A Survey of the Hodge-Arakelov Theory of Elliptic Curves I

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Abstract:

The purpose of the present manuscript is to give a survey of the *Hodge-Arakelov theory of elliptic curves* (cf. [Mzk1,2]) — i.e., a sort of "Hodge theory of elliptic curves" analogous to the classical *complex and p-adic* Hodge theories, but which exists in the *global arithmetic framework of Arakelov theory* — as this theory existed at the time of the workshop on "Galois Actions and Geometry" held at the Mathematical Sciences Research Institute (MSRI) at Berkeley, USA, in October 1999. Since then, various further important developments have occurred in this theory (cf. [Mzk3,4,5], etc.), but we shall not discuss these developments in detail in the present manuscript.

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Section 1: The Discretization of Local Hodge Theories

§1.1. The Main Theorem

The fundamental result of the *Hodge-Arakelov theory of elliptic curves* is a *Comparison Theorem* (cf. Theorem A below) for elliptic curves, which states roughly that:

> The space of "polynomial functions" of degree (roughly) $< d$ on the *universal extension of an elliptic curve* maps isomorphically via restriction to the space of (set-theoretic) functions on the *d-torsion points* of the universal extension.

This rough statement is essentially precise for (smooth) elliptic curves over fields of characteristic zero (cf. Theorem $\mathrm{A}^{\mathrm{simple}}$). For elliptic curves in mixed characteristic and degenerating elliptic curves, this statement may be made precise (i.e., the restriction map becomes an isomorphism) if one *modifies the "integral structure" on the space of polynomial functions* in an appropriate fashion (cf. Theorem A). Similarly, in the case of elliptic curves over the complex numbers, one can ask whether or not one obtains an *isometry* if one puts natural Hermitian metrics on the spaces involved. In [Mzk1], we also compute *what modification to these metrics is necessary to obtain an isometry* (or something very close to an isometry).

In characteristic zero, the universal extension of an elliptic curve may be regarded as the *de Rham cohomology of the elliptic curve*, with coefficients in the sheaf of invertible functions on the curve. On the other hand, the torsion points of the elliptic curve may be regarded as a portion of the *étale cohomology of the elliptic curve*. Thus, one may regard this Comparison Theorem as a sort of isomorphism between the de Rham and étale cohomologies of the elliptic curve, given by considering *functions* on each of the respective cohomology spaces. When regarded from this point of view, *this Comparison Theorem may be thought of as a sort of discrete or Arakelov-theoretic analogue of the usual comparison theorems between de Rham and étale/singular cohomology* in the complex and p-adic cases. This analogy with the "classical" local comparison theorems can be made very precise, and is one of the main topics of [Mzk1], Chapter IX.

> Using this point of view, we apply the Comparison Theorem to construct a *global/Arakelov-theoretic analogue* for elliptic curves over number fields of the Kodaira-Spencer morphism of a family of elliptic curves over a geometric base.

This *arithmetic Kodaira-Spencer morphism* will be discussed in detail in §1.4.

Suppose that E is an *elliptic curve over a field K of characteristic zero*. Let d be a positive integer, and $\eta \in E(K)$ a torsion point of order not dividing d. Write

$$\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(d \cdot [\eta])$$

for the line bundle on E corresponding to the divisor of multiplicity d with support at the point η. Write

$$E^\dagger \to E$$

for the *universal extension of the elliptic curve*, i.e., *the moduli space of pairs* $(\mathcal{M}, \nabla_{\mathcal{M}})$ *consisting of a degree zero line bundle* $\mathcal{M}$ *on* E*, together with a connection* $\nabla_{\mathcal{M}}$. Thus, $E^\dagger$ is an affine torsor on E under the module ω_E of invariant differentials on E. In particular, since $E^\dagger$ is (Zariski locally over E) the spectrum of a polynomial algebra in one variable with coefficients in the sheaf of functions on E, it makes sense to speak of the "relative degree over E" – which we refer to in this paper as the *torsorial degree* – of a function on $E^\dagger$. Note that (since we are in characteristic zero) the subscheme $E^\dagger[d] \subseteq E^\dagger$ of d-torsion points of $E^\dagger$ maps isomorphically to the subscheme $E[d] \subseteq E$ of d-torsion points of E. Then in its simplest form, the main theorem of [Mzk1] states the following:

Theorem $\mathbf{A}^{\text{simple}}$. *Let E be an elliptic curve over a field K of characteristic zero. Write $E^\dagger \to E$ for its universal extension. Let d be a positive integer, and $\eta \in E(K)$ a torsion point whose order does not divide d. Write $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta])$. Then the natural map*

$$\Gamma(E^\dagger, \mathcal{L})^{<d} \to \mathcal{L}|_{E^\dagger[d]}$$

given by restricting sections of $\mathcal{L}$ over $E^\dagger$ whose torsorial degree is $< d$ to the d-torsion points $E^\dagger[d] \subseteq E^\dagger$ is a **bijection** *between K-vector spaces of dimension d^2.*

The remainder of the main theorem essentially consists of specifying precisely *how one must modify the integral structure of* $\Gamma(E^\dagger, \mathcal{L})^{<d}$ *over more general bases* in order to obtain an isomorphism at the finite and infinite primes of a number field, as well as for degenerating elliptic curves.

Let us first consider the integral structures on the left-hand (de Rham side) of the isomorphism of Theorem $\mathrm{A}^{\text{simple}}$ necessary to make this isomorphism extend to an isomorphism at *finite primes* and *for degenerating elliptic curves* (i.e., as the "q-parameter" goes to zero). These integral structures may be described as follows. Let us work over a formal neighborhood of the point at infinity on the moduli stack of elliptic curves — i.e., say, the spectrum of a ring of power series of the form

$$S \stackrel{\text{def}}{=} \mathrm{Spec}(\mathcal{O}[[q^{\frac{1}{N}}]])$$

where $\mathcal{O}$ is a Dedekind domain of mixed characteristic, q an indeterminate, and N a positive integer. Write $\widehat{S}$ for the completion of S with respect to the q-adic topology, and $E \to S$ for the *tautological degenerating elliptic curve* (more precisely: one-dimensional semi-abelian scheme), with "q-parameter" equal to q. Then we have natural isomorphisms

$$E|_{\widehat{S}} \cong \mathbb{G}_m; \quad \omega_E = \mathcal{O}_S \cdot d\ \log(U); \quad E^\dagger|_{\widehat{S}} \cong \mathbb{G}_m \times \mathbb{A}^1$$

where we write U for the usual multiplicative coordinate on $\mathbb{G}_m$ (cf. [Mzk1], Chapter III, Theorem 2.1; the discussion of [Mzk1], Chapter III, §6, for more details). If we write "T" for the standard coordinate on this affine line, then near infinity, the *standard integral structure* on $E^\dagger$ may be described as that given by

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot T^r$$

On the other hand, the *"étale integral structure"* on $E^\dagger$ — i.e., the integral structure on $E^\dagger$ that makes the restriction morphism of Theorem $\mathrm{A}^{\mathrm{simple}}$ an isomorphism in *mixed characteristic* — is given by

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot \binom{d \cdot \{T - (i_\chi/n)\}}{r}$$

where $i_\chi/n \in \mathbb{Q}$ is an invariant determined by the torsion point η, and $\binom{X}{r} \stackrel{\mathrm{def}}{=} \frac{1}{r!}X(X-1) \cdot \ldots \cdot (X-(r-1)))$. Although the above definition of this integral structure is only valid near infinity, this integral structure may, in fact, be extended over the *entire moduli stack of elliptic curves* (over $\mathbb{Z}$). This fact is discussed in [Mzk1], Chapter V, §3, and [Mzk3], §9. Near infinity, one must further modify this integral structure by introducing certain poles — which we refer to as *Gaussian poles* (since they essentially look like "the exponential of a quadratic function") —

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot \binom{d \cdot \{T - (i_\chi/n)\}}{r} \cdot q^{-\mathbf{a}_r}$$

where

$$\mathbf{a}_r \approx \frac{r^2}{8d}$$

(for a precise discussion of $\mathbf{a}_r$, cf. [Mzk1], Chapter VI, Theorem 3.1, (3); [Mzk1], Chapter V, §4).

We are now ready to state the more general version of the comparison isomorphism (albeit in a somewhat "digested form"). For a more precise (and much more lengthy and technical statement!), we refer to [Mzk1], Introduction, Theorem A. In the remainder of this paper, we will use the following

Notation and Conventions:

We will denote by $(\overline{\mathcal{M}}_{\mathrm{ell}}^{\log})_{\mathbb{Z}}$ the *log moduli stack of log elliptic curves over* $\mathbb{Z}$ (cf. [Mzk1], Chapter III, Definition 1.1), where the log structure is that defined by the divisor at infinity. The open substack of $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$ parametrizing (smooth) elliptic curves will be denoted by $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Z}} \subseteq (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$.

Theorem A. (The Hodge-Arakelov Comparison Isomorphism) *If one equips the left-hand side (de Rham side) of the restriction morphism of Theorem A^{simple} with the "étale integral structure" and "Gaussian poles" just discussed, then this restriction morphism becomes an isomorphism* **over the entire moduli stack** *$(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$, except possibly over the schematic intersection of the torsion point η with the scheme $E[d] \subseteq E$ of d-torsion points. Moreover, at* **archimedean primes**, *the left and right hand sides of this restriction morphism admit natural Hermitian metrics with respect to which the deviation of the restriction morphism from being an isometry may be estimated using Hermite, Legendre, and binomial coefficient polynomials (cf. the discussion below, as well as [Mzk1], Introduction, Theorem A, for more details).*

Before continuing, we remark that the factor of $\frac{1}{8d}$ appearing in the *Gaussian poles* in the exponent may be justified by the following calculation: On the one hand, if one thinks in terms of degrees of vector bundles on $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{C}}$, the degree of the left-hand side of the comparison isomorphism (without Gaussian poles!) goes roughly as

$$\sum_{j=0}^{d-1} \deg(\tau_E^{\otimes j}) = \sum_{j=0}^{d-1} j \cdot \deg(\tau_E) \approx \frac{1}{2}d^2 \cdot \deg(\tau_E) = -\frac{1}{2}d^2 \cdot \frac{1}{12}\log(q) = -\frac{d^2}{24} \cdot \log(q)$$

where "$\log(q)$" is a symbol that stands for the element in $\mathrm{Pic}((\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{C}})$ defined by the divisor at infinity. (It turns out that the contribution to the degree by the line bundle $\mathcal{L}$ is negligible.) On the other hand, the sum of the degrees resulting from the Gaussian poles is

$$\sum_{j=0}^{d-1} \frac{j^2}{8d} \cdot \log(q) \approx \frac{1}{3}d^3 \cdot \frac{1}{8d} \cdot \log(q) = \frac{d^2}{24} \cdot \log(q)$$

In other words, the factor of $\frac{1}{8d}$ is just enough to make the total degree 0. Since the restriction of $\mathcal{L}$ to the torsion points is (essentially) a "torsion line bundle" (i.e., some tensor power of it is trivial), the degree of the range of the evaluation map is zero – i.e., *the factor of $\frac{1}{8d}$ is just enough to make the degrees of the domain and range of the evaluation map of Theorem A equal* (which is natural, since we want this evaluation map to be an isomorphism). In fact,

> *It turns out that the original proof (cf. [Mzk1], Chapter VI, §3) of the characteristic zero portion of Theorem A (i.e., Theorem A^{simple}), is based on precisely this sort of "summation of degrees" argument.*

In this proof, however, in order to get an exact isomorphism, it is necessary to compute all the degrees involved precisely. This computation requires a substantial amount of work (involving, for instance, the theory of [Zh]) and is carried out in [Mzk1], Chapters IV, V, VI. An alternative — and, in the opinion of the author, much *simpler and more elegant* (albeit somewhat less explicit!) — proof of the characteristic zero portion of Theorem A using *characteristic* p *methods* (i.e., "Frobenius and Verschiebung") is given in [Mzk5].

On the other hand, the key point of the archimedean portion of Theorem A is the comparison of what we refer to as the *étale and de Rham metrics* $|| \sim ||_{\text{et}}$, $|| \sim ||_{\text{DR}}$ (which are naturally defined on the range and domain of the comparison isomorphism, respectively). Unfortunately, we are unable to prove a simple sharp result that they always coincide (relative to the comparison isomorphism). Instead, we choose three natural "domains of investigation" – which we refer to as *models* – where we *estimate the difference between these two metrics* using a particular system of functions which are well-adapted to the domain of investigation in question. One of the most important features of these three models is that they each have natural *scaling factors* associated to them. The three models, along with their natural scaling factors, and natural domains of applicability are as follows:

$$\underline{\text{Hermite Model}}\ (\text{scaling factor} = d^{\frac{1}{2}}) : \text{nondegenerating } E, \text{ fixed } r < d$$

$$\underline{\text{Legendre Model}}\ (\text{scaling factor} = d) : \text{nondegenerating } E, \text{ varying } r < d$$

$$\underline{\text{Binomial Model}}\ (\text{scaling factor} = 1) : \text{degenerating } E$$

It is interesting to observe that the exponents appearing in these scaling factors, i.e., $0, \frac{1}{2}, 1$, which we refer to as *slopes*, are precisely the *same as the slopes that appear when one considers the action of Frobenius on the crystalline cohomology of an elliptic curve at a finite prime* – cf. the discussions at the end of [Mzk1], Chapter VII, §3, 6, for more on this analogy.

§1.2. Technical Roots

Let K be an *algebraically closed field of characteristic* 0. Let E be an *elliptic curve over* K. Let $\mathcal{L}$ be the line bundle of Theorem A^{simple}. Then instead of considering sections of $\mathcal{L}$ over $E^\dagger$, one can consider sections of $\mathcal{L}$ over E. Such sections may be restricted to $\mathcal{L}|_{E[d]}$. Moreover, by the theory of *algebraic theta functions* (cf. [Mumf1,2,3]), the restriction $\mathcal{L}|_{E[d]}$ of $\mathcal{L}$ to the d-torsion points $E[d] \subseteq E$ admits a *canonical trivialization*

$$\mathcal{L}|_{E[d]} \cong \mathcal{L}|_{0_E} \otimes_K \mathcal{O}_{E[d]}$$

(where $0_E \in E(K)$ is the zero element) — at least when d is *odd*. Thus, by composing the restriction morphism with this trivialization, we obtain a morphism (as in [Mumf1,2,3]):

$$\Gamma(E, \mathcal{L}) \hookrightarrow \mathcal{L}|_{E[d]} \cong \mathcal{L}|_{0_E} \otimes_K \mathcal{O}_{E[d]}$$

i.e., one may think of sections of $\mathcal{L}$ over E as *functions* on $E[d]$. These functions are Mumford's "algebraic theta functions."

Now let us observe that $\dim_K(\Gamma(E, \mathcal{L})) = d$, while $\dim_K(\mathcal{L}|_{0_E} \otimes_K \mathcal{O}_{E[d]}) = d^2$. That is to say, *Mumford's theory only addresses a fraction (more precisely: $\frac{1}{d}$) of the functions in $\mathcal{L}|_{0_E} \otimes_K \mathcal{O}_{E[d]}$*. Thus, it is natural to ask:

> *Is there a natural extension of Mumford's theory that allows one to give meaning to all the functions of $\mathcal{L}|_{0_E} \otimes_K \mathcal{O}_{E[d]}$ as some sort of "global" sections of $\mathcal{L}$?*

Theorem $\mathrm{A}^{\mathrm{simple}}$ provides a natural, affirmative answer to this question: i.e., it states that these functions may be interpreted naturally as the *sections of $\mathcal{L}$ over the universal extension $E^\dagger$ of torsorial degree $< d$.*

In more classical terms, to consider the universal extension amounts essentially to considering *the derivatives of (classical) theta functions* (cf., e.g., [Katz1], Appendix C). For instance, if one takes $K = \mathbb{C}$, and writes

$$\theta_\tau(z) \stackrel{\mathrm{def}}{=} \sum_{n \in \mathbb{Z}} e^{\pi i \tau \cdot n^2} \cdot e^{2\pi i z \cdot n}$$

for the "*standard theta function*" (where $z \in \mathbb{C}$, $\tau \in \mathfrak{H} \stackrel{\mathrm{def}}{=} \{w \in \mathbb{C} \mid \mathrm{Im}(w) > 0\}$), then up to the operation of taking the Fourier expansion, this theta function is essentially a "*Gaussian $e^{\pi i \tau \cdot n^2}$*," and its derivatives $P(\frac{\partial}{\partial z}) \cdot \theta_\tau(z)$ (where $P(-)$ is a polynomial with coefficients in $\mathbb{C}$) are given by *polynomial multiples of (which are equivalent to derivatives of) the Gaussian*:

$$P(2\pi i \cdot n) \cdot e^{\pi i \tau \cdot n^2}$$

Just as theta functions are the "fundamental functions on an elliptic curve" (more precisely: generate the space of sections of $\mathcal{L}$ over E), these derivatives are the *"fundamental functions on the universal extension of the elliptic curve"* (more precisely: generate the space of sections of $\mathcal{L}$ over $E^\dagger$). This point of view is discussed in more detail in [Mzk1], Chapter III, §5, 6, 7; [Mzk1], Chapter VII, §6. As one knows from elementary analysis, the most natural polynomial multiples/derivatives of a Gaussian are those given by the *Hermite polynomials*. It is thus natural to expect that the Hermite polynomials should appear naturally in the portion of this theory concerning the behavior of the comparison isomorphism at archimedean primes. This intuition is made rigorous in the theory of [Mzk1], Chapters IX, X. In fact, more generally:

> *The essential model that permeates the Hodge-Arakelov theory of elliptic curves is that of the Gaussian and its derivatives. This*

model may be seen especially in the "Gaussian poles," as well as in the "Hermite model" at the infinite prime (cf. §1.1, Theorem A).

In the classical theory over $\mathbb{C}$, the most basic derivative of the theta function is the so-called *Weierstrass zeta function* (cf. [Katz1], Appendix C). It should thus not be surprising to the reader that various generalizations of the Weierstrass zeta function – which we refer to as *Schottky-Weierstrass zeta functions* (cf. [Mzk1], Chapter III, §6, 7) – play a fundamental role in [Mzk1].

So far, in the above discussion, we concentrated on *smooth* elliptic curves. On the other hand, when one wishes to consider *degenerating elliptic curves*, Zhang has constructed a theory of *metrized line bundles* on such degenerating elliptic curves (cf. [Zh]). In this theory, one can consider the *curvatures* of such metrized line bundles, as well as intersection numbers between two metrized line bundles in a fashion entirely similar to Arakelov intersection theory. Using Zhang's theory of metrized line bundles, it is not difficult to *extend Mumford's theory of algebraic theta functions* in a natural fashion to metrized ample line bundles on degenerating elliptic curves (cf., e.g., [Mzk1], Chapter IV, §5, for more details). Unfortunately, however, just as Mumford's theory only addresses sections over the original elliptic curve (as opposed to over the universal extension, as discussed above), Zhang's theory also only deals with the theory of metrized line bundles over the original (degenerating) elliptic curve. Thus, it is natural to ask whether one can generalize Theorem $\mathrm{A}^{\mathrm{simple}}$ to the case of degenerating elliptic curves in such a way that the resulting generalization of the portion of $\Gamma(E^\dagger, \mathcal{L})$ arising from sections over E is compatible with Zhang's theory of metrized line bundles (and their sections) over E. In other words, it is natural to ask:

Can one "de Rham-ify" the theory of [Zh], so that it addresses the "metric" behavior of sections of $\mathcal{L}$ not only over E, but over $E^\dagger$, as well?

An affirmative answer to this question is given by the theory of *Gaussian poles*, or "*analytic torsion at the divisor at infinity*" – cf. [Mzk1], Chapters V, VI.

Another way to view the relation to Zhang's theory is the following. One consequence of the theory of [Zh] is the construction of a natural "metric" (or integral structure) on the space ω_E of invariant differentials on a (degenerating) elliptic curve. If we regard ω_E as a line bundle on the compactified moduli space of elliptic curves, then Zhang's "admissible metric" on ω_E essentially amounts to the (metrized) line bundle $\omega_E(-\frac{1}{12} \cdot \infty)$ (where ∞ is the divisor at infinity of the moduli space), i.e., the line bundle ω_E with integral structure at infinity modified by tensoring with $\mathcal{O}(-\frac{1}{12} \cdot \infty)$. Moreover, it follows from Zhang's theory that there is a natural *trivialization*

$$\omega_E(-\frac{1}{12} \cdot \infty) \cong \mathcal{O}$$

of this metrized line bundle over the moduli space. The 12-th tensor power of this trivialization is the cuspidal modular form usually denoted "Δ" ([KM], Chapter 8, §8.1). Similarly, Theorem A states that by allowing "Gaussian poles" in the sections of $\Gamma(E^\dagger, \mathcal{L})$, one gets a natural isomorphism between $\Gamma(E^\dagger, \mathcal{L})$ (with this modified integral structure) and a vector bundle which is trivial (in characteristic zero) over some finite log étale covering of the compactified moduli (log) stack of elliptic curves. That is to say,

> *One may regard the theory of Gaussian poles/analytic torsion at the divisor at infinity in Theorem A as a sort of "GL_2-analogue" of the isomorphism of line bundles (i.e., $\mathbb{G}_m$-torsors) $\omega_E(-\frac{1}{12} \cdot \infty) \cong \mathcal{O}$ — or, alternatively, a GL_2-analogue of the modular form Δ.*

Note: The reason that we mention "GL_2" is that the vector bundle on the "étale side" of the comparison isomorphism of Theorem A arises naturally (at least in characteristic zero) from a representation (defined by the Galois action on the d-torsion points) of the fundamental group of the moduli stack of elliptic curves into GL_2, whereas the isomorphism $\omega_E(-\frac{1}{12} \cdot \infty) \cong \mathcal{O}$ naturally corresponds to an *abelian* representation (i.e., a representation into $\mathbb{G}_m$ which is, in fact, of order 12) of this fundamental group – cf. [KM], Chapter 8, §8.1.

§1.3. Conceptual Roots

§1.3.1 From Absolute Differentiation to Comparison Isomorphisms

Let K be either a *number field* (i.e., a finite extension of $\mathbb{Q}$) or a *function field in one variable* over some coefficient field k (which we assume to be algebraically closed in K). Let S be the unique one-dimensional regular scheme whose closed points s correspond naturally (via Zariski localization of S at s) to the set of all discrete valuations of K (where in the function field case we assume that the elements of $k^\times$ are units for the valuations). We shall call S the *complete model* of K. Of course, in the number field case, it is natural to "formally append" to S the set of archimedean valuations of K.

Let

$$E \to S$$

be a *one-dimensional, generically proper semi-abelian scheme* over S, i.e., $E_K \stackrel{\text{def}}{=} E \times_S K$ is an *elliptic curve* over K with semi-stable reduction everywhere. Then E defines a *classifying morphism*

$$\alpha : S \to (\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Z}}$$

to the compactified moduli stack of (log) elliptic curves over $\mathbb{Z}$. It is natural to endow S with the log structure arising from the set of closed points at which $E \to S$ has bad reduction. Then α extends to a morphism $\alpha^{\log} : S^{\log} \to (\overline{\mathcal{M}}_{\mathrm{ell}}^{\log})_{\mathbb{Z}}$ in the logarithmic category.

Now, *in the function field case, if we differentiate* α*, we obtain the* **Kodaira-Spencer morphism** *of* $E \to S$:

$$\kappa_E : \omega_E^{\otimes 2} \cong \alpha^* \Omega_{(\overline{\mathcal{M}}_{\mathrm{ell}}^{\log})_{\mathbb{Z}}} \to \Omega_{S^{\log}/k}$$

(where ω_E is the restriction to the identity section of $E \to S$ of the relative cotangent bundle of E over S). Since ω_E is naturally the pull-back via α of an *ample line bundle on* $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$, and κ_E is typically nonzero (for instance, it is always nonzero if K is of characteristic zero and $E \to S$ is not isotrivial (i.e., trivial after restriction to a finite covering of S)), the existence of the Kodaira-Spencer morphism κ_E gives rise to a *bound* on the height of $E \to S$ by the degree of $\Omega_{S^{\log}/k}$. The thrust of a family of conjectures due to Vojta (cf. [Lang], [Vojta]) is that *this bound* (or at least, a bound roughly similar to this bound) in the geometric case (i.e., the case when K is a function field) *also holds in the "arithmetic case"* (i.e., the case when K is a number field). Thus,

> *In order to prove Vojta's Conjecture in the arithmetic case, it is natural to attempt to construct some sort of arithmetic analogue of the Kodaira-Spencer morphism.*

Indeed, this point of view of approaching the verification of some inequality by first trying to construct "the theory underlying the inequality" is reminiscent of the approach to proving the Weil Conjectures (which may be thought of as *inequalities* concerning the number of rational points of varieties over finite fields) by attempting to construct a "Weil cohomology theory" for varieties over finite fields which has enough "good properties" to allow a natural proof of the Weil Conjectures.

Of course, if one tries to construct any sort of naive analogue of the Kodaira-Spencer morphism in the arithmetic case, one immediately runs into a multitude of fundamental obstacles. In some sense, these obstacles revolve around the fact that the ring of rational integers $\mathbb{Z}$ does not admit *"a field of absolute constants"* $\mathbb{F}_1 \subseteq \mathbb{Z}$. If such a field of absolute constants existed, then one could consider "*absolute differentials* $\Omega_{\mathbb{Z}/\mathbb{F}_1}$," or

$$\text{"}\Omega_{\mathcal{O}_K/\mathbb{F}_1}\text{"}$$

Moreover, since moduli spaces tend to be rather absolute and fundamental objects, it is natural to imagine that if one had a field of absolute constants "$\mathbb{F}_1$," then $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$ should *descend naturally* to an object $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{F}_1}$ over $\mathbb{F}_1$, so that one could *differentiate the classifying morphism* $\alpha : S \to (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$ *in the arithmetic case, as well, to obtain an* **arithmetic Kodaira-Spencer morphism**

$$\text{“}\kappa_E : \omega_E^{\otimes 2} \cong \alpha^* \Omega_{(\overline{\mathcal{M}}_{\mathrm{ell}}^{\log})_{\mathbb{F}_1}} \to \Omega_{S^{\log}/\mathbb{F}_1}\text{”}$$

and then use this arithmetic Kodaira-Spencer morphism to prove Vojta's Conjecture concerning the heights of elliptic curves. (Note: In this case, Vojta's Conjecture is also referred to as “Szpiro's Conjecture.”)

Unfortunately, this sort of “absolute field of constants $\mathbb{F}_1$” does not, of course, exist in any naive sense. Thus, it is natural to look for a more indirect, abstract approach. In the geometric case, when $k = \mathbb{C}$, the algebraic curve S defines a *Riemann surface* S^{an}. Let us write $U_S \subseteq S$ for the open subobject where the log structure of $S^{\log}$ is trivial. Then the *first singular cohomology module of the fibers of* $E \to S$ naturally forms a local system

$$H^1_{\mathrm{sing}}(E/S, \mathbb{Z})$$

on the Riemann surface U_S^{an}. One the other hand, the *first de Rham cohomology module* of the fibers of $E \to S$ forms a rank two vector bundle

$$H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$$

on U_S^{an}. This de Rham cohomology admits a *Hodge filtration*, which may be thought of as a natural exact sequence:

$$0 \to \omega_E \to H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \to \tau_E \to 0$$

Moreover, this vector bundle $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ on U_S^{an} admits a connection ∇_{DR} — *called the Gauss-Manin connection* — which allows one to differentiate sections of $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$. Using this connection ∇_{DR} to *differentiate the Hodge filtration* gives rise to a natural morphism

$$\Theta_{S^{\log}/k} \to \tau_E^{\otimes 2}$$

(where $\Theta_{S^{\log}/k}$ is the dual to $\Omega_{S^{\log}/k}$) which is dual to the Kodaira-Spencer morphism κ_E. Thus,

> *Another way to think of our search for “$\mathbb{F}_1$” or “a notion of absolute differentiation” is as the search for an arithmetic analogue of the Gauss-Manin connection ∇_{DR} on the de Rham cohomology $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$.*

This is the first step towards raising our search for an arithmetic Kodaira-Spencer morphism to a more abstract level.

Next, let us recall that the *de Rham isomorphism* defines a natural isomorphism

$$H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \cong H^1_{\mathrm{sing}}(E/S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}_{U_S^{\mathrm{an}}}$$

(in the complex analytic category) over U_S^{an}. Moreover, the sections of $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ defined (via this isomorphism) by sections of $H^1_{\mathrm{sing}}(E/S, \mathbb{Z})$ are *horizontal* for ∇_{DR}. Thus, we conclude that:

> *To construct the Gauss-Manin connection* ∇_{DR}*, it is enough to know the de Rham isomorphism between de Rham and singular cohomology.*

The de Rham isomorphism is a special case of the general notion of a *Comparison Isomorphism* between de Rham and singular/étale cohomology. In the last few decades, this sort of Comparison Isomorphism has been constructed over p-adic bases, as well (cf., e.g., [Falt1,2], [Hyodo]). In the arithmetic case, we would like to construct some sort of analogue of the Kodaira-Spencer morphism over S which also has natural integrality properties at the archimedean places, as well (since we would like to use it conclude inequalities concerning the height of the elliptic curve E_K). Put another way, we would like to construct some sort of arithmetic Kodaira-Spencer morphism in the context of *Arakelov theory.* Thus, in summary, the above discussion suggests that:

> *In order to construct this sort of arithmetic Kodaira-Spencer morphism, a natural approach is to attempt to construct some sort of* **Comparison Isomorphism in the "Arakelov theater,"** *analogous to the well-known complex and p-adic Comparison Isomorphisms between de Rham and étale/singular cohomology.*

The construction of such an Arakelov-theoretic Comparison Theorem is the main goal of the Hodge-Arakelov theory of elliptic curves. *To a certain extent, this goal is achieved by Theorem A* (cf. §1.1). For a detailed explanation of the sense in which the Comparison Isomorphism of Theorem A is analogous to the well-known complex and p-adic Comparison Isomorphisms, we refer to [Mzk1], Chapter IX. Unfortunately, however, for various technical reasons, the arithmetic Kodaira-Spencer morphism that naturally arises from Theorem A is not well enough understood at the time of writing to allow its application to a proof of Vojta's Conjectures (for more on these "technical reasons," cf. §1.5.1 below). In the remainder of the present §1.3, we would like to explain in detail how we were led to Theorem A as a global, Arakelov-theoretic analogue of the well-known "local Comparison Isomorphisms."

§1.3.2 A Function-Theoretic Comparison Isomorphism

In §1.3.1, we saw that one way to think about absolute differentiation or an absolute/arithmetic Kodaira-Spencer morphism is to regard such objects as natural consequences of a *"global Hodge theory,"* or *Comparison Isomorphism* between the de Rham and étale cohomologies of an elliptic curve. The question then arises:

> *Just what form should such a Global Comparison Isomorphism — i.e., in suggestive notation*
>
> $$H^1_{\mathrm{DR}}(E) \otimes ?? \;\cong\; H^1_{\mathrm{et}}(E) \otimes ??$$
>
> *— take?*

For instance, over $\mathbb{C}$, such a comparison isomorphism exists naturally over $\mathbb{C}$, i.e., when one takes $?? = \mathbb{C}$. In the p-adic case, one must introduce rings of *p-adic periods* such as $\mathbb{B}_{dR}$, $\mathbb{B}_{crys}$ (cf., e.g., [Falt2]) in order to obtain such an isomorphism. Thus, we would like to know over if there is some sort of natural "ring of global periods" over which we may expect to obtain our global comparison isomorphism.

In fact, in the comparison isomorphism obtained in [Mzk1] (cf. §1.1, Theorem A), unlike the situation over complex and p-adic bases, we do not work over some "global ring of periods." Instead, the situation is somewhat more complicated. Roughly speaking, what we end up doing is the following:

> *In the Hodge-Arakelov Comparison Isomorphism, we obtain a comparison isomorphism between the de Rham and étale cohomologies of an elliptic curve by considering functions on the de Rham and étale cohomologies of the elliptic curve and then constructing an isomorphism between the two resulting* **function spaces** *which is (essentially) an* **isometry** *with respect to natural metrics on these function spaces at all the primes of the base.*

Indeed, for instance over a number field, the de Rham cohomology and étale cohomology are finite modules over very different sorts of rings (i.e., the ring of integers of the number field in the de Rham case; the profinite completion of $\mathbb{Z}$, or one of its quotients in the étale case), and it is difficult to imagine the existence of a natural "global arithmetic ring" containing both of these two types of rings. (Note here that unlike the case with Shimura varieties, the adèles are not a natural choice here for a number of reasons. Indeed, to consider the adèles here roughly amounts to simply forming the direct product of the various local (i.e., complex and p-adic) comparison isomorphisms, which is not very interesting in the sense that such a simple direct product does not result in any natural *global structures*.) Thus:

> *The idea here is to abandon the hope of obtaining a global* **linear** *isomorphism between the de Rham and étale cohomology* **modules**, *and instead to look for an isomorphism (as mentioned above) between the corresponding function spaces which does not necessarily arise from a linear morphism between modules.*

In the present §, we explain how we were led to look for such a "function-theoretic comparison isomorphism," while in §1.3.3 below, we examine the meaning of the nonlinearity of this sort of comparison isomorphism.

In order to understand the motivating circumstances that naturally lead to the introduction of this sort of function-theoretic point of view, we must first return to the discussion of the case over the complexes in §1.3.1 above. Thus, in the following discussion, we use the notation of the discussion of the complex case in §1.3.1. One more indirect way to think about the existence of the Kodaira-Spencer morphism is the following. Recall the exact sequence

$$0 \to \omega_E \to H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \to \tau_E \to 0$$

which in fact exists naturally over S^{an}. The Gauss-Manin connection ∇_{DR} acts on the middle term of this exact sequence (as a connection with logarithmic poles at the points of bad reduction), but does not preserve the image of ω_E. One important consequence of this fact is that:

> *(If one imposes certain "natural logarithmic conditions" on the splitting at points of bad reduction, then) this exact sequence does not split.*

Indeed, if this exact sequence split, then one could use this splitting to obtain a *connection on* ω_E induced by ∇_{DR}. Moreover, if the "natural logarithmic conditions" are satisfied, it would follow that this connection on ω_E has zero monodromy at the points of bad reduction, i.e., that the connection is regular over all of S^{an}. But since ω_E is the pull-back to S of an *ample* line bundle on $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$, it follows (so long as $E \to S$ is not isotrivial) that $\deg(\omega_E) \neq 0$, hence that the line bundle ω_E cannot admit an everywhere regular connection. That is, we obtain a contradiction.

Another (essentially equivalent) way to think about the relationship between the fact that the above exact sequence does not split and the existence of the Kodaira-Spencer morphism is the following. If one considers the $\omega_E^{\otimes 2}$-*torsor* of splittings of the above exact sequence (together with the "natural logarithmic conditions" at the points of bad reduction), we obtain a class

$$\eta \in H^1_{\mathrm{c}}(S, \omega_E^{\otimes 2})$$

where the subscript "c" stands for "cohomology with compact support." (The reason that we get a class with compact support is because of the "natural logarithmic conditions" at the points of bad reduction.) On the other hand, if we apply the functor $H^1_{\mathrm{c}}(-)$ to the Kodaira-Spencer morphism, we obtain a morphism

$$H^1_{\mathrm{c}}(S, \omega_E^{\otimes 2}) \to H^1_{\mathrm{c}}(S, \Omega_{S^{\log}/k}) = H^1_{\mathrm{c}}(S, \Omega_{S/k}) \cong k$$

Moreover, the image of η under this morphism can easily be shown to be the element of $H^1_{\mathrm{c}}(S, \Omega_{S/k}) \cong k$ which is the degree of the classifying morphism $\alpha : S \to (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{C}}$, i.e., $\deg(\alpha) \in \mathbb{Z} \subseteq \mathbb{C} = k$, which is nonzero (so long as $E \to S$ is not isotrivial). This implies that η is nonzero.

Thus, in summary,

> *An indirect way to "witness the existence of the Kodaira-Spencer morphism" is to observe that the above exact sequence does not split, i.e., that the $\omega_E^{\otimes 2}$-torsor of splittings of this sequence is nontrivial.*

This point of view is discussed in more detail in [Mzk7], Introduction, §2.3. Also, we observe that this nonsplitting of the above exact sequence may also be regarded as a sort of *"stability of the (vector bundle plus connection) pair"* $(H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E), \nabla_{\mathrm{DR}})$. This type of stability of a bundle equipped with connection is referred to as *"crys-stability"* in [Mzk7] — cf. [Mzk7], Introduction, §1.3; [Mzk7], Chapter I, for more details.

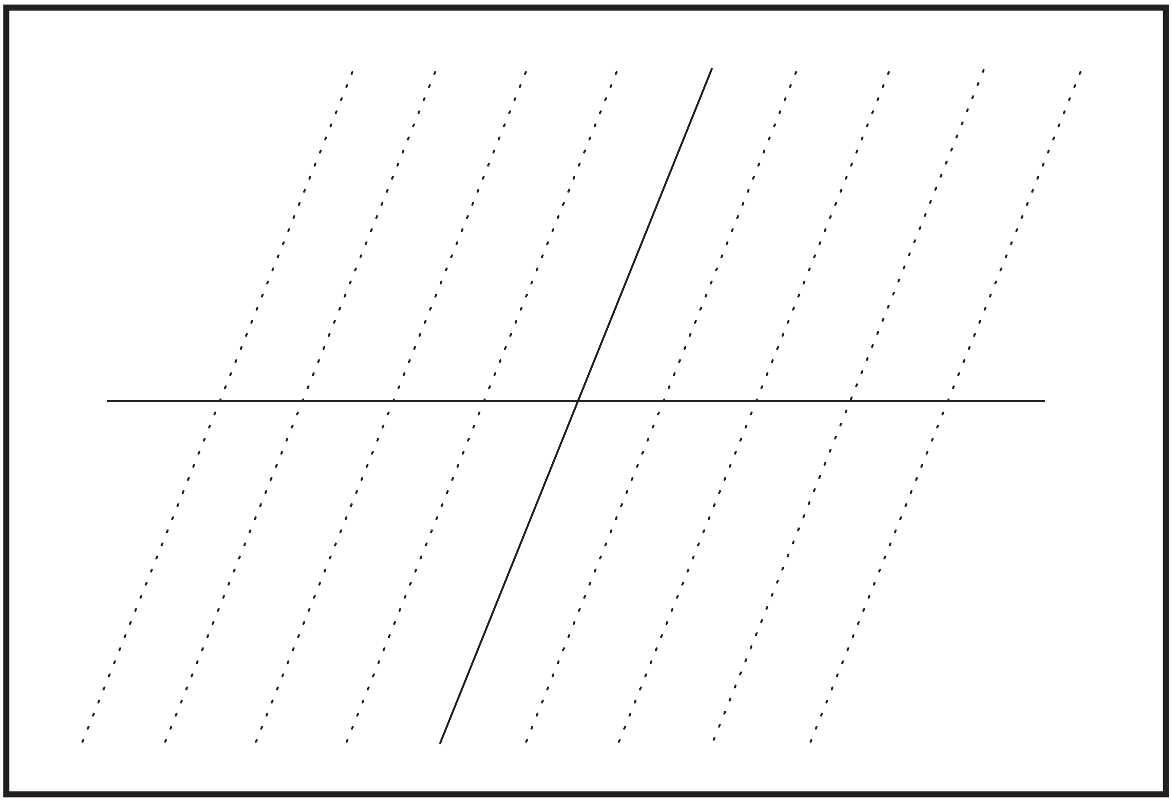

Fig. 1: The split case.

Now let us return to the *arithmetic case*. In this case, $S = \mathrm{Spec}(\mathcal{O}_K)$ (where $\mathcal{O}_K$ is the ring of integers of a number field K). Moreover, we have a natural exact sequence of $\mathcal{O}_K$-modules:

$$0 \to \omega_E \to H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \to \tau_E \to 0$$

which extends naturally to an *exact sequence of $\mathcal{O}_K$-modules with Hermitian metrics at the infinite primes*, i.e., an exact sequence of *arithmetic vector bundles on* $\overline{S}$ (where $\overline{S}$ denotes the formal union of S with the set of infinite primes of K) in the sense of Arakelov theory. Then one can consider *whether or not this exact sequence of arithmetic vector bundles splits*. Moreover, just as in the complex case, one can think of this issue as the issue of whether or not a certain *Arakelov-theoretic* $\omega_E^{\otimes 2}$*-torsor* splits. (The notion and basic properties of torsors in Arakelov-theory are discussed in [Mzk1], Chapter I.) Thus,

> *One way to regard the issue of constructing an arithmetic Kodaira-Spencer morphism is as the issue of constructing a theory that proves that/explains why this Arakelov-theoretic* $\omega_E^{\otimes 2}$*-torsor does not split.*

Indeed, the nonsplitting of this torsor is very closely related to the Conjectures of Vojta and Szpiro — in fact, the existence of (for instance, an infinite number of) counterexamples to these conjectures would imply (in an infinite number of cases) the splitting of this torsor (cf. [Mzk1], Chapter I, Theorem 2.4; [Mzk1], Chapter I, §4).

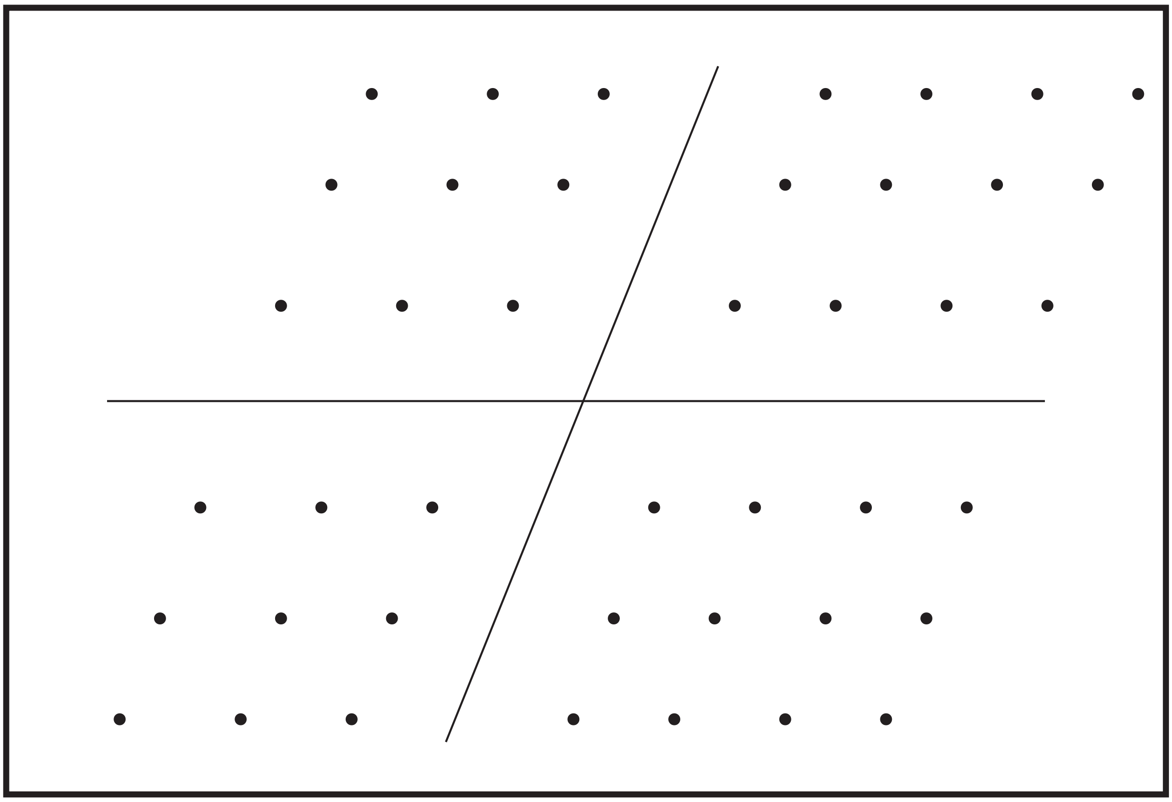

Fig. 2: The non-split case.

From a more elementary point of view, the nonsplitting of this torsor may be thought of in the following fashion. First of all, an arithmetic vector bundle over $\overline{S}$

may be thought of as an ($\mathcal{O}_K$-) *lattice* in a real or complex vector space. Thus, for instance, in the case $K = \mathbb{Q}$, one may think of $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ as a lattice in $\mathbb{R}^2$, while the arithmetic line bundles ω_E and τ_E may be thought of as lattices in $\mathbb{R}$. In the case of arithmetic line bundles, to say that the *degree* of the arithmetic line bundle (cf. [Mzk1], Chapter I, §1) is *large* (respectively, small) amounts to saying that the points of the lattice are *rather densely* (respectively, sparsely) *distributed.* Note that since ω_E is the pull-back to $\overline{S}$ of an ample line bundle $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$, its degree tends to be rather *large.* Thus, to say that the torsor in question splits is to say that $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ looks rather like the lattice of Fig. 1, i.e., it is dense in one direction and sparse in another (roughly orthogonal) direction. We would like to show that $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ looks more like the lattice in Fig. 2, i.e.:

> We would like to show that the lattice corresponding to $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ is *roughly equidistributed* in all directions.

Since we are thinking about *comparison isomorphisms*, it is thus tempting to think of the comparison isomorphism as something which *guarantees that the "distribution of matter" in the lattice $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ is as even in all directions as the "distribution of matter" in the étale cohomology of E.* Also, it is natural to think of the "distribution of matter issues" involving the étale cohomology, or torsion points, of E as being related to the action of Galois. (The action of Galois on the torsion points is discussed in [Mzk4].) Thus, in summary:

> *It is natural to expect that the global comparison isomorphism should be some sort of equivalence between "distributions of matter" in the de Rham and étale cohomologies of an elliptic curve.*

Typically, in analytic number theory, probability theory, and other field of mathematics where "distributions of matter" must be measured precisely, it is customary to measure them by thinking about functions — i.e., so-called *"test functions"* — (and the resulting function spaces) on the spaces where these distributions of matter occur. It is for this reason that the author was led to the conclusion that:

> *The proper formulation for a global comparison isomorphism should be some sort of isometric (for metrics at all the primes of a number field) isomorphism between spaces of functions on the de Rham and étale cohomologies of an elliptic curve.*

This is precisely what is obtained in Theorem A.

§**1.3.3 The Meaning of Nonlinearity**

In §1.3.2 above, we saw that one of the *central ideas of [Mzk1]* is that to obtain a "global Hodge theory," one must *sacrifice linearity = additivity*, and instead look for isometric isomorphisms between spaces of functions on the de Rham and étale cohomology. Put another way, this approach amounts to abandoning the idea that the de Rham and étale cohomologies are *modules*, and instead thinking of them as *(nonlinear) geometric objects*. Thus, one appropriate name for this approach might be *"geometric motive theory."* This approach contrasts sharply with typical approaches to the "theory of motives" or to "global Hodge theories" which tend to *revolve around additivity/linearization*, and involve such "linear" techniques as the introduction of derived categories that contain motives. In some sense, since we set out to develop a global Hodge theory in the context of *Arakelov theory*, the nonlinearity of the Comparison Isomorphism of Theorem A is perhaps not so surprising: Indeed, many objects in Arakelov theory which are analogues of "linear" objects in usual scheme theory become "nonlinear" when treated in the context of Arakelov theory. Perhaps the most basic example of this phenomenon is the fact that in Arakelov theory, the space of global sections of an arithmetic line bundle is not closed under addition. This makes it difficult and unnatural (if not impossible) to do homological algebra (e.g., involving derived categories) in the context of Arakelov theory.

Another way to think about the nonlinearity of the Hodge-Arakelov Comparison Isomorphism is that it is natural considering that the fundamental algebraic object that encodes the "symmetries of the Gaussian," namely, the *Heisenberg algebra* — i.e., the Lie algebra generated by 1, x, y, and the relations $[x, y] = 1$, $[x, 1] = [y, 1] = 0$ (the *étale counterpart* of which is the *theta groups* of Mumford) — is closely related to *nonlinear* geometries. In the field of noncommutative geometry, the object that represents these symmetries is known as the *noncommutative torus*. Since the Gaussian and its derivatives lie at the technical heart of the theory of [Mzk1], it is thus not surprising that the nonabelian nature of the symmetries of the Gaussian should manifest itself in the theory. In fact, in the portion of the theory of [Mzk1] at archimedean primes, it turns out that the Comparison Isomorphism in some sense amounts to a function-theoretic splitting of the exact sequence $0 \to d \cdot \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/d \cdot \mathbb{Z} \to 0$, i.e., a function-theoretic version of a bijection

$$(\mathbb{Z}/d \cdot \mathbb{Z}) \times (d \cdot \mathbb{Z}) \cong \mathbb{Z}$$

This sort of splitting is somewhat reminiscent of the "splitting" inherent in regarding $\mathbb{Z}$ as being some sort of "polynomial algebra over $\mathbb{F}_1$." Moreover, this archimedean portion of the theory is also (not surprisingly) closely related to the derivatives of the Gaussian and the symmetries encoded in the Heisenberg algebra. Thus, in summary:

> *It is as if the symmetries/twist inherent in the inclusion "$\mathbb{F}_1 \subseteq \mathbb{Z}$" are precisely the symmetries/twist encoded in the noncommutative torus (of noncommutative geometry).*

We refer to the discussion of [Mzk1], Chapter VIII, §0, for more details on this point of view.

§1.3.4 Hodge Theory at Finite Resolution

So far, we have discussed the idea that the appropriate way to think about Comparison Isomorphisms is to regard them as (isometric) isomorphisms between spaces of functions on the de Rham and étale cohomologies of an elliptic curve. The question then arises: How does one define such a natural isomorphism? The *key idea* here is the following:

> *Comparison Isomorphisms should be defined as* **evaluation maps**, *given by evaluating functions on the universal extension of an elliptic curve — which is a sort of "$H^1_{\mathrm{DR}}(E, \mathcal{O}_E^\times)$," i.e., a kind of de Rham cohomology of the elliptic curve — at the torsion points of the universal extension (which may naturally be identified with the étale cohomology of the elliptic curve).*

In fact, one of the main observations that led to the development of the theory of [Mzk1] is the following:

> *The classical Comparison Isomorphisms over complex and p-adic bases may be formulated precisely as evaluation maps of certain functions on the universal extension at the torsion points (or the "singular cohomology analogue of torsion points") of the universal extension.*

This *key observation* is discussed in detail in [Mzk1], Chapter IX, §1, 2. In the complex case, it amounts to an essentially trivial reformulation of the classical theory. Perhaps the best way to summarize this reformulation is to state that the *subspace of functions* on the "singular cohomology analogue of torsion points" *arising from the theta functions* on the elliptic curve is itself a sort of *"function-theoretic" representation of the Hodge filtration* induced (by the de Rham isomorphism) on the singular cohomology with complex coefficients of an elliptic curve. In fact, this observation more than any other played an essential role in convincing the author that (roughly speaking) "theta functions naturally define the Comparison Isomorphism" (hence that any global Comparison Isomorphism should involve theta functions). In the p-adic case, this "key observation" amounts to what is usually referred to as the *p-adic period map* (cf., e.g., [Coln], [Colz1,2], [Font], [Wint]; the beginning of [Mzk1], Chapter IX, §2) of elliptic curves (or abelian varieties).

Thus, in summary, *the complex, p-adic, and Hodge-Arakelov Comparison Isomorphisms may all be formulated along very similar lines, i.e., as evaluation maps of functions on the universal extension at the torsion points of the universal extension.* Of course, the difference between the Hodge-Arakelov Comparison Isomorphism and its local (i.e., complex and p-adic) counterparts is that unlike in the local case, where the spaces of torsion points involved are "completed at some prime," in the Hodge-Arakelov case, we work with a *discrete set of torsion points.* It is for this reason that we find it natural to think of the theory of [Mzk1] as a *"discretization"* of the well-known local comparison isomorphisms. Another way that one might think of the theory of [Mzk1] is as a *"Hodge theory at finite resolution"* (where we use the term "resolution" as in discussions of the number of "pixels" (i.e., "picture elements, dots") of a computer screen).

At this point, the reader might feel motivated to pose the following question:

> If the Hodge-Arakelov Comparison Isomorphism is indeed a comparison isomorphism analogous to the complex and p-adic comparison isomorphisms, then what sorts of "global periods" does it give rise to?

For instance, in the case of the complex comparison isomorphism, the most basic period is the period of the Tate motive, i.e., of H^1 of $\mathbb{G}_m$, namely, $2\pi i$. In the p-adic case, the corresponding period is the copy of $\mathbb{Z}_p(1)$ that sits naturally inside $\mathbb{B}_{crys}$. The analogous "period" resulting from the theory of Theorem A, then, is the following: Let U be the standard multiplicative coordinate on $\mathbb{G}_m$. Then $U - 1$ forms a section of some ample line bundle on $\mathbb{G}_m$, hence may be thought of as a sort of "theta function" (cf. especially, the Schottky uniformization of an elliptic curve, as in [Mumf4], §5). Then, roughly speaking, *the "discretized Hodge theory" of [Mzk1] amounts essentially* — from the point of view of *periods* — *to thinking of the period "$2\pi i$" as*

$$\lim_{n \to \infty} \ n \cdot (U - 1)|_{U = \exp(2\pi i/n)}$$

i.e., the evaluation of a theta function at an n-torsion point, for some large n. For the *elliptic curve analogue (at archimedean primes) of this representation of* $2\pi i$, we refer especially to [Mzk1], Chapter VII, §5, 6.

In fact, another way to interpret the theory of [Mzk1] is the following. First, let us observe that the classical complex comparison isomorphism (i.e., the de Rham isomorphism) is centered around "differentiation" and "integration," i.e., *calculus* on the elliptic curve. Moreover, in some sense, the most fundamental aspect of calculus as opposed to algebraic geometry on the elliptic curve is the use of *real analytic functions* on the elliptic curve. In the present context, however, we wish to keep everything "arithmetic" and "global" over a number field. Thus, instead of performing calculus on the underlying real analytic manifold of a (complex) elliptic curve, we approximate this classical sort of calculus by *performing calculus on a finite (but "large") set of torsion points* of the elliptic curve. That is to say:

> *We regard the set of torsion points as an* **approximation** *of the underlying real analytic manifold of an elliptic curve.*

Indeed, this notion of **"discrete torsion calculus"** is one of the key ideas of [Mzk1]. For instance, the universal extension of a complex elliptic curve has a *canonical real analytic splitting* (cf. [Mzk1], Chapter III, Definition 3.2), which is fundamental to the Hodge theory of the elliptic curve (cf., e.g., [Mzk7], Introduction, §0.7, 0.8). Since this splitting passes through the torsion points of the universal extension (and, in fact, is equal to the closure of these torsion points in the complex topology), it is thus natural to regard the torsion points of the universal extension as a "discrete torsion calculus approximation" to the canonical real analytic splitting (cf. Remark 1 following [Mzk1], Chapter III, Definition 3.2). This "discrete torsion calculus" point of view may also be seen in the use of the operator "δ" in [Mzk1], Chapter III, §6,7 (cf. also [Mzk1], Chapter V, §4), as well as in the discussion of the "discrete Tchebycheff polynomials" in [Mzk1], Chapter VII, §3.

§1.3.5 Relationship to Ordinary Frobenius Liftings and Anabelian Varieties

Finally, before proceeding, we present one more approach to thinking about "absolute differentiation over $\mathbb{F}_1$." Perhaps the most *naive* approach to defining the *derivative of a number* $n \in \mathbb{Z}$ (cf. [Ihara]) is to fix a prime number p, and then to compare n with its *Teichmüller representative* $[n]_p \in \mathbb{Z}_p$. The idea here is that Teichmüller representatives should somehow represent something analogous to a "field of constants" inside $\mathbb{Z}_p$. Thus, we obtain a correspondence

$$p \mapsto \frac{1}{p}(n - [n]_p)$$

Unfortunately, if one starts from this naive point of view, it seems to be very difficult to prove interesting global results concerning this correspondence, much less to apply it to proving interesting results in diophantine geometry.

Thus, it is natural to attempt to recast this naive approach in a form that is more amenable to globalization. To do this, let us first note that to consider Teichmüller representatives is very closely related to considering the natural *Frobenius morphism*

$$\Phi_A : A \to A$$

on the ring of Witt vectors $A \stackrel{\mathrm{def}}{=} W(\overline{\mathbb{F}}_p)$. In fact, the Teichmüller representatives in A are precisely the elements which satisfy the equation:

$$\Phi_A(a) = a^p$$

Put another way, if a is a *unit*, then it may be thought of as an element $\in \mathbb{G}_m(A)$. Moreover, $\mathbb{G}_m$ is equipped with its own natural *Frobenius action* $\Phi_{\mathbb{G}_m}$, given by $U \mapsto U^p$ (where U is the standard multiplicative coordinate on $\mathbb{G}_m$). Thus, the Teichmüller representatives are given by those elements of $a \in \mathbb{G}_m(A)$ such that

$$\Phi_A(a) = \Phi_{\mathbb{G}_m}(a)$$

In fact, this sort of situation where one has a natural Frobenius action on a p-adic (formal) scheme, and one considers natural p-adic liftings of points on this scheme modulo p which are characterized by the property that they are taken to their Frobenius (i.e., Φ_A) conjugates by the given action of Frobenius occurs elsewhere in arithmetic geometry. Perhaps the most well-known example of this situation (after $\mathbb{G}_m$) is *the Serre-Tate theory of liftings of ordinary abelian varieties* (cf., e.g., [Katz2], as well as [Mzk7], Introduction, §0). Recently, this theory has been generalized to the case of moduli of hyperbolic curves ([Mzk6,7]). We refer to the Introductions of [Mzk6,7] for more on this phenomenon. In the theory of [Mzk6,7], this sort of natural Frobenius action on a p-adic (formal) scheme is referred to as an *ordinary Frobenius lifting.*

The theory of ordinary Frobenius liftings is itself a special (and, in some sense, the simplest) case of *p-adic Hodge theory.* Thus, in summary, from this point of view, the naive approach discussed above (involving the correspondence $p \mapsto \frac{1}{p}(n - [n]_p)$) may be thought of as the approach given by *"looking at the p-adic Hodge theory of* $\mathbb{G}_m$ *at each prime p."* In particular, the relationship between the approach of [Mzk1] and the above naive approach may be thought of as the difference between **discretizing** *the various local p-adic Hodge theories into a global Arakelov-theoretic theory* (as discussed in §1.3.1 – 1.3.4 above) and *looking at the full* **completed** *p-adic Hodge theories individually.*

In fact, there is another important difference between the approach of [Mzk1] and the above naive approach — namely, the *difference between* $\mathbb{G}_m$ *and* $\overline{\mathcal{M}}_{\mathrm{ell}}^{\log}$ (the log moduli stack of log elliptic curves). That is to say, unlike the example discussed above, which is essentially concerned with rational points of $\mathbb{G}_m$, the theory of [Mzk1] concerns "absolute differentiation for points of $\overline{\mathcal{M}}_{\mathrm{ell}}^{\log}$." At the present time, the author does not know of an analogous approach to "globally discretizing" the local Hodge theories of $\mathbb{G}_m$ (i.e., of doing for $\mathbb{G}_m$ what is done for $\overline{\mathcal{M}}_{\mathrm{ell}}^{\log}$ in [Mzk1]). Also, it is interesting to observe that, unlike many theories for elliptic curves which generalize in a fairly straightforward manner to abelian varieties of higher dimension, it is not so clear how to generalize the theory of [Mzk1] to higher-dimensional abelian varieties (cf. §1.5.2 below). Thus, it is tempting to conjecture that perhaps the existence of the theory of [Mzk1] in the case of $\overline{\mathcal{M}}_{\mathrm{ell}}^{\log}$ is somehow related to the *anabelian nature of* $\overline{\mathcal{M}}_{\mathrm{ell}}^{\log}$ (cf. [Mzk8], [IN]). That is to say, one central feature of anabelian varieties is a certain *"extraordinary rigidity"* exhibited by their p-adic Hodge theory (cf. [Groth]; the Introduction to [Mzk8]). In particular, it is tempting to suspect that this sort of rigidity or *coherence* is what allows one to discretize the various local Hodge theories into a coherent global theory. Another interesting observation in this direction is that the *theta groups* that play an essential role in [Mzk1] are essentially the same as/intimately related to the quotient

$$\pi_1(E - \mathrm{pt.})/[\pi_1(E - \mathrm{pt.}), [\pi_1(E - \mathrm{pt.}), \pi_1(E - \mathrm{pt.})]]$$

of the fundamental group $\pi_1(E - \mathrm{pt.})$ of an elliptic curve with one point removed (which is itself an anabelian variety). This sort of quotient of the fundamental group plays a central role in [Mzk8].

Finally, we remark that one point of view related to the discussion of the preceding paragraph is the following: One fundamental obstacle to "differentiating an integer $n \in \mathbb{Z}$ (or $\mathbb{Q}$-rational point of $\mathbb{G}_m$) over $\mathbb{F}_1$" is that the residue fields $\mathbb{F}_p$ at the different points of $\mathrm{Spec}(\mathbb{Z})$ differ, thus making it difficult to compare the value of n at distinct points of $\mathrm{Spec}(\mathbb{Z})$. On the other hand, the theory of [Mzk1] — which involves differentiating $\mathbb{Z}$-valued points of the moduli stack of log elliptic curves $\overline{\mathcal{M}}_{\mathrm{ell}}^{\log}$ — gets around this problem effectively by taking the set of d-torsion points as one's *absolute constants* that do not vary even as the residue field varies. Note that relative to the discussion of §1.3.4, this set of torsion points should be regarded as a discrete analogue/approximation to the underlying real analytic manifold (which, of course, remains constant) of a family of complex elliptic curves.

§1.4. The Arithmetic Kodaira-Spencer Morphism

§1.4.1 Construction

In this §, we observe that one can apply *the Hodge-Arakelov Comparison Theorem* (§1.1, Theorem A) to construct an *arithmetic version of the well-known Kodaira-Spencer morphism* of a family of elliptic curves $E \to S$:

$$\kappa_E^{\mathrm{arith}} : \widetilde{\Pi}_S \to \mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}})(S)$$

Roughly speaking, this arithmetic Kodaira-Spencer morphism is a canonical map from the *algebraic fundamental groupoid* $\widetilde{\Pi}_S$ of $S_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} S \otimes_{\mathbb{Z}} \mathbb{Q}$ — i.e., the étale local system on $S_{\mathbb{Q}}$ whose fiber at a geometric point $\overline{s}$ is the fundamental group $\pi_1(S_{\mathbb{Q}}, \overline{s})$, with basepoint at $\overline{s}$ — to a flag variety of *filtrations* of a module which is a certain analogue of the *de Rham cohomology* of the elliptic curve.

This arithmetic Kodaira-Spencer morphism has certain *remarkable integrality properties (in the Arakelov sense)* at all the primes (both finite and infinite) of a number field; it is constructed in detail in [Mzk1], Chapter IX, §3 (cf. also [Mzk4], §1, where a certain technical error made in [Mzk1], Chapter IX, §3, is corrected). This construction will be surveyed in the remainder of the present §. In §1.4.2 below, we give a construction of the Kodaira-Spencer morphism in the *complex case* which is entirely analogous to the construction given in the arithmetic case in [Mzk1], Chapter IX, §3, but which shows quite explicitly how this construction is related (in the complex case) to the "classical Kodaira-Spencer morphism" that appears in the theory of moduli of algebraic varieties. Also, we remark that a similar treatment of the classical Kodaira-Spencer morphism may be given in the *p-adic context* (cf. [Mzk1], Chapter IX, §2), but we will not discuss this aspect of the theory in the present manuscript.

Conceptually speaking, the *main point* in all of these constructions consists, as depicted in the following diagram:

Kodaira-Spencer morphism:

motion in base-space $\mapsto$ *induced deformation of Hodge filtration*

of the idea that the Kodaira-Spencer morphism is the map which associates to a *"motion" in the base-space* of a family of elliptic curves, the *deformation in the Hodge filtration* of the de Rham cohomology of the elliptic curve induced by the motion. More concretely, the main idea consists of a certain *"recipe" for constructing "Kodaira-Spencer-type morphisms" out of "comparison isomorphisms between de Rham and étale/singular cohomology."* In the present §, we carry out this recipe in the case when the comparison isomorphism is the *the Hodge-Arakelov Comparison Isomorphism* (§1.1, Theorem A); in §1.4.2 below, we discuss a certain approach to the well-known comparison isomorphism for elliptic curves in the complex case, which, on the one hand, makes the connection with the classical Kodaira-Spencer morphism explicit, and, on the other hand, shows how the Hodge-Arakelov Comparison Isomorphism is entirely analogous to the well-known complex comparison isomorphism.

For simplicity, in the following discussion, we let S be a $\mathbb{Q}$-*scheme of finite type*, and

$$E \to S$$

a *family of elliptic curves* over S. Also, we use the notation

$$\widetilde{\Pi}_S$$

for the étale local system on S determined by the association

$$\overline{s} \mapsto \pi_1(S, \overline{s})$$

(where $\overline{s}$ is a geometric point of S).

Next, let us observe that (for $d \geq 1$ an integer) we have a *natural ($\mathcal{O}_S$-linear) action* of $\widetilde{\Pi}_S$ on

$$E[d] \to S$$

(where $E[d] \subseteq E$ is the closed subscheme of d-torsion points of $E \to S$). Indeed, if $\overline{s}$ is a geometric point of S, then the restriction of this action to the fibers at $\overline{s}$ is simply the usual action arising from the classical theory of the *algebraic fundamental group*. That is to say, according to this theory, one has a natural action of $\widetilde{\Pi}_S|_{\overline{s}} = \pi_1(S, \overline{s})$ on $E[d]|_{\overline{s}}$ arising from the fact that $E[d]|_{\overline{s}}$ is the result of applying the fiber functor defined by $\overline{s}$ to the finite étale covering $E[d] \to S$. The natural action of the algebraic fundamental groupoid on $E[d]$ is discussed in detail in [Mzk4], §1. Also, we remark that the dependence of this natural action on the basepoint $\overline{s}$ is *essential* (a fact overlooked in [Mzk1], Chapter IX, §3), which is why one must use the local system $\widetilde{\Pi}_S$, as opposed to the "static" fundamental group for a fixed basepoint.

Remark. One interesting observation relative to the appearance of the *algebraic fundamental groupoid* (i.e., as opposed to *group*) in the correct formulation of the arithmetic Kodaira-Spencer morphism (cf. [Mzk4], Corollary 1.6) is the following. In the *analogy* asserted in [Mzk1] between the arithmetic Kodaira-Spencer morphism of [Mzk1], Chapter IX, §3, and the usual geometric Kodaira-Spencer morphism, the *Galois group/fundamental group(oid) of the base plays the role of the tangent bundle of the base.* On the other hand, the tangent bundle of the base (typically) does not admit a canonical global trivialization, but instead *varies from point to point* — i.e., at a given point, it consists of *infinitesimal motions originating from that point.* Thus, it is natural that the arithmetic analogue of the tangent bundle should be not the "static" fundamental group, but instead the fundamental groupoid, which *varies* from point to point, and indeed, at a given point, consists of *paths* (which may be thought of as a sort of "motion") *originating from that point.*

Now suppose (cf. §1.1, Theorem $\mathrm{A}^{\mathrm{simple}}$) that we are also given *a torsion point* $\eta \in E(K)$ whose order does not divide d. Write

$$\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(d \cdot [\eta])$$

In the following discussion, we would also like to assume (for simplicity) that d is *odd.* Then by *Mumford's theory of theta groups*, one has a *natural isomorphism*

$$\mathcal{L}|_{E[d]} \cong \mathcal{L}|_{0_E} \otimes_{\mathcal{O}_S} \mathcal{O}_{E[d]}$$

(where $0_E \in E(S)$ is the origin of E) — which we shall refer to as the *"theta trivialization"* of $\mathcal{L}$ over $E[d]$. In particular, since the natural action of $\widetilde{\Pi}_S$ on $E[d]$ is $\mathcal{O}_S$*-linear*, one thus obtains (via this "theta trivialization") a *natural action* of $\widetilde{\Pi}_S$ on $\mathcal{L}|_{E[d]}$.

On the other hand (cf. §1.1, Theorem $\mathrm{A}^{\mathrm{simple}}$), we have a natural isomorphism

$$\mathcal{H}_{\mathrm{DR}} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}|_{E^\dagger})^{<d} \xrightarrow{\sim} \mathcal{L}|_{E[d]}$$

(where, by abuse of notation, all structure morphisms to S are denoted by f). Thus, the action of $\widetilde{\Pi}_S$ on $\mathcal{L}|_{E[d]}$ gives rise to a *natural action of* $\widetilde{\Pi}_S$ *on* $\mathcal{H}_{\mathrm{DR}}$. Moreover, since $\mathcal{H}_{\mathrm{DR}}$ is also equipped with a *Hodge filtration* $F^r(\mathcal{H}_{\mathrm{DR}})$ (given by considering sections of $\mathcal{L}|_{E^\dagger}$ of torsorial degree $< r$), one may consider the *extent to which this Hodge filtration is fixed by* $\widetilde{\Pi}_S$. If we write

$$\mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}}) \to S$$

for the *flag variety over* S of filtrations of $\mathcal{H}_{\mathrm{DR}}$ of the "same type" (i.e., such that the subobject of index r has the same rank as $F^r(\mathcal{H}_{\mathrm{DR}})$), then the correspondence

$$\sigma \mapsto \{F^r(\mathcal{H}_{\mathrm{DR}})\}^{\sigma}$$

(where σ is an étale local section of $\widetilde{\Pi}_S$, and "$\{F^r(\mathcal{H}_{\mathrm{DR}})\}^{\sigma}$" denotes the filtration of $\mathcal{H}_{\mathrm{DR}}$ obtained by acting on the Hodge filtration of $\mathcal{H}_{\mathrm{DR}}$ by σ) defines a morphism

$$\kappa_E^{\mathrm{arith}} : \widetilde{\Pi}_S \to \mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}})(S)$$

— which (cf. the discussion in the complex case in §1.4.2 below) we refer to as the *arithmetic Kodaira-Spencer morphism* of the family of elliptic curves $E \to S$.

When S is *flat over* $\mathbb{Z}$ — i.e., for instance, when S is the spectrum of a (Zariski localization) of the *ring of integers* $\mathcal{O}_F$ *of a number field* F — this morphism $\kappa_E^{\mathrm{arith}}$ has various strong *integrality properties* at both nonarchimedean (i.e., "p-adic") and archimedean primes, inherited from the integrality properties of the comparison isomorphism of §1.1, Theorem A. These integrality properties are discussed in more detail in [Mzk1], Chapter IX, §3.

§1.4.2 Complex Analogue

In this §, we review the de Rham isomorphism of a complex elliptic curve, showing how this isomorphism may be regarded as being analogous in a fairly precise sense to the *Comparison Isomorphism* of §1.1, Theorem A. We then discuss the theory of the Kodaira-Spencer morphism of a family of complex elliptic curves in the universal case, but we formulate this theory in a somewhat novel fashion, showing how the Kodaira-Spencer morphism may be derived directly from the de Rham isomorphism in a rather geometric way. This formulation will allow us to make the connection with the global arithmetic theory of §1.4.1; [Mzk1], Chapter IX, §3.

We begin our discussion by considering a single *elliptic curve* E over $\mathbb{C}$. Frequently in the following discussion, we shall also write "E" for the complex manifold defined by the original algebraic curve. Recall (cf. [Mzk1], Chapter III, §3) that we have a commutative diagram

$$\begin{array}{ccccccc}
H^1_{\mathrm{DR}}(E, \mathcal{O}_E) = \widetilde{E}^{\dagger} & \cong & H^1_{\mathrm{sing}}(E, \mathbb{C}) & \supseteq & H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbb{R}) & \supseteq & H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbb{Z}) \\
\downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\
H^1_{\mathrm{DR}}(E, \mathcal{O}_E^{\times}) = E^{\dagger} & \cong & H^1_{\mathrm{sing}}(E, \mathbb{C}^{\times}) & \supseteq & H^1_{\mathrm{sing}}(E, \mathbb{S}^1) = E_{\mathbb{R}} & \supseteq & \text{identity elt.}
\end{array}$$

Here, the horizontal isomorphisms are the *de Rham isomorphisms* relating de Rham cohomology to singular cohomology. Note that in characteristic zero, line bundles with connection are necessarily of degree zero, so $E^\dagger$ may be naturally identified with $H^1_{\mathrm{DR}}(E, \mathcal{O}_E^\times)$, the group of line bundles equipped with a connection on E. (Similarly, the (topological) universal cover $\widetilde{E}^\dagger$ of $E^\dagger$ may be identified with $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$.) The vertical maps are the morphisms induced on cohomology by the exponential map; $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ is the unit circle (equipped with its usual group structure). Finally, $E_\mathbb{R} \subseteq E^\dagger$ is the real analytic submanifold (discussed in [Mzk1], Chapter III, §3) which is equal to the closure of the torsion points of $E^\dagger$ and maps bijectively onto E via the natural projection $E^\dagger \to E$.

Here, we would like to consider the issue of *precisely how the de Rham isomorphisms of the above diagram are defined.* Of course, there are many possible definitions for these isomorphisms, but the point that we would like to make here is the following:

> *If one thinks of $H^1_{\mathrm{sing}}(E, \mathbb{S}^1) = E_\mathbb{R}$ (respectively, $T_v(E) \stackrel{\mathrm{def}}{=} H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbb{R})$) as the "$v$-divisible group of torsion points of E" (respectively, the "v-adic Tate module") — where $v = \infty$ is the archimedean prime of $\mathbb{Q}$ — then, roughly speaking, one may think of the de Rham isomorphisms as being given by the diagram*

$$\begin{array}{ccc} \text{Hol. fns. on } H^1_{\mathrm{DR}}(E, \mathcal{O}_E) = \widetilde{E}^\dagger & \text{“}\cong\text{”} & \text{Real an. fns. on } H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbb{R}) \\ \bigcup & & \bigcup \\ \text{Hol. fns. on } H^1_{\mathrm{DR}}(E, \mathcal{O}_E^\times) = E^\dagger & \text{“}\cong\text{”} & \text{Real an. fns. on } H^1_{\mathrm{sing}}(E, \mathbb{S}^1) = E_\mathbb{R} \end{array}$$

> *where the horizontal isomorphisms "$\cong$" are given by restricting holomorphic functions on $E^\dagger$, $\widetilde{E}^\dagger$ to real analytic functions on the "∞-adic torsion points/Tate module" $E_\mathbb{R}$, $T_\infty(E) \stackrel{\mathrm{def}}{=} H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbb{R})$.*

Here, we say "roughly speaking" (and write "$\cong$") for the following reason: Although this restriction morphism is *injective*, the correspondence between holomorphic functions on the de Rham objects $E^\dagger$, $\widetilde{E}^\dagger$ and real analytic functions on the v-adic torsion point objects $E_\mathbb{R}$, $T_\infty(E)$ is, strictly speaking, only true on an "infinitesimal neighborhood" of $E_\mathbb{R} \subseteq E^\dagger$, $T_\infty(E) \subseteq \widetilde{E}^\dagger$. (That is to say, although a real analytic function on $E_\mathbb{R}$ always corresponds to a holomorphic function on some open neighborhood of $E_\mathbb{R}$ in $E^\dagger$, whether or not this holomorphic function extends to a holomorphic function defined over all of $E^\dagger$ involves subtle convergence issues and, in fact, is not always the case.) Thus, here, in order to get a precise statement, we shall work with *polynomial functions* on $\widetilde{E}^\dagger$ and $T_\infty(E)$. Then one sees immediately that *the de Rham isomorphism of the first commutative diagram of this § may be formulated as the isomorphism*

$$\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger) \quad \cong \quad \mathrm{Real\ An}^{\mathrm{Poly}}(T_\infty(E))$$

given by restricting holomorphic polynomials on $\widetilde{E}^\dagger$ *to the* ∞*-adic torsion points so as to obtain real analytic polynomials on* $T_\infty(E)$. When formulated from this point of view, one sees that the Comparison Isomorphism of §1.1, Theorem A, is analogous in a very direct sense to the classical de Rham isomorphism in the complex case, i.e.:

> *Both may be thought of as being bijections between algebraic/holomorphic functions on de Rham-type objects and arbitrary/real analytic functions on torsion points – bijections given by restricting algebraic/holomorphic functions on de Rham-type objects to the torsion points lying inside those de Rham-type objects.*

This observation may be thought of as the philosophical starting point of the theory of [Mzk1].

Remark. Note that the collection of "holomorphic functions on $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$" includes, in particular, the *theta functions* (cf., e.g., [Mumf4], §3) associated to the elliptic curve E. Moreover, these functions are "fairly representative of" (roughly speaking, "generate") the set of *all* holomorphic functions on $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$ that arise by pull-back via the projection $H^1_{\mathrm{DR}}(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$ (defined by the *Hodge filtration*) from holomorphic functions on $H^1(E, \mathcal{O}_E)$. *This observation played a fundamental motivating role in the development of the theory of [Mzk1].*

Next, we shift gears and discuss various versions of the *Kodaira-Spencer morphism for the universal family of complex elliptic curves.* First, let us write $\mathfrak{H} \stackrel{\mathrm{def}}{=} \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ for the upper half-plane, and

$$E_{\mathfrak{H}} \to \mathfrak{H}$$

for the *universal family of complex elliptic curves* over $\mathfrak{H}$. That is to say, over a point $z \in \mathfrak{H}$, the fiber E_z of this family is given by $E_z = \mathbb{C}/ < 1, z >$ (where $< 1, z >$ denotes the $\mathbb{Z}$-submodule generated by $1, z$).

Let us fix a "base-point" $z_0 \in \mathfrak{H}$. Write

$$H^1_{\mathrm{DR}}(E_{z_0}) \stackrel{\mathrm{def}}{=} H^1_{\mathrm{DR}}(E_{z_0}, \mathcal{O}_{E_{z_0}})$$

Thus, $H^1_{\mathrm{DR}}(E_{z_0})$ is a two-dimensional complex vector space. Recall that, in fact, the correspondence $z \mapsto H^1_{\mathrm{DR}}(E_z)$ defines a rank two vector bundle $\mathcal{E}$ on $\mathfrak{H}$ equipped with a natural (integrable) connection (the "Gauss-Manin connection"). Since the underlying topological space of $\mathfrak{H}$ is *contractible*, parallel transport via this connection thus gives rise to a natural trivialization of this rank two vector bundle $\mathcal{E}$, i.e., a natural isomorphism

$$\mathcal{E} \cong \mathfrak{H} \times H^1_{\mathrm{DR}}(E_{z_0})$$

Recall that the *Hodge filtration* of de Rham cohomology defines a subbundle $F^1(\mathcal{E}) \subseteq \mathcal{E}$ of rank one. This subbundle induces a natural holomorphic morphism

$$\kappa_{\mathfrak{H}} : \mathfrak{H} \to P \stackrel{\text{def}}{=} \mathbb{P}^1(H^1_{\text{DR}}(E_{z_0}))$$

that maps a point $z \in \mathfrak{H}$ to the subspace of $H^1_{\text{DR}}(E_{z_0}) = H^1_{\text{DR}}(E_z)$ defined by $F^1(\mathcal{E}) \subseteq \mathcal{E}$ at z.

Now let us recall that we have a *natural action* of $SL_2(\mathbb{R})$ on $\mathfrak{H}$ given by linear fractional transformations. This action allows us to define a morphism

$$\kappa_{SL_2} : SL_2(\mathbb{R}) \to P$$

by letting $\kappa_{SL_2}(\gamma) \stackrel{\text{def}}{=} \kappa_{\mathfrak{H}}(\gamma \cdot z_0)$ (for $\gamma \in SL_2(\mathbb{R})$). If we then differentiate κ_{SL_2} at z_0, we obtain a *morphism on tangent spaces* that fits into a commutative diagram:

$$\begin{array}{ccc} sl_2(\mathbb{R}) & \xrightarrow{\kappa_{sl_2}} & \tau_{P,p_0} \\ \downarrow & & \uparrow \kappa_\tau \\ sl_2(\mathbb{R})/so_2 & \cong & \tau_{\mathfrak{H},z_0} \end{array}$$

Here, *the vertical morphism on top is the derivative of* κ_{SL_2} *at the origin of* $SL_2(\mathbb{R})$; τ_{P,p_0} (respectively, $\tau_{\mathfrak{H},z_0}$) is the tangent space to P (respectively, $\mathfrak{H}$) at $p_0 \stackrel{\text{def}}{=} \kappa_{\mathfrak{H}}(z_0)$ (respectively, z_0). This vertical morphism clearly factors through the quotient $sl_2(\mathbb{R}) \to sl_2(\mathbb{R})/so_2$, where $sl_2(\mathbb{R})$ (respectively, so_2) is the Lie algebra associated to $SL_2(\mathbb{R})$ (respectively, the subgroup of $SL_2(\mathbb{R})$ that fixes z_0). Moreover, all tangent vectors to $z_0 \in \mathfrak{H}$ are obtained by acting by various elements of $sl_2(\mathbb{R})$ on $\mathfrak{H}$ at z_0; thus, one may identify $sl_2(\mathbb{R})/so_2$ with $\tau_{\mathfrak{H},z_0}$ (the vertical isomorphism on the bottom).

Definition. We shall refer to κ_{SL_2} (respectively, κ_{sl_2}; κ_τ) as the *group-theoretic* (respectively, *Lie-theoretic*; *classical*) *Kodaira-Spencer morphism (of the family* $E_{\mathfrak{H}} \to \mathfrak{H}$ *at* z_0*)*.

Thus, the "classical Kodaira-Spencer morphism" κ_τ is obtained (cf. the above commutative diagram) simply by using the fact that κ_{sl_2} factors through the quotient $sl_2(\mathbb{R}) \to sl_2(\mathbb{R})/so_2$. One checks easily that this morphism is indeed the usual Kodaira-Spencer morphism associated to the family $E_{\mathfrak{H}} \to \mathfrak{H}$. In particular, κ_τ is an isomorphism.

The reason that we feel that it is natural also to regard κ_{SL_2} and κ_{sl_2} as "Kodaira-Spencer morphisms" is the following: The essence of the notion of a "Kodaira-Spencer morphism" is that of a *correspondence* that associates to a *motion* in the base-space the *induced deformation of the Hodge filtration of the de Rham cohomology*, i.e., symbolically,

<u>Kodaira-Spencer morphism</u>:

motion in base-space $\mapsto$ *induced deformation of Hodge filtration*

In the case of the "group-theoretic Kodaira-Spencer morphism" (respectively, "Lie-theoretic Kodaira-Spencer morphism"; "classical Kodaira-Spencer morphism"), this motion is a motion given by the "Lie group $SL_2(\mathbb{R})$ of motions of $\mathfrak{H}$" (respectively, the Lie algebra associated to this Lie group of motions; a tangent vector in $\mathfrak{H}$). That is to say, all three types of Kodaira-Spencer morphism discussed here fit into the general pattern just described.

It turns out that the *group-theoretic Kodaira-Spencer morphism is the version which is most suited to generalization to the arithmetic case* (cf. the discussions of §2,3 below).

Finally, *we make the connection between the theory of the Kodaira-Spencer morphism just discussed and the function-theoretic approach to the de Rham isomorphism discussed at the beginning of this* §. First of all, let us observe that the action of $SL_2(\mathbb{R})$ on $\mathfrak{H}$ lifts naturally to an action on $\mathcal{E}$. Moreover, if one thinks of $SL_2(\mathbb{R})$ as the group of unimodular (i.e., with determinant $= 1$) $\mathbb{R}$-linear automorphisms of the two-dimensional $\mathbb{R}$-vector space $T_\infty(E_{z_0})$, that is, if one makes the identification

$$SL_2(\mathbb{R}) = SL(T_\infty(E_{z_0}))$$

then the action of $SL_2(\mathbb{R})$ on $\mathcal{E} \cong \mathfrak{H} \times H^1_{\mathrm{DR}}(E_{z_0})$ corresponds to the natural action of $SL(T_\infty(E_{z_0}))$ on $H^1_{\mathrm{DR}}(E_{z_0}) \cong T_\infty(E_{z_0}) \otimes_{\mathbb{R}} \mathbb{C}$ (where the isomorphism here is the de Rham isomorphism). It thus follows that *the group-theoretic Kodaira-Spencer morphism κ_{SL_2} may also be defined as the morphism*

$$SL_2(\mathbb{R}) = SL(T_\infty(E_{z_0})) \to P = \mathbb{P}(H^1_{\mathrm{DR}}(E_{z_0}))$$

given by $\gamma \mapsto \gamma \cdot p_0$, where the expression "$\gamma \cdot p_0$" is relative to the natural action of $SL(T_\infty(E_{z_0}))$ on $P = \mathbb{P}(H^1_{\mathrm{DR}}(E_{z_0})) \cong \mathbb{P}(T_\infty(E_{z_0}) \otimes_{\mathbb{R}} \mathbb{C})$ (where the isomorphism here is that derived from the de Rham isomorphism). This approach to defining κ_{SL_2} shows that:

> *The group-theoretic Kodaira-Spencer morphism κ_{SL_2} may essentially be defined directly from the de Rham isomorphism.*

This observation brings us one step closer to the discussion of the arithmetic case in §3. In particular, in light of the above "function-theoretic approach to the de Rham isomorphism," it motivates the following *point of view*:

Note that (in the notation of the discussion at the beginning of this §) the space $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$ of holomorphic polynomials on $\widetilde{E}^\dagger$ has a *Hodge filtration*

$$\dots F^d(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)) \subseteq \dots \subseteq \mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$$

given by letting $F^d(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)) \subseteq \mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$ denote the subspace of polynomials whose "torsorial degree" (cf. [Mzk1], Chapter III, Definition 2.2), i.e., degree as a polynomial in the relative variable of the torsor $\widetilde{E}^\dagger \to \widetilde{E}$ (where $\widetilde{E}$ is the universal covering space of E), is $< d$. Note that relative to the "function-theoretic de Rham isomorphism" $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger) \cong \mathrm{Real\ An}^{\mathrm{Poly}}(T_\infty(E))$, $F^d(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$ corresponds to the subspace of $\mathrm{Real\ An}^{\mathrm{Poly}}(T_\infty(E))$ annihilated by $\overline{D}^d$. (Here, $\overline{D}$ is the usual "del-bar" operator of complex analysis on $T_\infty(E) = \widetilde{E}_{\mathbb{R}}$, relative to the complex structure on $\widetilde{E}_{\mathbb{R}}$ defined by $\widetilde{E}$.) Let us write

$$\mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$$

for the (infinite-dimensional) flag-manifold of $\mathbb{C}$-linear filtrations $\{F^d\}_{d \in \mathbb{Z}_{\geq 0}}$ of $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$ such that $F^0 = 0$. Then the "Hodge filtration" just defined determines a point

$$p_E^{\mathrm{func}} \in \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$$

Similarly, any one-dimensional complex quotient $\widetilde{E}^\dagger \to Q$ defines a filtration of $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$ (given by looking at the degree with respect to the variable corresponding to the kernel of $\widetilde{E}^\dagger \to Q$). In particular, we get an immersion

$$\mathbb{P}(\widetilde{E}^\dagger) \hookrightarrow \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$$

Thus, returning to the discussion of the *group-theoretic Kodaira-Spencer morphism*, we see that we may think of the composite

$$\kappa_{SL_2}^{\mathrm{func}} : SL_2(\mathbb{R}) = SL(T_\infty(E_{z_0})) \to \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$$

of κ_{SL_2} with the inclusion $\mathbb{P}(\widetilde{E}^\dagger) \hookrightarrow \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$ as being defined as follows:

> *The natural action of* $SL(T_\infty(E_{z_0}))$ *on* $\mathrm{Real\ An}^{\mathrm{Poly}}(T_\infty(E_{z_0}))$ *induces, via the "function-theoretic de Rham isomorphism"*
>
> $$\mathrm{Real\ An}^{\mathrm{Poly}}(T_\infty(E_{z_0})) \cong \mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger_{z_0})$$
>
> *an action of* $SL(T_\infty(E_{z_0}))$ *on* $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger_{z_0})$*; then the* "function-theoretic version of the group-theoretic Kodaira-Spencer morphism"

$$\kappa_{SL_2}^{\text{func}} : SL(T_\infty(E_{z_0})) \to \mathfrak{Filt}(\text{Holom}^{\text{Poly}}(\widetilde{E}_{z_0}^\dagger))$$

is defined by $\gamma \mapsto \gamma \cdot p_{E_{z_0}}^{\text{func}}$*, where* $p_{E_{z_0}}^{\text{func}} \in \mathfrak{Filt}(\text{Holom}^{\text{Poly}}(\widetilde{E}_{z_0}^\dagger))$ *is the natural point defined by the Hodge filtration on* $\text{Holom}^{\text{Poly}}(\widetilde{E}_{z_0}^\dagger)$.

It is this point of view that forms the basis of our approach to the arithmetic case, as surveyed in §1.4.1.

Remark. The theory discussed above (cf. [Mzk1], Chapter IX, §1,2) generalizes immediately to the case of *higher-dimensional abelian varieties*. Since, however, the Hodge-Arakelov Comparison Theorem (§1.1, Theorem A) is only available (at the time of writing) for elliptic curves, we restricted ourselves in the above discussion to the case of elliptic curves.

§1.5. Future Directions

§1.5.1 Gaussian Poles and Diophantine Applications

In some sense, the most fundamental outstanding problem left unsolved in [Mzk1] is the following:

How can one get rid of the **Gaussian poles** *(cf. §1)?*

For instance, if one could get rid of the Gaussian poles in Theorem A, there would be *substantial hope of applying Theorem A to the ABC (or, equivalently, Szpiro's) Conjecture.*

The main idea here is the following: Assume that we are given an *elliptic curve* E_K *over a number field* K, with everywhere semi-stable reduction. Also, let us assume that all of the d-torsion points of E_K are defined over K. The *arithmetic Kodaira-Spencer morphism* (cf. §1.4) essentially consists of applying some sort of *Galois action* to an Arakelov-theoretic vector bundle $\mathcal{H}_{\text{DR}}$ on $\text{Spec}(\mathcal{O}_K)$ and seeing what effect this Galois action has on the natural *Hodge filtration* $F^r(\mathcal{H}_{\text{DR}})$ on $\mathcal{H}_{\text{DR}}$. If one *ignores the Gaussian poles*, the subquotients $(F^{r+1}/F^r)(\mathcal{H}_{\text{DR}})$ of this Hodge filtration essentially ("as a function of r") look like

$$\tau_E^{\otimes r}$$

(tensored with some object which is essentially irrelevant since it is *independent* of r). Thus, as long as the "arithmetic Kodaira-Spencer is nontrivial" (which it most surely is!), the Galois action on $\mathcal{H}_{\text{DR}}$ would give rise to *nontrivial globally integral* (in the sense of Arakelov theory) *morphisms*

$$\mathcal{O}_K = \tau_E^{\otimes 0} \approx F^1(\mathcal{H}_{\mathrm{DR}}) \rightarrow (F^{r+1}/F^r)(\mathcal{H}_{\mathrm{DR}}) \approx \tau_E^{\otimes r}$$

for some $r > 1$. That is to say, we would get a *global integral section of some positive power of* $\tau_E = \omega_E^\vee$, which would imply a *bound on the Arakelov-theoretic degree of* ω_E, i.e., a bound on the *height* $ht(E_K)$ *of the elliptic curve* E_K (since the height $\mathrm{ht}(E_K)$ is equal to twice the degree of ω_E). Of course, one should not expect to be able to get rid of the Gaussian poles without paying some sort of *"tax."* If, for instance, this tax is of the order of *denominators of size* $\approx D^{\frac{r}{2}}$ *on* (F^{r+1}/F^r) (where $D > 0$ is some real number independent of r), then we would obtain an inequality

$$\mathrm{ht}(E_K) \leq \log(D)$$

Thus, if

$$D \approx \log(\mathrm{disc}_{K/\mathbb{Q}})$$

(the log of the discriminant of K over $\mathbb{Q}$), then we would obtain precisely the inequality asserted in the ABC (or Szpiro's) Conjecture. (Here, we note that since the d-torsion points are assumed to be rational over K, and we typically expect that we will want to take d of the order of $\mathrm{ht}(E_K)$ (cf. [Mzk1], Chapter II), typically $\log(\mathrm{disc}_{K/\mathbb{Q}})$ will be approximately equal to the log discriminant of the minimal field of definition of E_K plus the logarithm of the primes of bad reduction of E_K. This is the form in which the ABC (or Szpiro's) Conjecture is usually stated (cf. [Lang], [Vojta]).)

The above sketch of an argument (i.e., that one might be able to apply Theorem A to the ABC Conjecture if only one could get rid of the Gaussian poles) provides, in the opinion of the author, *strong motivation for investigating the issue of whether or not one can somehow eliminate the Gaussian poles from Theorem A*.

One approach to eliminating the Gaussian poles is the following: As discussed in §1.2, in some sense, one may regard the theory of [Mzk1] as the theory of the *Gaussian and its derivatives.* The classical example of the theory of the Gaussian and its derivatives is the theory of *Hermite functions.* The Hermite functions, which are various derivatives of the Gaussian, are not themselves polynomials, but rather of the form:

$$\text{(polynomial)} \cdot \text{(Gaussian)}$$

Thus, it is natural to *divide the Hermite functions by the Gaussian*, which then gives us polynomials which are called the *Hermite polynomials.* In the theory of [Mzk1], the original Gaussian corresponds (relative to taking the Fourier expansion) to the algebraic theta functions of Mumford (i.e., before we consider derivatives); *the "unwanted" Gaussian that remains in the Hermite functions corresponds to* *the Gaussian poles*. Moreover, since multiplication and division correspond (after taking the Fourier expansion) to *convolution*, it is natural to imagine that the image of the "domain without Gaussian poles" of the Comparison Isomorphism (of Theorem A) should correspond to those functions on the torsion points that are in the image of (the morphism given by) *convolution with the (original) theta function* — which we refer to as the *"theta convolution"* for short. Thus, it is natural to conjecture that:

> *By studying the theta convolution, one might be able to construct a Galois action like the one needed in the argument above, i.e., a "Galois action without Gaussian poles."*

In [Mzk2], we study this theta convolution, and obtain, in particular, a *theta-convoluted comparison isomorphism*, which has the property that, in a neighborhood of the divisor at infinity, when one works with an *étale* (i.e., isomorphic to $\mathbb{Z}/d\mathbb{Z}$) Lagrangian subgroup, and a *multiplicative* (i.e., isomorphic to $\boldsymbol{\mu}_d$) restriction subgroup, then the Gaussian poles *vanish*, as desired (cf. [Mzk2], especially Remark 1 following Theorem 10.1, for more details). Even in the case of the theta-convoluted comparison isomorphism, however, the Gaussian poles fail to vanish (in a neighborhood of the divisor at infinity) if either the restriction subgroup fails to be multiplicative or the Lagrangian subgroup fails to be étale.

At the time of writing, however, the author no longer believes the theory of [Mzk2] to be a strong candidate for eliminating the Gaussian poles in the case of elliptic curves over a number field. Instead, it is the sense of the author that *the approach of [Mzk4] provides the most promising candidate for achieving this goal*. Nevertheless, the theory of [Mzk2] is of interest in its own right, and will be surveyed in §2 of the present paper.

§1.5.2 Higher Dimensional Abelian Varieties and Hyperbolic Curves

Once results such as Theorem A (of §1.1) have been established for elliptic curves, it is natural to attempt to generalize such results to *higher dimensional abelian varieties* and *hyperbolic curves*. Unfortunately, however, even in the case of abelian varieties, where one expects the generalization to be relatively straightforward, one immediately runs into a number of problems. For instance, if one considers sections of an ample line bundle $\mathcal{L}$ over the universal extension of an abelian variety of dimension g, the dimension over the base field of the space of global sections of torsorial degree $< d$ is:

$$\binom{d-1+g}{g} \cdot d^g < d^{2g}$$

(where we assume that the dimension of the space of global sections of $\mathcal{L}$ over the abelian variety itself is equal to d^g) if $g > 1$. Thus, the naive generalization of Theorem $\mathrm{A}^{\mathrm{simple}}$ cannot possibly hold (i.e., since the two spaces between which one must construct an isomorphism have different dimensions). Nevertheless, it would be interesting if someday this sort of technical problem could be overcome, and the theory of [Mzk1] could be generalized to arbitrary abelian varieties. If such a generalization could be realized, it would be interesting if, for instance, just as the various models (Hermite, Legendre, Binomial) that occur in the archimedean theory of the present paper correspond naturally to the possible slopes of the action of Frobenius (on the first crystalline cohomology module of an elliptic curve) at finite primes (cf. the discussion of this phenomenon in §1), it were the case that the corresponding models in the archimedean theory for arbitrary abelian varieties correspond to the possible *Newton polygons* of the action of Frobenius (on the first crystalline cohomology module of an abelian variety of the dimension under consideration) at finite primes.

Another natural direction in which to attempt to generalize the theory of this paper would be to extend it to a global/Arakelov-theoretic Hodge theory of hyperbolic curves. Indeed, the "complex Hodge theory of hyperbolic curves," which revolves around the Köbe uniformization of (the Riemann surfaces corresponding to) such curves by the upper half-plane, has already been extended to the p-adic case ([Mzk6,7]). Moreover, the theory of [Mzk6,7] may also be regarded as the "hyperbolic curve analogue" of Serre-Tate theory. Thus, it is natural to attempt to construct a "global Arakelov version" of [Mzk6,7], just as the theory of the present paper in some sense constitutes a globalization of the Serre-Tate theory/p-adic Hodge theory of elliptic curves (cf. Chapter IX, §2).

Section 2: The Theta Convolution

§2.1. Background

In [Mzk2], we continue our development of the theory of the Hodge-Arakelov Comparison Isomorphism of [Mzk1]. Our main result concerns the *invertibility of the coefficients of the Fourier transform of an algebraic theta function.* Using this result, we obtain a modified version of the Hodge-Arakelov Comparison Isomorphism of [Mzk1], which we refer to as the *Theta-Convoluted Comparison Isomorphism.* The significance of this modified version is that the principle obstruction to the application of the theory of [Mzk1] to diophantine geometry — namely, the *Gaussian poles* — partially vanishes in the theta-convoluted context.

Perhaps the simplest way to explain the *main idea* of [Mzk2] is the following: The theory of [Mzk1] may be thought of as a sort of discrete, scheme-theoretic version of the theory of the *classical Gaussian* e^{-x^2} *(on the real line) and its derivatives* (cf. [Mzk1], Introduction, §2). More concretely, the theory of [Mzk1] may, in essence, be thought of as the theory of the *theta function*

$$\Theta \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} q^{n^2} \cdot U^n$$

(where q is the q-parameter, and U is the standard multiplicative coordinate on $\mathbb{G}_m$) and *its derivatives* — i.e., functions of the form

$$\sum_{n \in \mathbb{Z}} q^{n^2} \cdot P(n) \cdot U^n$$

for some polynomial $P(-)$ with constant coefficients. From this point of view, the "Gaussian poles" — which, as remarked above, constitute the principle obstruction to the application of the theory of [Mzk1] to diophantine geometry — arise from the factors of

$$q^{n^2}$$

appearing in the above series. Thus, if one could somehow "magically replace" the above series by series of the form

$$\sum_{n \in \mathbb{Z}} P(n) \cdot U^n$$

then there would be some hope of obtaining a theory without Gaussian poles. Put another way,

> *One would like to divide the coefficient of U^n in the usual series for Θ and its derivatives by the factor q^{n^2}.*

If we think of these series as *Fourier series* (on the group $\mathbb{G}_m$), then the "magical replacement" referred to above may be expressed as *convolution* of the given series

$$\sum_{n \in \mathbb{Z}} q^{n^2} \cdot P(n) \cdot U^n$$

with the series

$$\sum_{n \in \mathbb{Z}} q^{-n^2} \cdot U^n$$

— where we recall that, at the level of Fourier series, *convolution of functions corresponds to multiplication of Fourier coefficients.*

Thus, in summary, if we refer to the map of functions

$$\phi \mapsto \phi * \Theta$$

(where "$*$" denotes convolution) given by convolution with Θ as the *theta convolution*, we see that what we would really like to do is to apply to a given series

$$\sum_{n \in \mathbb{Z}} q^{n^2} \cdot P(n) \cdot U^n$$

the *inverse of the theta convolution.*

In order to do this, one must first resolve a number of *technical obstacles*:

(1) First of all, in order to consider the inverse of the theta convolution, one must first show that the theta convolution is *invertible.* The above formal discussion shows that the theta convolution is invertible in a "formal neighborhood of infinity" (of the moduli stack of elliptic curves). In fact, however, in the context of [Mzk1], one needs to know invertibility over *the entire moduli stack of elliptic curves* (not just near infinity).

(2) In the context of [Mzk1], instead of working with the group $\mathbb{G}_m$, one is effectively working with the torsion subgroup $\boldsymbol{\mu}_d \subseteq \mathbb{G}_m$ (of d-torsion points). Thus, one must prove that the rather transparent argument given above in the case of $\mathbb{G}_m$ may also be carried out for the *torsion subgroup* $\boldsymbol{\mu}_d$ — i.e., instead of dealing with the *"classical theta function Θ,"* we must deal with *Mumford's algebraic theta functions*, in which case the coefficients contain (in addition to the easy to understand terms discussed above) other less transparent terms arising from the discrepancy between $\boldsymbol{\mu}_d$ and $\mathbb{G}_m$.

These technical obstances are resolved in the main theorem ([Mzk2], Theorem 9.1) of [Mzk2] on the *invertibility of the Fourier coefficients of Mumford's algebraic theta functions.* This theorem will be discussed in more detail in §2.2 below.

Before proceeding, we would like to explain several ways to think about the contents of [Mzk2]. First of all, the fact that the Fourier coefficients of Mumford's algebraic theta functions are invertible away from the divisor at infinity appears (to the knowledge of the author) to be *new* (i.e., it does not seem to appear in the classical theory of theta functions). Thus, one way to interpret [Mzk2], Theorem 9.1, is as a result which implies the existence of certain *interesting, new modular units.* It would be interesting to see if this point of view can be pursued further (cf. [Mzk2], the Remark following Theorem 9.1). For instance, it would be interesting to try to compute these new modular units as (multiplicative) linear combinations of well-known modular units, such as *Siegel modular units.*

Another way to think about the contents of [Mzk2] is the following. The classical Fourier expansion of Θ discussed above arises from the Fourier expansion of the restriction of the theta function to a certain *particular cycle* (or copy of the circle $\mathbb{S}^1$) on the elliptic curve E in question. More explicitly, if one thinks of this elliptic curve E as being

$$E = \mathbb{C}^\times / q^{\mathbb{Z}}$$

then this special cycle is the image of the natural copy of $\mathbb{S}^1 \subseteq \mathbb{C}^\times$. In the present context, we are considering discrete analogues of this classical complex theory, so instead of working with this $\mathbb{S}^1$, we work with its *d-torsion points* (for some fixed positive integer d), i.e., $\boldsymbol{\mu}_d \subseteq \mathbb{S}^1 \subseteq \mathbb{C}^\times$.

On the other hand, in order to obtain a theory valid over the entire moduli stack of elliptic curves, we must consider Fourier expansions of theta functions restricted not just to this special cyclic subgroup of order d, i.e., $\boldsymbol{\mu}_d \subseteq E$, (with respect to which the Fourier expansion is particularly simply and easy to understand), but rather with respect to an *arbitrary* cyclic subgroup of order d. Viewed from the classical complex theory, considering Fourier expansions arising from more general restriction subgroups amounts to considering the *functional equation of the theta function*. In the classical complex theory, Gauss sums arise naturally in the functional equation of the theta function. Thus, *it is not surprising that Gauss sums (and, in particular, their invertibility) also play an important role in the theory of [Mzk2]*.

In fact, returning to the theory of the Gaussian on the real line, one may recall that one "important number" that arises in this theory is the *integral of the Gaussian* (over the real line). This integral is (roughly speaking) $\sqrt{\pi}$. On the other hand, in the theory of [Mzk2], Gaussians correspond to "discrete Gaussians" (cf. [Mzk2], §2), so integrals of Gaussians correspond to *"Gauss sums."* That is to say, *Gauss sums may be thought of as a sort of discrete analogue of* $\sqrt{\pi}$. Thus, the appearance of Gauss sums in the theory of [Mzk2] is also natural from the point of view of the analogy of the theory of [Mzk1] with the classical theory of Gaussians and their derivatives (cf. §1.2).

Indeed, this discussion of discrete analogues of Gaussians and $\sqrt{\pi}$ leads one to suspect that there is also a natural *p-adic analogue* of the theory of [Mzk2] involving the *p-adic ring of periods* $\mathbb{B}_{crys}$. Since this ring of periods contains a certain copy of $\mathbb{Z}_p(1)$ which may be thought of as a "p-adic analogue of π," it is thus natural to suspect that in a p-adic analogue of the theory of [Mzk2], some "square root of this copy of $\mathbb{Z}_p(1)$" — and, in particular, its *invertibility* — should play an analogously important role to the role played by the invertibility of Gauss sums in [Mzk2].

§2.2. Statement of the Main Theorem

Let E/K, d, η, and $\mathcal{L}$ be as in Theorem $\mathrm{A}^{\mathrm{simple}}$. Also, for simplicity, we assume in the present discussion that d is *odd*, and that we are given two subgroup schemes $G, H \subseteq E[d]$ which are étale locally isomorphic to $\mathbb{Z}/d\mathbb{Z}$ and which satisfy the condition

$$G \times H = E[d]$$

Since d is odd, it follows from Mumford's theory of algebraic theta functions (cf., e.g., [Mzk1], Chapter IV, §1) that we obtain *natural actions* of G, H on the pair $(E, \mathcal{L})$.

In the context of [Mzk2], we refer to G (respectively, H) as the *restriction subgroup* (respectively, *Lagrangian subgroup*) since we use it as the subgroup to which we restrict sections of $\mathcal{L}$ (respectively, via which we descend sections of $\mathcal{L}$ to E/H — cf. the theory of theta groups). Note, in particular, that since G acts on the pair $(E, \mathcal{L})$, we obtain a *theta trivialization*

$$\mathcal{L}|_G \cong \mathcal{L}|_{0_E} \otimes_K \mathcal{O}_G$$

We are now ready to state the main result of [Mzk2]:

Theorem B. (Invertibility of the Fourier Coefficients of an Algebraic Theta Function) *Let E/K, d, η, $\mathcal{L}$, G, and H be as above. In particular, we assume here (for simplicity) that d is* **odd**. *Let $s \in \Gamma(E, \mathcal{L})^H$ be a generator of the one-dimensional K-vector space of H-invariant sections of $\mathcal{L}$ over E. By Mumford's theory of algebraic theta functions, restricting s to $G \subseteq E$ and applying the theta trivialization (reviewed above) gives rise to an* **algebraic theta function**

$$\Theta_s \in \mathcal{L}|_G \cong \mathcal{L}|_{0_E} \otimes_K \mathcal{O}_G$$

i.e., a function on the finite group scheme G. With this notation, the Fourier coefficients of the function Θ_s on G are all inverbile.

In [Mzk2], Theorem 9.1, the case of even d is also addressed, as well as the extent to which these Fourier coefficients are invertible in *mixed characteristic* and *near infinity* (i.e., as the elliptic curve in question degenerates).

Theorem B is proven by comparing the degree of the line bundle (on the moduli stack of elliptic curves) of which the *norm* of the Fourier transform (i.e., the product of its coefficients) is a section to the degree of the zero locus of this norm in a neighborhood of the divisor at infinity. A rather complicated (but entirely elementary, "high school level") calculation reveals that these two degrees coincide. This *coincidence of degrees* implies that the norm is therefore invertible (in characteristic 0) away from the divisor at infinity. This proof is thus reminiscent of the proof of Theorem A given in [Mzk1], although the calculation used to prove the "coincidence of degrees" in [Mzk2] is somewhat more complicated than the calculation that appears in [Mzk1] in the proof of Theorem A.

Finally, in [Mzk2], §10, we apply Theorem B as discussed in §2.1: That is, now that we know (by Theorem B) that the theta convolution is *invertible*, we *compose* the comparison isomorphism of Theorem A with the inverse of the theta convolution to obtain a new *"theta-convoluted comparison isomorphism."* Unfortunately, however, even this new comparison isomorphism is *not entirely free of Gaussian poles.* More precisely, of the various "points at infinity" of the moduli stack of elliptic curves equipped with the data appearing in Theorem B (i.e., η, G, H), those points at infinity for which the restriction subgroup G is *multiplicative* (i.e., coincides with the subgroup $\boldsymbol{\mu}_d \subseteq \mathbb{G}_m \twoheadrightarrow \mathbb{G}_m/q^{\mathbb{Z}} = E$) satisfy the property that in a neighborhood of such points at infinity, *the theta-convoluted comparison* *isomorphism is free of Gaussian poles.* At the other points of infinity (which, incidentally, constitute the *overwhelming majority* of points at infinity), however, even the theta-convoluted comparison isomorphism fails to be free of Gaussian poles. Thus, the theory of [Mzk2] is still not sufficient to allow one to apply the theory of [Mzk1] to diophantine geometry (cf. also the discussion of §1.5.1).

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