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GAFA Geometric And Functional Analysis

PDE AS A UNIFIED SUBJECT

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Introduction

Given that one of the goals of the conference is to address the issue of the unity of Mathematics, I feel emboldened to talk about a question which has kept bothering me all through my scientific career: Is there really a unified subject of Mathematics which one can call PDE? At first glance this seems easy: we may define PDE as the subject which is concerned with all partial differential equations. According to this view, the goal of the subject is to find a general theory of all, or very general classes of PDE's. This "natural" definition comes dangerously close to what M. Gromov had in mind, I believe, when he warned us, during the conference, that objects, definitions or questions which look natural at first glance may in fact "be stupid". Indeed, it is now recognized by many practitioners of the subject that the general point of view, as a goal in itself, is seriously flawed. That it ever had any credibility is due to the fact that it works quite well for linear PDE's with constant coefficients, in which case the Fourier transform is extremely effective. It has also produced significant results for some general special classes of linear equations with variable coefficients.[1] Its weakness is most evident in connection to nonlinear equations. The only useful general result we have is the Cauchy-Kowalevsky theorem, in the quite boring class of analytic solutions. In the more restrictive frameworks of elliptic, hyperbolic, or parabolic equations, some important local aspects of nonlinear equations can be treated with a considerable degree of generality. It is the passage from local to global properties which forces us to abandon any generality and take full advantage of the special features of the important equations.

The fact is that PDE's, in particular those that are nonlinear, are too subtle to fit into a too general scheme; on the contrary each important

[1]Linear equations with variable coefficients appear naturally by linearizing nonlinear equations around specific solutions. They also appear in the study of specific operators on manifolds, in Several Complex Variables, and Quantum Mechanics. The interaction between the $\bar{\partial}$ operator in SCV and its natural boundary value problems have led to very interesting linear equations with exotic features, such as lack of solvability.

PDE seems to be a world in itself. Moreover, general points of view often obscure, through unnecessary technical complications, the main properties of the important special cases. A useful general framework is one which provides a simple and elegant treatment of a particular phenomenon, as is the case of symmetric hyperbolic systems in connection to the phenomenon of finite speed of propagation and the general treatment of local existence for nonlinear hyperbolic equations. Yet even when a general framework is useful, as symmetric hyperbolic systems certainly are, one would be wrong to expand the framework beyond its natural role. Symmetric hyperbolic systems turn out to be simply too general for the study of more refined questions concerning the important examples of hyperbolic equations.

As the general point of view has lost its appeal many of us have adopted a purely pragmatic point of view of our subject; we chose to be concerned only with those PDE's or classes of PDE's which are considered important. And indeed the range of applications of specific PDE's is phenomenal, many of our basic equations being in fact at the heart of fully fledged fields of Mathematics or Physics such as Complex Analysis, Several Complex Variables, Minimal Surfaces, Harmonic Maps, Connections on Principal Bundles, Kahlerian and Einstein Geometry, Geometric Flows, Hydrodynamics, Elasticity, General Relativity, Electrodynamics, Nonrelativistic Quantum Mechanics, etc. Other important subjects of Mathematics, such as Harmonic Analysis, Probability Theory and various areas of Mathematical Physics are intimately tied to elliptic, parabolic, hyperbolic or Schrödinger type equations. Specific geometric equations such as Laplace–Beltrami and Dirac operators on manifolds, Hodge systems, Pseudo-holomorphic curves, Yang–Mills and recently Seiberg–Witten, have proved to be extraordinarily useful in Topology and Symplectic Geometry. The theory of Integrable systems has turned out to have deep applications in Algebraic Geometry; the spectral theory Laplace–Beltrami operators as well as the scattering theory for wave equations are intimately tied to the study of automorphic forms in Number Theory. Finally, Applied Mathematics takes an interest not only in the basic physical equations but also on a large variety of phenomenological PDE's of relevance to engineers, biologists, chemists or economists.

With all its obvious appeal the pragmatic point of view makes it difficult to see PDE as a subject in its own right. The deeper one digs into the study of a specific PDE the more one has to take advantage of the particular features of the equation and therefore the corresponding results may make sense only as contributions to the particular field to which that PDE is relevant. Thus each major equation seems to generate isolated islands of mathematical activity. Moreover, a particular PDE may be studied from largely different points of view by an applied mathematician, a physicist, a geometer or an analyst. As we lose perspective on the common features of our main equations we see PDE less and less as a unified subject. The field of PDE, as a whole, has all but ceased to exist, except in some old fashioned textbooks. What we have instead is a large collection of loosely connected subjects.

In the end I find this view not only somewhat disconcerting but also, intellectually, as unsatisfactory as the first. There exists, after all, an impressive general body of knowledge which would certainly be included under the framework of a unified subject if we only knew what that was. Here are just a few examples of powerful general ideas:[2]:

1) *Well-posedness*: First investigated by Hadamard at the beginning of this century well-posed problems are at the heart of the modern theory of PDE. The issue of well-posedness comes about when we distinguish between analytic and smooth solutions. This is far from being an academic subtlety, without smooth, non-analytic solutions we cannot talk about *finite speed of propagation*, the distinctive mark of relativistic physics. Problems are said to be well posed if they admit unique solutions for given *smooth* initial or boundary conditions. The corresponding solutions have to depend continuously on the data. This leads to the classification of linear equations into elliptic, hyperbolic and parabolic with their specific boundary value problems. Well-posedness also plays a fundamental role in the study of nonlinear equations, see a detailed discussion in the last section of this paper. The counterpart of well-posedness is also important in many applications. *Ill-posed problems* appear naturally in Control Theory, Inverse Scattering, etc., whenever we have a limited knowledge of the desired solutions. *Unique continuation* of solutions to general classes of PDE's is intimately tied to ill-posedness.

[2]I failed to mention, in the few examples given above, the development of topological methods for dealing with global properties of elliptic PDE's as well as some of the important functional analytic tools connected to Hilbert space methods, compactness, the implicit function theorems, etc. I also failed to mention the large body of knowledge with regard to spaces of functions, such as Sobolev, Schauder, BMO and Hardy, etc., or the recent important developments in nonlinear wave and dispersive equations connected to restriction theorems in Fourier Analysis. For a more in depth discussions of many of the ideas mentioned below, and their history, see the recent survey [BreB].

2) *A priori estimates, boot-strap and continuity arguments*: A priori estimates allow us to derive crucial information about solutions to complicated equations without having to solve the equations. The best known examples are energy estimates, maximum principle or monotonicity type arguments. Carleman type estimates appear in connection to ill-posed problems. The a priori estimates can be used to actually construct the solutions, prove their uniqueness and regularity, and provide other qualitative information. The boot-strap type argument is a powerful general philosophy to derive a priori estimates for nonlinear equations. According to it we start by making assumptions about the solutions we are looking for. This allows us to think of the original nonlinear problem as a linear one whose coefficients satisfy properties consistent with the assumptions. We may then use linear methods, a priori estimates, to try to show that the solutions to the new linear problem behave as well, or better, than we have postulated. A continuity type argument allows us to conclude the original assumptions are in fact true. This "conceptual linearization" of the original nonlinear equation lies at the heart of our most impressive results for nonlinear equations.

3) *Regularity theory for linear elliptic equations*: We have systematic methods for deriving powerful regularity estimates for linear elliptic equations. The L^∞ estimates are covered by Schauder theory. The more refined L^p theory occupies an important part of modern Real and Harmonic Analysis. The theory of singular integrals and pseudodifferential operators are intimately tied to the development of L^p-regularity theory.

4) *Direct variational methods*: The simplest example of a direct variational method is the Dirichlet Principle. Though first proposed by Dirichlet as a method of solving the Poisson equation $\Delta\phi = f$ and later used by Riemann in his celebrated proof of the *Riemann Mapping Theorem* in complex analysis, it was only put on a firm mathematical ground in this century. The method has many deep applications to elliptic problems. It allows one to first solve the original problem in a"generalized sense", and then use regularity estimates, to show that the generalized solutions are in fact classical. The ultimate known expression of this second step is embodied in the Nash–De-Giorgi method which allows one to derive full regularity estimates for the generalized solutions of nonlinear, scalar, elliptic equations. This provides, in particular the solution to the famous problem of the regularity of minimal hypersurfaces, as graphs over convex, or mean convex, domains, in all dimensions. Other important applications of the Nash–De-Giorgi method were found in connection with such diverse situa-

tions as the Calabi problem in Kahler Geometry, R. Hamilton's Ricci flow and free boundary value problem arising in Continuum Mechanics.

5) *Energy type estimates*: The energy estimates provide a very general tool for deriving a priori estimates for hyperbolic equations. Together with Sobolev inequalities, which were developed for this reason, they allow us to prove local in time existence, uniqueness and continuous dependence on the initial data for general classes of nonlinear hyperbolic equations, such as symmetric hyperbolic, similar to the classical local existence result for ordinary differential equations. A more general type of energy estimates, based on using the symmetries of the linear part of the equations, allows one to also prove global in time, perturbation results, such as the global stability of the Minkowski space in General Relativity.

6) *Microlocal analysis, parametrices and paradifferential calculus*: One of the fundamental difficulties of hyperbolic equations consists of the interplay between geometric properties, which concern the physical space, and properties intimately tied to oscillations, which are best seen in Fourier space. Microlocal analysis is a general, still developing, philosophy according to which one isolates the main difficulties by careful localizations in physical or Fourier space, or in both. An important application of this point of view is the construction of parametrices, as Fourier integral operators, for linear hyperbolic equations and their use in propagation of singularities results. The paradifferential calculus can be viewed as an extension of this philosophy to nonlinear equations. It allows one to manipulate the form of a nonlinear equation, by taking account of the way large and small frequencies interact, to achieve remarkable technical versatility.

7) *Generalized solutions*: The idea of a generalized solution appears already in the work of D'Alembert (see [Lu]) in connection with the one dimensional wave equation (vibrating string). A systematic and compelling concept of generalized solutions has developed in connection with the Dirichlet principle; more generally via the direct variational method. The construction of fundamental solutions to linear equations led also to various types of such solutions. This and other developments in linear theory led to the introduction of distributions by L. Schwartz. The theory of distributions provides a most satisfactory framework to generalized solutions in linear theory. The question of what is a good concept of a generalized solution in nonlinear equations, though fundamental, is far more murky. For elliptic equations the solutions derived by the direct variational methods have proved very useful. For nonlinear, one dimensional, conser-

vation laws the concept of a generalized solution has been discussed quite early in the works of J.J. Stokes (see [St]), Rankine, Hugoniot, Riemann, etc. For higher dimensional evolution equations the first concept of a *weak solution* was introduced by J. Leray. I call weak a generalized solution for which one cannot prove any type of uniqueness. This unsatisfactory situation may be temporary, due to our technical inabilities, or unavoidable in the sense that the concept itself is flawed. Leray was able to produce, by a compactness method, a weak solution of the initial value problem for the Navier–Stokes equations. The great advantage of the compactness method (and its modern extensions which can, in some cases, cleverly circumvent lack of compactness) is that it produces global solutions for all data. This is particularly important for supercritical, or critical, nonlinear evolution equations where we expect that classical solutions develop finite time singularities. The problem, however, is that one has very little control of these solutions, in particular we don't know how to prove their uniqueness.[3] Similar types of solutions were later introduced for other important nonlinear evolution equations. In most of the interesting cases of supercritical evolution equations, such as Navier–Stokes, the usefulness of the type of weak solutions used so far remains undecided.

8) *Scaling properties and classification of nonlinear equations*: Essentially all basic nonlinear equations have well-defined scaling properties. The relationship between the nonlinear scaling and the *coercive* a priori estimates[4] of the equations leads to an extremely useful classification between subcritical, critical and supercritical equations. The definition of criticality and its connection to the issue of regularity was first understood in the case of elliptic equations such as Harmonic Maps, the euclidean Yang–Mills or Yamabe problem. The same issue appears in connection with geometric heat flows and nonlinear wave equations.

Given that some PDE's are interesting from a purely mathematical point of view, while others owe their relevance to physical theories, one of the problems we face when trying to view PDE as a coherent subject is that of the fundamental ambiguity of its status; is it part of Mathematics or Physics or both? In the next section I will try to broaden the discussion by considering some aspects of the general relationship between Mathematics

[3]Leray was very concerned about this point. Though, like all other researchers after him, he was unable to prove uniqueness of his weak solution; he showed however that it must coincide with a classical one as long as the latter remains smooth.

[4]See the section "The Problem of Breakdown" for a more thorough discussion.

and Physics, relevant to our main concern. I will try to argue that we can redraw the boundaries between the two subjects in a way which allows us to view PDE as a core subject of Mathematics, with an important applied component. In the third section I will attempt to show how some of the basic principles of modern physics can help us organize the immense variety of PDE's into a coherent field. Equally important, in the fourth section, I will attempt to show that our main PDE's are not only related through their derivation; they also share a common fundamental problem, regularity or breakdown. I have tried to keep the discussion of the first four sections as general as possible, and have thus avoided giving more than just a few references. I apologize to all those who feel that their contributions, alluded to in my text, should have been properly mentioned. In the last section of the paper I concentrate on a topic of personal research interest, tied to the issue of regularity, concerning the problem of well-posedness for nonlinear wave equations. My main goal here is to discuss three precise conjectures which I feel are important, difficult and accessible to generate future developments in the field. Even in this section, however, I only provide full references to works directly connected to these conjectures.

Many of the important points I make below, such as the unified geometric structure of the main PDE's, the importance of the scaling properties of the equations and its connection to regularity and well-posedness, have been discussed in similar ways before and are shared by many of my friends and collaborators. My only claim to originality in this regard is the form in which I have assembled them. The imperfections, errors and omissions are certainly my own.

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Between Mathematics and Physics

In search of a unified point of view for our subject it pays to look at the broader problem of Mathematics as a whole. Isn't Mathematics also in danger of becoming a large collection of loosely connected subjects? Our cherished intellectual freedom to pursue whatever problems strike our imagination as worthwhile is a great engine of invention, but, in the absence of unifying goals, it seems to lead to an endless proliferation of subjects.

This is precisely, I believe, what Poincaré [P] had in mind in the following passage, contained in his address to the first International Congress of Mathematicians, more than a hundred years ago.

"...The combinations that can be formed with numbers and symbols are an infinite multitude. In this thicket how shall we choose those that are worthy of our attention? Shall we be guided only by whimsy?...... [This] would undoubtedly carry us far from each other, and we would rapidly cease to understand each other. But that is only the minor side of the problem. Not only will physics perhaps prevent us from getting lost, but it will also protect us from a more fearsome danger ...... turning around forever in circles. History [shows that] physics has not only forced us to choose [from the multitude of problems which arise], but it has also imposed on us directions that would never have been dreamed of otherwise....... What could be more useful!

The full text of [P] is a marvelous analysis of the complex interactions between Mathematics and Physics. Poincaré argues not only that Physics provides us with a great source of inspiration and cohesiveness but that itself, in return, owes its language, sense of beauty and order to Mathematics. Yet Poincaré's viewpoint concerning the importance of close relations with Physics was largely ignored during most of this century by a large segment of the mathematical community. One reason is certainly due to the fact that traditional areas of Mathematics such as Algebra, Number Theory and Topology have, or seemed to have,[5] relatively little to gain from direct interactions with Physics. Another, more subtle, reason may have to do with the remarkable and unexpected effectiveness of pure mathematical structures in the formulation of the major physical theories of the century: Special and General Relativity, Quantum Mechanics and Gauge Theories. This has led to the popular point of view, coined by Wigner [Wi] as "The unreasonable effectiveness of Mathematics," according to which mathematical objects or ideas developed originally without any reference to Physics turn out to be at the heart of solutions to deep physical problems. Einstein, himself, wrote that any important advance in Physics will have to come in the wake of major new developments in Mathematics. This very seductive picture has emboldened us mathematicians to believe that anything we do may turn out, eventually, to have real applications and has thus, paradoxically, contributed to the problem of ignoring the physical world Poincaré

[5]The situation has changed dramatically in the last 25 years with the advent of Gauge fields and String Theories.

has warned us against.

But this is only a minor paradox by comparison to the one which seems to arise from the above discussion relative to the remarkable symbiosis between Physics and Mathematics. On one hand, as Poincaré argues very convincingly in [P], Mathematics needs, to keep itself together, unifying goals and principles; Physics, due, I guess, to the perceived unity of the Physical World, is in a perfect position to provide them for us. On the other hand, Physics owes to Mathematics the very tools which makes it possible to uncover and formulate the unified features of physical reality; it is indeed the search for a selfconsistent mathematical formalism which seems to be at the core of the current attempts to find that *unified theory of everything* which, as theoretical physicists often declare, is Physics' ultimate goal. The paradox is due, of course, to the artificial distinctions we make between the two subjects. We imagine them as separated when in fact they have a nontrivial intersection. Can we identify that intersection? The naive picture would be of two sets which intersect in an area, somewhat peripheral to both, which we might call Mathematical Physics. But this picture does not help to solve the paradox we have mentioned above, which concerns the core of both subjects. A central intersection, however, could imply some form of equality or inclusion between the two subjects, which is definitely not the case. Mathematics pursues goals which are not necessarily suggested by the physical sciences. A research direction is deemed important by mathematicians if it leads to elegant developments and unexpected connections. Physics, on the other hand, cannot allow itself the luxury of being carried away by elegant mathematical theories; in the final analysis it has to subject itself to the tough test of real experiments. Moreover the difference between the work practice and professional standards of mathematicians and modern theoretical physicists cannot be more striking. We mathematicians find ourselves constrained by rigor and are often reluctant to proceed without a systematic analysis of all obstacles in our path. In their quest for the ultimate truth theoretical physicists have no time to waste on unexpected hurdles and unpromising territory. Clearly the relationship between the two subjects is far more complex than may seem at first glance.

The task of defining PDE as a unified subject is tied to that of clarifying, somehow, this ambiguous relationship between Mathematics and Physics. The very concept of partial differential equations has its roots in Physics or, more appropriately Mathematical Physics; there were no clear distinctions at the time of D'Alembert, Euler, Poisson, Laplace, between the two subjects. Riemann was the first, I believe, to show how one can use PDE's to attack problems considered pure mathematical in nature, such as conformal mappings in Complex Analysis. The remarkable effectiveness of PDE's as a tool to solve problems in Complex Analysis, Geometry and Topology has been confirmed many times during this century.

One can separate all mathematicians and other scientists concerned with the study of PDE's into four[6] groups, according to their main interests. In the first group I include those developing and using PDE methods to attack problems in Differential Geometry, Complex Analysis, Symplectic Geometry, Topology and Algebraic Geometry. In the second I include those whose main motivation is the development of rigorous mathematical methods to deal with the PDE's arising in the physical theories. In the third group I include mathematicians, physicists or engineers interested in understanding the main consequences of the physical theories, governed by PDE's, using a variety of heuristic, computational or experimental methods. It is only fair to define yet a fourth category[7] which include all those left out of the groups defined above. According to the common preconceptions about the proper delimitations between Mathematics and Physics only the first group belongs unambiguously within Mathematics. The third group is considered, correctly in my view, as belonging either to Applied Mathematics or Applied Physics. The second group however has an ill defined identity. Since the ultimate goals are not directly connected to specific applications to the traditional branches of Mathematics, many view this group as part of either Applied Mathematics or Mathematical Physics. Yet, apart from the original motivation, it is hard to distinguish the second group from the first. Both groups are dedicated to the development of rigorous analytic techniques. They are tied by many similar concerns, concepts and methods. They are both intimately tied to subjects considered pure, mainly Real and Fourier Analysis but also Geometry, Topology and Algebraic Geometry.

In view of the above ambiguities it helps to take a closer look at the role played by Mathematics in developing the consequences of the *established*

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[6]My classification is mainly rhetorical. There are, of course, many mathematicians who can cross these artificial boundaries. I will in fact argue below that the first two groups should be viewed as one.

[7]This includes, in particular, PDE's appearing in Biology or Economics. Exotic PDE's, not necessarily connected with any specific application, should also be included in this class.

physical theories.[8] I have heard theoretical physicists and also, alas, mathematicians, expressing the view that the consequences of an established physical theory are of lesser importance and may properly be relegated to Engineering or Chemistry. Nothing, in my mind, can be further from the truth. The first successful physical theory, that of space, was written down by Euclid more than two thousand years ago. Undoubtedly Euclidean Geometry was used by engineers to design levers, pulleys and many other marvelous applications, but does anybody view the further development of the subject as Engineering? Geometry is the primary example of a "physical theory" developed for centuries as a pure mathematical discipline, without too much new input from the physical world, which grew to have deep, mysterious, completely unexpected consequences to the point that pre-eminent physicists talk today of a complete "geometrization" of modern physics, see [N].

But this is not all; the Principle of Least Action was developed by mathematicians such as Fermat, Leibnitz, Maupertius, the Bernoulli brothers and Euler from the analysis of simple geometric and physical problems (see [HT] for a very good presentation of the early history of the principle). Their work led to a comprehensive reformulation of the laws of Mechanics by Lagrange who showed how to derive them from a simple Variational Principle. Today the Lagrangian point of view, together with its Hamiltonian reformulation and the famous result of E. Noether concerning the relation between the symmetries of the Lagrangian and conservation laws, is a foundational principle for all Physics. Connected to these are the continuous groups of symmetries attached to the name of S. Lie.

Fourier Analysis was initiated in works by D'Alembert, Euler and D. Bernoulli in connection with the study of the initial-boundary value problem for the one dimensional wave equation (vibrating string). Bernoulli's idea of approximating general periodic functions by sums of sines and cosines was later developed by J. Fourier in connection with the Heat Equation. Further mathematical developments made the theory into a fun-

[8] I distinguish between the quest to uncover the basic laws of Nature, which defines the core of theoretical Physics, and the scientific activities concerned with deriving the consequences of a given, established, theory which involve applied physicists, engineers, chemists, applied mathematicians and, as I argue below, "pure" mathematicians. Needless to say, mathematicians have often had direct, fundamental, contributions to theoretical Physics. But more often, I believe, the most impressive contributions came from inner developments within Mathematics of subjects with deep roots in the physical world, such as Geometry, Newtonian Mechanics, Electromagnetism, Quantum Mechanics, etc.

damental tool throughout all of Science.

There are plenty of other examples. I suspect that many, if not most, of the examples of the "unreasonable effectiveness of mathematics" are in fact of this type.[9] There are also many other examples of ideas which originate in Mathematical Physics, and turn out to have a deep, mysterious, impact on the traditional subjects of Mathematics, such as Topology, Geometry or even Number Theory.

All this seems to point to the fact that the further development of the established physical theories ought to be viewed as a genuine and central goal of Mathematics itself. In view of this I think we need to reevaluate our current preconception about what subjects we consider as belonging properly within Mathematics. We may gain, consistent with Poincaré's point of view, considerably more unity by enlarging the boundaries of Mathematics to include, on equal footing with all other more traditional fields, physical theories such as Classical and Quantum Mechanics and Relativity Theory, which are expressed in clear and unambiguous mathematical language. We may then develop them, if we wish, on pure mathematical terms asking questions we consider fundamental, which may not coincide, at any given moment, with those physicists are most interested in, and providing full rigor to our proofs. Of course this has happened to a certain extent, Mathematical Physics and parts of Applied Mathematics fulfill precisely this role. Yet their status remains ambiguous and somewhat peripheral. Many mathematicians assume that subjects like Classical General Relativity[10] or

[9] A clear example of this type, this century, is the discovery of the soliton and the "integrable method." Though they both emerged in connection with simple nonlinear partial differential equations the integrability method has found deep applications way beyond the original PDE context. There are other examples which do not quite fit into my description. The extraordinary role played by complex numbers in the formulation of Quantum Mechanics is certainly one which has its roots in Algebra rather than Geometry or Mathematical Physics.

[10] The formulation of General Relativity, by A. Einstein, following the work of Gauss and Riemann in Geometry, and that of Lorentz, Poincaré, Einstein and Minkowski on special relativity can be viewed as one of the most impressive triumphs of Mathematics. Following the recent experiments with double pulsars, GR is considered the most accurate of all physical theories. Research in General Relativity involves, in a fundamental way, all aspects of traditional mathematics; Differential Geometry, Analysis, Topology, Group Representation, Dynamical systems, and of course PDE's. Assuming that the further development of the subject is covered by physics departments is misleading; most theoretical physicists view classical GR as a completely understood physical theory, their main goal now is to develop a quantum theory of gravity. Given their lack of interest and the rich mathematical content of the subject, is there any reason why we should not

Quantum Mechanics belong properly to Physics departments while Physicists often consider them as perfectly well understood, closed, subjects. They are indeed closed, or so it seems, in so far as theoretical physicists are concerned. From their perspective Geometry may have become a closed book more than 2000 years ago, with the publication of Euclid's Elements. But they present us, mathematicians, with wonderful, fundamental challenges formulated in the purest mathematical language. Should we relegate subjects such as Classical and Quantum Mechanics or General Relativity to the periphery of Mathematics, despite their well defined and rich mathematical structures, only because they happen to describe important aspects of the physical world? Is it reasonable to hesitate to include General Relativity as a subject of Mathematics simply because it concerns itself with Lorentz rather than Riemannian metrics? Or because it does not seem to have any applications to Topology? (There are in fact proposals to tie GR to the geometrization conjecture of 3D manifolds, see [FM].)

My proposal is not just to accept these disciplines as some applied appendices to pure Mathematics, but to give them the central role[11] they deserve. This would force us to broaden our outlook and would give us fresh energy and cohesion in the spirit envisioned by Poincaré. It would help us, in particular, to clarify the ambiguous status of subjects such as PDE's and Mathematical Physics and their relations with Applied Mathematics. It would also set more natural boundaries between Mathematics and Physics. As theoretical physicists are primarily interested in understanding new physical phenomena, the further mathematical developments of a confirmed physical theory becomes one of our tasks.[12] Though our pure mathematical considerations may lead us into seemingly esoteric directions, we should hold our ground for with time physicists may come to admit, once more, to the unexpected effectiveness of our Science.

Finally I want to distinguish my proposal from another, more radical, point of view, discussed in this conference, according to which Mathematics ought to become fully engaged with the great problems of Chemistry, Biology, Computing, Economics and Engineering. Though I strongly suspect that one day some, still to be discovered, deep mathematical structure

take the opportunity and embrace it fully, as our own?

[11] An easy step, which will go a long way in this direction, would be to add, as a requirement for mathematics majors, or graduate students, a course containing a comprehensive discussion of the mathematical structures which underly the main physical theories.

[12] This does not exclude the possibility that the same subject may be pursued, *in different ways*, in both Mathematics and Physics departments.

will help explain some of the important features of complex biological systems, we are very far from that. It is certainly to be hoped that individual mathematicians will make significant contributions to these fields but it is unrealistic to think that Mathematics can fully embrace these areas while maintaining its inner continuity, coherence, and fundamental commitment to rigor. We have to distinguish between the core of Mathematics, where I believe the basic physical theories ought to belong, and various problems of Science and Engineering where mathematicians can play a very useful role.

The Main Equations

To return to PDE, I want to sketch a way of looking at the subject from simple first principles which happen to coincide with some of the underlying geometric principles of modern Physics. It turns out that most of our basic PDE's can be derived in this fashion. Thus the main objects of our subject turn out to be in no way less "pure mathematical" in nature than the other fundamental objects[13] studied by mathematicians: numbers, functions and various types of algebraic and geometric structures. But most importantly, these simple principles provide a unifying framework[14] for our subject and thus help endow it with a sense of purpose and cohesion. It also explains why a very small number of linear differential operators, such as the Laplacian and D'Alembertian, are all pervasive; they are the simplest approximations to equations naturally tied to the two most fundamental geometric structures, Euclidean and Minkowskian. The Heat equation is the simplest paradigm for diffusive phenomena while the Schrödinger equa-

[13]Some pure mathematicians distrust the basic physical PDE's, as proper objects of Mathematics, on the spurious notion that they are just imperfect approximations to an ultimate physical reality of which we are still ignorant. On the basis of this analysis groups, C^* algebras, topological vector spaces or the $\overline{\partial}$ operator are perfect mathematical objects, as long as they have no direct relations to Physics, while Hamiltonian systems, the Maxwell, Euler, Schrödinger and Einstein equations are not!

[14]The scheme I present below is only an attempt to show that, in spite of the enormous number of PDE's studied by mathematicians, physicists and engineers, there are nevertheless simple basic principles which unite them. I don't want, by any means, to imply that the equations discussed below are the only ones worthy of our attention. It would be also foolish to presume that we can predict which PDE's are going to lead to the most interesting developments. Certainly, nobody could have predicted 100 years ago the emergence on the scene of the Einstein and Yang–Mills equations, or the remarkable mathematical structure behind the seemingly boring KdV equation.

tion can be viewed as the Newtonian limit of a lower order perturbation of the D'Alembertian. The geometric framework of the former is Galilean space which, itself, is simply the Newtonian limit of the Minkowski space, see [M].

Starting with the Euclidean space $\mathbb{R}^n$, the Laplacian Δ is the simplest differential operator invariant under the group of isometries, or rigid transformations, of $\mathbb{R}^n$. The heat, Schrödinger, and wave operators $\partial_t - \Delta$, $\frac{1}{i}\partial_t - \Delta$ and $\partial_t^2 - \Delta$ are the simplest evolution operators which we can form using Δ. The wave operator $\Box = -\partial_t^2 + \Delta$ has a deeper meaning; it is associated to the Minkowski space $\mathbb{R}^{n+1}$ in the same way that Δ is associated to $\mathbb{R}^n$. Moreover, the solutions to the equation $\Delta\phi = 0$ can be viewed as special, time independent solutions, to $\Box\phi = 0$. The Schrödinger equation can also be obtained, by a simple limiting procedure, from the Klein–Gordon operator $\Box - m^2$. Appropriate, invariant, and local definitions of square roots of Δ and $\Box$, or $\Box - m^2$, corresponding to spinorial representations of the Lorentz group, lead to the associated Dirac operators.

In the same vein we can associate to every Riemannian, or Lorentzian, manifold (M, g) the operators Δ_g, resp. $\Box_g$, or the corresponding Dirac operators. These equations inherit in a straightforward way the symmetries of the spaces on which they are defined. There exists a more general, *unreasonably effective*, scheme of generating equations with prescribed symmetries. The variational Principle allows us to associate to any Lagrangian L a system of partial differential equations, called the Euler–Lagrange equations, which inherit the symmetries *built in L*. In view of Noether's principle, to any continuous symmetry of the Lagrangian there corresponds a conservation law for the associated Euler–Lagrange PDE. Thus, the Variational Principle generates equations with desired conservation laws such as Energy, Linear and Angular Momenta, etc. The general class of Lagrangian equations, plays the same selected role among all PDE's as that played by Hamiltonian systems among ODE's. Calculus of Variations is by itself a venerable and vast subject of Mathematics. The main equations of interest in both Geometry[15] and Physics, however, are not just variational; they

[15]There are, however, important geometric problems, such as prescribed curvature and isometric embeddings in Riemannian Geometry or Lewy flat surfaces in Complex Geometry, without an obvious variational structure. The real and complex Monge Ampere equations are typical examples. The Pseudo-holomorphic Curves, used by Gromov in the study of symplectic manifolds, provides another example. Nevertheless these equations have a rich geometric structure and share with the variational PDE's many common characteristics. Moreover on closer inspection they may turn out to have a nontrivial,

are obtained from Lagrangians constructed from simple geometric objects such as:

1) *Lorentz or Riemannian metrics*: On a Lorentzian manifold (M, g) the Lagrangian given by the scalar curvature $R(g)$ of the metric leads, through variations of the metric, to the Einstein-Vacuum (EV) equations of General Relativity. A similar procedure leads to Einstein metrics in Riemannian geometry. The restriction of the Einstein functional $\int R(g)dv_g$ to a conformal class of metrics leads to the well-known Yamabe equation.

2) *Connections on a principal bundle*: The quadratic scalar invariant formed by the curvature of a connection defines the Yang–Mills Lagrangian. The Yang–Mills (YM) equations are obtained through variations of the connection. The Maxwell equations correspond to the case of a trivial bundle over the Minkowski space with structure group $U(1)$. The standard model of particle physics corresponds to the group $SU(3) \times SU(2) \times U(1)$. The YM equations used in Topology correspond to Riemannian connections with nonabelian group $SU(2)$.

3) *Scalar equations*: Are derived for scalar functions $\phi : M \to \mathbb{R}, \mathbb{C}$. The Lagrangian is $L = g^{\mu\nu}\phi_\mu\phi_\nu + V(\phi)$, with $V(\phi) \geq 0$. When $V = 0$ we derive $\Delta_g\phi = 0$, in the Riemannian case, and $\Box_g\phi = 0$ in the Lorentzian case. The case $V(\phi) = \frac{1}{2}m^2|\phi|^2$ corresponds to the Klein–Gordon equation, $V(\phi) = \frac{1}{4}|\phi|^4$ leads to the well-known cubic wave equation. We will refer to this type of equations as nonlinear scalar wave equations (NSWE).

4) *Mappings between two manifolds*: Consider mappings $\phi : (M, g) \to (N, h)$ between the pseudoriemannian domain manifold M of dimension $d + 1$ and Riemannian target N of dimension n. Let ϕ^*h be the symmetric 2-tensor on M obtained by taking the pull-back of the metric h of N. Let $\lambda_0, \lambda_1, \dots, \lambda_d$ be the eigenvalues of ϕ^*h relative to the metric g and $S_0, S_1, \dots, S_d$ the corresponding elementary symmetric polynomials in $\lambda_0, \lambda_1, \dots, \lambda_d$. Any symmetric function of $\lambda_0, \lambda_1, \dots, \lambda_d$, or equivalently, any function $L(S_0, S_1, \dots, S_d)$, can serve as a Lagrangian. By varying the action integral $\int_M Ldv_g$ relative to ϕ, with dv_g the volume element of the metric g, we obtain a vast class of interesting equations. Here are some examples:[16]

hidden, variational structure. This is the case, for example, of the Monge Ampere equations, see [Br].

[16] I want to thank D. Christodoulou for his help in the presentation of this section. Most of the examples below, and much more, are discussed in detail in his book [Chr1].

(i) The Harmonic and Wave Maps (WM) are obtained in the particular case $L = \mathrm{tr}_g(\phi^* h)$. The only distinction between them is due to the character, Riemannian respectively Lorentzian, of the metric g.

(ii) The basic equations of Continuum Mechanics are obtained from a general Lagrangian, as described above, in the particular case when g is Lorentzian, $n = d = 3$ and the additional assumptions that ϕ has maximal rank at every point and the curves $\phi^{-1}(p)$ are time-like for all $p \in N$. Since the dimension of N is one less than the dimension of M one of the eigenvalues, say λ_0, is identically zero. Elasticity corresponds to general choices of L as a symmetric function of $\lambda_1, \dots, \lambda_d$. Fluid Mechanics corresponds to the special case when L depends only on the product $\lambda_1 \cdot \lambda_2 \cdots \lambda_d$. One can also derive the equations of Magneto-hydrodynamics (MHD) by assuming an additional structure on N given by a 2-form Ω. The 2-form $F = \phi^*\omega$ defines the electromagnetic field on M. The Lagrangian of MHD is obtained by adding the Maxwell Lagrangian $\frac{1}{2}F_{\mu\nu} \cdot F^{\mu\nu}$ to the fluid Lagrangian described above.

(iii) The minimal surface equation is derived from the Lagrangian $L = \sqrt{\det_g \phi^* h}/\sqrt{\det(g)}$ in the case when g is Riemannian and $m = d + 1 < n$. The case when g is Lorentzian leads to a quasilinear wave equation.

5) *Lagrangian leading to higher order equations*: While the main equations of Physics are all first or second order, there is no reason why one should avoid higher order equations for applications to Geometry. It is natural, for example, to consider equations associated to conformally invariant Lagrangians. Many of the known Lagrangians, which lead to second order equations such as Harmonic Maps, are conformally invariant only in dimension 2. To produce a larger class of conformally invariant equations, in even dimensions, it pays to look for higher order theories such as biharmonic maps in 4D, see [CWY]. The variational problem associated to the zeta functional determinant of the Laplace–Beltrami operator, of a higher dimensional Riemannian metric, also leads to higher order equations. Finally the Willmore problem for closed surfaces in $\mathbb{R}^3$ provides another interesting example of a fourth order equation.

6) *Composite Lagrangians*: By adding various Lagrangians we derive other basic equations. This is true, most remarkably, for the gravitational Lagrangian, given by the scalar curvature of the metric. In combination with the Lagrangian of a matter theory, in fact any other relativistic La-

grangians described above, it leads to the famous Einstein Field Equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$, with T the energy momentum tensor of the matter Lagrangian. The Lagrangian of the Seiberg–Witten equations are obtained by coupling the Lagrangian of the Maxwell theory with that of the Dirac equation.

The equations derived by the above geometric constructions are elliptic, if the metric g on M is Riemannian, and hyperbolic if g is Lorentzian. In the hyperbolic case we distinguish between the Field Theories, for which the only characteristics of the corresponding PDE's are given by the Lorentz metric g, and the other equations; Fluids, Continuum Mechanics, MHD, etc., which have additional characteristics. The YM, WM and the EV are all field theories in the sense we have just defined. The EV equations is distinguished from the other field theories by being the only one for which the metric g itself is the solution. This fact gives the EV equations a quasilinear character. For all other field equations, since the metric g is fixed, the equations are semilinear.

With the exceptional case of EV, which does not have local conservation laws, all equations described above have associated to them, a well-defined energy-momentum tensor T which verifies the positive energy condition. I recall that the energy-momentum tensor of a Lagrangian theory is a rank 2 symmetric tensor $T_{\mu\nu}$ verifying the local conservation law $D^{\mu}T_{\mu\nu} = 0$. We say that T satisfies the positive energy condition if $T(X,Y) \geq 0$ for all time-like future oriented vectorfields X, Y.

Many other familiar equations can be derived from the *fundamental ones* described above by the following procedures:

(a) *Symmetry reductions*: Are obtained by assuming that the solutions we are looking for have certain continuous symmetries. They lead to much simpler equations than the original, often intractable ones. Another, somewhat more general, way of obtaining simpler equations is to look for solutions which verify a certain ansatz, such as stationary, spherically symmetric, equivariant, self-similar, traveling waves, etc.

(b) *The Newtonian approximation and other limits*: We can derive a large class of new equations, from the basic ones described above, by taking one or more characteristic speeds to infinity. The most important one is the Newtonian limit, which is formally obtained by letting the velocity of light go to infinity. At the level of the space-time manifold itself this limit, described in the seminal paper of Minkowski [M], takes a Lorentz manifold to the Galilean space-time of Newtonian mechanics. As we have mentioned above the Schrödinger equation itself can be derived, in this fashion, from the linear Klein–Gordon equation. In the same way we can formally derive the Lagrangian of nonrelativistic Elasticity (see [Z]), Fluids or MHD equations. The formal Newtonian limit of the full Einstein field equations leads to the various continuum mechanics theories in the presence of Newtonian gravity. The Newtonian potential is tied to the lapse function of the original space-time metric.

We should not be surprised that the better known nonrelativistic equations, look more messy than the relativistic ones. The simple geometric structure of the original equations gets lost in the limit. The remarkable simplicity of the relativistic equations is a powerful example of the importance of Relativity as a unifying principle.

Once we are in the familiar world of Newtonian physics we can perform other well-known limits. The famous incompressible Euler equations are obtained by taking the limit of the general nonrelativistic fluid equations as the speed of sound tends to infinity. Various other limits are obtained relative to other characteristic speeds of the system or in connection with specific boundary conditions, such as the boundary layer approximation in fluids. The equations of Elasticity, for example, approach in the limit, when all characteristic speeds tend to infinity, to the familiar equations of a rigid body in Classical Mechanics. Another important type of limit, leading to the well-known Hamilton–Jacobi equations of Classical Mechanics, is the high frequency or the geometric optics approximation.

Many of these very singular limits remain purely formal. While some of them have been rigorously derived, many more present serious analytic difficulties.

(c) *Phenomenological assumptions*: Even after taking various limits and making symmetry reductions, the equations may still remain unyielding. In various applications it makes sense to assume that certain quantities are small and may be neglected. This leads to simplified equations which could be called *phenomenological*[17] in the sense that they are not derivable from first principles. They are used to illustrate and isolate important physical phenomena present in complicated systems. A typical way of generating interesting phenomenological equations is to try to write down the simplest model equation which describes a particular feature of the original system.

[17] I use this term here quite freely, it is typically used in a somewhat different context. Also some of the equations which I call phenomenological below, e.g., dispersive equations, can be given formal asymptotics derivations by Applied Math. techniques.

Thus, the self-focusing, plane wave effects of compressible fluids, or elasticity, can be illustrated by the simple minded Burgers equation $u_t + uu_x = 0$. Nonlinear dispersive phenomena, typical to fluids, can be illustrated by the famous KdV equation $u_t + uu_x + u_{xxx} = 0$. The nonlinear Schrödinger equations provide good model problems for nonlinear dispersive effects in Optics. The Ginzburg–Landau equations provide a simple model equation for symmetry breaking phase transitions. The Maxwell–Vlasov equations is a simplified model for the interactions between Electomagnetic forces and charged particles, used in Plasma Physics.

When well chosen, a model equation leads to basic insights into the original equation itself. For this reason simplified model problems are also essential in the day to day work of the rigorous PDE mathematician. We all test our ideas on such carefully selected model problems. It is crucial to emphasize that good results concerning the basic physical equations are rare; a very large percentage of important rigorous work in PDE deals with simplified equations selected, for technical reasons, to isolate and focus our attention on some specific difficulties present in the basic equations.

It is not at all a surprise that the equations derived by symmetry reductions, various limits and phenomenological assumptions have additional symmetries and therefore additional conservation laws. It is however remarkable that some of them have infinitely many conserved quantities or turn out to be even integrable. The discovery of the integrability of the KdV equation and, later, that of other integrable PDE's is one of the most impressive achievements of the field of PDE's in this century. It remains also the model case of a beneficial interaction between numerical experiments, heuristic applied mathematics arguments algebra and rigorous analysis. Together they have led to the creation of a beautiful mathematical theory with extensive and deep applications outside the field of PDE's where they have originated from. We have to be aware, however, of the obvious limitations of integrable systems; with few exceptions (the KP-I and KP-II equations are, sort of, 2-dimensional) all known integrable evolution equations are restricted to one space dimension.

In all the above discussion we have not mentioned diffusive equations such as the Navier–Stokes. They are in fact not variational and, therefore, do not fit at all in the above description. They provide a link between the microscopic, discrete, world of Newtonian particles and the continuous macroscopic one described by Continuum Mechanics. Passing from discrete to continuous involves some loss of information hence the continuum equations have diffusive features. The best known examples of diffusive effects are the "heat conduction," which appears in connection with the dissipation of energy in compressible fluids, and "viscosity," corresponding to dissipation of momentum, in Fluids. Another example is that of "electrical resistivity" for the electrodynamics of continuum media. The Navier–Stokes equation appears in the incompressible limit. The incompressible Euler equations are the formal limit of the Navier–Stokes equations as the viscosity tends to zero. Because of the loss of information involved in their derivation the diffusive equations have probabilistic interpretations.

Diffusive equations turn out to be also very useful in connection with geometric problems. Geometric flows such as mean curvature, inverse mean curvature, Harmonic Maps, Gauss Curvature and Ricci flows are some of the best known examples. Some can be interpreted as the gradient flow for an associated elliptic variational problem. They can be used to construct nontrivial stationary solutions to the corresponding stationary systems, in the limit as $t \to \infty$, or to produce foliations with remarkable properties, such as that used recently in the proof of the Penrose conjecture.

REMARK. *The equations which are obtained by approximations or by phenomenological assumptions present us with an interesting dilemma. The dynamics of such equations may lead to behavior which is incompatible with the assumptions made in their derivation. Should we continue to trust and study them, nevertheless, for pure mathematical reasons or should we abandon them in favor of the original equations or a better approximation? Whatever one may feel about this in a specific situation it is clear that the problem of understanding, rigorously, the range of validity of various approximations is one of the fundamental problems in PDE.*

The Problem of Breakdown

The most basic mathematical question in PDE is, by far, that of regularity. In the case of elliptic equations, or subelliptic in Complex Analysis, the issue is to determine the regularity of the solutions to a geometric variational problem. In view of the modern way of treating elliptic equations, one first constructs a generalized solution by using the variational character of the equations. The original problem, then, translates to that of showing that the generalized solution has additional regularity. This is a common technique for both linear and nonlinear problems. Moreover the technique can be extended to scalar, fully nonlinear, nonvariational problems, such as Monge-Ampere equations, with the help of the viscosity method. In linear cases as well as in some famous nonlinear cases, such as the minimal hypersurfaces as graphs over mean convex domains, one can show that the generalized solutions are smooth. The solutions to the general Plateau problem, however, may have singularities. In this case the main issue becomes the structure of the singular set of a given nonsmooth solutions. Geometric Measure Theory provides sophisticated analytical tools to deal with this problem. Singularities are also known to occur in the case of higher dimensional harmonic maps, for positively curved target manifolds such as spheres.

In the case of evolution equations the issue is the possible spontaneous, finite time (in view of results concerning local in time existence, the breakdown can only occur after a short time interval), breakdown of solutions, corresponding to perfectly nice initial conditions. This is a typical nonlinear, multidimensional phenomenon.[18] It can be best illustrated in the case of the Burgers equation $u_t + uu_x = 0$. Despite the presence of infinitely many positive conserved quantities, $\int |u(t,x)|^{2k}dx$, $k \in \mathbb{N}$, all solutions, corresponding to smooth, compactly supported, nonzero initial data at $t = 0$, breakdown in finite time. The breakdown corresponds, physically, to the formation of a shock wave. Similar examples of breakdown can be constructed for compressible fluids or Elasticity, see [J], [Si]. Singularities are also known to form, in some special cases, for solutions to the Einstein field equations in General Relativity. Moreover, one expects this to happen, in general, in the presence of strong gravitational fields. It is also widely expected that the general solutions of the incompressible Euler equations in three space dimensions, modeling the behavior of inviscid fluids, breakdown in finite time. Some speculate that the breakdown may have something to do with the onset of turbulence for incompressible fluids with very high Reynolds numbers. These fluids are in fact described by the Navier–Stokes equations. In this case the general consensus is that the evolution of all smooth, finite energy, initial data lead to global in time, smooth, solutions. The problem is still widely open. *It is conceivable that there are in fact plenty of solutions which break down but are unstable, and thus impossible to detect numerically or experimentally.*

Breakdown of solutions is also an essential issue concerning nonlinear

[18]For smooth, one dimensional, Hamiltonian systems with positive energy, solutions are automatically global in time. This the case, for example, of the nonlinear harmonic oscillator $\frac{d^2}{dt^2}x + V'(x) = 0$, $V \geq 0$.

geometric flows, such as the mean and inverse mean curvature flows, Ricci flow, etc. As singularities do actually form in many important geometric situations, one is forced to understand the structure of singularities and find ways to continue the flow past them. Useful constructions of generalized flows can lead to the solution of outstanding geometric problems, as in the recent case of the Penrose conjecture [HuI].

The problem of possible breakdown of solutions to interesting, nonlinear, geometric and physical systems is not only the most basic problem in PDE; it is also the most conspicuous unifying problem, in that it affects all PDE's. It is intimately tied to the basic mathematical question of understanding what we actually mean by solutions and, from a physical point of view, to the issue of understanding the very limits of validity of the corresponding physical theories. Thus, in the case of the Burgers equation, for example, the problem of singularities can be tackled by extending our concept of solutions to accommodate "shock waves," i.e. solutions discontinuous across curves in the t, x space. One can define a functional space of generalized solutions in which the initial value problem has unique, global solutions. Though the situation for more realistic physical systems is far less clear and far from being satisfactorily solved, the generally held opinion is that shock wave type singularities can be accommodated without breaking the boundaries of the physical theory at hand. The situation of singularities in General Relativity is radically different. The type of singularities expected here is such that no continuation of the solutions is possible without altering the physical theory itself. The prevaling opinion, in this respect, is that only a quantum field theory of Gravity could achieve this.

One can formulate a general philosophy to express our expectations with regard to regularity. To do that we need to classify our main equations according to the strength of their nonlinearities relative to that of the known *coercive* conservation laws or other a priori estimates. Among the basic conservation laws that provided by the Energy is coercive, because it leads to an absolute, local, space-time bound on the size of solutions, or their first derivatives. The others, such as the linear and angular momentum, do not provide any additional information concerning local regularity. For the basic evolution equations, discussed in the previous section, the energy integral provides the best possible a priori estimate and therefore the classification is done relative to it. This raises a question of fundamental importance; *are there other, stronger, local a priori bounds which cannot be* *derived from Noether's Principle?* There are methods which can rule out the existence of some exact conserved quantities, different from the physical ones, yet there is no reason, I believe, to discount other, more subtle bounds. A well-known *Morawetz multiplier method* leads, for some classes of nonlinear wave equations, to bounded space-time quantities which do not correspond to any conservation law. The Morawetz quantity, however, has the same scaling properties as the energy integral; it only provides additional information in the large. *The discovery of any new bound, stronger than that provided by the energy, for general solutions of any of our basic physical equations would have the significance of a major event.*

In other cases, when there are additional symmetries, one often has better a priori estimates. For many elliptic equations, for example, one can make use of the maximal principle or some monotonicity arguments to derive far more powerful a priori estimates than those given by the energy integral. Integrable equations, such as KdV, also have additional, coercive, conservation laws. As explained above, the Burgers equation has infinitely many positive conserved quantities. The incompressible Euler equations in dimension $n = 2$ have, in addition to the energy, a pointwise a priori estimate for the vorticity. It is for this reason that we can prove global regularity for 2D Euler equations. In all these cases the classification has to be done relative to the optimal available a priori estimate.

In what follows I will restrict myself to the case I find, personally, most interesting, that of the basic evolution equations for which there are no better known, a priori estimates than those provided by the Energy integral. These include all relativistic field theories, Fluids, Continuum Mechanics and Magentofluidynamic, in three space dimensions and the absence of any additional symmetries. In these cases the classification is done by measuring the scaling properties of the energy integral relative to those of the equations. To illustrate how this is done consider the nonlinear scalar equation $\square\phi - V'(\phi) = 0$ with $V(\phi) = \frac{1}{p+1}|\phi|^{p+1}$. The energy integral is given by $\int \big(\frac{1}{2}|\partial\phi(t,x)|^2 + |\phi|^{p+1}(t,x)\big)dx$. If we assign to the space-time variables the dimension of length, L^1, then $\square$ has the dimension of L^{-2} and ϕ acquires, from the equation, the dimension $L^{\frac{2}{1-p}}$. Thus the energy integral has the dimension L^e, $e = n - 2 + \frac{4}{1-p}$. We say that the equation is subcritical if $e < 0$, critical for $e = 0$ and supercritical for $e > 0$. The same analysis can be done for all the other basic equations. YM is subcritical for $n \leq 3$, critical for $n = 4$ and supercritical for $n > 4$. WM is subcritical for $n = 1$, critical for $n = 2$, and supercritical for all other dimensions.

The same holds true for the Einstein Vacuum equations. Most of our basic equations, such as EV, Euler, Navier–Stokes, Compressible Euler, Elasticity, etc., turn out to be supercritical in the physical dimension $n = 3$. A PDE is said to be regular if all smooth, finite energy, initial conditions lead to global smooth solutions.

The general philosophy is that subcritical equations are regular while supercritical equations may develop singularities. Critical equations are important[19] borderline cases. For the particular case of field theories, as defined in the previous section, one can formulate a more precise conjecture:

GENERAL CONJECTURE. (i) *All basic, subcritical, field theories are regular for all smooth data.*

(ii) *Under well defined restrictions on their geometric set-up the critical field theories are regular for all smooth data.*

(iii) *"Sufficiently small" solutions to the supercritical field theories are regular. There exist solutions, corresponding to large, smooth, finite energy data, which develop singularities in finite time.*

The part (iii) of the Conjecture is the most intriguing. The fact that all small solutions are regular seems to be typical to field theories; it may fail for fluids or the general elasticity equations. The issue of existence of singular solutions for supercritical equations is almost entirely open. In the case of supercritical, defocusing NSWE, $\square\phi - V'(\phi) = 0$ for positive power law potential V, most analysts, familiar with the problem, expect that global regularity still prevails. Numerical calculations seem to support that view. *It is however entirely possible that singular solutions exist but are unstable and therefore difficult to construct analytically and impossible to detect numerically.* A similar phenomenon may hold true in the case of the 3D Navier–Stokes equations, which would contradict the almost universal assumption that these equations are globally regular.

If this worst case scenario is true, the big challenge for us would be to prove that *almost all* solutions to such equations are globally regular. At the opposite end of possible situations is that for which almost all solutions form singularities. The 3D incompressible Euler equations are a good candidate for this situation. Moreover it is not inconceivable that this *most*

[19] Some of the most exciting advances in Geometric PDE's in the last twenty five years involve the study of PDE's which are critical relative to the optimal available a priori estimates. This is the case of the Yamabe problem (related to the critical exponent of the Sobolev inequality), Weakly Harmonic Maps in 2D, Yang–Mills connections in 4D, the Wilmore problem in 2D. See [S] for a beautiful survey and [Y] for his updated list of problems in Geometry.

unstable of all known equations would exhibit the following perverse scenario: *The set of all initial data which lead to global regular solutions has measure zero, yet, it is dense in the set of all regular initial conditions, relative to a reasonable topology.* Such a possibility, which cannot be ruled out, would certainly explain why it is so difficult to make any progress on the 3D Euler equations with our present techniques. It would also explain, in particular, why it is so difficult to produce specific examples, or numerical evidence, of the widely expected finite time breakdown of solutions.

REMARK. *The development of methods which would allow us to prove generic, global, results may be viewed as one of the great challenges for the subject of PDE's in the next century.*

It is expected that the global structure of singularities in General Relativity will have to be phrased in terms of generic conditions (see [AM] and [W] for up to date surveys concerning Cosmic Censorship and recent mathematical progress on it). Understanding the problem of turbulence for the Navier–Stokes equations would almost certainly require a statistical approach. The effectiveness of many geometric flows is hindered by the presence of bad, seemingly nongeneric, type of singularities. So far the subject of nonlinear PDE's has been dominated by methods well suited for the study of individual solutions; we have had very little success in dealing with families of solutions. By comparison in the case of finite dimensional Hamiltonian systems the natural Liouville measure, defined in the space phase, allows one to prove nontrivial generic results[20] such as Poincaré's recurrence theorem.

The Problem of Well-posedness for Nonlinear Equations

With the exception of the a priori estimates derived from conservation laws, or monotonicity and maximum principle for elliptic or parabolic equations, almost all methods currently used to deal with nonlinear PDE's depend on elaborate comparison arguments between solutions to the original system and those of an appropriate linearization of it. It is essential to have very precise estimates for the linear system, in tune with the a priori estimates and the scaling properties of the nonlinear equations. In the case of elliptic and parabolic problems we have a large and powerful arsenal of such esti-

[20] There exist some interesting generic results in PDE also, based on the construction of Gibbs measures on the space of solutions, see [B1,2]. Unfortunately the class of equations for which such measures can be constructed is extremely limited.

mates, almost all developed during the course of this century, see [BreB]. Our knowledge of linear estimates for hyperbolic and dispersive equations is far less satisfactory.

The need for a well adapted linear theory, for evolution equations, can be best understood from the perspective of the problem of optimal well posedness. In what follows I will limit my discussion to field theories such as the nonlinear scalar wave equation (NSWE), Yang–Mills (YM), Wave-Maps (WM) and the Einstein Vacuum (EV) equations. My goal is to write down three specific conjectures, WP1–WP3, which are, I feel, just beyond the boundary of what can be obtained with present day techniques. They are thus both accessible and important to generate interesting mathematics.

The initial value problem for an evolutionary system of equations is said to be well posed (WP) relative to a Banach, functional, space X if, for any data in X, there exist uniquely defined local in time solutions belonging to X for $t \neq 0$, and depending continuously on the data. The problem is said to be strongly WP if the dependence on the data is analytic and weakly WP if the dependence is merely continuous or differentiable. In the case of hyperbolic equations, especially quasilinear, there is a natural, apparently unique, choice for X. Locally, it has to coincide with the Sobolev[21] space $H^s(\mathbb{R}^n)$. This is due to the fact that L^p norms are not preserved by the linear evolution in dimension $n > 1$ while norms defined in Fourier space are meaningless for quasilinear equations. Taking into account the scaling properties of the basic field equations and proceeding in the same manner as in the previous section, one can define the critical WP exponent s_c to be that value of s for which the H^s norm of initial data is dimensionless. With this definition we can formulate the following:

GENERAL WP CONJECTURE. i) *For all basic field theories the initial value problem is locally, strongly well posed for any data in H^s, $s > s_c$.*

ii) *The basic field theories are weakly, globally well posed for all initial data with small H^{s_c} norm.*

iii) *There can be no well defined solutions*[22] *for $s < s_c$.*

The proof of the WP conjecture for $s \geq s_c$ will provide us with an essential tool for the problem of regularity discussed in the previous sec-

[21]We talk of a space H^s rather than a pair H^s, H^{s-1}. Thus, in the case of the IVP for the wave equation $\Box\phi = 0$, $\phi(0) = f$, $\partial_t\phi(0) = g$, $(f,g) \in H^s$ means $f \in H^s$, $g \in H^{s-1}$.

[22]Weak solutions may exist below the s_c threshold but are, completely unstable and have weird properties. In other words weak solutions, corresponding to $s < s_c$, are mathematical "ghosts".

tion. So far the conjecture was proved only in the case of NSWE (see [K] and [ShS]); it is based entirely on Strichartz type inequalities. Semilinear equations whose nonlinear terms involve derivatives, such as YM and WM, are far more difficult, see discussion below. The case $s < s_c$ is interesting for a philosophical reason. There are supercritical cases (in the case of the supercritical NSWE see [Str], for the case of WM see [Sh], [MuS]), for which one can prove the existence of a weak solution corresponding to any, finite energy, initial conditions. Part iii) of the above conjecture asserts that these solutions are unstable (it is easy in fact to see that they are linearly unstable) and therefore not particularly useful. It is interesting to remark, in this respect, the recent remarkable result of Schaeffer [S], see also [Shn]. Schaeffer has constructed examples of weak solutions for the 3D Euler equations which are compactly supported in space-time! The result is reminiscent of the famous result of Nash [N], see also Kuiper [Ku], on C^1 isometric imbeddings, which turn out to be plentiful, dense in the set of all smooth functions, and a lot more pliant than the more regular ones.[23] Another remarkable example of how bad weak solutions can sometimes be is that of Rivière, concerning weak harmonic maps from a three dimensional space to S^2 with a dense set of singularities [R]. This is in sharp contrast to the case of minimizers [SchU], or stationary solutions [E] for the same equations. I suspect that similar, *unacceptable properties of weak solutions* type results can be proved for solutions to nonlinear wave equations, below the critical regularity. Moreover, short of additional regularity assumptions on the initial data, there may exist no *entropy type conditions* which would stablize the solutions.

In the case of subcritical equations, for which the energy norm is stronger than H^{s_c}, part i) of the conjecture would imply well-posedness in the energy norm, and therefore, by energy conservation, global well-posedness and regularity. In other words the solutions preserve the H^s regularity of the data for any $s > s_c$. This would thus settle the first part of the General Conjecture stated in the previous section.

In the case of critical equations, part ii) of the WP conjecture will imply the following:

SMALL ENERGY CONJECTURE. *For all basic critical field theories all small energy solutions are globally regular.*

[23]This phenomenon has been called the h-Principle and discussed in a very general set-up by M. Gromov, see [Gr].

The small energy conjecture is an essential step in the proof of the general regularity conjecture for critical field theories. In the case of wave equations, whose nonlinearities do not depend on derivatives or in the case of spherical symmetric solutions, one can prove it directly. In the case of equations like YM or WM, with derivatives appearing in the nonlinear terms, it is now believed that the only way to settle the small energy conjecture is to prove the much stronger part ii) of the WP conjecture. In what follows I will give a more precise formulation of it for the special case of the WM and YM equations.

CONJECTURE WP1. *The Wave Maps equation, defined from the Minkowski $\mathbb{R}^{n+1}$ to a complete, Riemannian, target manifold, is globally well posed for small initial data in $H^{n/2}$, $n \geq 2$.*

CONJECTURE WP2. *The Yang–Mills equation, for $SO(N), SU(N)$ structure groups, is globally well posed for small initial data in $H^{\frac{n-2}{2}}$, $n \geq 4$.*

To understand the difficulties involved in WP1, I will summarize below what are the most significant known results in connection to it.

1) The conjecture is true in the case of equivariant wave maps, see [ShZ], in which case the nonlinear terms do not depend on derivatives. In [ChrZ] the small energy conjecture was proved for the special case of spherically symmetric solutions. Their approach avoids the proof of the WP1 conjecture, which is still not known, even in the spherically symmetric case, by proving directly, in this case, the small energy conjecture. In the general case it does not seem possible to prove the small energy conjecture independent of Conjecture WP. This has to do with the lack of any space-time L^p, $p \neq 2$, first derivative estimates (see [Wo1]) for solutions to $\Box\phi = F$.

2) In [KlM3] and [KlS] one proves local well-posedness for all data in H^s, $s > s_c = n/2$, $n \geq 2$ (see also [KeT] for $n = 1$). The result depends heavily on bilinear estimates. This was further improved in [Ta], who has established well-posedness (his result is in fact global in time, in view of the scaling properties of the equations) for small data in the Besov space $B_2^{n/2,1}$. Both above mentioned results fail to to take into account the completeness of the target manifold.

3) We know, from simple examples, that we may not have $H^{n/2}$-well-posedness if the target manifold is not complete.

4) The dependence of solutions on the data, with respect to the $H^{n/2}$ norm, cannot be twice differentiable.

The methods which have been used to tackle the case $s > s_c$ depend heavily on an iterative procedure in which one estimates the $H^{s,\delta}$ norm of each iterate, for $s > n/2$, $\delta > 1/2$, in terms of the $H^{s,\delta}$ norms of the previous iterates. These norms, defined with respect to the space-time Fourier transform, are intimately tied to the symbol of $\Box$ and to bilinear estimates, see [KlM1,3], [KlS] and [FoK]. Similar norms where introduced by J. Bourgain [B3], see also [KenPV], in connection with nonlinear dispersive equations.

To treat the critical case one needs to overcome two difficulties. The first has to do with improving the estimates at each iterative step, to make them optimal. The second is an important conceptual difficulty, which has to do with the iterative process itself. Any iterative procedure, if successful, would imply not only well posedness but also analytic dependence on the data in the $H^{n/2}$ norm. This is however wrong, according to the observation (3) above. To understand this effect consider the Hilbert space $X = H^{n/2}(\mathbb{R}^n)$, u a function in X, and let $\Phi(t) = e^{itu}$. It is known that $\Phi(t)$ is a C^1 function of t with values in X but, since X is not closed under multiplications, it is not in C^2 (see [KeT]). The reason $e^{itu} \in X$ is due to the fact that the function e^{iu} is bounded, it cannot be guessed by just considering the Taylor expansion $e^{iu} = \sum_{n\geq 0} \frac{1}{n!}(iu)^n$ in which all terms are divergent.

In the case of the WM equations any iterative procedure loses the crucial information about the completeness of the target manifold and therefore leads to logarithmic divergences. To see that consider WM solutions of the form $\phi = \gamma(u)$ where $\Box u = 0$ with data in $H^{n/2}$ and γ is a geodesic of the target manifold M. Since the L^∞ norm of u is not controlled, $\gamma(u)$ makes sense only if the geodesic is globally defined. A standard iteration fails to distinguish between complete and incomplete geodesics.

This situation seems to call for a "renormalization" procedure. More precisely, one may hope that by understanding the nature of the logarithmic divergences of each iterate, we can overcome them by a clever regularization and limiting procedure. In view of the simple minded model problem studied in [KlM4] one may hope that such an approach is not impossible.

I will only make a few remarks concerning the WP2 conjecture. The optimal known result, in dimension $n \geq 4$, is small data well-posedness for $s > s_c$, see [KlM5] and [KlT]. In the case $s = s_c$ it can be shown that any iteration procedure leads to logarithmic divergences. The situation seems thus similar to that described in the previous conjecture. In dimension $n = 3$ we have global well posedness in the energy norm $s = 1$, see [KlM2] and the discussion in connection to WP3 below, and local well-posedness for $s > 3/4$. It is not at all known what happens for $s_c = 1/2 < s \leq 3/4$.

The case of the Einstein Vacuum equations is far more difficult than that of WM or YM. Written relative to wave coordinates the EV equations take the form, $g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = N(g, \partial g)$, where g is a Lorentz metric and N is a nonlinear term quadratic in the first derivatives of g. This form of the Einstein leads to the study of quasilinear wave equations of the form:

$$\square_{g(\phi)}\phi = \Gamma(\phi)Qr(\partial\phi, \partial\phi)\,, \tag{1}$$

with $g(\phi)$ a Lorentz metric depending smoothly on ϕ, Γ smooth function of ϕ and $Qr(\partial\phi, \partial\phi)$ quadratic in $\partial\phi$. Other types of quasilinear wave equations, such as those appearing in Elasticity or Compressible Fluids, depending only on $\partial\phi$ can be written as systems of wave equations of type (1).

Using energy estimates and Sobolev inequalities one can prove the "classical local existence" result, or local well-posedness, for H^s initial data with $s > s_c + 1 = \frac{n}{2} + 1$. This result leads, in the case of the EV expressed relative to wave coordinates, to the well-known local existence result of Y.C. Bruhat. (Bruhat's result, see [Bru], requires in fact more derivatives of the data. The optimal $3+1$ dimensional result, $s > s_c + 1 = 5/2$ was proved in [FM].)

Getting close to the critical exponent $s = s_c = 3/2$ is entirely out of reach. I believe, however, that the intermediate result, $s = 2$, is both very interesting and accessible.

Conjecture WP3. *The Einstein Vacuum equations are strongly, locally, well posed for initial data sets*[24] (Σ, g, k) *for which* $Ric(g) \in L^2(\Sigma)$ *and* $k \in H^1(\Sigma)$.

The conjecture can be viewed, in a sense, as a far more difficult analogue of the well-posedness result, see [KlM2], for the $3+1$ YM equations in the energy norm. Writing the YM equations in the Lorentz gauge, which is the precise analogue of wave coordinates, one is led to a system of equations of the form

$$\square\phi = Qr(\phi, \partial\phi) + C(\phi)\,, \tag{2}$$

with Qr quadratic in $\phi, \partial\phi$ and linear in $\partial\phi$ and C cubic in ϕ. In this case the scaling exponent is $s_c = \frac{n-1}{2}$. The classical local existence result, based on energy estimates and the $H^\sigma \subset L^\infty, \sigma > n/2$ Sobolev estimate, requires data in H^s, $s > s_c + 1$. One can improve the result to $s > s_c + \frac{1}{2}$ for $n = 3$

[24] (Σ, g) is a Riemannian 3D manifold and k a symmetric 2-tensor, verifying the constraint equations.

and $s = s_c + \frac{1}{2}$ in higher dimensions by using the classical Strichartz[25] type inequalities for solutions to the inhomogeneous standard wave equation $\Box\phi = F$, Moreover one can show, see[Lind], that for $n = 3$ the result $s > s_c + \frac{1}{2} = 1$ is optimal for general equations of type (2). Therefore to prove the H^1 well-posedness result for the Yang–Mills equations one needs to take advantage of some additional cancellations present in the nonlinear terms. One can do that by using the "gauge covariance" of the Yang–Mills equations, according to which a solution of YM is a class of equivalence of solutions relative to gauge transformations. In view of this one is free to pick the particular gauge conditions best suited to the problem at hand. In [KlM2] the choice of the Coulomb gauge leads to a coupled system of elliptic-hyperbolic equations which satisfies the "null condition". This means, very roughly, that the hyperbolic part of the YM (Coulomb) system has the form

$$\Box\phi = Q(\phi, \phi) + \text{better behaved terms.}$$

with $Q(\phi, \phi)$ a nonlocal "null" quadratic form. To deal with the cancellations present in the null quadratic forms Q one has developed the so called bilinear estimates, see [KlM1], [FoK].

In trying to implement a similar strategy to EV one encounters fundamental difficulties due the quasilinear character of the Einstein equations. For example, to improve Bruhat's classical local existence result from $s > s_c + 1$ to $s > s_c + \frac{1}{2}$, in wave coordinates, one needs to prove a version of the classical Strichartz estimates for $\Box$ replaced with the wave operator $\Box_g$, where g is a rough (assuming we fix ϕ, the metric $g(\phi)$ will have the "expected" regularity of ϕ) Lorentz metric.

Until recently this seemed to be an intractable problem. In fact it is known that, if the coefficients of a linear wave equation have less regularity than $C^{1,1}$, some of the main Strichartz inequalities may fail, see [SmS]. H. Smith, see [Sm], was also able to show that all the Strichartz type inequalities hold true if the coefficients are at least $C^{1,1}$ and $n \leq 3$, see [Ta] for $n \geq 3$. The $C^{1,1}$ condition, however, is much too strong to apply to nonlinear equations.

Recently J.Y. Chemin and H. Bahouri, see [ChB], have succeeded in deriving the first improvement over the classical result. They have proved local WP for equations of type 1 provided that $s > s_c + \frac{3}{4}$ for $n \geq 3$ and

[25] The Strichartz type inequalities are intimately tied to restriction results in Fourier Analysis. Together with the more recent bilinear estimates they exemplify the strong, modern, ties between Harmonic Analysis and nonlinear wave equations.

$s > s_c + \frac{7}{8}$ for $n = 2$. The same result was proved also by D. Tataru [T2] using a somewhat different method. Both Chemin-Bahouri and Tataru have later obtained some further improvements but fall short of the expected optimal result. (The optimal known result, $s > s_c + \frac{2}{3}$ for $n \geq 3$ and $s > s_c + \frac{5}{6}$ is proved in [Ta].) In dimension $n = 3$ we also have examples, due to H. Linblad [L2], which show that one cannot have well-posedness, in general, for $s \leq 2$.

Even if the Strichartz based methods initiated by Chemin-Bahouri and Tataru can be made optimal they will still fall short of proving the desired H^2 result, conjectured by WP3. To obtain such a result one needs to take into account the "null structure" of the EV equations. We know, indirectly from the proof of stability of the Minkowski space, [ChrK], that written in appropriate form, i.e. using their general covariance, the equations must exhibit such a structure. Yet the indirect method of [ChrK], based on the Bianchi identities and a careful decomposition of all geometric components appearing in the equation relative to a null frame, cannot be used in this case. One needs instead a method similar to the one we have sketched above for YM. In other words we need a "gauge condition," similar to the Coulomb one in YM, relative to which all quadratic terms of the Einstein equations exhibit a null bilinear structure. Once this is done we need to develop techniques to prove bilinear estimates,[26] similar to those of [KlM1], [FoK], in a quasilinear set-up. A good warm-up problem, in this respect, would be the study of the Minkowski space analogue of the minimal surface equation, for which the null structure, in the sense of [Kl1,2], [Chr2], is obvious.

To summarize, the study of Conjecture WP3 requires:

1) To develop new analytic techniques to improve the results of Chemin-Bahouri to the optimal regularity possible for Strichartz based methods.
2) To investigate quasilinear equations which verify the null condition, and develop bilinear estimates for linear equations with very rough coefficients.
3) To investigate, in a direct way, the null structure of the Einstein equations.

[26]The bilinear estimates of [KlM1] have been recently derived, by Smith and Sogge [Sm-So2], for $C^{1,1}$ coefficients.

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[27]With few exceptions most of the references given below are in regard with the last section of the paper, more precisely in connection to the conjectures WP1–WP3. I apologize for giving only a very limited number of references in connection to the first four sections.

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