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Restricted partitions and generalized Catalan numbers

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Abstract. The *generalized Catalan numbers* w_n are given by the recurrence $w_n = 2w_{n-1} + \sum_{i=1}^{n-2} w_i w_{n-2-i}$ if $n \geq 2$, with $w_0 = w_1 = 1$, and count a restricted subset of the Catalan paths having semilength n. In this paper, we provide new combinatorial interpretations of these numbers in terms of finite set partitions. In particular, we identify five classes of the partitions of size n, all of which have cardinality w_n and each avoiding a set of two classical patterns of length four. We use both combinatorial and algebraic arguments to establish our results, applying the kernel method in a couple of the apparently more difficult cases.

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1 Introduction

Let w_n denote the *generalized Catalan number* defined by the recurrence

$$w_n = 2w_{n-1} + \sum_{i=1}^{n-2} w_i w_{n-2-i}, \qquad n \geq 2, \tag{1}$$

with $w_0 = w_1 = 1$. The right side of (1) is sometimes written as $\sum_{i=-1}^{n-1} w_i w_{n-2-i}$, if one assumes $w_{-1} = 2$. The first few w_n values for $n \geq 0$ are $1, 1, 2, 5, 13, 35, 97, 275, \ldots$. Using (1), one can show that the w_n have generating function given by

$$\sum_{n\geq 0} w_n x^n = \frac{(1-x)^2 - \sqrt{1-4x+2x^2+x^4}}{2x^2}. \tag{2}$$

The numbers w_n count, among other things, the Catalan paths of semilength n having no occurrences of $DDUU$ (see [8]), or, equivalently, the Catalan paths of semilength $n+1$ having no $UUDD$. They also count the subset of the permutations of size n avoiding the two generalized patterns 1-3-2 and 12-34 (see Example 2.10 of [6]). For further information on these numbers, see also A025242 of [10]. Here, we provide new combinatorial interpretations for the w_n in terms of finite set partitions, showing that they enumerate certain two-pattern avoidance classes.

We'll use the following notational conventions: $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{P} := \{1, 2, \dots\}$, $[0] := \varnothing$, and $[n] := \{1, \dots, n\}$ for $n \in \mathbb{P}$. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. If $n \in \mathbb{P}$, then a *partition* of $[n]$ is any collection of non-empty, disjoint subsets, called *blocks*, whose union is $[n]$. (If $n = 0$, then there is a single empty partition which has no blocks.) A partition with k blocks is also called a *k-partition* and is denoted by $B_1/B_2/\cdots/B_k$, where the blocks are arranged in the standard order: $\min(B_1) < \cdots < \min(B_k)$. The set of k-partitions of $[n]$ will be denoted by $P_{n,k}$ and the set of all partitions of $[n]$ by P_n. In what follows, we will represent $\Pi = B_1/B_2/\cdots/B_k \in P_{n,k}$, equivalently, by the *canonical sequential form* $\pi = \pi_1\pi_2\cdots\pi_n$ wherein $j \in B_{\pi_j}$, $1 \le j \le n$, and in such case we will write $\Pi = \pi$. Note that the word $\pi = \pi_1\pi_2\cdots\pi_n$ is a *restricted growth function* from $[n]$ onto $[k]$ (see, e.g., [12] for details). For example, the partition $\Pi = 1,5/2,3,7,8/4/6 \in P_{8,4}$ has the canonical sequential form $\pi = 12231422$.

A *classical pattern* τ is a member of $[\ell]^m$ which contains all of the letters in $[\ell]$. We say that a word $\sigma \in [k]^n$ *contains* the classical pattern τ if σ contains a subsequence isomorphic to τ. Otherwise, we say that σ *avoids* τ. For example, a word $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ avoids the pattern 231 if it has no subsequence $\sigma_i\sigma_j\sigma_k$ with $i < j < k$ and $\sigma_k < \sigma_i < \sigma_j$. The pattern avoidance question is a well studied problem in enumerative combinatorics, starting with Knuth [5], who showed that the number of elements of S_n avoiding the pattern τ is the n-th Catalan number C_n for all $\tau \in S_3$. Simion and Schmidt [9] later extended this result by determining the number of elements of S_n avoiding the patterns in any subset of S_3. Comparable work has been done more recently concerning pattern avoidance on set partitions; we refer the reader to the papers by Klazar [4], Sagan [7], and Jelínek and Mansour [3] and to the references therein.

In what follows, we will represent set partitions as words using the canonical sequential form and consider the problem of avoidance of certain classical patterns by these words. If $\{w_1, w_2, \dots\}$ is a set of classical patterns, then let $P_n(w_1, w_2, \dots)$ and $P_{n,k}(w_1, w_2, \dots)$ denote the subsets of P_n and $P_{n,k}$, respectively, which avoid all of the patterns. We will denote the cardinalities of $P_n(w_1, w_2, \dots)$ and $P_{n,k}(w_1, w_2, \dots)$ by $p_n(w_1, w_2, \dots)$ and $p_{n,k}(w_1, w_2, \dots)$, respectively. Note that $p_n(w_1, w_2, \dots) = \sum_{k\ge0} p_{n,k}(w_1, w_2, \dots)$.

In this paper, we identify five classes of partitions each avoiding a set of two classical patterns of length four and each enumerated by the generalized Catalan number w_n. This addresses some particular cases of a general question raised by Goyt at the end of [2] concerning the avoidance by finite set partitions of two or more patterns of length four. Our main result is the following theorem which we prove in the next section as a series of propositions.

THEOREM 1.1 *If $n \ge 0$, then $p_n(u, v) = w_n$ for the following pairs (u, v):*

(1) $(1211, 1212)$ (2) $(1121, 1212)$ (3) $(1121, 1221)$
(4) $(1112, 1123)$ (5) $(1122, 1123)$.

Furthermore, in the first three cases above, we in fact have $p_{n,k}(u, v)$ the same for all n and k and we supply two different explicit formulas for it, thereby obtaining a seemingly new identity involving Catalan numbers and binomial coefficients.

2 Pattern avoidance and generalized Catalan numbers

Theorem 1.1 above will follow from combining the results in the sections below. We start with the patterns $\{1211, 1212\}$ and $\{1121, 1212\}$.

2.1 The cases $\{1211, 1212\}$ and $\{1121, 1212\}$

Define the polynomials $w_n(q)$ by

$$w_n(q) = \sum_{k=1}^{n} p_{n,k}(1211, 1212)q^k, \qquad n \in \mathbb{P},$$

with $w_0(q) = 1$. We consider the following cases regarding members π of $P_n(1211, 1212)$, where $n \geq 2$:

1. $\pi = 1\pi'$, where π' contains no 1's.
2. $\pi = 1^i\pi'$, where $i \geq 2$ and π' contains no 1's.
3. $\pi = 1^i\pi'1\pi''$, where $i \geq 2$, π' and π'' contain no 1's, and π' is non-empty.
4. $\pi = 1\pi'1\pi''$, where π' and π'' contain no 1's and π' is non-empty.

In all of the cases above, note that π' and π'' avoid the patterns 1211 and 1212, with each letter of π'' greater than each letter of π' in the last two cases. Note that we get a contribution towards the total weight of $qw_{n-1}(q)$ in the first case. The partitions in the second and third cases, combined, contribute weight $w_{n-1}(q)$, for removing the first 1 yields a bijection between these members of $P_n(1211, 1212)$ and members of $P_{n-1}(1211, 1212)$. For the fourth case, first observe that π' and π'' are both partitions on their respective sets and have no letters in common. Thus, the members of $P_n(1211, 1212)$ in this case have weight $q\sum_{i=1}^{n-2} w_i(q)w_{n-2-i}(q)$, upon considering the length i of π'. Combining all of the cases yields the following recurrence for $w_n(q)$.

Proposition 2.1 *If $n \geq 2$, then*

$$w_n(q) = (1+q)w_{n-1}(q) + q\sum_{i=1}^{n-2} w_i(q)w_{n-2-i}(q), \tag{3}$$

with $w_0(q) = 1$ and $w_1(q) = q$.

Recurrence (3) reduces to (1) when $q = 1$, which implies $p_n(1211, 1212) = w_n$. Let $f(x; q)$ denote the generating function

$$f(x; q) = \sum_{n \geq 0} w_n(q)x^n.$$

Multiplying both sides of (3) by x^n and summing over $n \geq 2$ implies that $f(x; q)$ satisfies

$$qx^2f^2(x; q) - (1-qx)(1-x)f(x; q) + (1-x) = 0,$$

and is thus given by

$$f(x;q) = \frac{(1-qx)(1-x) - \sqrt{[(1-qx)(1-x)]^2 - 4qx^2(1-x)}}{2qx^2}, \tag{4}$$

which generalizes (2).

The generating function $f(x;q)$ can be written as

$$f(x;q) = \frac{1}{1-qx} C\left(\frac{qx^2}{(1-x)(1-qx)^2}\right),$$

where $C(t)$ is the ordinary generating function for the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, that is, $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ (see, e.g., [11]). Thus,

$$\begin{aligned} f(x;q) &= \sum_{i\geq 0} C_i \frac{q^i x^{2i}}{(1-x)^i(1-qx)^{2i+1}} \\ &= \sum_{i,j\geq 0} C_i \binom{2i+j}{j} \frac{x^{2i+j}}{(1-x)^i} q^{i+j}, \end{aligned}$$

which implies that the coefficient of $x^n q^k$ in the generating function $f(x;q)$ is given by

$$\sum_{i=1}^{k} C_i \binom{k+i}{2i}\binom{n-1-k}{i-1}, \qquad n > k \geq 1.$$

Replacing 1211 with 1121 in the preceding and making some slight modifications shows that the statistic recording the number of blocks on $P_n(1121, 1212)$ has the same distribution as it does on $P_n(1211, 1212)$. Note that the third case above should instead concern partitions of the form $\pi = 1\pi'1^i\pi''$, where neither π' nor π'' contains the letter 1, π' is non-empty, and $i \geq 2$. In particular, we obtain the following result.

Proposition 2.2 *If $n \in \mathbb{N}$, then $p_n(1121, 1212) = p_n(1211, 1212)$.*

One may also prove Proposition 2.2 by defining bijections $f_k = f_{n,k}$ between $P_{n,k}(1211, 1212)$ and $P_{n,k}(1121, 1212)$ for each k and combining them. We construct the f_k in a recursive fashion, letting f_0 be the empty mapping and $f_1 = \iota_d$. If $k \geq 2$ and $\pi \in P_{n,k}(1211, 1212)$, then let

$$f_k(\pi) = 1^i f_{k-1}(\pi'),$$

if $\pi = 1^i\pi'$ and $i \geq 1$ (the mapping f_{k-1} here is understood to be applied to partitions on the letters $\{2, 3, \ldots\}$); let

$$f_k(\pi) = 1 f_r(\pi') 1^i f_s(\pi''),$$

if $\pi = 1^i\pi'1\pi''$, where $r + s = k - 1$ and $i \geq 2$; and let

$$f_k(\pi) = 1 f_r(\pi') 1 f_s(\pi''),$$

if $\pi = 1\pi'1\pi''$, where $r + s = k - 1$. For example, if $\pi = 11222343215565 \in P_{14,6}(1211, 1212)$, then $f_6(\pi) = 12343222115655 \in P_{14,6}(1121, 1212)$. The above bijection can be extended to show, more generally, that

$$p_{n,k}(1^j21, 1212) = p_{n,k}(121^j, 1212), \qquad n, k \in \mathbb{N},$$

where $j \geq 1$.

2.2 The case $\{1121, 1221\}$

We will show that the partitions avoiding $\{1121, 1221\}$ are equivalent to those which avoid $\{1121, 1212\}$, implying $p_n(1121, 1221) = w_n$. In what follows, we will call a (maximal) sequence of identical consecutive letters a *run* (of the letter).

PROPOSITION 2.3 *If $n \in \mathbb{N}$, then $p_n(1121, 1221) = p_n(1121, 1212)$.*

Proof. We will show $p_{n,k}(1121, 1221) = p_{n,k}(1121, 1212)$ for all n and k. To do this, we first describe an inductive procedure for generating the members of $P_{n,k}(1121, 1212)$, starting with the largest letter. First note that the k's occurring within a member $\pi \in P_{n,k}(1121, 1212)$ are limited to a single run of the letter. The possible positions of the $k-1$'s relative to the k's in π are then

$$\underbrace{(k-1)(k-1)\cdots(k-1)}_{r \text{ times}}\underbrace{kk\cdots k}_{s \text{ times}}$$

or

$$(k-1)\underbrace{kk\cdots k}_{s \text{ times}}\underbrace{(k-1)(k-1)\cdots(k-1)}_{r \text{ times}},$$

where r and s are positive integers. One can then subsequently add the letters $k-2, k-3, \ldots,$ at each point deciding whether a letter i occurs as a single run of letters or as two runs, the first of which has length one. Given a partition $\pi = \pi_1\pi_2\cdots\pi_m$ on the letters $\{2, 3, \ldots, k\}$ having $k-1$ blocks and avoiding the patterns 1121 and 1212, let us call the letter π_i an *active site* (in accordance with the generating tree methodology described in [13]) if one may write a non-empty sequence of the letter 1 directly to the right of π_i within the partition 1π without creating an occurrence of 1121 or 1212.

Given $k \geq 2$, let $a_{n,k,t}$ denote the number of partitions of length n using the letters $\{2, 3, \ldots, k\}$, avoiding the patterns 1121 and 1212, and having exactly t active sites. Note that we always have $t \geq 1$, since the last letter is always an active site. From the definitions, we may write

$$p_{n,k}(1121, 1212) = \sum_{t\geq 1} a_{n,k+1,t}, \tag{5}$$

for all $n \geq k \geq 1$.

We now find a recurrence for the numbers $a_{n,k,t}$. Let $\mathcal{A}_{n,k,t}$ denote the class of partitions enumerated by $a_{n,k,t}$ defined above. If $k \geq 1$, we first observe that a member $\pi \in \mathcal{A}_{n,k+1,t}$ may be obtained by writing a sequence of i 1's just before a member $\alpha \in \mathcal{A}_{n-i,k,t-1}$ for some $i \geq 1$ and then adding one to each letter of the resulting partition. Note that all of the active sites of α remain intact in π and that there is an active site that has been created corresponding to the right-most 2 in π. Hence, there are $\sum_{i\geq 1} a_{n-i,k,t-1}$ possible members of $\mathcal{A}_{n,k+1,t}$ in this case.

Alternatively, one may add the 1's in the procedure described in the prior paragraph as two separate runs, the first of which has length one. Note that this is the only other option since we are to avoid 1121. If one were to write a 1 before some $\beta \in \mathcal{A}_{m,k,j}$, add a non-empty run of 1's just after the t-th active site of β from the right (where $t \leq j$), and then add one to each letter, the resulting partition γ would belong to $\mathcal{A}_{m+i,k+1,t}$ for some $i \geq 2$. To see this, note that all of the active sites in β to the left of (and including) the t-th right-most one are deactivated in γ, while those to the right of it remain active in γ, with the final 2 in γ now also active. Furthermore, no other active sites are lost or created in the transition from β to γ. Thus, the members of $\mathcal{A}_{n,k+1,t}$ having i 2's for some i occurring as two runs number $\sum_{i\geq 2}\sum_{j\geq t} a_{n-i,k,j}$. Combining this case with the prior one, we obtain the recurrence

$$a_{n,k+1,t} = \sum_{i=1}^{n-k+1} a_{n-i,k,t-1} + \sum_{i=2}^{n-k+1}\sum_{j=t}^{n-i} a_{n-i,k,j}, \qquad n \geq k \geq 2, \quad t \geq 1, \tag{6}$$

with the initial values $a_{n,2,1} = 1$ and $a_{n,2,m} = 0$ if $m \geq 2$ for all $n \geq 1$, where $a_{n,k,t} = 0$ if $n < k-1$ or if $t > n$ or if $t = 0$ with $n, k \in \mathbb{P}$.

We now turn to the case $\{1121, 1221\}$. Note that the positions of the $k-1$'s relative to the k's within some member of $P_{n,k}(1121, 1221)$ are given either by

$$\underbrace{(k-1)(k-1)\cdots(k-1)}_{r \text{ times}}\underbrace{kk\cdots k}_{s \text{ times}}$$

or

$$(k-1)k\,\underbrace{(k-1)(k-1)\cdots(k-1)}_{r \text{ times}}\;\underbrace{kk\cdots k}_{s-1 \text{ times}}\,,$$

where r and s are positive integers. We proceed similarly to the case $\{1121, 1212\}$, defining *active site* just as before, with 1221 in place of 1212 in the definition. Make the same replacement in the definition of $a_{n,k,t}$ given above, letting $b_{n,k,t}$ represent the new sequence, and denote the set of partitions that result by $\mathcal{B}_{n,k,t}$. Note that the active sites now correspond to the first occurrences of letters instead of final occurrences. From the definitions, we may write

$$p_{n,k}(1121, 1221) = \sum_{t\geq 1} b_{n,k+1,t},$$

for all $n \geq k \geq 1$.

To complete the proof that $p_{n,k}(1121, 1221) = p_{n,k}(1121, 1212)$, it then suffices to show that $a_{n,k,t} = b_{n,k,t}$ for all possible values, and for this, we show that the sequence $b_{n,k,t}$ satisfies recurrence (6) since it clearly satisfies the same initial conditions. Similar reasoning applies. For the first sum, suppose $\alpha \in \mathcal{B}_{n-i,k,t-1}$, where $i \geq 1$. If we write a 1 before α, write a run of $i-1$ 1's just after its right-most active site, and then add one to each letter, the resulting partition π belongs to $\mathcal{B}_{n,k+1,t}$. Note that all of the active sites in α remain active in π and that an additional site occurs just after the first 2 of π (which is its first letter). The contribution in this case is then $\sum_{i\geq 1} b_{n-i,k,t-1}$.

On the other hand, suppose we add 1's to an active site which is not the right-most one. In this case, we can form $\gamma \in \mathcal{B}_{n,k+1,t}$ by writing a 1 just before $\beta \in \mathcal{B}_{n-i,k,j}$, writing a run of $i-1$ 1's after the $(t-1)$-st left-most active site of β, and then adding one to each letter, where $i \geq 2$ and $j \geq t$. (If $t = 1$, we simply add a run of i 1's just before some member of $\mathcal{B}_{n-i,k,j}$, where $j \geq 1$.) Note that the $t-1$ left-most active sites of β remain active in γ, with an additional site created after the first 2 in γ. Thus, the contribution in this case is $\sum_{i\geq 2}\sum_{j\geq t} b_{n-i,k,j}$, which completes the proof. □

Using recurrence (6), one may derive explicit formulas for the coefficients $a_{n,k,t}$ as follows. First, we replace n by $n-1$ in (6) and subtract to get the recurrence

$$a_{n,k+1,t} = a_{n-1,k+1,t} + a_{n-1,k,t-1} + \sum_{j=t}^{n-2} a_{n-2,k,j}, \qquad n \geq k \geq 2, \quad t \geq 1. \tag{7}$$

If one defines $A_{n,k}(u) = \sum_{t=1}^{n} a_{n,k,t}u^t$, then multiplying (7) by u^t and summing over $t = 1, 2, \dots, n$ yields

$$A_{n,k+1}(u) = A_{n-1,k+1}(u) + uA_{n-1,k}(u) + \frac{u}{1-u}(A_{n-2,k}(1) - A_{n-2,k}(u)), \qquad n \geq k \geq 2. \tag{8}$$

In order to solve this, we define the generating function $A_k(x; u) = \sum_{n \geq k-1} A_{n,k}(u)x^n$. Multiplying (8) by x^n and summing over $n \geq k$ yields

$$A_{k+1}(x; u) = xA_{k+1}(x; u) + uxA_k(x; u) + \frac{ux^2}{1-u}(A_k(x; 1) - A_k(x; u)), \qquad k \geq 2,$$

which is equivalent to

$$A_{k+1}(x; u) = \frac{ux}{1-x}A_k(x; u) + \frac{ux^2}{(1-u)(1-x)}(A_k(x; 1) - A_k(x; u)), \qquad k \geq 2, \tag{9}$$

with the initial value $A_2(x; u) = \frac{ux}{1-x}$. If we define $A_1(x; u) = 1$, then one sees (9) holds for $k = 1$ as well. Now, we define the generating function $A(x, q; u) = \sum_{k \geq 1} A_k(x; u)q^k$. Multiplying (9) by q^{k+1} and summing over $k \geq 1$ yields

$$A(x, q; u) - q = \frac{uxq}{1-x}A(x, q; u) + \frac{ux^2q}{(1-u)(1-x)}(A(x, q; 1) - A(x, q; u)),$$

which is equivalent to

$$\left(1 - \frac{uxq}{1-x} + \frac{ux^2q}{(1-u)(1-x)}\right)A(x, q; u) = q + \frac{ux^2q}{(1-u)(1-x)}A(x, q; 1). \tag{10}$$

This type of functional equation can be solved systematically using the *kernel method* (see [1]). In this case, if we assume that $u = u_0$, where $u_0 = u_0(x, q)$ satisfies $1 - \frac{u_0xq}{1-x} + \frac{u_0x^2q}{(1-u_0)(1-x)} = 0$, i.e.,

$$u_0(x, q) = \frac{(1 + xq)(1 - x) - \sqrt{(1 + xq)^2(1 - x)^2 - 4xq(1 - x)}}{2xq},$$

then

$$\begin{aligned} A(x, q; 1) &= \frac{(1-x)(u_0 - 1)}{u_0x^2} \\ &= \frac{(1 - xq)(1 - x) - \sqrt{[(1 + xq)(1 - x)]^2 - 4xq(1 - x)}}{2x^2}. \end{aligned} \tag{11}$$

(Note that $A(0, 0; 1) = 0$ dictates our choice of root for $u_0(x, q)$ above.)

Observe that $A(x, q; 1) = qf(x; q)$, in accordance with (5), where $f(x; q)$ is given by (4) above. Substituting $q = 1$ into (11) gives the generating function (2). Furthermore, substituting the expression for $A(x, q; 1)$ given in (11) into (10) yields an explicit formula for the generating function $A(x, q; u)$, from which one can recover the coefficients $a_{n,k,t}$.

Remark. Rewriting $A(x,q;1)$ as

$$A(x,q;1)=-\frac{q(1-x)}{x}+\frac{q}{x(1+xq)}C(t),$$

where $t=\frac{xq}{(1-x)(1+xq)^2}$ and $C(t)=\sum_{n\geq 0}C_nt^n=\frac{1-\sqrt{1-4t}}{2t}$, we see that its x^nq^{k+1} coefficient is given by

$$\sum_{i=1}^{k}(-1)^{k-i}C_i\binom{k+i}{2i}\binom{n+i-k}{i-1},$$

for all $n\geq k\geq 1$. Since this is also the value of $p_{n,k}(1121,1212)$, comparison with the expression found in the prior section for the x^nq^k coefficient of $f(x;q)$ yields the following Catalan number identity which seems to be new:

$$\sum_{i=1}^{k}C_i\binom{k+i}{2i}\binom{n-1-k}{i-1}=\sum_{i=1}^{k}(-1)^{k-i}C_i\binom{k+i}{2i}\binom{n+i-k}{i-1},\qquad n,k\in\mathbb{P}. \tag{12}$$

2.3 The case $\{1112, 1123\}$

Let $P_k(x)$ be the generating function for the number of set partitions of $[n]$ having exactly k blocks that avoid the patterns 1112 and 1123, where k is fixed. Suppose $k\geq 2$ and $\pi\in P_{n,k}(1112,1123)$. Then π may be uniquely expressed as $\pi=123\cdots(k-1)\pi'$, where π' is a k-ary word that contains the letter k and avoids 112 and 123. This implies the relation

$$P_k(x)=x^{k-1}(W_k(x)-W_{k-1}(x)),\qquad k\geq 2, \tag{13}$$

where $W_k(x)$ is the generating function for the number of k-ary words of length n that avoid 112 and 123. Note that (13) also holds when $k=1$ since $P_1(x)=\frac{x}{1-x}$.

In order to find an explicit formula for the generating function $W_k(x)$, we refine it as follows. Let $W_{k,m}(x)$ denote the generating function for the number of k-ary words $\pi=\pi_1\pi_2\cdots\pi_n$ of length n that avoid 112 and 123 such that π contains each of the letters $1,2,\ldots,m$ exactly once with these letters occurring from right to left in increasing order.

Our first step in finding an explicit formula for $W_k(x)$ is to establish a recurrence relation for $W_{k,m}(x)$.

LEMMA 2.4 *Let $k\geq 1$. Then $W_{k,k}(x)=x^k$ and for all $m=1,2,\ldots,k-1$,*

$$W_{k,m}(x)=W_{k-1,m}(x)+\frac{1}{1-x}W_{k,m+1}(x)+\sum_{j=1}^{m}\frac{x^{m+1-j}}{(1-x)^{m+2-j}}W_{k-m-1+j,j}(x). \tag{14}$$

Proof. Clearly, $W_{k,k}(x)=x^k$. We find a recurrence for $W_{k,m}(x)$ when $m=1,2,\ldots,k-1$. Let $\pi=\pi^{(m+1)}m\pi^{(m)}(m-1)\cdots\pi^{(2)}1\pi^{(1)}$ denote any k-ary word of length n that avoids 112 and 123 such that π contains each of the letters $1,2,\ldots,m$ exactly once with these letters occurring from right to left in increasing order. We now consider the appearance of the letter $m+1$ in π, distinguishing the following cases:

1. π does not contain the letter $m+1$: The contribution from this case is $W_{k-1,m}(x)$.

2. $\pi^{(m+1)}$ contains the letter $m+1$ exactly once: We consider the following subcases:

   - $\pi^{(j)}$ does not contain the letter $m+1$ for all $j = 2, 3, \ldots, m$ and $\pi^{(1)} = \pi'(m+1)\cdots(m+1)$. Note that π' can only have letters from the set $\{m+2, m+3, \ldots, k\}$. The contribution from this subcase is then $\frac{1}{1-x}W_{k,m+1}(x)$.
   - $\pi^{(j)}$ does not contain the letter $m+1$ for all $j = i+1, i+2, \ldots, m$ and $\pi^{(i)}$ contains the letter $m+1$, which implies that

   $$\begin{aligned}&\pi^{(i)}(i-1)\cdots\pi^{(2)}1\pi^{(1)}\\&= \pi'(m+1)\cdots(m+1)(i-1)(m+1)\cdots(m+1)\cdots1(m+1)\cdots(m+1),\end{aligned}$$

   where π' has letters greater than $m+1$ and the sequence of $m+1$'s directly following it is non-empty. The contribution is then $\frac{x^i}{(1-x)^i}W_{k+1-i,m+2-i}(x)$.

   Thus the total contribution in this case is given by

   $$\frac{1}{1-x}W_{k,m+1}(x) + \sum_{j=2}^{m}\frac{x^{m+2-j}}{(1-x)^{m+2-j}}W_{k-m-1+j,j}(x).$$

3. $\pi^{(m+1)}$ contains the letter $m+1$ at least twice: In this case, one may write π as

   $$\pi = \pi'(m+1)\pi''(m+1)\cdots(m+1)m(m+1)\cdots(m+1)\cdots1(m+1)\cdots(m+1),$$

   where π' and π'' have only letters greater than $m+1$ and all of the sequences of the letter $m+1$ coming after π'' are possibly empty, except for the first. Thus, the contribution for this case is given by $\frac{x^{m+1}}{(1-x)^{m+1}}W_{k-m,1}(x)$.

4. π contains at least one letter $m+1$, but $\pi^{(m+1)}$ does not: Suppose $\pi^{(i)}$ contains $m+1$ for some $i \in [m]$, with i maximal. Then each $\pi^{(j)}$ is a (possibly empty) sequence of the letter $m+1$ for each $j \in [i-1]$. That is, π may be written as

   $$\begin{aligned}\pi = \pi^{(m+1)}m\pi^{(m)}(m-1)\cdots\pi^{(i+1)}i\pi'(m+1)\cdots(m+1)\\(i-1)(m+1)\cdots(m+1)\cdots1(m+1)\cdots(m+1),\end{aligned}$$

   where π', $\pi^{(i+1)}$, $\pi^{(i+2)}$, ..., $\pi^{(m+1)}$ are words on $\{m+2, \ldots, k\}$ and all but the first sequence of $m+1$'s is possibly empty. Thus the total contribution in this case is given by

   $$\sum_{i=1}^{m}\frac{x^i}{(1-x)^i}W_{k-i,m-i+1}(x) = \sum_{i=1}^{m}\frac{x^{m+1-i}}{(1-x)^{m+1-i}}W_{k-m-1+i,i}(x).$$

Adding all the above contributions implies that the generating function $W_{k,m}(x)$ satisfies the recurrence relation

$$\begin{aligned}W_{k,m}(x) &= W_{k-1,m}(x) + \frac{1}{1-x}W_{k,m+1}(x) + \sum_{j=2}^{m}\frac{x^{m+2-j}}{(1-x)^{m+2-j}}W_{k-m-1+j,j}(x)\\ &\quad + \frac{x^{m+1}}{(1-x)^{m+1}}W_{k-m,1}(x) + \sum_{i=1}^{m}\frac{x^{m+1-i}}{(1-x)^{m+1-i}}W_{k-m-1+i,i}(x)\\ &= W_{k-1,m}(x) + \frac{1}{1-x}W_{k,m+1}(x) + \sum_{j=1}^{m}\frac{x^{m+1-j}}{(1-x)^{m+2-j}}W_{k-m-1+j,j}(x),\end{aligned}$$

for all $m = 1, 2, \ldots, k-1$, as required. □

We now define the generating function $W_k(x,u) = \sum_{m=1}^{k} W_{k,m}(x)u^{m-1}$. Multiplying (14) by u^{m-1} and summing over $m = 1, 2, \ldots, k-1$, we obtain

$$\begin{aligned}&W_k(x,u) - x^k u^{k-1}\\ &= W_{k-1}(x,u) + \frac{1}{(1-x)u}\sum_{m=1}^{k-1}W_{k,m+1}(x)u^m + \sum_{m=1}^{k-1}\sum_{j=1}^{m}\frac{x^{m+1-j}}{(1-x)^{m+2-j}}W_{k-m-1+j,j}(x)u^{m-1}\\ &= W_{k-1}(x,u) + \frac{1}{(1-x)u}(W_k(x,u) - W_k(x,0)) + \sum_{j=1}^{k-1}\frac{x^{k-j}}{(1-x)^{k+1-j}}W_j(x,u)u^{k-1-j},\end{aligned}$$

for all $k \geq 1$. In order to solve the above recurrence relation, we define the further generating function $W(x,u,y) = \sum_{k\geq 1} W_k(x,u)y^k$. Multiplying the above recurrence relation by y^k, summing over $k \geq 1$, and interchanging summation yields

$$\begin{aligned}&W(x,u,y) - \frac{xy}{1-xyu}\\ &= yW(x,u,y) + \frac{1}{(1-x)u}(W(x,u,y) - W(x,0,y)) + \frac{\frac{x}{(1-x)^2}}{1-\frac{x}{1-x}yu}\sum_{k\geq 1}W_k(x,u)y^{k+1},\end{aligned}$$

which is equivalent to

$$\begin{aligned}W(x,u,y) &= \frac{xy}{1-xyu} + yW(x,u,y) + \frac{1}{(1-x)u}(W(x,u,y) - W(x,0,y))\\ &\quad + \frac{\frac{xy}{(1-x)^2}}{1-\frac{x}{1-x}yu}W(x,u,y).\end{aligned}$$

We solve this functional equation using the kernel method. If we assume that $u = u_0$, where $u_0 = u_0(x,y)$ satisfies the equation $1 = y + \frac{1}{(1-x)u_0} + \frac{\frac{xy}{(1-x)^2}}{1-\frac{x}{1-x}yu_0}$, i.e.,

$$u_0(x,y) = \frac{(1-x)(1-y) - \sqrt{[(1-x)(1-y)]^2 - 4xy(1-y)}}{2xy(1-y)},$$

then we get

$$W(x,0,y) = \frac{x(1-x)yu_0(x,y)}{1-xyu_0(x,y)}. \tag{15}$$

(Note that $W(0,0,0)=0$ dictates our choice of root for $u_0(x,y)$ above.)

Now we may state the main result of this section.

PROPOSITION 2.5 *The generating function for the number of partitions of $[n]$, $n \geq 0$, that avoid the patterns* 1112 *and* 1123 *is given by*

$$\frac{(1-x)^2 - \sqrt{1-4x+2x^2+x^4}}{2x^2}.$$

Proof. First note that every k-ary word π that avoids the patterns 112 and 123 and contains the letter 1 may be expressed uniquely as $\pi = \pi' 1 \pi'' 11 \cdots 1$, where π' and π'' are k-ary words that avoid the patterns 112 and 123 but do not contain the letter 1. Thus, the generating function $W_k(x)$ satisfies the recurrence

$$W_k(x) = W_{k-1}(x) + \frac{1}{1-x} W_{k,1}(x), \qquad k \geq 1,$$

where $W_0(x) = 1$. Therefore, relation (13) gives

$$P_k(x) = \frac{x^{k-1}}{1-x} W_{k,1}(x) = \frac{x^k}{x(1-x)} W_k(x,0).$$

Summing this over all $k \geq 1$ then yields

$$\sum_{k\geq 1} P_k(x) = \frac{1}{x(1-x)} W(x,0,x),$$

which, by (15), implies

$$1 + \sum_{k\geq 1} P_k(x) = 1 + \frac{(1-x)^2 - \sqrt{1-4x+2x^2+x^4}}{x(1-x^2+\sqrt{1-4x+2x^2+x^4})} = \frac{(1-x)^2 - \sqrt{1-4x+2x^2+x^4}}{2x^2},$$

as required. □

2.4 The case $\{1122, 1123\}$

Suppose the distinct letters in some word α are $a_1 < a_2 < \cdots < a_t$ and that $\alpha = a_{i_1} a_{i_2} \cdots a_{i_s}$ for some positive integers s and t with $1 \leq i_j \leq t$ for each j. Let $r(\alpha)$ denote the word given by $a_{t+1-i_s} a_{t+1-i_{s-1}} \cdots a_{t+1-i_1}$. We now define an explicit bijection ρ between $P_{n,k}(1112, 1123)$ and $P_{n,k}(1122, 1123)$, where we may clearly assume $k \geq 2$. As previously noted, any member of the set $P_{n,k}(1112, 1123)$ can be written uniquely as $\pi = 12 \cdots (k-1)\pi'$, where π' is a k-ary word avoiding the patterns 112 and 123 and containing the letter k. We define $\rho(\pi)$ to be the word $12 \cdots (k-1) r(\pi')$. Note that $r(\pi')$ avoids the patterns 122 and 123 and contains the letter k if and only if π' avoids 112 and 123 and contains k. Since each member of $P_{n,k}(1122, 1123)$ can be expressed uniquely as $12 \cdots (k-1)\pi''$, where π'' is a k-ary word that avoids the patterns 122 and 123 and contains the letter k, we see that ρ is a bijection. Proposition 2.5 then gives

PROPOSITION 2.6 *The generating function for the number of partitions of* $[n]$, $n \geq 0$, *that avoid the patterns* 1122 *and* 1123 *is given by*

$$\frac{(1-x)^2 - \sqrt{1-4x+2x^2+x^4}}{2x^2}.$$

Remark. The bijection above shows further that $p_n(1123, 1\tau) = p_n(1123, 1r(\tau))$, where τ is any classical pattern. The specific case described above is $\tau = 122$.

We conclude by noting that the structure of this paper can be extended to study the number of partitions of $[n]$ avoiding other sets of patterns. In our study, we have already linked the number of set partitions satisfying certain conditions to generalized Catalan numbers (as done in this paper), Fibonacci numbers, Catalan numbers, sequence A054391 in [10] counting a restricted set of permutations, and sequence A005773 in [10] counting Motzkin left factors. These connections we will describe in forthcoming papers.

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