3\varsigma\vec{\nabla}\cdot\vec{v} \quad \text{(Stokes)} \tag{4.2.15}$$

$$\vec{q} = -\kappa\vec{\nabla}T \quad \text{(Fourier)} \tag{4.2.16}$$

$$\pi_{ij} = -2\eta\sigma_{ij} \quad \text{(Newton)} \tag{4.2.17}$$

We may identify the following thermodynamical coefficients using the Newtonian laws (4.2.16)-(4.2.17)

- $\varsigma(\rho, n)$ is the bulk viscosity
- $\kappa(\rho, n)$ is the thermal conductivity
- $\eta(\rho, n)$ is the shear viscosity

With the linear constitutive equations (4.2.12)-(4.2.14), we can express the entropy production rate as

$$S^{\alpha}{}_{;\alpha} = \frac{\Pi^2}{\varsigma T} + \frac{q_\alpha q^\alpha}{\kappa T^2} + \frac{\pi_{\alpha\beta}\pi^{\alpha\beta}}{2\eta T} \tag{4.2.18}$$

$S^{\alpha}{}_{;\alpha}$ will be non-negative if

$$\varsigma \geq 0, \quad \kappa \geq 0, \quad \eta \geq 0 \tag{4.2.19}$$

are satisfied.

Several applications of irreversible thermodynamics in general relativity thus far have used the Eckart theory as presented above. There are shortcomings to this approach, however. One such problem is the algebraic nature of the Eckart constitutive equations. If a thermodynamic force is suddenly set equal to zero in this theory, then the corresponding thermodynamic flux vanishes instantaneously. This violates relativistic causality, since the signal propagates through the fluid at an infinite speed. We discuss the shortcomings of this theory in the next chapter.

4.3 Causal thermodynamics

Causal theories of dissipative fluids, both relativistic and non-relativistic, were postulated in order to avoid some undesired effects of the conventional Eckart-type theories. There are several advantages to taking a causal approach, such as: (i) Causal propagation of dissipative signals for stable fluid configurations, (ii) Unlike Eckart-type theories, there is no generic short-wavelength secular instability in causal theories, (iii) The perturbations have a reasonably posed initial value problem, even in the case of rotating fluids.

The central idea in causal theories is to extend the space of variables of conventional theories by incorporating the dissipative quantities concerned (such as heat flux, particle currents, shear and bulk stresses) in it. Hence these quantities are treated similarly to the conserved variables (such as energy density, particle numbers, etc.). This leads to a more involved theory with a larger number of variables and parameters.

The Eckart postulate for R^α is an oversimplified one. Kinetic theory shows that R^α is in fact second-order in the dissipative fluxes. By truncating at first order, the Eckart assumption removes terms that are key in order to provide causality and stability. We choose the most general form for R^α which is at most second-order in the dissipative fluxes, which gives

$$\begin{aligned} S^\mu &= Snu^\mu + \frac{q^\mu}{T} - \left(\beta_0\Pi^2 + \beta_1 q_\nu q^\nu + \beta_2 \pi_{\nu\kappa}\pi^{\nu\kappa}\right)\frac{u^\mu}{2T} \\ &\quad + \frac{\alpha_0 \Pi q^\mu}{T} + \frac{\alpha_1 \pi^{\mu\nu} q_\nu}{T} \end{aligned} \tag{4.3.1}$$

In the above equation, $\beta_A(\rho, n) \geq 0$ are thermodynamic coefficients for scalar, vector and tensor dissipative contributions to the entropy density, and $\alpha_A(\rho, n)$ are thermodynamic viscous/ heat coupling coefficients. Thus the effective entropy density measured by a comoving observer is

$$-u_\mu S^\mu = Sn - \frac{1}{2T}\left(\beta_0\Pi^2 + \beta_1 q_\mu q^\mu + \beta_2 \pi_{\mu\nu}\pi^{\mu\nu}\right) \tag{4.3.2}$$

which is independent of α_0, α_1. At equilibrium, the entropy density is at a maximum. We proceed to set the viscous/ heat coupling to zero for simplicity. With this, the divergence of the extended current (4.3.1) follows from the Gibbs equation (4.2.11) and the conservation equations (4.2.1), (4.2.8) and (4.2.8):

$$
\begin{aligned}
TS^{\alpha}{}_{;\alpha} &= -\Pi\left[3H + \beta_0\dot{\Pi} + \frac{1}{2}T\left(\frac{\beta_0}{T}u^{\alpha}\right)_{;\alpha}\Pi\right] \\
&\quad -q^{\alpha}\left[D_{\alpha}\ln T + \dot{u}_{\alpha} + \beta_1\dot{q}_{\alpha} + \frac{1}{2}T\left(\frac{\beta_1}{T}u^{\mu}\right)_{;\mu}q_{\alpha}\right] \\
&\quad -\pi^{\alpha\mu}\left[\sigma_{\alpha\mu} + \beta_2\dot{\pi}_{\alpha\mu} + \frac{1}{2}T\left(\frac{\beta_2}{T}u^{\nu}\right)_{;\nu}\pi_{\alpha\mu}\right] \qquad (4.3.3)
\end{aligned}
$$

Following a similar approach as in the standard theory, we impose linear relationships between the thermodynamical fluxes and forces (extended). This is the easiest way to satisfy the second law of thermodynamics. We obtain the following constitutive or transport equations:

$$
\begin{aligned}
\tau_0\dot{\Pi} + \Pi &= -3\varsigma H - \left[\frac{1}{2}\varsigma T\left(\frac{\tau_0}{\varsigma T}u^{\alpha}\right)_{;\alpha}\Pi\right] \qquad (4.3.4) \\
\tau_1 h_{\alpha}{}^{\beta}\dot{q}_{\beta} + q_{\alpha} &= -\kappa(D_{\alpha}T + T\dot{u}_{\alpha}) - \left[\frac{1}{2}\kappa T^2\left(\frac{\tau_1}{\kappa T^2}u^{\beta}\right)_{;\beta}q_{\alpha}\right] \qquad (4.3.5) \\
\tau_2 h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}\dot{\pi}_{\mu\nu} &= -2\eta\sigma_{\alpha\beta} - \left[\eta T\left(\frac{\tau_2}{2\eta T}u^{\nu}\right)_{;\nu}\pi_{\alpha\beta}\right] \qquad (4.3.6)
\end{aligned}
$$

where the relaxation times $\tau_A(\rho, n)$ are given by

$$
\tau_0 = \varsigma\beta_0, \quad \tau_1 = \kappa T\beta_1, \quad \tau_2 = 2\eta\beta_2 \qquad (4.3.7)
$$

In many instances, the terms in square brackets on the right side of (4.3.4) - (4.3.6) are omitted. This corresponds to an assumption that these terms are negligible when compared to the other terms in the equations. With the no-coupling assumption, the truncated equations are of covariant relativistic Maxwell-Cattaneo form:

$$\tau_0\dot{\Pi} + \Pi = -3\varsigma H \tag{4.3.8}$$

$$\tau_1 {h_\alpha}^\beta \dot{q}_\beta + q_\alpha = -\kappa(D_\alpha T + T\dot{u}_\alpha) \tag{4.3.9}$$

$$\tau_2 {h_\alpha}^\mu {h_\beta}^\nu \dot{\pi}_{\mu\nu} = -2\eta\sigma_{\alpha\beta} \tag{4.3.10}$$

The evolution terms, with the relaxation time coefficients τ_A, are needed for causality, as well as for modelling high-frequency or transient phenomena, where 'fast' variables and relaxation effects are important. Frequently in the analysis of the gravitational collapse of fluid spheres, the thermal relaxation time τ is ignored, since it is usually very small in comparison with the typical time scales of collapse for gravitating systems (for phonon-electron interaction $\tau \sim 10^{-11}$ sec, and for phonon-phonon and free electron interaction $\tau \sim 10^{-13}$ sec at room temperature). However, situations arise for which τ cannot be disregarded against the gravitational collapse. For instance, in the cores of evolved stars, the quantum cells of phase space are filled up and the electron mean-free path increases substantially, as does τ. Furthermore, events prior to relaxation may significantly influence the subsequent evolution of the system (i.e. for times longer than τ). It has been shown for spherically symmetric stars with radial heat flow, the temperature gradient which appears as a result of the perturbation, and hence the luminosity, and highly dependent on the product of the relaxation time by the period of the oscillation of the star.

4.4 Thermodynamics of radiating stars

We are primarily interested in heat transport in relativistic astrophysics and hence (4.3.9) plays a significant role in determining the evolution of the causal temperature profile of our models. For the line element (2.3.1) the causal transport equation (4.3.9) becomes

$$\tau(qB)_{,t} + A(qB) = -\kappa\frac{(AT)_{,r}}{B} \tag{4.4.1}$$

which governs the behaviour of the temperature. Setting $\tau = 0$ in (4.4.1) we obtain the familiar Fourier heat transport equation

$$A(qB) = -\kappa \frac{(AT)_{,r}}{B} \tag{4.4.2}$$

which predicts reasonable temperatures when the fluid is close to quasi–stationary equilibrium.

For a physically reasonable model, we use the thermodynamic coefficients for radiative transfer outlined in Martínez (1996). We consider the situation where energy is carried away from the stellar core by massless particles that are thermally generated with energies of the order of kT. The thermal conductivity takes the form

$$\kappa = \gamma T^3 \tau_{\rm c} \,, \tag{4.4.3}$$

where γ (≥ 0) is a constant and $\tau_{\rm c}$ is the mean collision time between the massless and massive particles. Based on this treatment we assume the power–law behaviour

$$\tau_{\rm c} = \left(\frac{\alpha}{\gamma}\right) T^{-\omega} \,, \tag{4.4.4}$$

where α (≥ 0) and ω (≥ 0) are constants. With $\omega = \frac{3}{2}$ we regain the case of thermally generated neutrinos in neutron stars. The mean collision time decreases with growing temperature, as expected. For the special case $\omega = 0$, the mean collision time is constant. This special case can only give a reasonable model for a limited range of temperature. Following Martínez (1996), we assume that the velocity of thermal dissipative signals is comparable to the adiabatic sound speed which is satisfied if the relaxation time is proportional to the collision time:

$$\tau = \left(\frac{\beta\gamma}{\alpha}\right) \tau_{\rm c} \,, \tag{4.4.5}$$

where τ (≥ 0) is a constant. We can think of τ as the 'causality' index, measuring the strength of relaxational effects, with $\tau = 0$ giving the noncausal case.

Using the above definitions for τ and κ, (4.4.1) takes the form

$$\beta(qB)_{,t} T^{-\omega} + A(qB) = -\alpha \frac{T^{3-\omega}(AT)_{,r}}{B} \,. \tag{4.4.6}$$

When $\beta = 0$, we can find all noncausal solutions of (4.4.6), *viz.*

$$(AT)^{4-\omega} = \frac{\omega-4}{\alpha}\int A^{4-\omega}qB^2\mathrm{d}r + F(t) \qquad \omega \neq 4 \tag{4.4.7}$$

$$\ln(AT) = -\frac{1}{\alpha}\int qB^2\mathrm{d}r + F(t) \qquad \omega = 4, \tag{4.4.8}$$

where $F(t)$ is an arbitrary function of integration. This is fixed by the expression for the temperature of the star at its surface Σ. The above expressions for the temperature T were first generated by Govender and Govinder (2001b) extending the model of Govender *et al* (1998) on causal radiating collapse.

In the case of constant mean collision time, *ie.* $\omega = 0$, the causal transport equation (4.4.6) is simply integrated to yield

$$(AT)^4 = -\frac{4}{\alpha}\left[\beta\int A^3B(qB)_{,t}\mathrm{d}r + \int A^4qB^2\mathrm{d}r\right] + F(t) \tag{4.4.9}$$

while one solution valid for a less limited range of temperature can be found for $\omega = 4$, which corresponds to variable collision time (Govender and Govinder 2001b):

$$\begin{aligned}(AT)^4 &= -\frac{4\beta}{\alpha}\exp\left(-\int\frac{4qB^2}{\alpha}\mathrm{d}r\right)\int A^3B(qB)_{,t}\exp\left(\int\frac{4qB^2}{\alpha}\mathrm{d}r\right)\mathrm{d}r \\ &\quad + F(t)\exp\left(-\int\frac{4qB^2}{\alpha}\mathrm{d}r\right).\end{aligned} \tag{4.4.10}$$

In order to investigate the relaxational effects due to shear we utilize (4.3.10) as a definition for the relaxation time for the shear stress. For the metric (2.3.1) the shear transport equation (4.3.10) reduces to

$$\tau_1 = \frac{-P}{\dot{P} + \frac{8}{15}r_0\sigma T^4}, \tag{4.4.11}$$

where we have used the coefficient of shear viscosity for a radiative fluid (Maartens 1996b)

$$\eta = \frac{4}{15}r_0T^4\tau_1, \tag{4.4.12}$$

where $P = \frac{1}{3}(p_T - p_R)$ and r_0 is the radiation constant for photons. We have further assumed that $\tau_1 = \beta_1\tau_c$ where $\beta_1(\geq 0)$ is a constant. This allows for both cases where the relaxation time-scale is greater than the collision time ($\beta_1 > 1$, Maartens and

Triginer 1998), and the case of pertubative deviations from the quasi-stationary case ($\beta_1 < 1$). The degenerate case $\beta_1 = 0$ recovers the non-causal transport law.

Chapter 5

Thermal evolution of a radiating anisotropic star with shear

5.1 Introduction

In this chapter we study the effects of pressure anisotropy and heat dissipation in a spherically symmetric radiating star undergoing gravitational collapse. An exact solution of the Einstein field equations is presented in which the model has a Friedmann-like limit when the heat flux vanishes. The behaviour of the temperature profile of the evolving star is investigated within the framework of causal thermodynamics. In particular, we show that there are significant differences between the relaxation time for the heat flux and the relaxation time for the shear stress.

It has been shown that the role of anisotropy can alter the evolution and subsequently the physical properties of stellar objects. As an example, investigations have shown that the maximal surface redshift for anisotropic stars may differ drastically from isotropic stars. The origin of anisotropy within the stellar core has received widespread attention amongst astrophysicists. A review of the origins and effects of local anisotropy in stellar objects was carried out by Herrera and Santos (1997a), and Chan *et al* (2003) and more recently by Herrera *et al* (2004b). The physical processes that are responsible for deviations from isotropy can be investigated in the high and low density regimes. Hartle *et al* (1975) have shown that pion condensation at nuclear densities ($0.2f^{-3} < \rho < 2.0f^{-3}$, $1f = 10^{-13}$ cm) can drastically reduce the pressure and hence impact on the evolution of the collapsing star. At higher densities the short range repulsion effects dominate which damp out the pionic effects giving rise to significantly different values for the pressure. As pointed out by Martinez (1996) viscosity effects due to neutrino trapping at nuclear densities can alter the gravitational collapse of a radiating, viscous star. Anisotropic velocity distributions and rotation can induce local anisotropy in low density systems.

The boundary of a radiating star divides spacetime into two distinct regions, the interior and the exterior region. In order to fully describe the evolution of such a system, one needs to satisfy the junction conditions for the smooth matching of the interior and exterior spacetimes. In the case of a shear-free, spherically symmetric star undergoing dissipative gravitational collapse these conditions were first derived by Santos (1985). Subsequently, many models of shear-free radiative collapse were developed and the physical viability of these models were studied in great detail (Kolassis *et al* 1988, Kramer 1992, Grammenos 1995, Govender *et al* 1998, Govender *et al* 1999). These junction conditions were later generalized to include pressure anisotropy (Chan 1994) and the electromagnetic field (Maharaj and Govender 2000).

There are a few exact solutions to the Einstein field equations for a bounded shearing matter configuration. There have been numerous attempts to produce models of radiative gravitational collapse which incorporate the effects of shear (Barreto *et al* 1992, Chan 2000, Herrera and Martínez 1998a). Most treatments to date are based on numerical results as the resulting temporal evolution equation derived from the junction conditions is highly nonlinear (Chan 2000). It is in this spirit that we seek an exact solution of the Einstein field equations which represents a spherically symmetric radiating star undergoing dissipative gravitational collapse with nonzero shear.

The assumption of local isotropy in the study of objects undergoing gravitational collapse is common. The fluid approximation used to describe the matter distribution of an object implies a Pascalian character, and is supported by a large amount of observational evidence that points towards isotropy under several circumstances. However, strong theoretical evidence suggest that for different density ranges, different kinds of physical phenomena may occur, leading to local anisotropy. In this chapter, we investigate the role of anisotropy in the thermal evolution of a spherically symmetric star undergoing dissipative gravitational collapse in the presence of shear.

In this treatment, we consider the relaxation effects due to the heat flux and shear separately. We show that earlier assumptions of the relaxation time being proportional to the corresponding collision time only hold for a limited regime of the evolution of the star. These results agree with earlier suggestions by Anile *et al* (1998). We are also in a position to integrate the causal heat transport equation and obtain the corresponding causal temperature profile in the interior of the star.

This chapter is organized as follows. In §5.2, we provide the Einstein field equations for the most general, nonrotating, spherically symmetric line element. The energy momentum tensor for the interior spacetime is that of an imperfect fluid with heat conduction and pressure anisotropy. In §5.3, we discuss the thermodynamics and physical considerations. The general solution for the special case of constant mean collision time in both the causal and noncausal theories are presented. A discussion of some relevant physical parameters is presented in §5.4. The results of this chapter have been published in Naidu *et al* (2006).

5.2 Radiating anisotropic collapse

We begin with the line element (2.3.1)

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2.1}$$

and set $A = 1$ to obtain

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2.2}$$

where the metric functions B and Y are yet to be determined. The energy-momentum tensor is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{5.2.3}$$

The temporal evolution of B and Y are obtained from the junction condition $(p_R)_\Sigma = (qB)_\Sigma$ which, for the line element (5.2.2) yields

$$\left[2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}Y\dot{Y}' - 2\frac{\dot{B}}{B^2}YY' + 1\right]_\Sigma = 0 \tag{5.2.4}$$

A particular solution of (5.2.4) is given by

$$Y = rt^{2/3} \tag{5.2.5a}$$

$$B = \left(\frac{1 + c_1(r)e^{\frac{3t^{1/3}}{r}}}{1 - c_1(r)e^{\frac{3t^{1/3}}{r}}}\right)t^{2/3} \tag{5.2.5b}$$

which yields the line element

$$ds^2 = -dt^2 + t^{4/3}\left[\left(\frac{1 + c_1(r)e^{\frac{3t^{1/3}}{r}}}{1 - c_1(r)e^{\frac{3t^{1/3}}{r}}}\right)^2 dr^2 + r^2 d\Omega^2\right] \tag{5.2.6}$$

With this form of the line element, the field equations (2.3.6a)–(2.3.6d) give

$$\begin{aligned}
\rho &= \frac{4}{3\,t^2}\left(1+\frac{1}{r^3}\left(\frac{r^2\,t^{\frac{1}{3}}}{1-e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)}+\frac{18\,t}{\left(1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)^3}\right.\right.\\
&\quad\left.-\frac{3\left(r\,t^{\frac{2}{3}}+9\,t\right)}{\left(1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)^2}+\frac{-\left(r^2\,t^{\frac{1}{3}}\right)+3\,r\,t^{\frac{2}{3}}+9\,t}{1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)}\right)\\
&\quad\left.-\frac{3\,e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,t^{\frac{2}{3}}\left(-1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)c_1'(r)}{r\left(1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)^3}\right) &\text{(5.2.7a)}\\
p_R &= \frac{-4\,e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)}{t^{\frac{4}{3}}\left(r+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,r\,c_1(r)\right)^2} &\text{(5.2.7b)}\\
p_T &= \frac{2}{3\,r^3\,t^{\frac{5}{3}}}\left(\frac{-3\,r\,t^{\frac{1}{3}}}{\left(-1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)^2}+\frac{r\left(2\,r-3\,t^{\frac{1}{3}}\right)}{-1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)}\right.\\
&\quad+\frac{9\,t^{\frac{1}{3}}\left(3\,t^{\frac{1}{3}}\,c_1(r)-r^2\,c_1'(r)\right)}{c_1(r)\left(1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)^2}+\frac{6\,t^{\frac{1}{3}}\left(-3\,t^{\frac{1}{3}}\,c_1(r)+r^2\,c_1'(r)\right)}{c_1(r)\left(1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)^3}\\
&\quad\left.+\frac{2\,r^2\,c_1(r)-9\,t^{\frac{2}{3}}\,c_1(r)+3\,r^2\,t^{\frac{1}{3}}\,c_1'(r)}{c_1(r)+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)^2}\right) &\text{(5.2.7c)}\\
q &= \frac{4\,e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\left(-1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)}{r^2\,t^2\left(1+e^{\frac{3\,t^{\frac{1}{3}}}{r}}\,c_1(r)\right)^3} &\text{(5.2.7d)}
\end{aligned}$$

for a shearing, expanding fluid in geodesic motion.

In the absence of heat flux ($c_1 = 0$) our model yields

$$Y = rt^{2/3} \tag{5.2.8a}$$

$$B = t^{2/3} \tag{5.2.8b}$$

$$\rho = \frac{4}{3t^2} \tag{5.2.8c}$$

$$p_R = p_T = 0 \tag{5.2.8d}$$

The above solution represents a dust sphere and the metric is described by the Einstein–de Sitter solution. We note that when $q = 0$ the pressure must vanish which allows for the matter to have free-fall motion. For $q \neq 0$, the pressure is non-vanishing so that it compensates for the outgoing heat flux thus allowing for free-fall motion. With this in mind we expect that the luminosity radius of the star in both the radiative and non-radiative cases should have the same temporal dependence. Calculating the luminosity radius for our radiating model, we obtain

$$Y_\Sigma = bt^{2/3} \tag{5.2.9}$$

which is independent of c_1. Hence the case $c_1 = 0$ reduces to the model investigated by Oppenheimer and Snyder (1939).

5.3 Physical considerations

In order to check the physical viability of our model, we investigate the evolution of the temperature profile within the framework of extended irreversible thermodynamics.

Utilising (4.4.9) and (5.2.6) we obtain

$$
\begin{aligned}
T^4 &= \frac{L_\infty}{(4\pi\delta Y^2)_\Sigma} + \frac{16\beta c_1}{3\alpha t^{5/3}}\left[\frac{e^{3t^{1/3}/b}}{b(1+c_1 e^{3t^{1/3}/b})} - \frac{e^{3t^{1/3}/r}}{r(1+c_1 e^{3t^{1/3}/r})}\right] \\
&+ \frac{16}{9\alpha t^2}\left\{\log\left[\left(\frac{-1+c_1 e^{3t^{1/3}/b}}{-1+c_1 e^{3t^{1/3}/r}}\right)^2\left(\frac{1+c_1 e^{3t^{1/3}/r}}{1+c_1 e^{3t^{1/3}/b}}\right)^3\right]\right\} \\
&+ \frac{16}{3\alpha t}\left[\tanh^{-1}(c_1 e^{2t^{1/3}/b}) - \tanh^{-1}(c_1 e^{2t^{1/3}/r})\right] \qquad (5.3.1)
\end{aligned}
$$

where L_∞ is given by

$$L_\infty(v) = -\left(\frac{dm}{dv}\right)_\Sigma = \frac{(p_R)_\Sigma}{2}\left[Y^2\left(\frac{Y'}{B}+\dot{Y}\right)^2\right]_\Sigma \qquad (5.3.2)$$

where $\frac{dm}{dv} \leq 0$ since L_∞ is positive. An observer with four–velocity $v^a = (\dot{v}, \dot{\mathsf{r}}, 0, 0)$ located on Σ has proper time η related to the time t by $d\eta = Adt$. The radiation energy density that this observer measures on Σ is

$$\epsilon_\Sigma = \frac{1}{4\pi}\left(-\frac{\dot{v}^2}{\mathsf{r}^2}\frac{dm}{dv}\right)_\Sigma \qquad (5.3.3)$$

and the luminosity observed on Σ can be written as

$$L_\Sigma = 4\pi \mathsf{r}^2 \epsilon_\Sigma \qquad (5.3.4)$$

The boundary redshift z_Σ of the radiation emitted by the star is given by

$$1 + z_\Sigma = \frac{dv}{d\eta} = \left(\frac{Y'}{B} + \dot{Y}\right)^{-1}_\Sigma \qquad (5.3.5)$$

which can be used to determine the time of formation of the horizon. The above expressions allow us to write

$$1 + z_\Sigma = \left(\frac{L_\Sigma}{L_\infty}\right)^{\frac{1}{2}} \qquad (5.3.6)$$

which relates the luminosities L_Σ to L_∞. The redshift for an observer at infinity diverges at the time of formation of the horizon which is determined from

$$\frac{Y'}{B} + \dot{Y} = 0 \qquad (5.3.7)$$

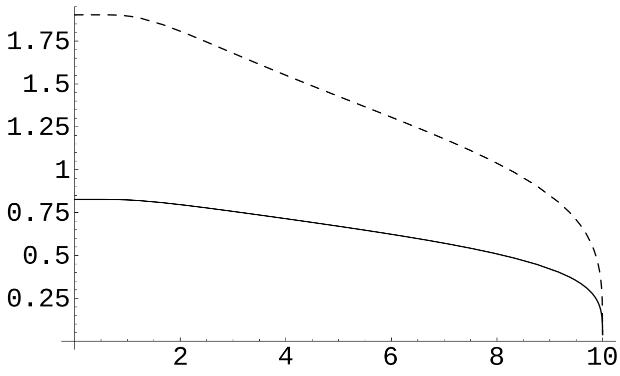

Figure 5.1: Causal temperature (dashed line), noncausal temperature (solid line) versus r.

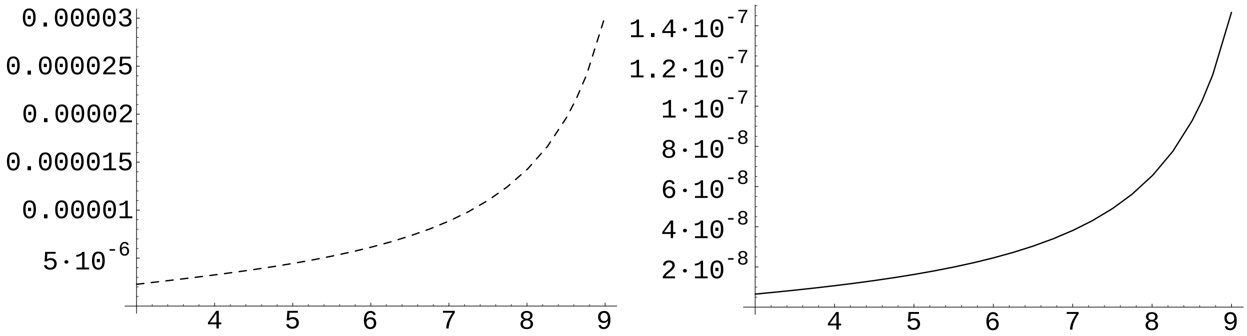

Figure 5.2: Relaxation time for the shear stress (close to equilibrium - dashed line), (far from equilibrium - solid line) versus r.

We note that the causal and the noncausal ($\beta = 0$) temperatures coincide at the boundary ($r = b$):

$$T(t, b) = \tilde{T}(t, b) \tag{5.3.8}$$

However, Figure 5.1 shows that at all interior points, the causal and non-causal temperatures differ. In particular, we observe that the causal temperature is greater than the non-causal temperature at each interior point of the star. For small values of β, the temperature profile is similar to that of the non-causal theory; but as β is increased, i.e. as relaxational effects grow, it is clear from Figure 5.1 that the temperature profile can deviate substantially from that of the non-causal theory. Similar results were obtained in the shear-free models studied by Herrera and Santos (1997b) and Govender *et al* (1998, 1999). Also, from the plots in Figure 5.2 the relaxation time for the shear stress exhibits substantially different behaviour when the fluid is close to hydrostatic equilibrium as opposed to late-time collapse. In particular, we find that

$$\frac{(\tau_1)_{early}}{(\tau_1)_{late}} \approx 100, \tag{5.3.9}$$

emphasizing the importance of relaxational effects during the different stages of collapse. We further note that while the relaxation time for the heat flux is taken to be constant, the relaxation time for the shear stress increases as the collapse proceeds. Making use of (2.3.1) and (5.2.5b) the proper radius can be written as

$$R = \int_0^{r_\Sigma} B dr = t^{2/3} \int_0^{r_\Sigma} \left(\frac{1 + c_1 e^{\frac{3t^{1/3}}{r}}}{1 - c_1 e^{\frac{3t^{1/3}}{r}}} \right) dr \tag{5.3.10}$$

Numerical integration of (5.3.10) (with $c_1(r) = -1$) shows that the proper radius is a decreasing function of time. This is expected as the star is contracting and losing mass.

5.4 Discussion

We have successfully provided an analytical model of a radiating star undergoing gravitational collapse with non-vanishing shear. This model has a Friedmann-like limit when the heat flux vanishes. We further showed that the causal temperature (representing the stellar fluid out of hydrostatic equilibrium) is higher than the noncausal temperature at all points of the star. Further analysis revealed that the relaxation time for the shear stress (taken to be proportional to the mean collision time) increases radially outwards, towards the surface of the star. This is expected, as the outer layers of the fluid are cooler than the central regions. Of particular significance is the result that the relaxation time for the heat flux (in our case taken to be constant) differs from the relaxation time for the shear stress. This is contrary to earlier treatments where it was assumed that $(\tau_r)_{heat} \approx (\tau_r)_{shear}$ (Martínez 1996, Herrera and Martínez 1998a).

To make a more realistic comparison of the relaxation times, one requires an analytic solution of the causal temperature equation for non-constant relaxation times. Future work in this direction will also require the comparison of the various relaxation times using the truncated and full transport equations.

Chapter 6

Causal temperature profiles in horizon-free collapse

6.1 Introduction

We investigate the causal temperature profiles in a recent model of a radiating star undergoing dissipative gravitational collapse without the formation of an horizon. It is shown that this simple exact model provides physically reasonable behaviour for the temperature profile within the framework of extended irreversible thermodynamics.

The Cosmic Censorship Conjecture has continued to occupy centre stage within the realms of relativistic astrophysics. The final outcome of the gravitational collapse of a star is still very much open to debate with the discovery of models admitting naked singularities (Harada *et al* 1998, Kudoh *et al* 2000). Various scenarios of gravitational collapse have been considered in which the energy momentum tensor is taken to be either a perfect fluid or an imperfect fluid with heat flux and anisotropic pressure (Bonnor *et al* 1989, Herrera and Santos 1997a, Naidu *et al* 2006). It is well known that the collapse of reasonable matter distributions always lead to the formation of a black hole in the absence of shear or in the case of homogeneous densities. It has been shown that shearing effects delay the formation of the apparent horizon by making the final stages of collapse incoherent thus leading to the generation of naked singularities (Joshi *et al* 2002). In this chapter we revisit a radiating stellar model proposed by Banerjee *et al* (2002), (hereafter referred to as the *BCD* model) in which the horizon is never encountered. The interior matter distribution is that of an imperfect fluid with heat flux and the exterior spacetime is described by the radiating Vaidya metric (Vaidya 1951). The junction conditions required for the smooth matching of the interior and exterior spacetimes across a four-dimensional time-like hypersurface are solved exactly.

We present the *BCD* model in §6.2. In §6.3 we investigate the physical viability of the *BCD* model. In particular, we analyse the relaxational effects on the temperature profiles within the framework of extended irreversible thermodynamics. We are in a position to obtain exact solutions to the causal heat transport equation for both the special case of constant collision time as well as variable collision time. Our results are in agreement with earlier thermodynamical investigations of radiating stellar models. We find that relaxational effects enhance the temperature at each interior point of the stellar configuration. Our investigations show that the *BCD* model displays physically reasonable temperature profiles throughout the evolution of the star. We discuss the results in §6.4. The results of this chapter have been published (Naidu and Govender 2007).

6.2 The *BCD* radiating model revisited

In the *BCD* model the following form of the metric for the interior spacetime is assumed to be (2.3.10):

$$ds^2 = -A^2(r,t)dt^2 + B^2(r,t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \tag{6.2.1}$$

in which the metric functions A and B are yet to be determined. The energy-momentum tensor for the interior matter distribution is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{6.2.2}$$

The heat flow vector q^a is orthogonal to the velocity vector so that $q^a u_a = 0$. To generate an exact model of radiative gravitational collapse the following ansatz was adopted for the metric functions in (6.2.1)

$$A = a(r) \tag{6.2.3}$$

$$B = b(r)R(t) \tag{6.2.4}$$

which reduces the Einstein field equations for the interior matter distribution to

$$\rho = \frac{1}{R^2}\left[\frac{3}{a^2}\dot{R}^2 - \frac{1}{b^2}\left(\frac{2b''}{b} - \frac{b'^2}{b^2} + \frac{4b'}{rb}\right)\right] \tag{6.2.5a}$$

$$p = \frac{1}{R^2}\left[-\frac{1}{a^2}(2R\ddot{R} + \dot{R}^2) + \frac{1}{b^2}\left(\frac{b'^2}{b^2} + \frac{2a'b'}{ab} + \frac{2}{r}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right)\right] \tag{6.2.5b}$$

$$q = -\frac{2a'\dot{R}}{R^3 a^2 b^2} \tag{6.2.5c}$$

The condition of pressure isotropy yields

$$\frac{a''}{a} + \frac{b''}{b} - 2\frac{b'^2}{b^2} - 2\frac{a'b'}{ab} - \frac{a'}{ra} - \frac{b'}{rb} = 0 \tag{6.2.6}$$

Since the star is radiating energy the exterior spacetime is described by the Vaidya metric given explicitly in the form

$$ds^2 = -\left(1 - \frac{2M(v)}{\bar{r}}\right)dv^2 - 2d\bar{r}dv + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.2.7}$$

where v is the retarded time and $M(v)$ is the exterior Vaidya mass. The junction conditions required for the smooth matching of the interior metric (Banerjee *et al* 2002) and the exterior Vaidya metric (6.2.7), across a timelike hypersurface Σ, are given by

$$(rB)_\Sigma = \bar{r}_\Sigma \tag{6.2.8a}$$

$$p_\Sigma = (qB)_\Sigma \tag{6.2.8b}$$

$$m_\Sigma = \left[\frac{r^3 B\dot{B}^2}{2A^2} - r^2 B' - \frac{r^3 B'^2}{2B}\right]_\Sigma \tag{6.2.8c}$$

where m_Σ represents the total mass of the stellar configuration of radius r inside Σ. Utilising (6.2.5b) and (6.2.5c) in the boundary condition (6.2.8c) yields

$$2R\ddot{R} + \dot{R}^2 + m\dot{R} = n \tag{6.2.9}$$

where m and n are constants. A simple particular solution of (6.2.9) is

$$R(t) = -Ct \tag{6.2.10}$$

where $C > 0$ is a constant of integration. As pointed out in Banerjee *et al* (2002), the mass-to-radius ratio, $m_\Sigma/\bar{r}_\Sigma$, is independent of time. A simple calculation yields

$$\frac{2m_\Sigma}{\bar{r}_\Sigma} = \frac{2m_\Sigma}{(rB)_\Sigma} = 2\left[\frac{C^2 r_0^2 b_0^2}{2a_0^2} - \frac{r_0 b_0'}{b_0} - \frac{r_0^2 b_0'^2}{2b_0^2}\right] \tag{6.2.11}$$

where $b(r_0) = b_0$ and r_0 defines the boundary of the stellar configuration. It is interesting to note that the parameters in (6.2.11) may be chosen so that $2m_\Sigma/\bar{r}_\Sigma < 1$ in order to avoid the appearance of horizon at the boundary.

6.3 Causal temperature profiles

In this section we consider the physical viability of the *BCD* model. In order to satisfy the condition of pressure isotropy (6.2.6), the *BCD* model assumes $b(r) = 1$ and

$$A = a(r) = (1 + \xi_0 r^2) \tag{6.3.1}$$

The fluid volume collapse rate is

$$\Theta = \frac{3}{A}\frac{\dot{B}}{B} = \frac{3}{(1+\xi_0 r^2)t} \tag{6.3.2}$$

which is the same in both the radial and tangential directions in the absence of shear. The proper stellar radius is given by

$$r_p(t) = \int_0^{b_0} B dr = -Ctb_0 \tag{6.3.3}$$

Since the star is collapsing we require that C be positive which corresponds to $-\infty < t < 0$. We further have

$$C^2 < 4\xi_0(1+\xi_0 r_0^2) \tag{6.3.4}$$

The Einstein field equations (6.2.5a)–(6.2.5c) reduce to

$$\rho = \frac{3}{t^2(1+\xi_0 r^2)^2} \tag{6.3.5a}$$

$$p = \frac{1}{t^2(1+\xi_0 r^2)^2}\left[\frac{4\xi_0}{C^2}(1+\xi_0 r^2) - 1\right] \tag{6.3.5b}$$

$$q = -\frac{4\xi_0 r}{(1+\xi_0 r^2)^2}\frac{1}{C^2 t^3} \tag{6.3.5c}$$

We note that all the above thermodynamical quantities diverge as $t \longrightarrow 0$. The regularity conditions $\rho > 0, p > 0$ and $\rho' < 0$, and $p' < 0$ together with the dominant energy condition, $(\rho - p) > 0$ and the more stringent requirement $(\rho + p) > 2|q|$ are all satisfied when

$$\left[1 - \frac{2\xi_0 r}{C}\right]^2 > -\frac{2\xi_0}{C^2}(1 - \xi_0 r^2) \tag{6.3.6}$$

We can now write

$$1 - \frac{2m_\Sigma}{\bar{r}_\Sigma} = \left[1 - \frac{C^2 r_0^2}{(1+\xi_0 r_0^2)^2}\right] \tag{6.3.7}$$

We note that that when

$$C^2 < 1/r_0^2 + \xi_0^2 r_0^2 + 2\xi_0 \tag{6.3.8}$$

the boundary surface can never reach the horizon (Banerjee *et al* 2002). Furthermore, the surface redshift is given by

$$1 + z_{\Sigma} = \left(1 + r_0 \frac{b_0'}{b_0} + r_0 \dot{b}_0\right)^{-1} \tag{6.3.9}$$

which diverges for an observer at infinity at the time of the appearance of the horizon. For the *BCD* model (6.3.9) reduces to

$$1 + z_{\Sigma} = (1 - C r_0)^{-1} \tag{6.3.10}$$

which diverges when $C = 1/r_0$. In order to avoid the divergence of the surface redshift we must have

$$1/r_0^2 < C^2 < 1/r_0^2 + \xi_0^2 r_0^2 + 2\xi_0 \tag{6.3.11}$$

where we have taken (6.3.8) into account. The luminosity of the star as perceived by an observer at infinity is given by

$$L = -\frac{dm}{dv} = \frac{c^3 r^3 (1 + \xi_0 r^2 - rC)}{(1 + \xi_0 r^2)^4} \tag{6.3.12}$$

which is independent of time. We now turn our attention to the evolution of the temperature profiles of the *BCD* model. To this end we employ the causal transport equation for the heat flux, which in the absence of rotation and viscous stress, is given by

$$\tau {h_a}^b \dot{q}_b + q_a = -\kappa (D_a T + T \dot{u}^a) \tag{6.3.13}$$

where τ is the relaxation time for the thermal signals. For constant collision time ($\omega = 0$), the causal temperature profile is given by

$$\begin{aligned} T^4(r,t) \quad = \quad & \frac{8\beta\xi_0 \left[2(r_0^2 - r^2) + \xi_0 (r_0^4 - r^4)\right]}{\alpha t^2 (1 + \xi_0 r^2)^4} \\ & + \frac{8\xi_0 \left[3(r^2 - r_0^2) + 3\xi_0 (r^4 - r_0^4) + \xi_0^2 (r^6 - r_0^6)\right]}{3\alpha t (1 + \xi_0 r^2)^4} \\ & + \left(\frac{L}{4\pi\delta}\right) \frac{1}{r_0^2 c^2 t^2} \left(\frac{1 + \xi_0 r_0^2}{1 + \xi_0 r^2}\right)^4 \end{aligned} \tag{6.3.14}$$

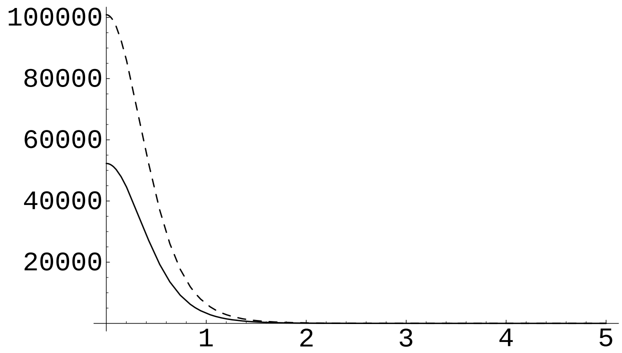

Figure 6.1: Temperature profiles for constant collision time, (close to equilibrium - solid line), (far from equilibrium - dashed line) versus r.

where L is given by (6.3.12) and δ is a constant. For $\omega = 4$, the causal temperature is given by

$$\begin{aligned} T^4(r,t) &= \frac{8\beta\xi_0}{\alpha t^2(1+\xi_0 r^2)^3}\left[\left(\frac{1+\xi_0 r_0^2}{1+\xi_0 r^2}\right)[8+\alpha t(1+\xi_0 r_0^2)]e^{\frac{8\xi_0}{\alpha t}\left(\frac{r^2-r_0^2}{(1+\xi_0 r^2)(1+\xi_0 r_0^2)}\right)}\right. \\ & \left. -[8+\alpha t(1+\xi_0 r^2)]\right] + \frac{512\beta\xi_0 e^{-\left(\frac{8}{\alpha t(1+\xi_0 r^2)}\right)}}{\alpha^2 t^3(1+\xi_0 r^2)^4}\left[\mathrm{E}i\left(\frac{8}{\alpha t(1+\xi_0 r^2)}\right)\right. \\ & \left. -\mathrm{E}i\left(\frac{8}{\alpha t(1+\xi_0 r_0^2)}\right)\right] + \left[\frac{1+\xi_0 r_0^2}{1+\xi_0 r^2}\right]^4 \frac{L}{(4\pi\delta)r_0^2 c^2 t^2} e^{\frac{8\xi_0}{\alpha t}\left[\frac{r^2-r_0^2}{(1+\xi_0 r^2)(1+\xi_0 r_0^2)}\right]} \end{aligned} \tag{6.3.15}$$

where the exponential integral $\mathrm{E}i(z)$ is defined as

$$\mathrm{E}i(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt \tag{6.3.16}$$

We note that the noncausal temperature ($\beta = 0$) and causal temperature are equal at the boundary ($r = r_0$).

Figure 6.1 shows that the relaxational effects are dominant when the stellar fluid is far from equilibrium (large values of β). In the case of variable collision time, figure 6.2, we see that the causal temperature is everywhere greater than the corresponding noncausal temperature within the stellar interior. Furthermore, figures 6.1 and 6.2 indicate that the causal temperatures at late times (large values of β) decrease more rapidly than the causal temperatures when the star is close to quasi-static equilibrium.

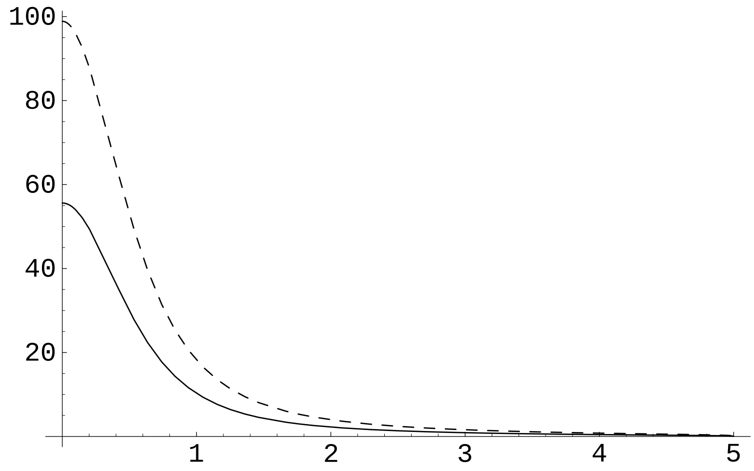

Figure 6.2: Temperature profiles for variable collision time, (close to equilibrium - solid line), (far from equilibrium - dashed line) versus r.

This is in agreement with the perturbative results of Herrera and Santos (1997b) as well as the acceleration-free model studied by Govender *et al* (1998).

6.4 Concluding remarks

We have investigated the physical viability of the *BCD* model within the framework of extended irreversible thermodynamics. We have shown that this simple model allows us greater insight into the evolution of the temperature for different collision times. More importantly, we were able to confirm earlier findings that the causal temperature dominates the Eckart temperature within the stellar core, even for variable collision time. As pointed out in earlier treatments, the constant collision time approximation is only valid for a limited period of the stellar evolution (Naidu *et al* 2006). One expects that the collision time between the particles making up the stellar fluid to change with temperature. Such effects on the evolution of the temperature profiles were clearly demonstrated with the variable collision time solution. It must be pointed out that the truncation of the transport equations leads naturally to an implicitly defined temperature law (Govender and Govinder 2001a). Such a temperature law may only be valid for a limited period of collapse. What remains is to investigate the behaviour of the temperature by employing the full transport equation for the heat flux as well as to include the effects of shear. The general framework for such an investigation has recently been provided by Herrera and Santos (2004a) for a spherically symmetric radiating star.

Chapter 7

Conclusion

In this study involving the dynamics of dissipative gravitational collapse, an analytical model for the anisotropic, shearing case was presented. The relaxation time for the heat flux was shown to differ from that of the shear flux. In addition, it was confirmed that the causal temperature dominates the Eckart temperature within the stellar core, even for variable collision time. The behaviour of the temperature within the framework of extended irreversible thermodynamics was studied. We also studied the thermodynamics of a radiating star undergoing collapse, with dissipation, avoiding the formation of a horizon. The matching conditions for spherically symmetric shearing and shear-free line elements that match to the Vaidya line element, were generated. Future work involves including the effects of a cosmological constant, rotation, and charge.

Previously, there was no solution to the Einstein field equations which described the gravitational collapse of a radiating star in the presence of shear and pressure anisotropy. Our model in chapter 5 represents the first original research in this direction and was published in 2006. This solution was later extended by Misthry *et al* (2008).

The analysis of the thermodynamics of a radiating star undergoing gravitational collapse without the formation of a horizon, was presented in chapter 6. These original results extend the model of Banerjee *et al* (2002), and they have been recently published (Naidu and Govender 2007).

These simple models of radiative collapse make transparent the physics at play and also serve as a check for more complicated numerical models. Radiating stellar models studied here can also act as precursor models for the study of the nature of the singularities found during collapse. Our models narrow the window on the initial conditions for the study of continued gravitational collapse.

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