The game semantics of game theory

Jules Hedges
(Max Planck Institute for Mathematics in the sciences, Leipzig)

A family feud in the mathematics of games

Zermello (the Patriarch)

von Neumann & Morgenstern

Nash

Arrow, Aumann, Kreps,
Harsanyi, Rubinstein, ...

Lorenz & Lorenzen

Borel

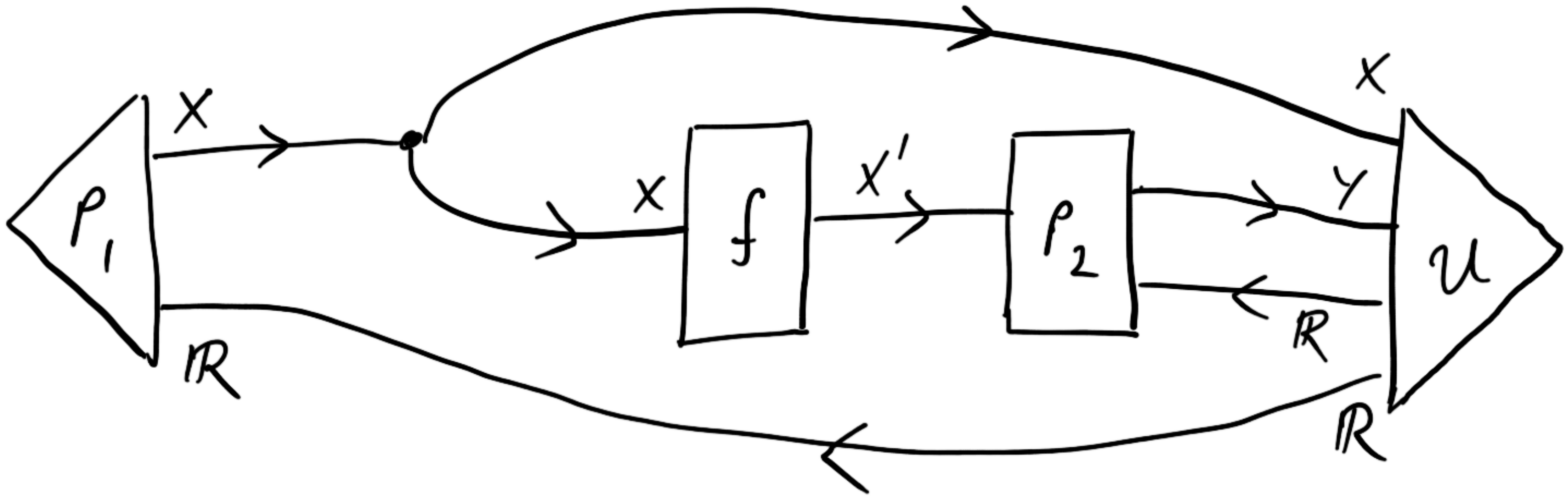
Conway

Blass

Abramsky & Jagadeesan

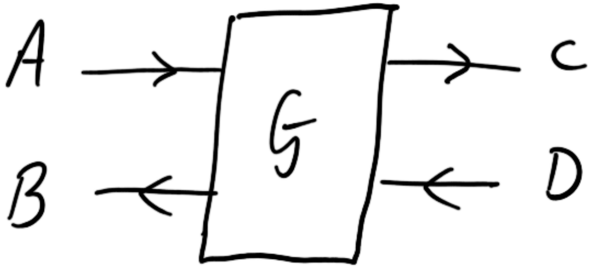
Hyland & Ong

Compositional game theory



(Inspired by game semantics, CQM, DisCoCat)

... has a mysterious semantics



Let A, B, C, D be sets.

An open game $G : \binom{A}{B} \longrightarrow \binom{C}{D}$ consists of:

(1) a set Σ_G of <u>strategy profiles</u>

(2) a <u>play function</u> $P_G : \Sigma_G \times A \longrightarrow C$

(3) a <u>coplay function</u> $C_G : \Sigma_G \times A \times D \longrightarrow B$

(4) an <u>equilibrium function</u> $E_G : A \times (C \to D) \longrightarrow \mathcal{P}(\Sigma_G)$

This talk: Making sense of this as a game semantics of game theory

A prophetic quote

"Note that the distinction between System and Environment and the corresponding designation as Player or Opponent depend on the point of view: If Tom, Tim and Tony converse in a room then from Tom's point of view he is the System, and Tim and Tony form the Environment; while from Tim's point of view, he is the System, and Tom and Tony form the Environment."

(Samson Abramsky, Semantics of Interaction)

A dialectica category

Objects: 2-move dialogues $A^+ ; B^-$ (P moves first!)

Morphisms $A^+ ; B^- \longrightarrow C^+ ; D^-$

P-strategies for $A^- ; C^+ ; D^- ; B^+$

(This interleaving is characteristic of dialectica)

(Valeria de Paiva, The dialectica categories
A game semantics for linear logic)

No winning conditions yet!

P-strategies and O-strategies

For the dialogue $A^- \, ; \, C^+ \, ; \, D^- \, ; \, B^+$

we get:

(1) a set of P-strategies

$$(A \to C) \times (A \times D \to B)$$

(2) a set of O-strategies

$$A \times (C \to D)$$

Look familiar?

The definition of open games again

An open game $\mathcal{G} : A^+ ; B^- \longrightarrow C^+ ; D^-$

consists of:

(1) a set $\Sigma_{\mathcal{G}}$ of strategy labels

(2) a function $\Sigma_{\mathcal{G}} \longrightarrow \{\text{P-strategies for } A^- ; C^+ ; D^- ; B^+\}$

(3) an equilibrium relation

$$E_{\mathcal{G}} \subseteq \Sigma_{\mathcal{G}} \times \{\text{O-strategies for } A^- ; C^+ ; D^- ; B^+\}$$

The Game theory of Game semantics

Inside

$n \geq 0$ non-cooperative players

Goal: maximise payoff $\in \mathbb{R}$

Player / System

Goal: Nash equilibrium

Inside

Outside

Context (history + continuation)

No strategic goal

Opponent / Environment

Goal: Prevent NE

Outside

Syntax of interaction : identities

Identity on $A^+ ; B^-$: copycat P-strategy

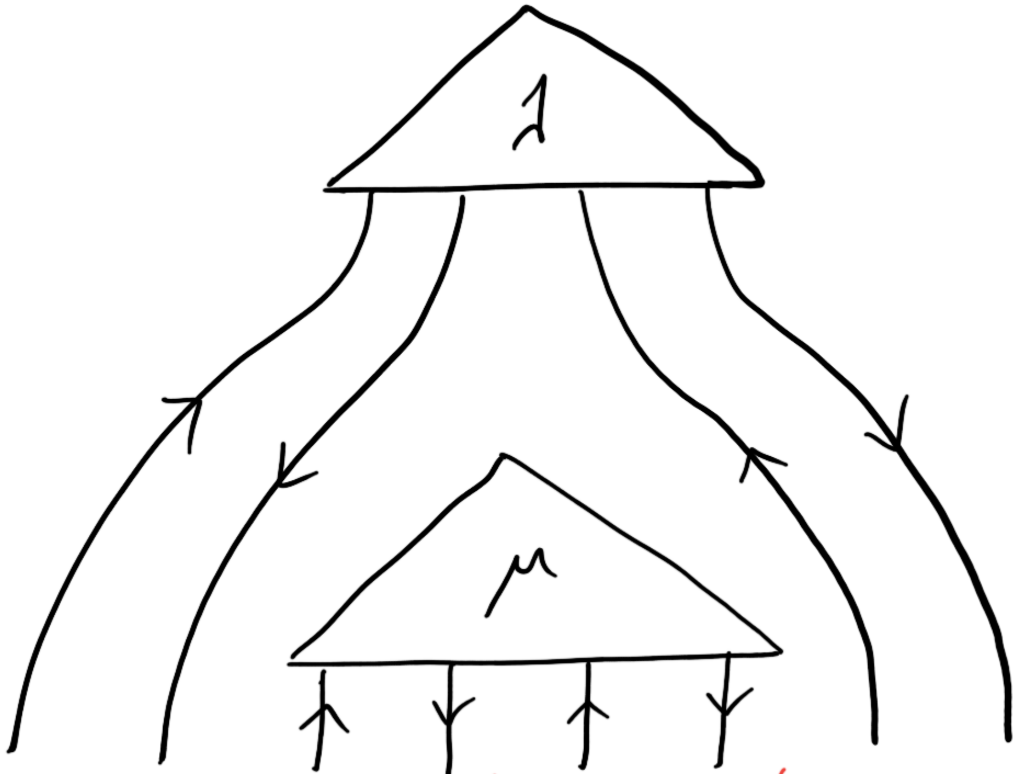
$$A^- \; ; \; A^+ \; ; \; B^- \; ; \; B^+$$

(Samson Abramsky, Semantics of Interaction)

but suspiciously similar to Lambek's type-logical grammar!

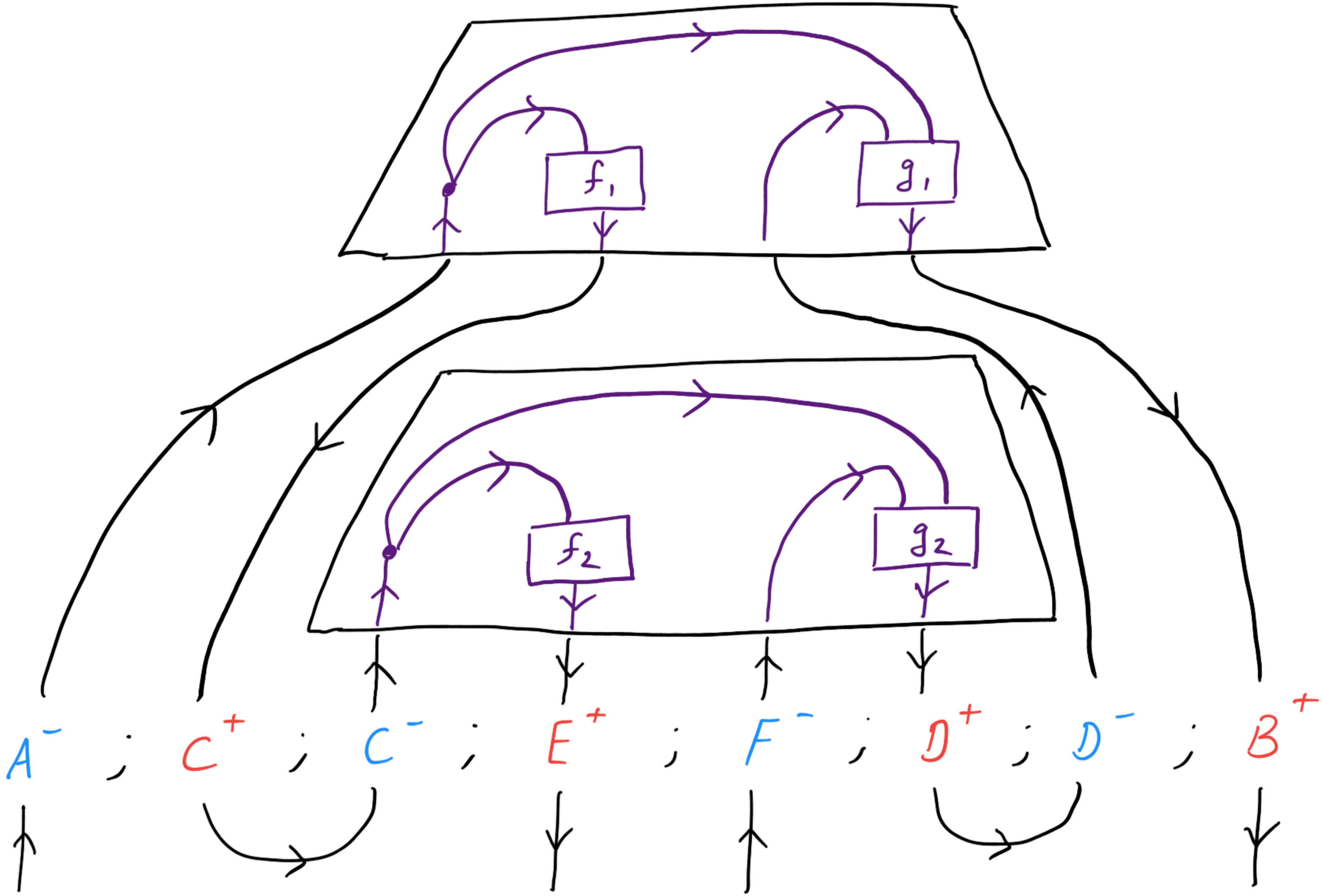
Syntax of interaction : composition

$$A^+ \,;\, B^- \xrightarrow{\lambda} C^+ \,;\, D^- \xrightarrow{\mu} E^+ \,;\, F^-$$

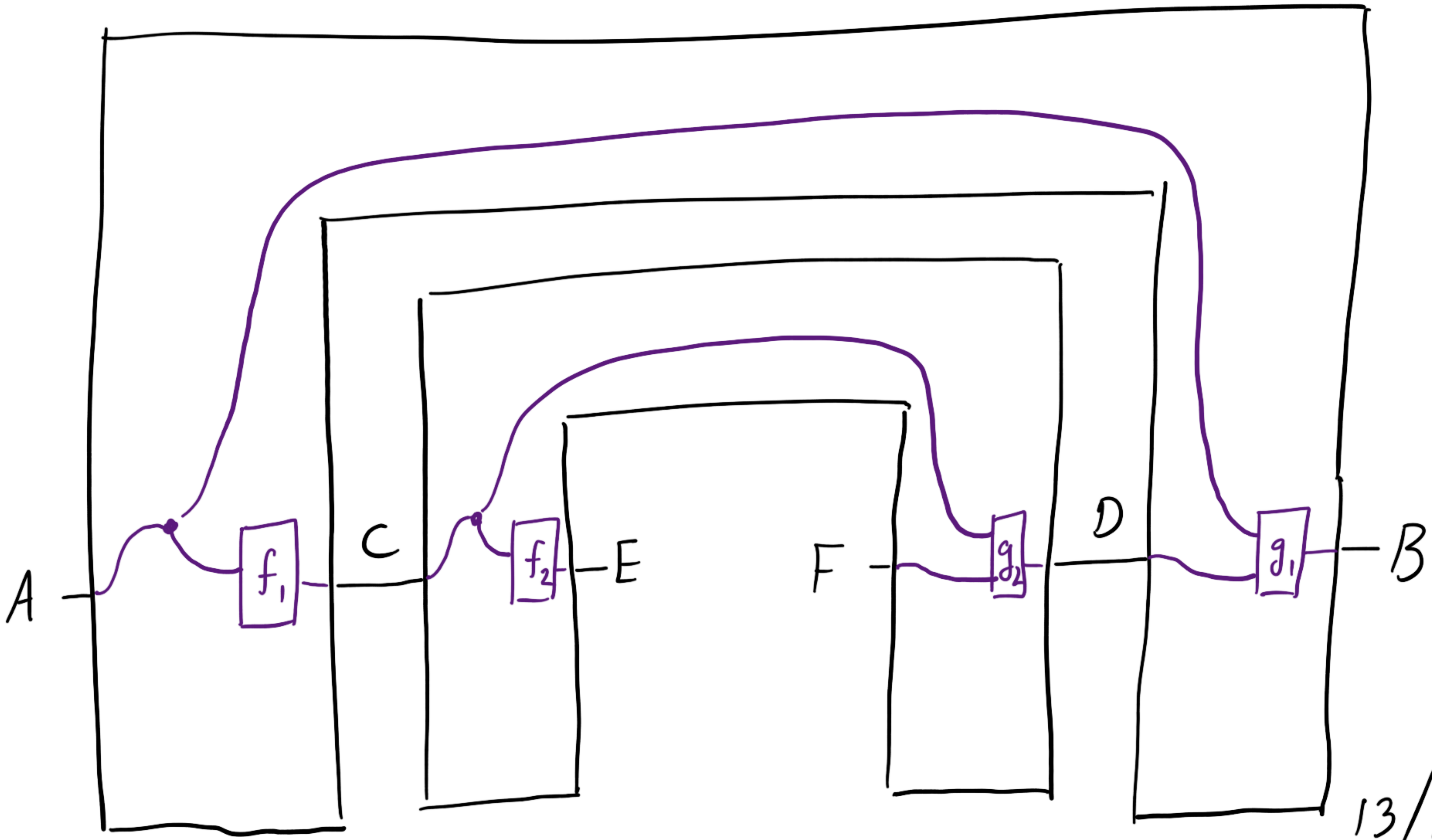


$$A^- \,;\, C^+ \,;\, C^- \,;\, E^+ \,;\, F^- \,;\, D^+ \,;\, D^- \,;\, B^+$$

Looking inside

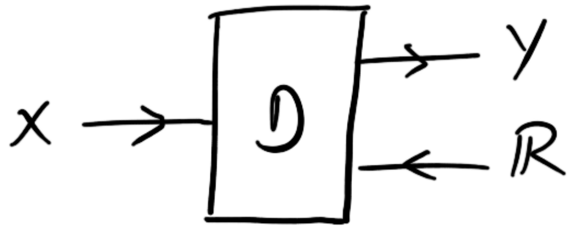


Combs

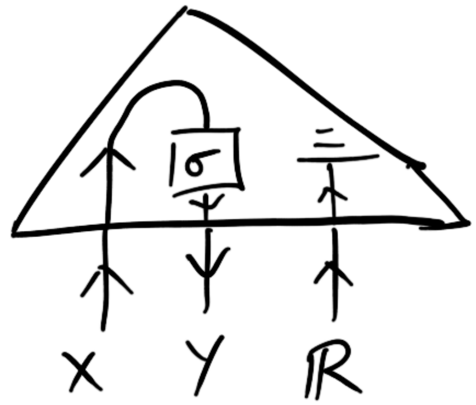


Enter game theory

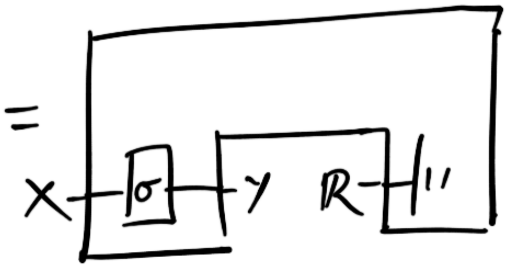
$$\mathcal{D} : \binom{X}{1} \longrightarrow \binom{Y}{\mathbb{R}} \qquad \Sigma_{\mathcal{D}} = \{\text{functions } X \longrightarrow Y\}$$

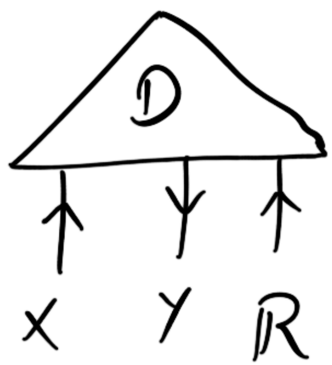


$\mathcal{D}(\sigma) =$

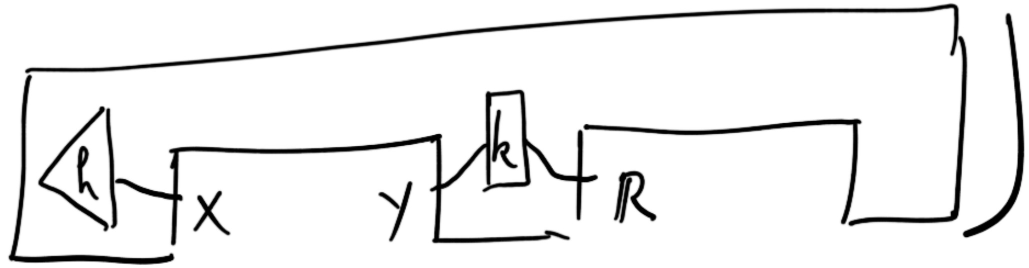


$=$



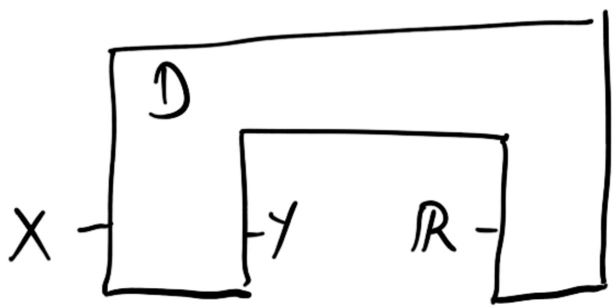


$E_{\mathcal{D}}\big(\sigma,$

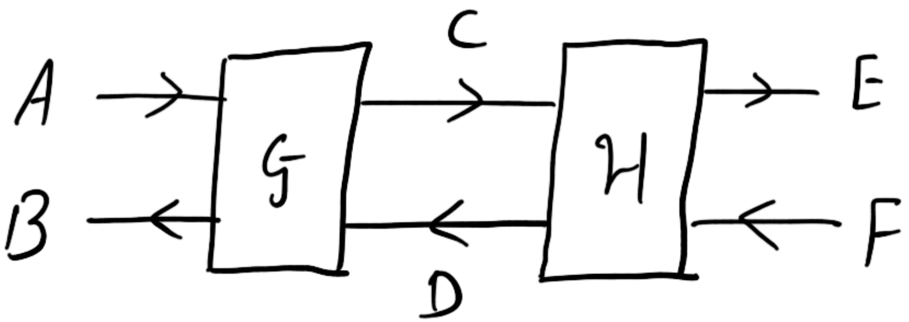


$\big)$

$$\Longleftrightarrow \sigma(h) \in \operatorname{argmax}(k)$$



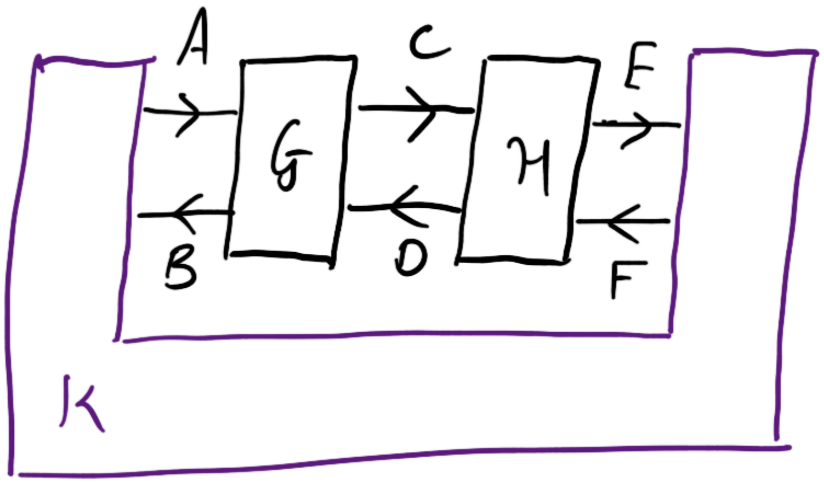
Composing open games

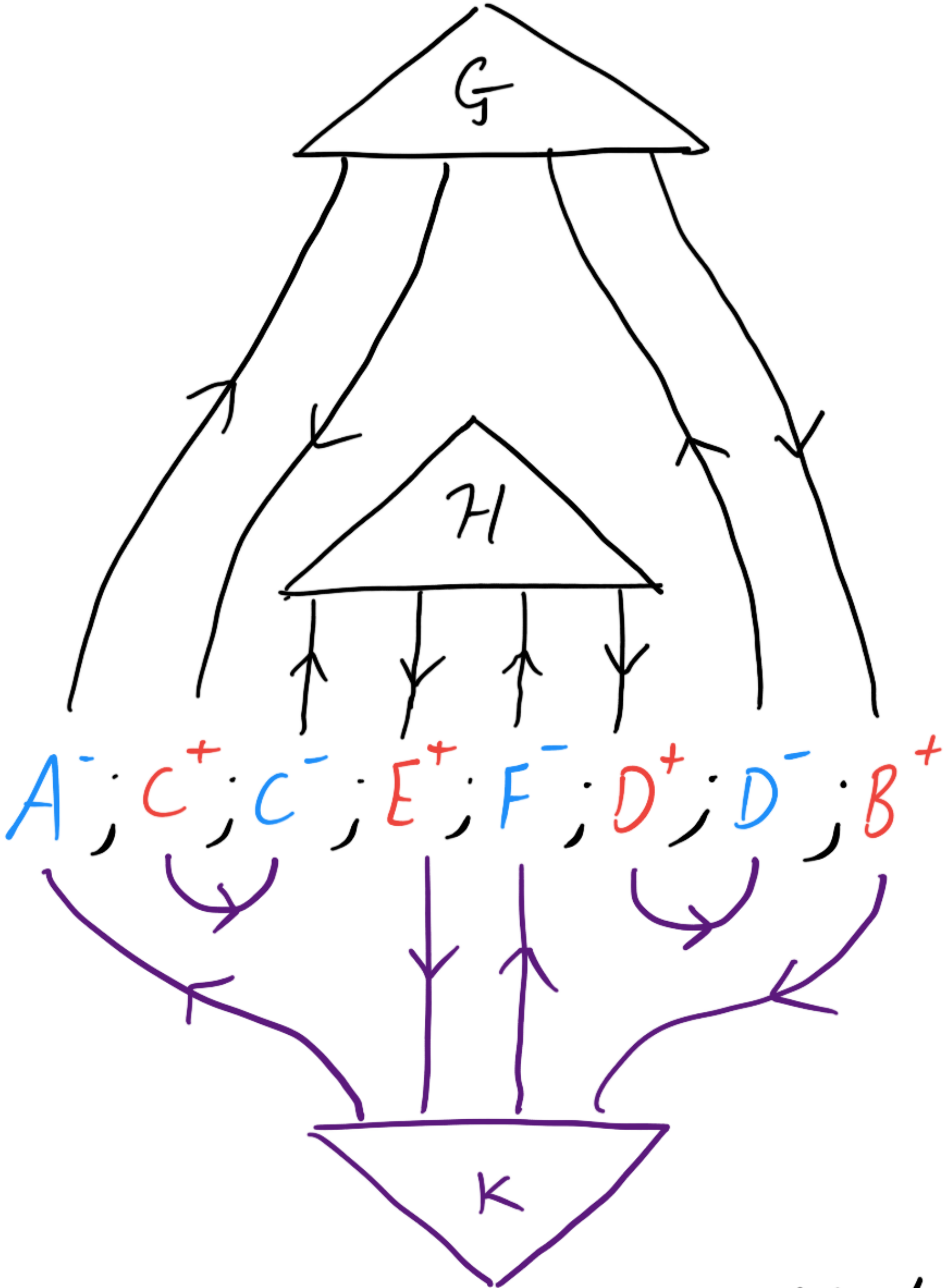


$$\Sigma_{\mathcal{G}\mathcal{H}} = \Sigma_{\mathcal{G}} \times \Sigma_{\mathcal{H}}$$

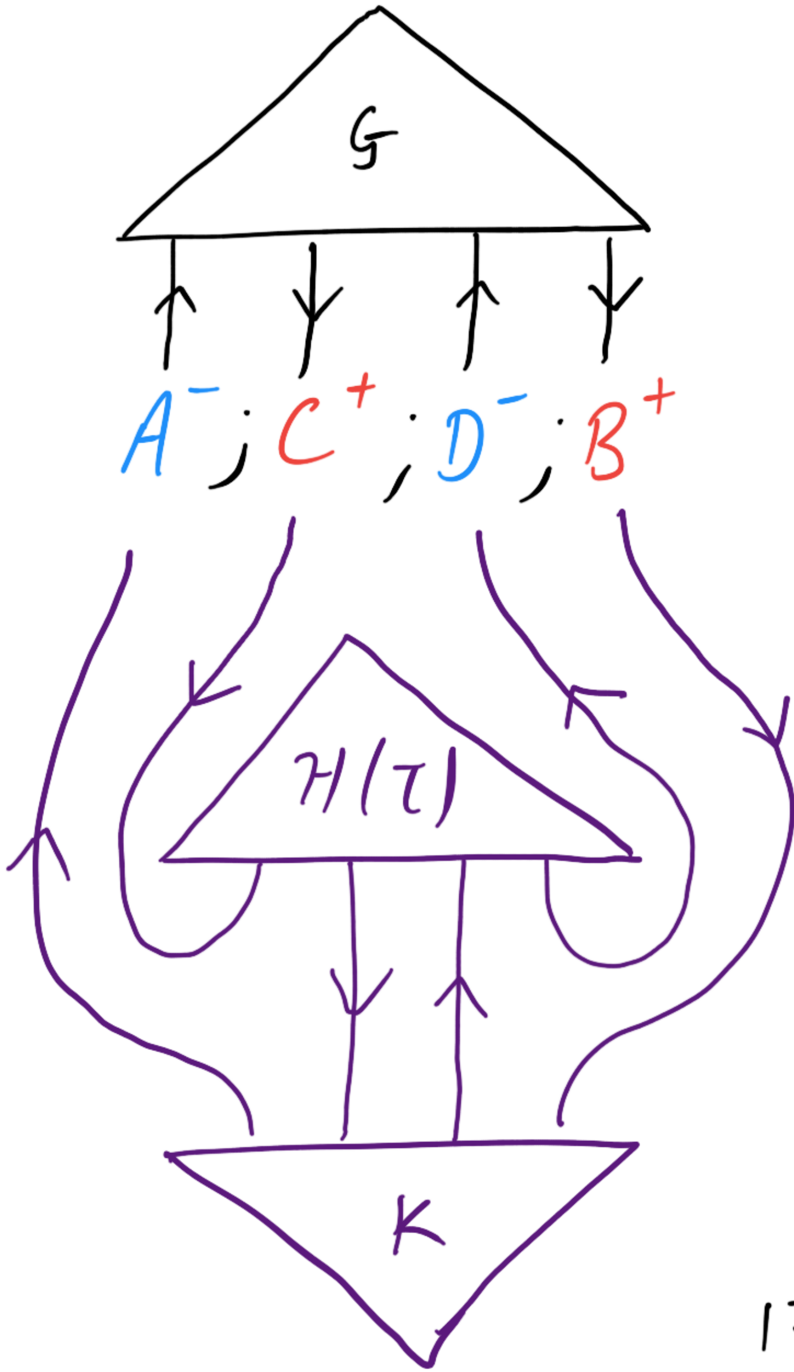
$$\begin{tikzcd}
A^+ ; B^- \arrow[d, "\mathcal{G}(\sigma)"'] \arrow[dr, "(\mathcal{G}\mathcal{H})(\sigma,\tau)"] & \\
C^+ ; D^- \arrow[r, "\mathcal{H}(\tau)"'] & E^+ ; F^-
\end{tikzcd}$$

When does a strategy profile (σ, τ) for $\mathcal{G}\mathcal{H}$ "win" (be in Nash eq.) against an O-strategy κ for $A^+ ; E^+ ; F^- ; B^+$?



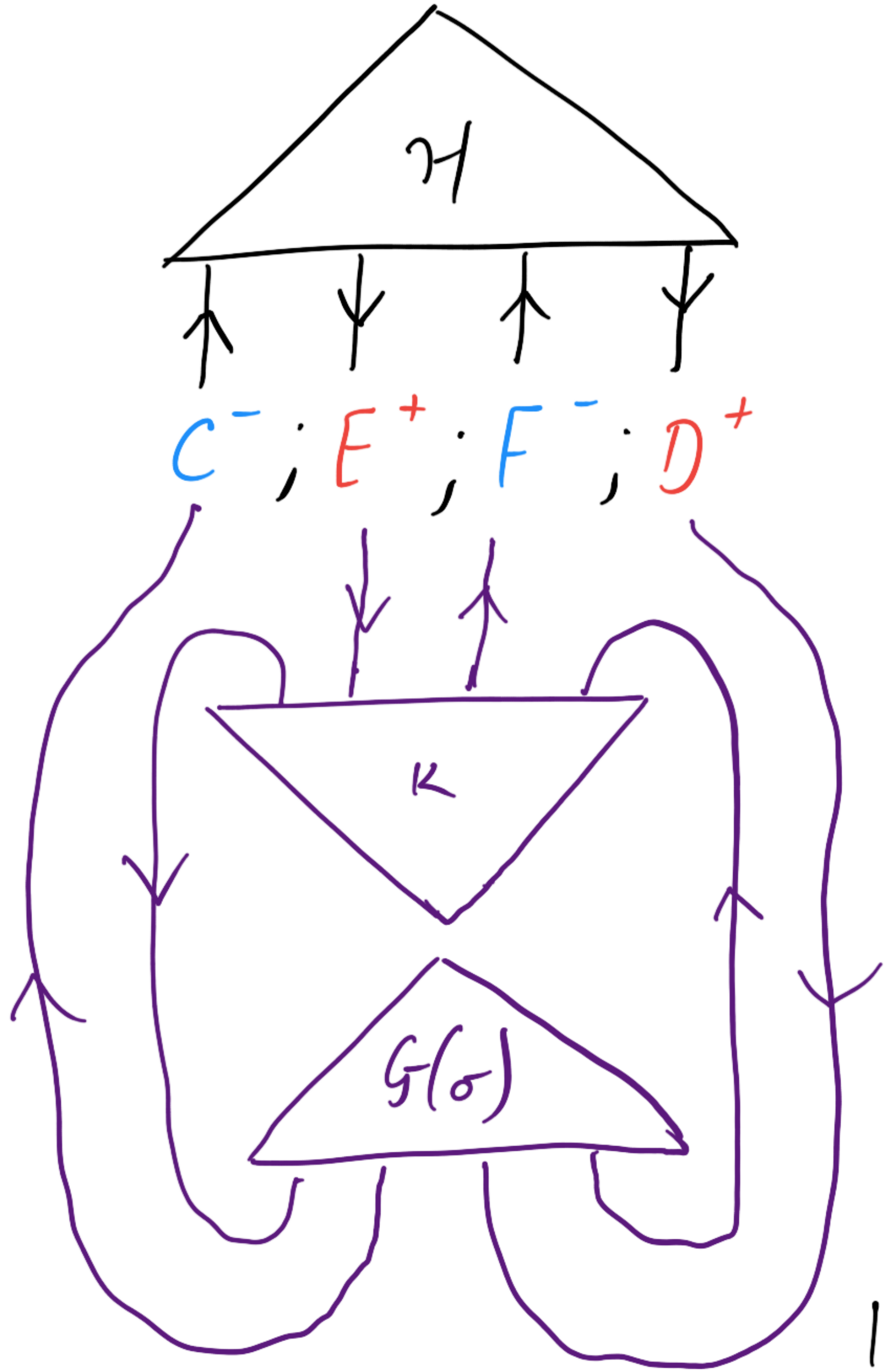


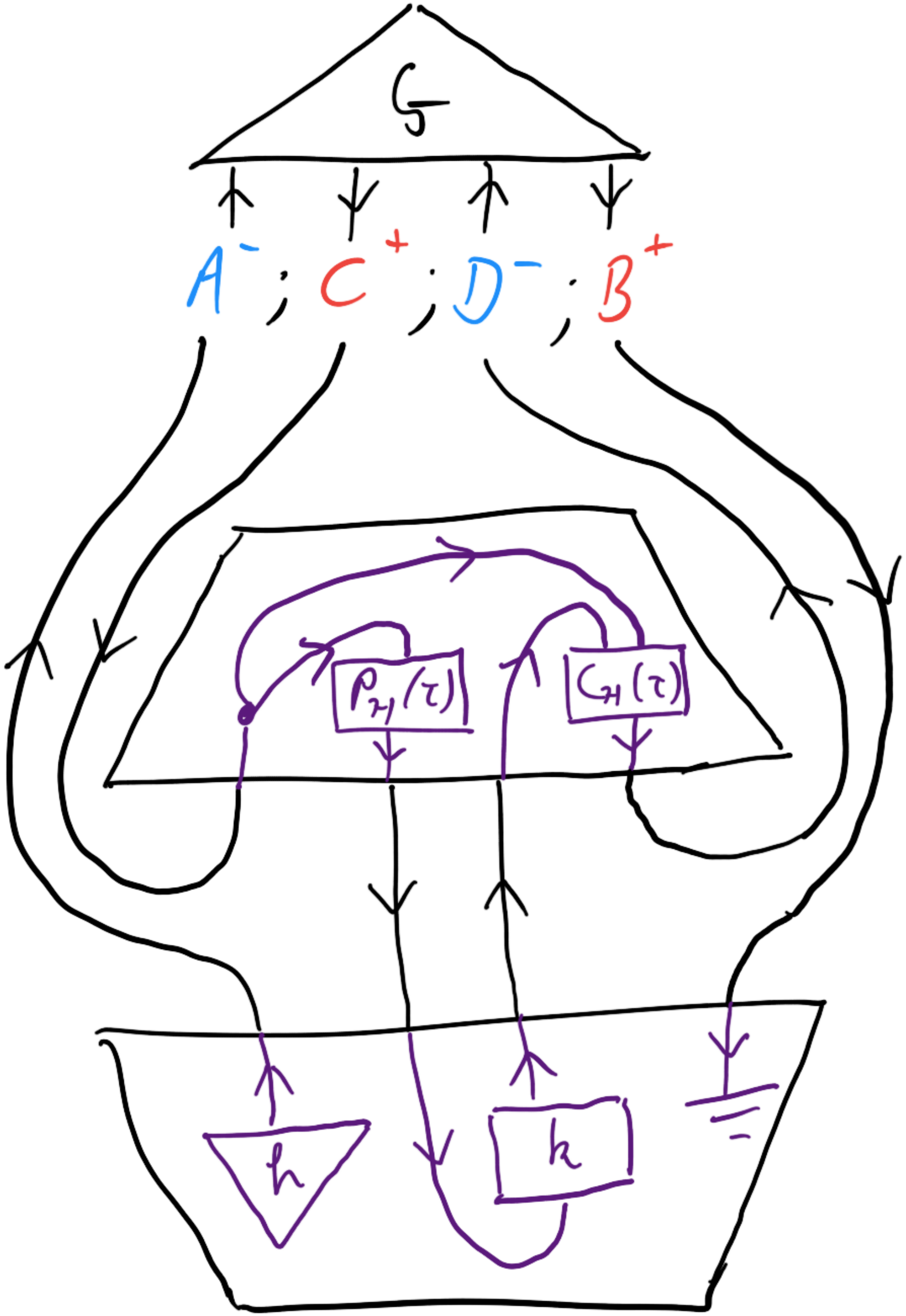
A: When (1) σ wins in $\mathcal{G}$ against an adjusted O-strategy made from $\mathcal{K}$ and $\mathcal{H}(\tau)$:



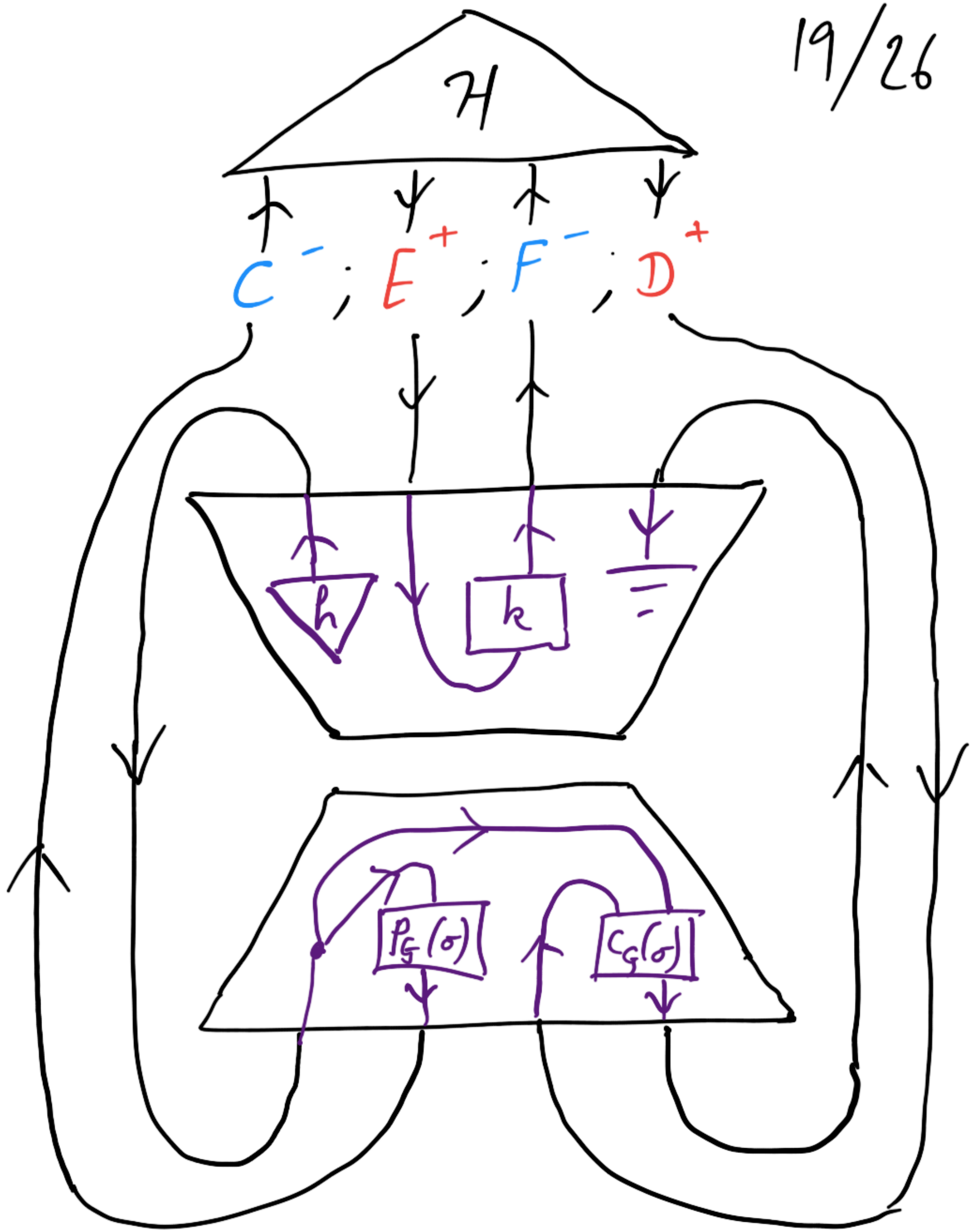
and...

… and (2), $\mathcal{T}$ is winning in $\mathcal{H}$ against an adjusted O-strategy made from $\mathcal{K}$ and $\mathcal{G}(\sigma)$:

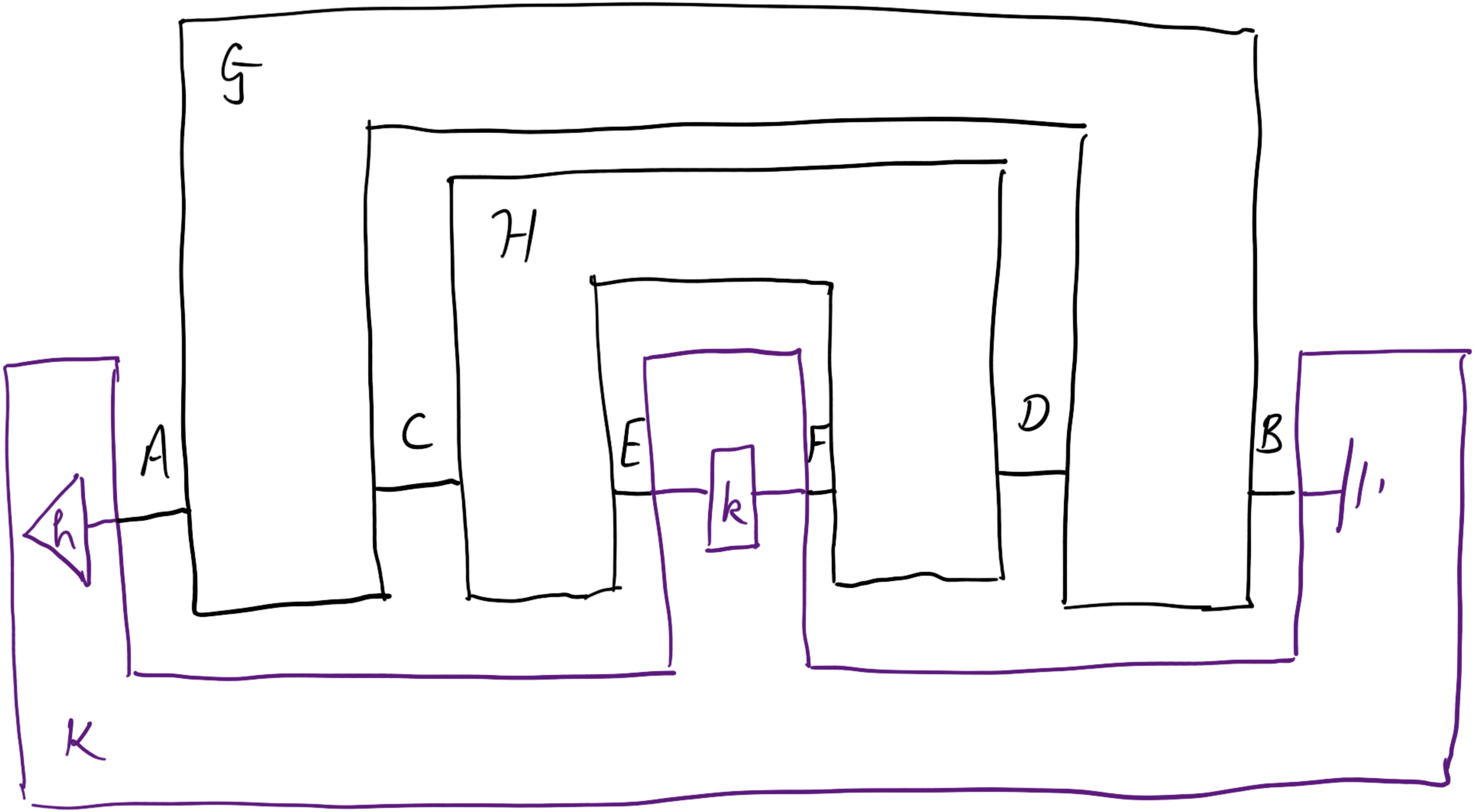




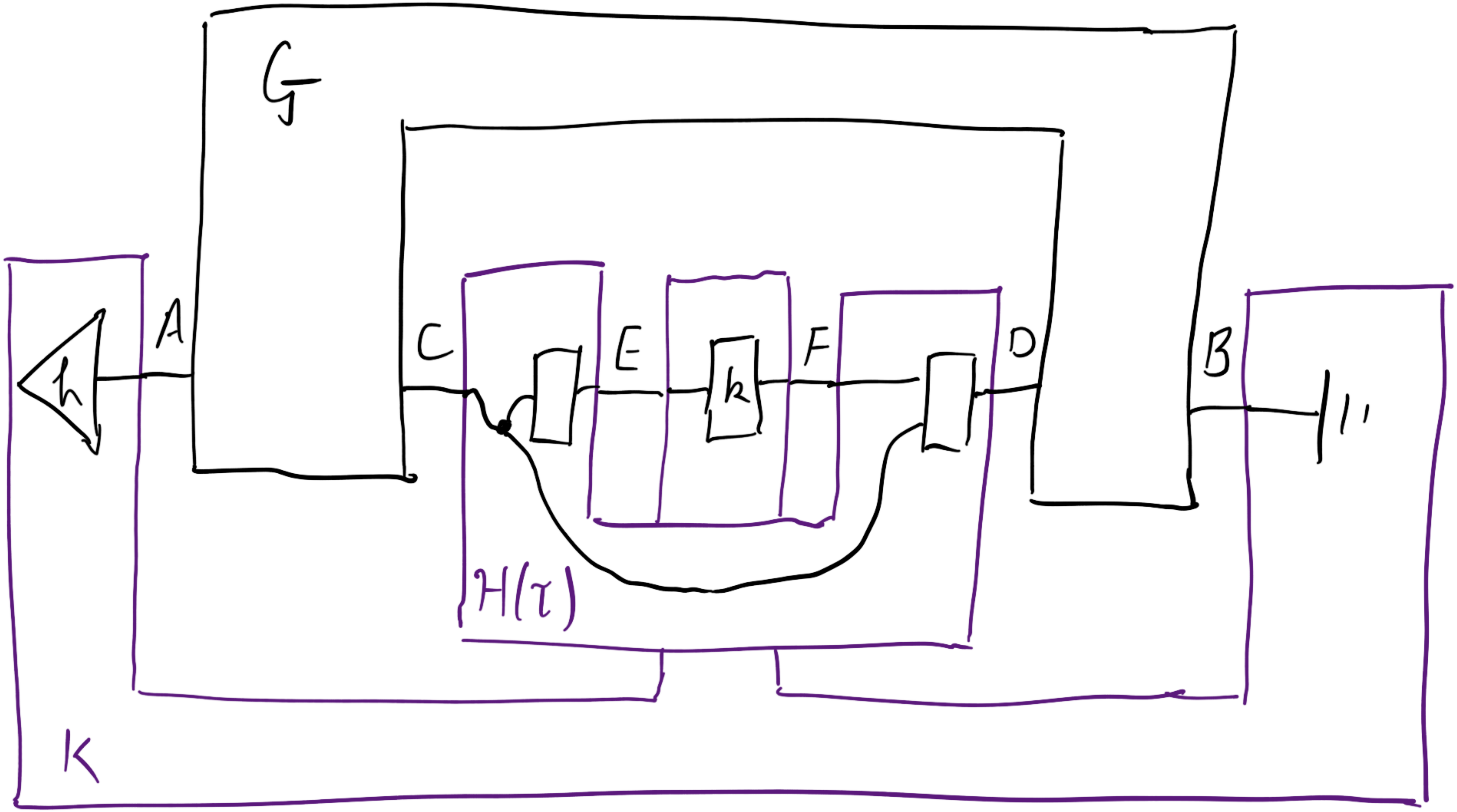
G
A⁻ ; C⁺ ; D⁻ ; B⁺
P_H(τ)
C_H(τ)
h
k

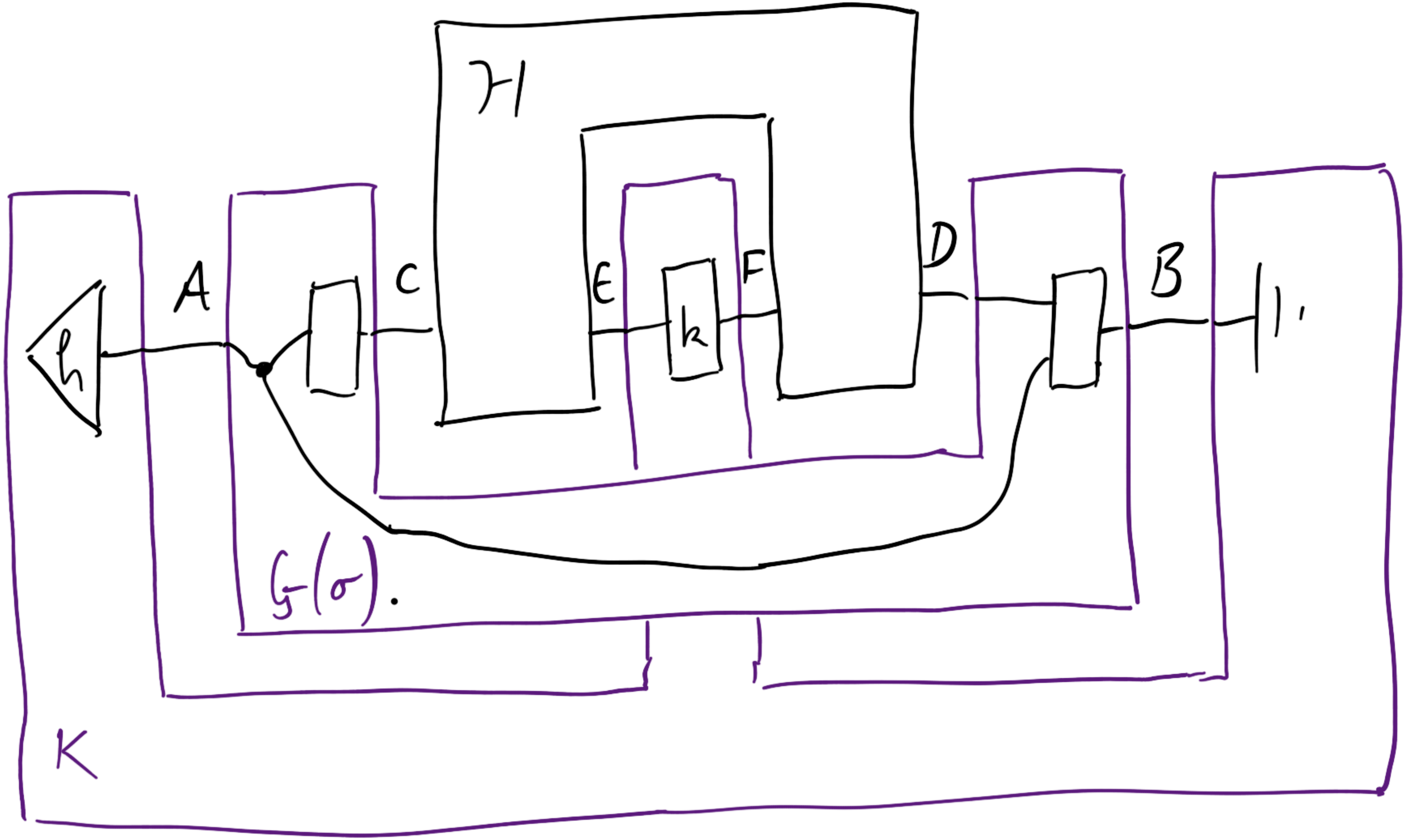


H
C⁻ ; E⁺ ; F⁻ ; D⁺
h
k
P_G(σ)
C_G(σ)



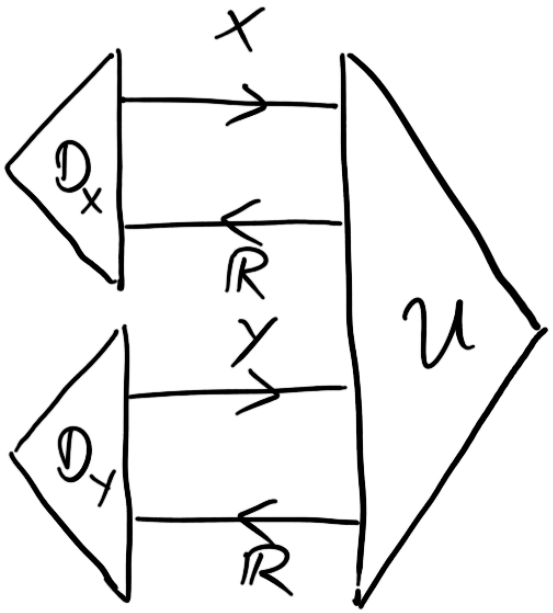
G
H
A
C
E
F
D
B
h
k
K

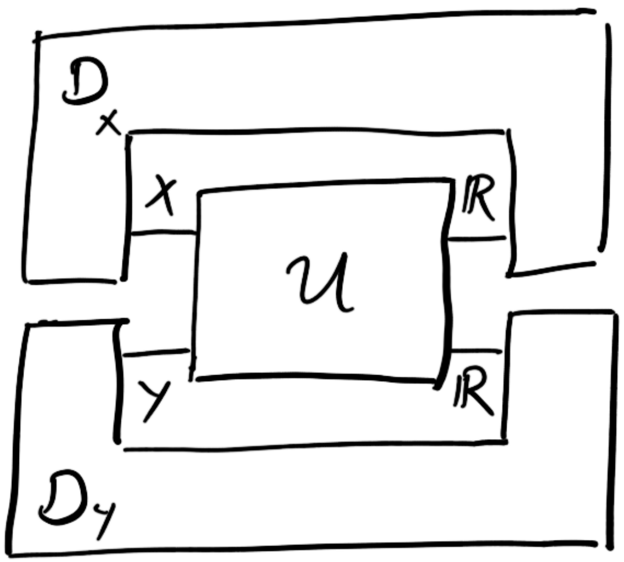

G
A
C
E
F
D
B
h
k
H(τ)
K



$\mathcal{H}$
A
C
E
k
F
D
B
h
$\mathcal{G}(\sigma)$.
K

Bimatrix game

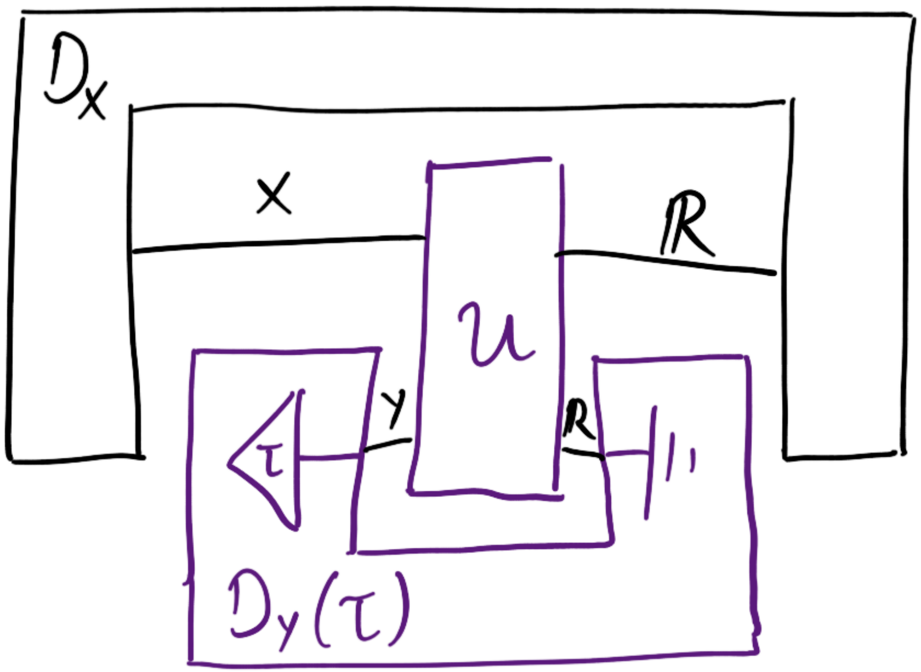


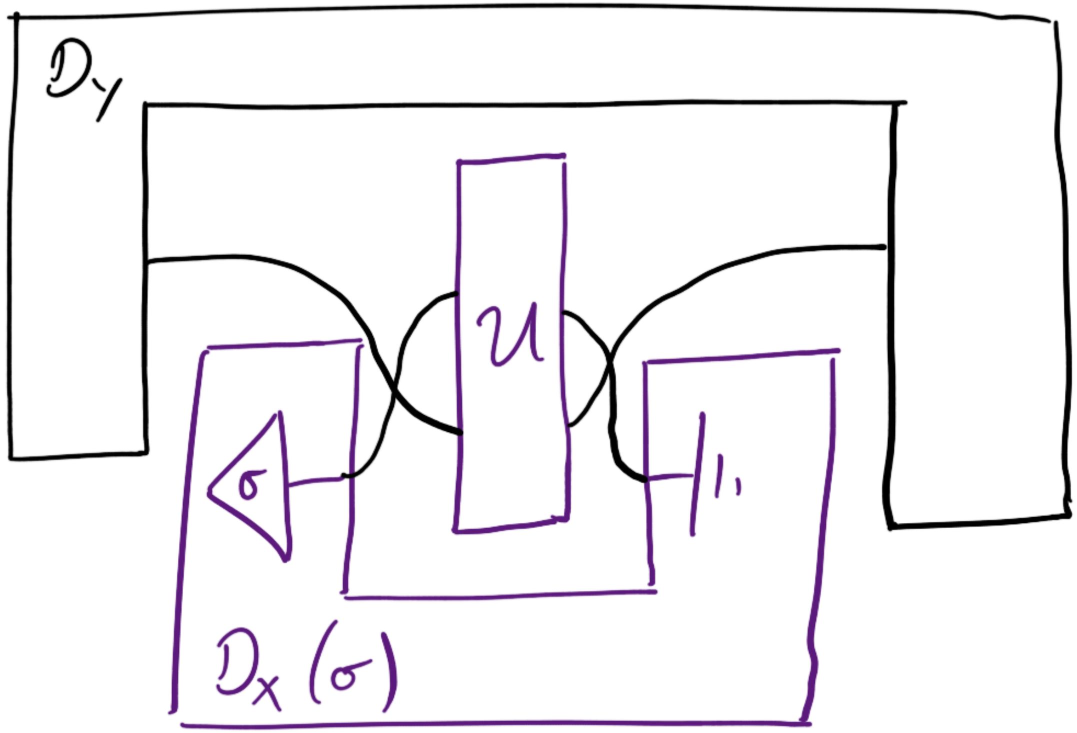


$$(D_X \otimes D_Y) \odot \mathcal{U} : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Sigma_{(D_X \otimes D_Y) \circ \mathcal{U}} = X \times Y$$

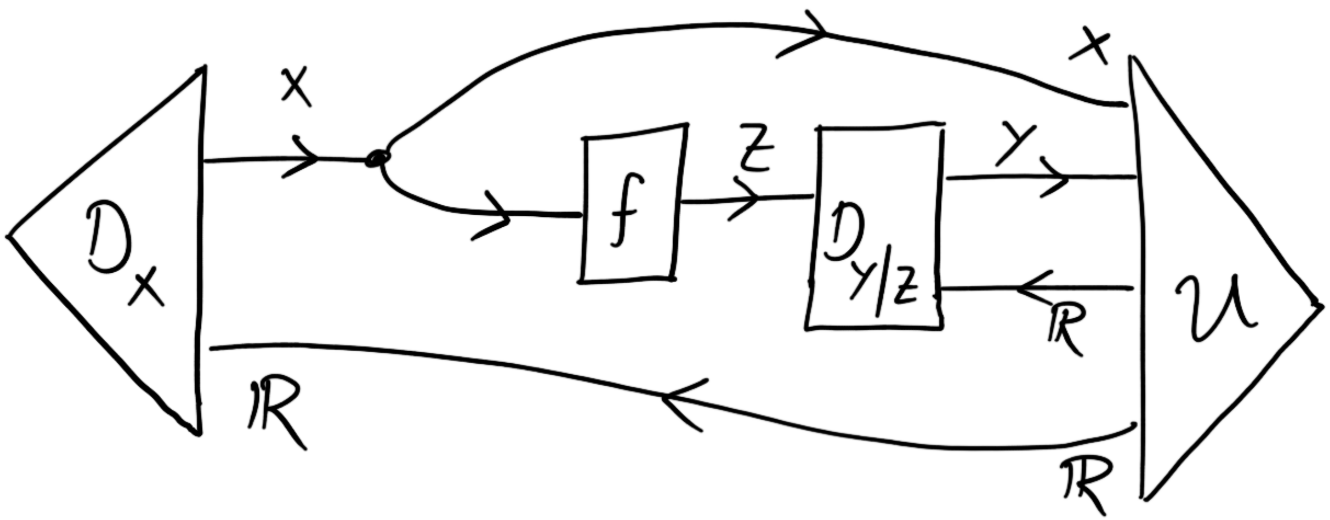
When is $(\sigma, \tau) \in X \times Y$ a Nash equilibrium?





$$\sigma \in \operatorname{argmax}_{x \in X} \mathcal{U}_1(x, \tau) \qquad \tau \in \operatorname{argmax}_{y \in Y} \mathcal{U}_2(\sigma, y)$$

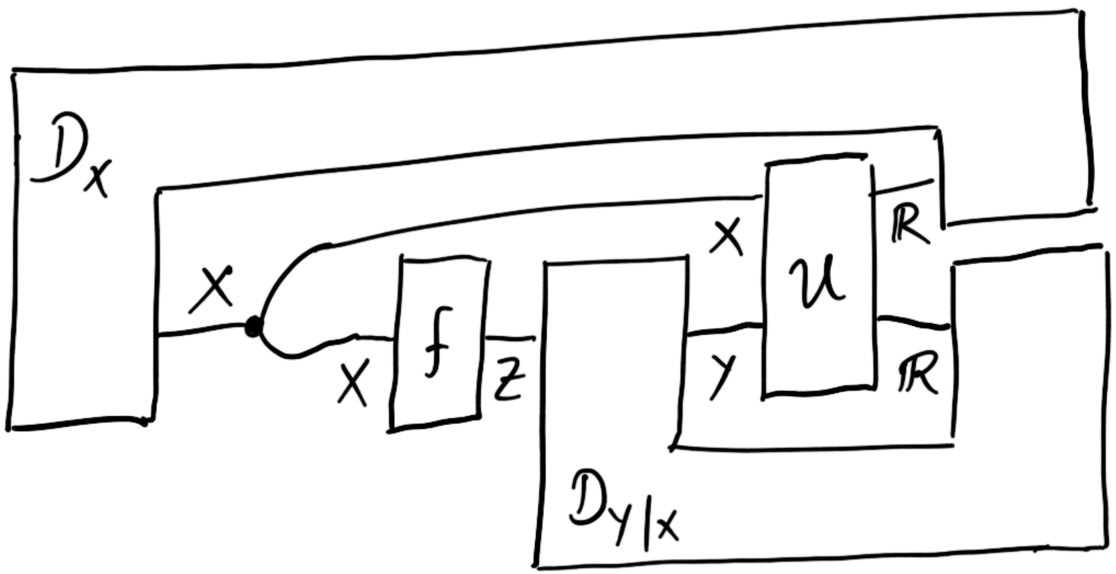
Imperfect information game



$$\mathcal{D}_X \odot \left(\left(\Delta_X \odot \left(\mathrm{id}_{\binom{X}{1}} \otimes (f \odot \mathcal{D}_{Y|X})\right)\right) \otimes \mathrm{id}_{\binom{1}{\mathbb{R}}}\right) \odot \mathcal{U} : \binom{1}{1} \to \binom{1}{1}$$

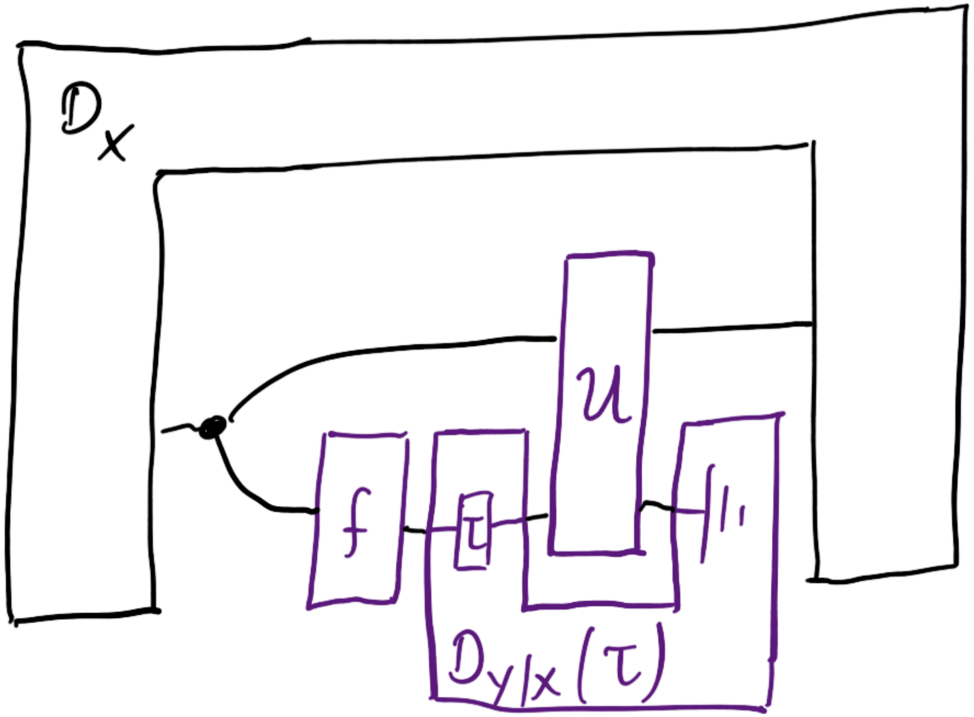
$$\shortparallel$$

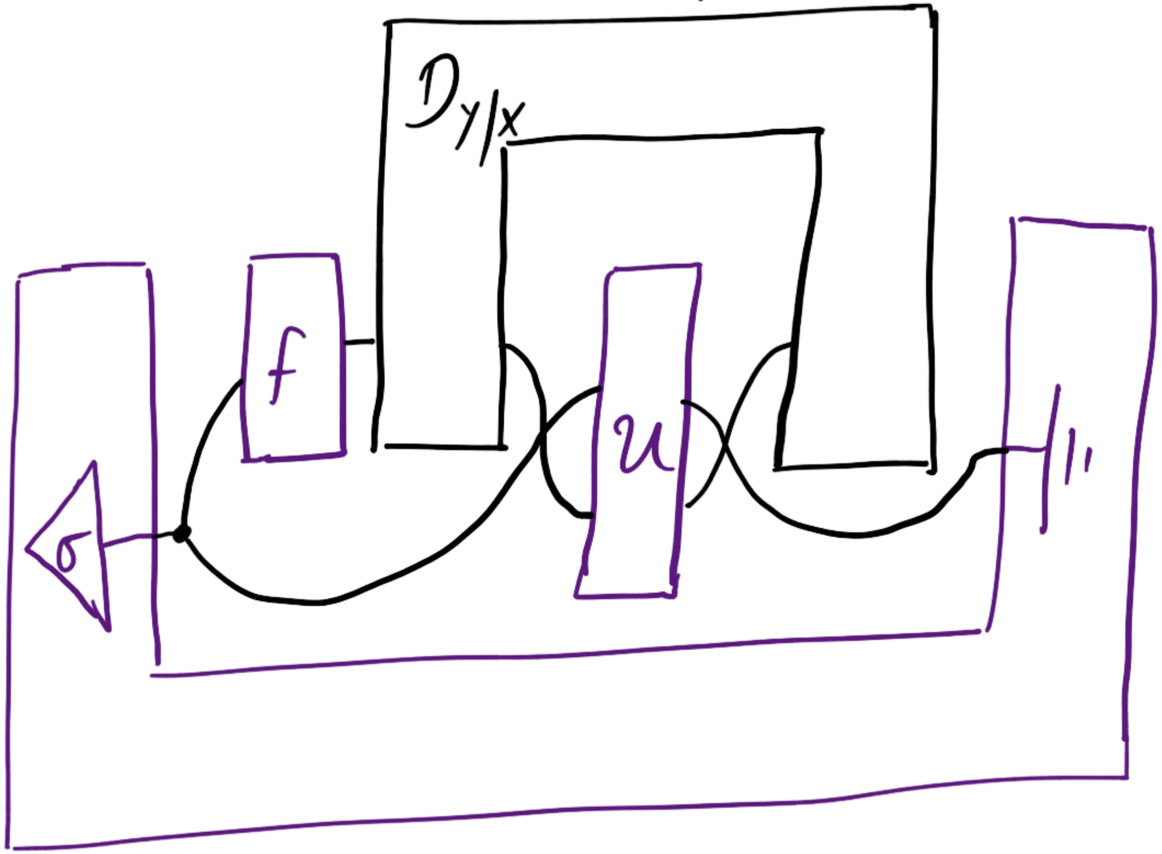
$$\mathcal{G}$$



$$\Sigma_{\mathcal{G}} = X \times (Z \to Y)$$

When is $(\sigma, \tau) \in X \times (Z \to Y)$ a Nash equilibrium ?





$$\sigma \in \operatorname*{argmax}_{x \in X} U_1(x, \tau(f(x)))$$

$$\tau(f(\sigma)) \in \operatorname*{argmax}_{y \in Y} U_2(\sigma, y)$$

Bonus content: Where next?

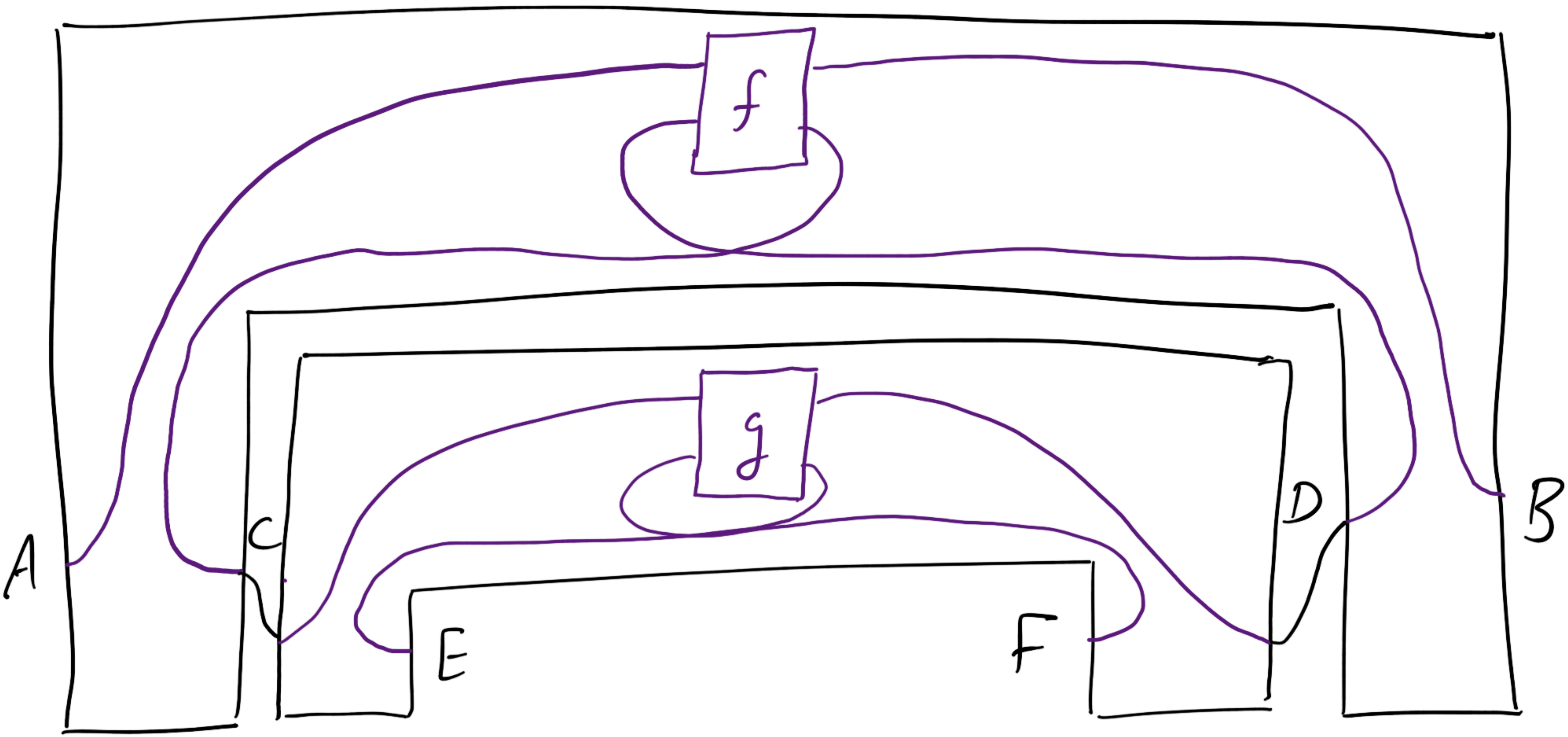
If we have a trace, we can upgrade the allowed P-strategies $A^+ ; B^- \to C^+ ; D^-$ of $A^- ; C^+ ; D^- ; B^+$ from

$$(A \to C) \times (A \times D \to B)$$

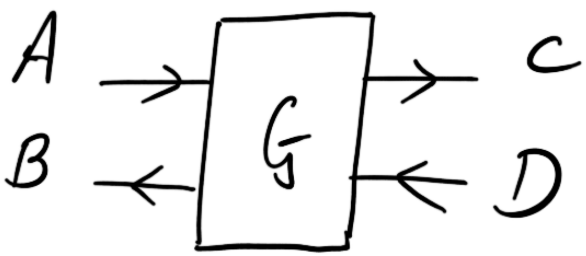
to

$$A \times D \to C \times B$$

Int as a ⊔-algebra



Computable compositional game theory



(1) Σ_G : Set

(2) $P_G : \Sigma_G \times A \times D \longrightarrow C$

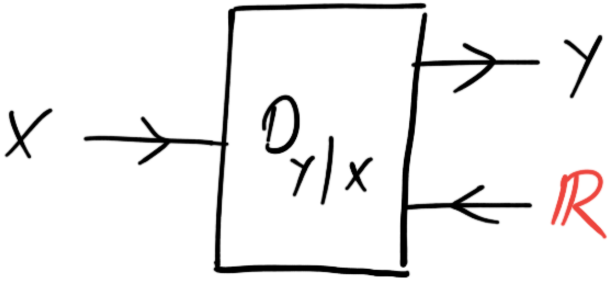
(3) $C_G : \Sigma_G \times A \times D \longrightarrow B$

(4) $E_G \subseteq \Sigma_G \times (C \times B \longrightarrow A) \times (C \times B \longrightarrow D)$

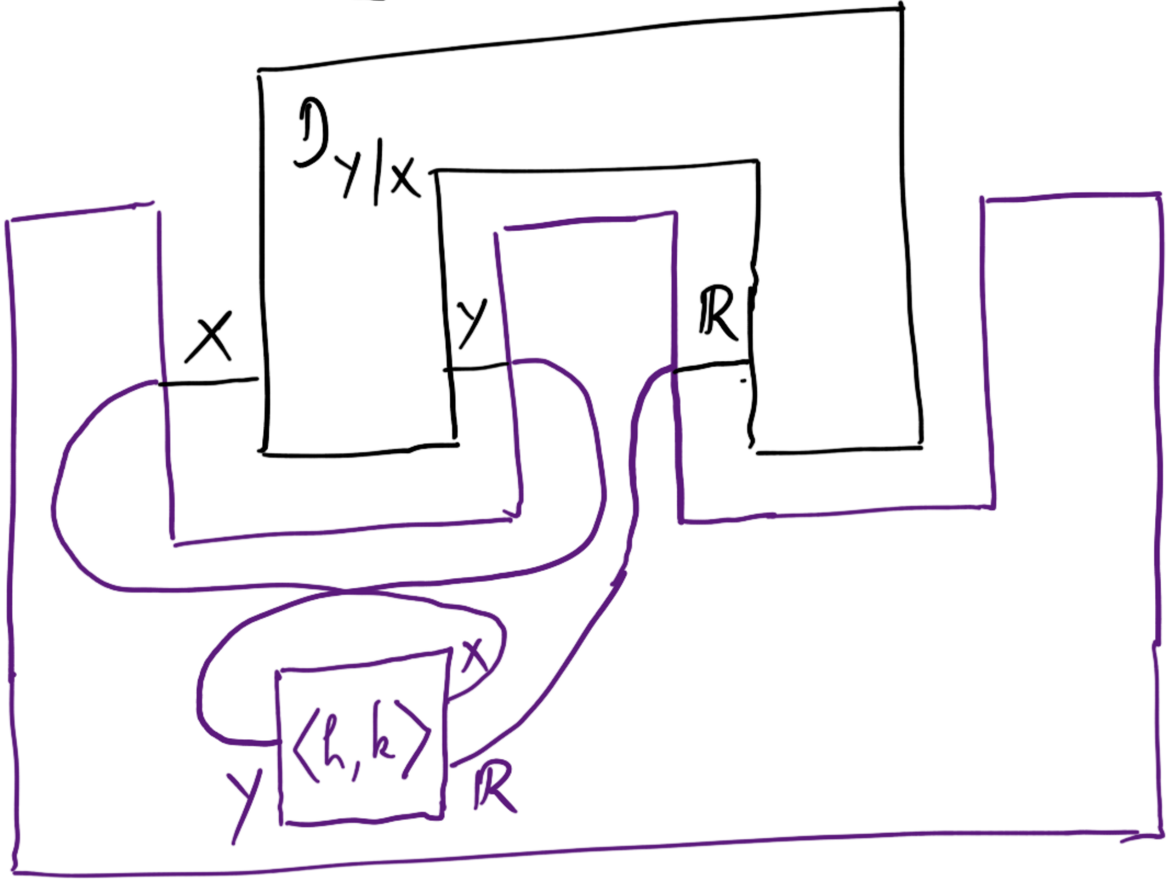
<u>Theorem</u> : This yields a compact closed category

<u>Theorem</u> : It's a conservative extension of the old thing

Enter game theory again



$\Sigma_{Y|X} = \{\text{computable functions } X \to Y\}$



Reasonable definition of $E_{D_{Y|X}}(\sigma, (h, k))$:

$\mu y.\, \sigma(h(y)) \in \operatorname{argmax}(k)$