Cotlar-Stein Almost Orthogonality Lemma

Andrew Comech

March 22, 2007

When deriving the estimates on integral operators one often uses the Almost Orthogonality principle of M. Cotlar and E.M. Stein, first proved by M. Cotlar in [Cot55]. This result is classical; our excuse for formulating it once again is a need to have its weighted form which sometimes allows to reduce the number of integrations by parts in half (hereby weakening smoothness requirements), and also to state explicitly the convergence of the series $\sum_i T_i$ in the strong operator topology.

Let E and F be the Hilbert spaces, and let T be a linear operator which acts from E to F. An often situation is that one can decompose the operator T into an infinite sum of operators $T = \sum_i T_i$, which satisfy certain estimates, and the question is, under which assumptions on T_i one can deduce an adequate estimate on T.

Definition 1 (Almost orthogonal operators)**.** *We will call a family of continuous operators*

$$T_i : E \to F, \qquad i \in \mathbb{Z}$$

almost orthogonal if they satisfy the following conditions:

$$\|T_i^* T_j\| \le a(i,j), \qquad \|T_i T_j^*\| \le b(i,j), \tag{1}$$

where $a(i,j)$ and $b(i,j)$ are non-negative symmetric functions on $\mathbb{Z} \times \mathbb{Z}$ which satisfy

$$\|a\|_{\infty,1/2}^{1/2} := \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} a^{1/2}(i,j) < \infty, \qquad \|b\|_{\infty,1/2}^{1/2} := \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} b^{1/2}(i,j) < \infty, \tag{2}$$

or, more generally,

$$\|a\|_{\infty,\mu}^{\mu} := \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} a^{\mu}(i,j) < \infty, \qquad \|b\|_{\infty,\nu}^{\nu} := \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} b^{\nu}(i,j) < \infty, \tag{3}$$

with some non-negative exponents $\quad 0 \le \mu, \nu \le 1, \quad \mu + \nu = 1$. *(If μ or ν is zero, then in the summations in (3) we leave out the terms with $a(i,j) = 0$ or $b(i,j) = 0$.)*

Theorem 1 (Cotlar-Stein Lemma)**.** *Let $T_i : E \to F$, $i \in \mathbb{Z}$ be a family of almost orthogonal operators that satisfy (1) and (2), or, more generally, (1) and (3). Then the formal sum $\sum_i T_i$ converges in the strong operator topology (but not necessarily in the uniform operator topology) to a continuous linear operator*

$$T : E \to F,$$

which is bounded by

$$\|T\| \le (\|a\|_{\infty,\mu}^{\mu} \|b\|_{\infty,\nu}^{\nu})^{1/2} = \left(\sup_i \sum_j a^{\mu}(i,j)\right)^{\frac{1}{2}} \left(\sup_i \sum_j b^{\nu}(i,j)\right)^{\frac{1}{2}}. \tag{4}$$

The proof is split into two steps. In Lemma 1 below, we prove that the norm of $\sum_{i\in I} T_i$ is uniformly bounded for any finite $I \subset \mathbb{Z}$. An immediate consequence is that the series $\sum_{i\in\mathbb{Z}} T_i$ converges in the weak operator topology. In Lemmas 2 and 3, we prove that the series $\sum_{i\in\mathbb{Z}} T_i$ converges in the strong operator topology.

For brevity, we denote

$$A := \|a\|_{\infty,\mu}^{\mu} = \sup_i \sum_j a^{\mu}(i,j), \qquad B := \|b\|_{\infty,\nu}^{\nu} = \sup_i \sum_j b^{\nu}(i,j). \tag{5}$$

Lemma 1. *For any finite subset $I \subset \mathbb{Z}$, the operator $T_I := \sum_{i\in I} T_i$ is bounded by $\|T_I\| \le \sqrt{AB}$, where A, B are defined in (5).*

Remark 1. The important part of this claim is that the estimate does not depend on the number of the summands, $|I|$.

Proof. We reproduce an elegant proof from the book of E.M. Stein [Ste93, Theorem 1 in Chapter VII §2]. For any $N \in \mathbb{N}$, we have:

$$\|(T_I^* T_I)^N\| = \Big\| \sum_{i_1\in I,\, j_1\in I,\dots i_N\in I,\, j_N\in I} T_{i_1}^* T_{j_1} \dots T_{i_N}^* T_{j_N} \Big\| \le \sum_{i_1\in I,\, j_1\in I,\dots i_N\in I,\, j_N\in I} \|T_{i_1}^* T_{j_1} \dots T_{i_N}^* T_{j_N}\|. \tag{6}$$

For each term $T_{i_1}^* T_{j_1} \dots T_{i_N}^* T_{j_N}$, we have two following bounds:

$$\|(T_{i_1}^* T_{j_1}) \dots (T_{i_N}^* T_{j_N})\| \le a(i_1, j_1) \dots a(i_N, j_N), \tag{7}$$

$$\|T_{i_1}^* (T_{j_1} T_{i_2}^*) \dots (T_{j_{N-1}} T_{i_N}^*) T_{j_N}\| \le b(i_1,i_1)^{1/2} b(j_1,i_2) \dots b(j_{N-1},i_N) b(j_N,j_N)^{1/2}. \tag{8}$$

Taking the weighted geometric mean of (7) and (8) and noting that $b^{\nu/2}(i_1,i_1) \le B^{\frac12}$ and $b^{\nu/2}(i_N,i_N) \le B^{\frac12}$ by (5), we bound $\|T_{i_1}^* T_{j_1} \dots T_{i_N}^* T_{j_N}\|$ by

$$\|T_{i_1}^* T_{j_1} \dots T_{i_N}^* T_{j_N}\| \le B^{\frac12} a^\mu(i_1,j_1) b^\nu(j_1,i_2) \dots b^\nu(j_{N-1},i_N) a^\mu(i_N,j_N) B^{\frac12}.$$

We first sum up in i_1; according to (3), $\sup_{j_1} \sum_{i_1} a^\mu(i_1,j_1) \le A$. Similarly, we sum up in $j_1, \dots, i_N$. Finally, summation in j_N results in the following bound on $\|(T_I^* T_I)^N\|$:

$$\sum_{i_1\in I,\, j_1\in I,\dots i_N\in I,\, j_N\in I} \|T_{i_1}^* T_{j_1} \dots T_{i_N}^* T_{j_N}\| \le B^{\frac12} A^N B^{N-1} \sum_{j_N\in I} B^{\frac12} = A^N B^N \sum_{j_N\in I} 1 = |I| A^N B^N, \tag{9}$$

where $|I| < \infty$ is the number of elements in I.

Now we assume that $N = 2^n$, for some integer n. Using (6) and (9), we get:

$$\|T_I\|^{2N} = \|T_I^* T_I\|^N = \|(T_I^* T_I)^2\|^{N/2} = \dots = \|(T_I^* T_I)^N\| \le |I| A^N B^N, \tag{10}$$

or $\|T_I\| \le |I|^{\frac{1}{2N}} A^{1/2} B^{1/2}$. Since this bound is true for any $N = 2^n$, we have: $\|T_I\| \le A^{1/2} B^{1/2}$. □

Now we need to know in what sense we can draw the conclusion about the norm of T which consists of infinite number of almost orthogonal pieces T_i.

Corollary 2. *The series $\sum_{i\in\mathbb{Z}} T_i$ converges in the weak operator topology.*

Proof. To prove the convergence of $\sum_{i\in\mathbb{Z}} T_i$ in the weak operator topology, we need to show that for any $u \in E$, $v \in F$, the series of numbers

$$\sum_{i\in\mathbb{Z}} \langle v, T_i u\rangle \tag{11}$$

converges. If it were not the case, then for any $C > 0$ there would exist a finite subset $I \subset \mathbb{Z}$ such that $|\sum_{i\in I} \langle v, T_i u\rangle| > C$. In particular, for some finite $I \subset \mathbb{Z}$,

$$|\sum_{i\in I} \langle v, T_i u\rangle| > \sqrt{AB}\|u\|\|v\|.$$

On the other hand, by Lemma 1, $|\sum_{i\in I} \langle v, T_i u\rangle| = |\langle v, T_I u\rangle| \le \sqrt{AB}\|u\|\|v\|$. This contradiction finishes the proof. □

In the next two lemmas, we prove that, as the matter of fact, $\sum_{i\in\mathbb{Z}} T_i$ converges not only in the weak operator topology, but also in the strong operator topology. We start with the following auxilliary result.

Lemma 2. *Let $m \in \mathbb{N}$, and let I_α, $1 \le \alpha \le m$, be non-intersecting finite subsets of $\mathbb{Z}$. Then*

$$\|\sum_{\alpha=1}^m T_{I_\alpha}^* T_{I_\alpha}\| \le AB, \qquad \text{where} \quad T_{I_\alpha} = \sum_{i\in I_\alpha} T_i.$$

Proof. We denote

$$\Delta := \bigcup_{\alpha=1}^{m} I_\alpha \times I_\alpha \subset \mathbb{Z} \times \mathbb{Z},$$

so that $\sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha} = \sum_{(i,j)\in\Delta} T_i^* T_j$. Then we have:

$$\|\Big(\sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha}\Big)^N\| = \Big\|\Big(\sum_{(i_1,j_1)\in\Delta} T_{i_1}^* T_{j_1}\Big)\dots\Big(\sum_{(i_N,j_N)\in\Delta} T_{i_N}^* T_{j_N}\Big)\Big\| \le \sum_{(i_1,j_1)\in\Delta}\dots\sum_{(i_N,j_N)\in\Delta} \|T_{i_1}^* T_{j_1}\dots T_{i_N}^* T_{j_N}\|$$

$$\le \sum_{i_1\in I,\, j_1\in I,\dots i_N\in I,\, j_N\in I} \|T_{i_1}^* T_{j_1}\dots T_{i_N}^* T_{j_N}\| \le |I| A^N B^N, \qquad \text{where} \quad I = \bigcup_{\alpha=1}^{m} I_\alpha.$$

In the last inequality, we used (9). We note that $\sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha}$ is self-adjoint. Therefore, as in (10), we assume that $N = 2^n$, $n \in \mathbb{N}$, and arrive at

$$\|\sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha}\|^N = \|\Big(\sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha}\Big)^2\|^{N/2} = \ldots = \|\Big(\sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha}\Big)^N\| \le |I| A^N B^N.$$

Since N could be arbitrarily large, the conclusion of the Lemma follows. □

Lemma 3. *The series $\sum_{i\in\mathbb{Z}} T_i$ converges in the strong operator topology.*

Proof. Pick a vector $u \in E$ and denote $v_i = T_i u$. We need to show that the series $\sum_{i\in\mathbb{Z}} v_i$ converges in F. Let us assume that this is not the case. Then there exists $\varepsilon > 0$ and infinitely many non-intersecting finite subsets $I_\alpha \in \mathbb{Z}$ such that $\|\sum_{i\in I_\alpha} v_i\| \ge \varepsilon$. Therefore, there exists $m \in \mathbb{N}$ such that

$$\sum_{\alpha=1}^{m} \|\sum_{i\in I_\alpha} v_i\|^2 > AB\|u\|^2. \tag{12}$$

On the other hand, by Lemma 2,

$$\sum_{\alpha=1}^{m} \|\sum_{i\in I_\alpha} v_i\|^2 = \sum_{\alpha=1}^{m} \Big\langle \sum_{i\in I_\alpha} T_i u, \sum_{j\in I_\alpha} T_j u\Big\rangle = \sum_{\alpha=1}^{m} \Big\langle T_{I_\alpha} u, T_{I_\alpha} u\Big\rangle = \Big\langle u, \sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha} u\Big\rangle \le AB\|u\|^2.$$

This contradicts (12), indicating that our assumption that for some $u \in E$ the series $\sum_{i\in\mathbb{Z}} T_i u$ does not converge in F was false. This finishes the proof of the Lemma. □

This concludes the proof of the Cotlar-Stein Lemma.

References

[Cot55] M. Cotlar, *A combinatorial inequality and its applications to L^2-spaces*, Rev. Mat. Cuyana **1** (1955), 41–55 (1956). MR 18,219a

[Ste93] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.