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Combinatorial results for semigroups of order-preserving mappings

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1. *Introduction*

Consider the finite set $X_n = \{1, 2, \ldots, n\}$ ordered in the standard way. Let T_n denote the full transformation semigroup on X_n, that is, the semigroup of all mappings α: $X_n \to X_n$ under composition. We shall call α order-preserving if $i \leqslant j$ implies $i\alpha \leqslant j\alpha$ for $i, j \in X_n$, and α is decreasing if $i\alpha \leqslant i$ for all $i \in X_n$. This paper investigates combinatorial properties of the semigroup $\mathcal{O}_n$ of all order-preserving mappings on X_n, and of its subsemigroup $\mathscr{C}_n$, which consists of all decreasing and order-preserving mappings.

Combinatorial properties of T_n have been studied over a long period under various guises and many interesting and delightful results have emerged (see, e.g. [**7**, **8**]). Two papers concerned with combinatorial results of $\mathcal{O}_n$ are [**11**] and [**3**], while little seems to have been written on $\mathscr{C}_n$. The inverse semigroup of partial one-to-one order-preserving maps has been studied by Garba [**2**]. Nevertheless both $\mathcal{O}_n$ and $\mathscr{C}_n$ have arisen naturally in language theory: any $\mathscr{J}$-trivial finite semigroup divides some $\mathscr{C}_n$ (see [**14**]) while a description of the class of semigroups that are divisors of some $\mathcal{O}_n$ remains an interesting open problem. It comes as a pleasant surprise that many classical results from elementary analysis and combinatorics emerge as keys to our combinatorial problems. For instance it transpires that $|\mathscr{C}_n| = C_n$, the nth Catalan number, and indeed the standard Catalan identity (Result 1·4) follows easily from a natural partition of $\mathscr{C}_n$. Many of our counting tasks can be formulated as certain 'ballot' problems and thus may be solved using André's Reflection Principle (Result 1·3). From single-variable calculus the Wallis Product arises during the calculation of the mean number of orbits of a randomly chosen $\alpha \in \mathcal{O}_n$ and allows us to prove that this number approaches $\frac{1}{2}\surd(n\pi)$ for large n. By way of contrast, the mean number of orbits of $\alpha \in \mathscr{C}_n$ never exceeds 3: we find the distribution of the orbit number explicitly in this case and obtain the limiting distribution.

We list some standard combinatorial results germane to our purposes. We shall write the binomial coefficient $\binom{n}{r}$ also as $C(n, r)$.

RESULT 1·1.
(i) $\sum_{k=0}^{n} C(n,k)\,C(m,k) = C(n+m, m)$ *for* $n \leqslant m$.
(ii) $\sum_{k=0}^{n} C(2k,k)\,C(2n-2k, n-k) = 4^n$.

To see (i), consider a collection of n red and m blue labelled balls. The number of ways of choosing m balls from this collection can be thought of as the sum, from $k = 0$ to n, of the number of ways of choosing k blue balls and $m-k$ red balls, from which the follows the identity.

To prove (ii) one can use the factorization $(1-4x)^{-1} = (1-4x)^{-\frac{1}{2}} \cdot (1-4x)^{-\frac{1}{2}}$. The coefficient of x^n in the binomial expansion of $(1-4x)^{-\frac{1}{2}}$ is $C(2n,n)$, while the corresponding coefficient in the geometric series for $(1-4x)^{-1}$ is 4^n. The coefficient in the product is then the required summation.

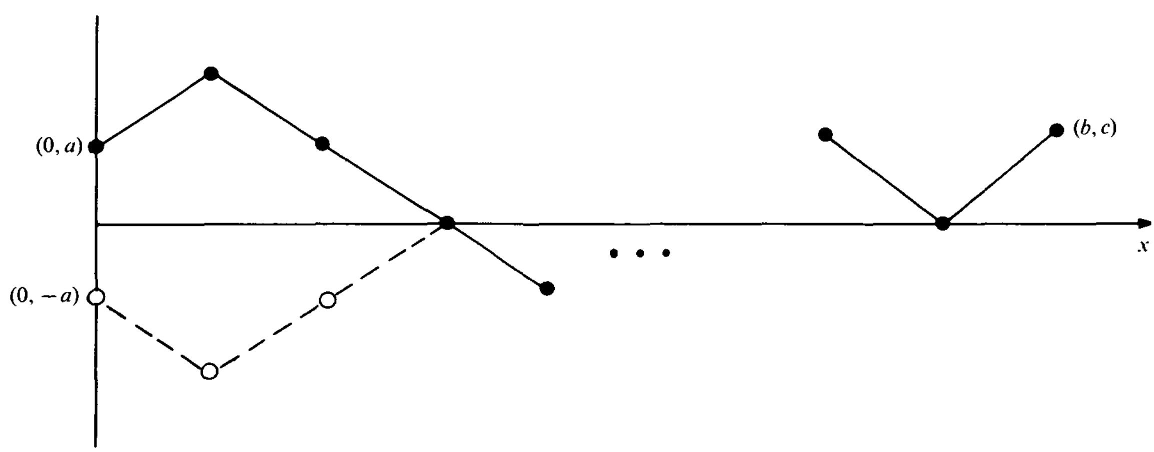
As explained in [**11**], the members of $\mathcal{O}_n$ are in a natural one-to-one correspondence with the sequences of n A's and n B's ending in B: given such a sequence the corresponding mapping $\alpha \in \mathcal{O}_n$ is defined by the rule $i\alpha = j$, where j is the number of the next B in the sequence following the ith A. From this we see

RESULT 1·2. $|\mathcal{O}_n| = C(2n-1, n-1)$.

Deleting the final B in such a sequence gives us an arbitrary sequence of n A's and $(n-1)$ B's which can be thought of as the result of counting a ballot in which the two candidates, A and B, receive n and $n-1$ votes respectively. Such a count can be represented as a lattice path from $(0,0)$ to $(2n-1, 1)$, where each point of the count corresponds to a lattice point (p,q), and the following point on the path is $(p+1, q\pm 1)$ according as A or B polls the next vote. The lattice point (p,q) then signifies a lead of q votes for A after p votes have been counted.

RESULT 1·3 (the reflection principle). *Let a, b and c be positive integers. Then the number of lattice paths from $(0,a)$ to (b,c) which touch or cross the x-axis equals the total number of lattice paths from $(0,-a)$ to (b,c).*

The principle follows from the observation that there is a bijection between the two types of path defined by reflecting in the x-axis the initial segment of any path of the first type up to the point where it first meets the axis (see Figure below).



The bold path represents a typical lattice path from $(0,a)$ to (b,c) which first meets the axis as indicated. The dotted path represents the reflection of the initial segment which, together with the remainder of the path, yields a lattice path from $(0,-a)$ to (b,c). In general there will be paths of either type if and only if there is some positive integer d (representing the number of down-directed edges of the path from $(0,-a)$) such that $2d+a+c=b$, that is, if and only if $b-(a+c)$ is an even positive integer. The number of such lattice paths is then $C(b,d)$, corresponding to the number of choices for selecting the positions of the d down-edges amongst the b edges of the lattice path.

The nth Catalan number C_n is $C(2n,n)/(n+1) = C(2n,n-1)/n$. Historically, C_n

was defined as the number of distinct bracketings of the $(n+1)$-fold product $x_1 x_2 \dots x_{n+1}$. Equally, C_n may be defined as the number of binary trees with n source nodes, and as the number of ways of triangulating a convex $(n+2)$-gon by means of non-intersecting diagonals (see [**10**]). The amenability of Catalan numbers to manipulation stems from the following convolution-style identity.

RESULT 1·4. $\sum_{k=1}^{n} C_{k-1} C_{n-k} = C_n$.

This result can be verified by partition arguments appropriate to the type of collection used to define C_n: we provide a new argument based on partitioning $\mathscr{C}_n$ in Section 3.

Section 2 deals with combinatorial aspects of $\mathcal{O}_n$ while corresponding results for $\mathscr{C}_n$ appear in Section 3. Some results for T_n are stated along the way for the sake of comparison. The content of Section 2 was in part the subject of the author's talk at the Lisbon Conference on Lattices, Semigroups and Universal Algebra, and appears in the proceedings of that meeting [**7**].

Notation. We shall write μ_X, σ_X, σ_X^2 for the mean, standard deviation, and variance respectively of a random variable X (we shall also write $\mathrm{Var}(X)$ for the latter quantity). Each $\alpha \in T_n$ can be pictured as a digraph on n vertices with ij an arc of α if $i\alpha = j$. Each component of such a digraph is functional, meaning that it consists of a unique cycle, together with a number of trees rooted around the points of the cycle. The components of the digraph of α correspond to the orbits of the mapping α, where we say that the orbit of $i \in X_n$ is $\{j \in X_n : i\alpha^r = j\alpha^s$ for some positive integers $r, s\}$. The arrows within a tree rooted on the cycle of some component of α point along the path towards the root of the tree. Hence if we follow the convention that the cycles of α are directed anti-clockwise, then the arrows may be deleted from the picture of α without loss of information, provided that one-cycles (which correspond to fixed points) are shaded to avoid ambiguity. For further background see [**8**].

PROPOSITION 1·5. *The cycles of the components of $\alpha \in \mathcal{O}_n$ each consist of a unique fixed point. Each orbit of $\alpha \in \mathcal{O}_n$ is convex in the ordered set X_n.*

Proof. Suppose that $i, j \in X_n$ are distinct members of some cycle of a component of α, so that $j = i\alpha^r$, $i = j\alpha^s$ for some positive integers r and s. Suppose that $i < i\alpha$. We then obtain an increasing sequence $i, i\alpha, i\alpha^2, \dots, i\alpha^r = j, i\alpha^{r+1}, \dots, i\alpha^{r+s}$. But $i\alpha^{r+s} = j\alpha^s = i$, giving equality throughout, and yielding the contradiction that $i = j$. The same contradiction results if $i > i\alpha$, so that all cycles of the digraph of $\alpha \in \mathcal{O}_n$ are trivial.

Next let C be an orbit of $\alpha \in \mathcal{O}_n$ with (unique) fixed point p say, and let c and d be the respective minimum and maximum members of C. Let r, s be the respective least non-negative integers such that $c\alpha^r = d\alpha^s = p$. Then

$$c < c\alpha < c\alpha^2 < \dots < c\alpha^r = p = d\alpha^s < d\alpha^{s-1} < \dots < d\alpha < d.$$

Let $k \in X_n$ lie between c and d. Then either $c\alpha^m \leqslant k \leqslant c\alpha^{m+1}$ for some $0 \leqslant m \leqslant r-1$, or $d\alpha^m \geqslant k \geqslant d\alpha^{m+1}$ for some $0 \leqslant m \leqslant s-1$. In the first case we obtain

$$p = c\alpha^r = c\alpha^{m+r} \leqslant k\alpha^r \leqslant c\alpha^{m+1+r} = p,$$

which gives $k\alpha^r = p$, so that $k \in C$. A similar argument and conclusion applies in the alternative case, and so we conclude that $C = \{k \in X_n : c \leqslant k \leqslant d\}$, as required.

Remark. It follows from Proposition 1·5 that the digraph of any $\alpha \in \mathcal{O}_n$ is a forest of rooted, labelled trees (each vertex is labelled by a member of X_n, and each component is a tree with a distinguished root vertex in the unique fixed point of the component). It is easy to verify that this is a necessary and sufficient condition on a subsemigroup S of T_n to ensure that S is aperiodic or combinatorial meaning that all the subgroups of S are trivial. This is also equivalent to Green's $\mathscr{H}$ relation being trivial.

2. *The semigroup of order-preserving mappings*

Let $S_r = \{m_1, m_2, \ldots, m_r\}$ with $1 \leqslant m_1 < m_2 < \ldots < m_r \leqslant n$. An order-preserving mapping $\alpha: S_r \to X_n$ is determined by the choice of $p_1, p_2, \ldots, p_r$ such that $1 \leqslant p_1 \leqslant p_2 \leqslant \ldots \leqslant p_r \leqslant n$, where $m_i\alpha = p_i$ $(i = 1, 2, \ldots, r)$. We can then represent α as a sequence of r A's and n B's, consisting of $p_1 - 1$ B's followed by an A; $p_2 - p_1$ B's followed by an A; ...; $p_r - p_{r-1}$ B's followed by an A; and finally $n - p_r + 1$ B's.

For example, if $n = 6$ and $S_r = \{1, 2, 4, 6\}$ then the map

$$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 1 & 3 & 3 & 5 \end{pmatrix}$$

is associated with the sequence *ABBAABBABB*. Any such sequence is determined by the choice of the r places among the $r+n-1$ available for the positioning of the A's. Hence we conclude that the number of order-preserving maps from S_r into X_n is $C(r+n-1, r) = C(r+n-1, n-1)$.

Let $Y_n(\alpha)$ stand for the random variable, the value of which is the rank of α (that is, $|\text{im}\,\alpha|$ where α is a randomly selected member of $\mathcal{O}_n$).

LEMMA 2·1. $\mathbb{E}(Y_n) = n^2/(2n-1)$, $\sigma_{Y_n} = \sqrt{((n-1)/2)} \cdot n/(2n-1)$.

Proof. We first show that the number of order-preserving mappings $\alpha: S_r \to X_n$ such that $\text{rank}\,\alpha = k$ is given by $C(n,k) \cdot C(r-1, k-1)$ $(1 \leqslant k \leqslant r)$: the first factor counts the choices for $\text{im}\,\alpha = \{i_1, i_2, \ldots, i_k\}$ say, while the second counts the choices for the set $P_\alpha = \{p_1, p_2, \ldots, p_k\}$, where p_j is the greatest member of S_r such that $p_j\alpha = i_j$ $(1 \leqslant j \leqslant k)$. (Note that p_k must equal m_r, and thus $p_j < m_r$, for all $j < k$). Since α is determined by the choice of the pair $(\text{im}\,\alpha, P_\alpha)$ and different pairs determine different mappings, the assertion is established.

Now take $\alpha \in \mathcal{O}_n$ with $\text{rank}\,\alpha = r$, while β is a randomly chosen member of $\mathcal{O}_n$. Denote the random variable with value $\text{rank}\,\alpha\beta$ by $Y_{n,r}$. From the foregoing we obtain

$$\Pr(Y_{n,r} = k) = \frac{C(n,k)\,C(r-1,k-1)}{C(r+n-1,n-1)}. \tag{2·2}$$

(Taking $r = n$ thus gives the distribution of Y_n.) Hence

$$\mathbb{E}(Y_{n,r}) = (C(r+n, n-1))^{-1} \sum_{k=1}^{r} kC(n,k)\,C(r-1,k-1)$$

$$= \frac{n}{C(r+n-1,n-1)} \sum_{k=0}^{r-1} C(n-1,k)\,C(r-1,k).$$

By Result 1·1 (i) this equals

$$\frac{n}{C(r+n-1, n-1)} \cdot C(r+n-2, n-1) = \frac{nr}{n+r-1}. \tag{2·3}$$

If one now takes $r = n$, (2·3) yields the stated result for $\mathbb{E}(Y_n) = \mathbb{E}(Y_{n,n})$. Again by using (2·2) one obtains

$$\begin{aligned}\mathbb{E}(Y_{n,r}(Y_{n,r}-1)) &= C(r+n-1, n-1)^{-1} \sum_{k=2}^{r} k(k-1)\, C(n,k)\, C(r-1,k-1) \\ &= \frac{n(r-1)}{C(r+n-1,n-1)} \sum_{k=1}^{r-1} C(n-1,k)\, C(r-2,k-1).\end{aligned}$$

We re-write this using the identity

$$C(r-2, k-1) = C(r-1,k) - C(r-2,k).$$

We obtain, using Result 1·1 (i),

$$\begin{aligned}&\mathbb{E}(Y_{n,r}(Y_{n,r}-1)) \\ &= \frac{n(r-1)}{C(r+n-1,n-1)} \left(\sum_{k=1}^{r-1} C(n-1,k)\, C(r-1,k) - \sum_{k=1}^{r-1} C(n-1,k)\, C(r-2,k) \right) \\ &= \frac{n(r-1)\,(C(n+r-2,r-1)-1-(C(n+r-3,r-2)-1))}{C(r+n,n-1)} \\ &= \frac{n(n-1)\,r(r-1)}{(n+r-1)\,(n+r-2)}.\end{aligned}$$

Then from the general fact that $\sigma_X^2 = \mathbb{E}(X(X-1)) - \mathbb{E}(X)\,(\mathbb{E}(X)-1)$ we obtain

$$\operatorname{Var}(Y_{n,r}) = \frac{n(n-1)\,r(r-1)}{(n+r-1)^2\,(n+r-2)} \tag{2·4}$$

and in particular $\operatorname{Var}(Y_{n,r}) < n$ for all values of $r = 1, 2, \ldots, n$. The value of the standard deviation of Y_n is obtained by putting $r = n$ and taking square roots, thus completing the proof.

THEOREM 2·5. *Let Z_k be the random variable with value $n^{-1}\operatorname{rank}(\alpha_1 \alpha_2 \ldots \alpha_k)$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are randomly selected members of $\mathcal{O}_n$. Then*

$$M_k = \lim_{n\to\infty} \mathbb{E}(Z_k) = 1/(1+k) \quad \textit{for all } k \geqslant 1.$$

Remark. If we replace $\mathcal{O}_n$ by T_n in this result then we obtain $M_k \sim 2/k$: see [**6**].

Proof. We prove by induction on k both the stated value for M_k and that $\sigma_{Z_k} \to 0$ as $n \to \infty$; that this is true for $k = 1$ follows from Lemma 2·1. Assume that our claim holds for some arbitrary value $k-1 \geqslant 1$ and let $\delta, \epsilon > 0$ be given. From Chebyshev's Inequality, for any positive number R we have

$$\Pr(|Z_{k-1} - \mu_{k-1}| \geqslant R\sigma_{Z_{k-1}}) \leqslant 1/R^2, \quad \text{where} \quad \mu_{k-1} = \mathbb{E}(Z_{k-1}).$$

Choose R so that $1/R^2 < \epsilon$. Then choose N sufficiently large so that $R\sigma_{Z_{k-1}} < \delta/2$ and $|\mu_{k-1} - M_{k-1}| < \delta/2$ for all $n \geqslant N$. Thus, for all $n \geqslant N$, $|Z_{k-1} - M_{k-1}| \geqslant \delta$ implies that $|Z_{k-1} - \mu_{k-1}| + |\mu_{k-1} - M_{k-1}| \geqslant \delta$, and hence $|Z_{k-1} - \mu_{k-1}| \geqslant \delta/2$. Therefore

$$\Pr(|Z_{k-1} - M_{k-1}| \geqslant \delta) \leqslant \Pr(|Z_{k-1} - \mu_{k-1}| \geqslant \delta/2) < \epsilon$$

for all $n \geqslant N$. We conclude that $\Pr(|Z_{k-1} - M_{k-1}| < \delta) \geqslant 1 - \epsilon$ for all sufficiently large values of n. Now, by (2·3),

$$\mathbb{E}(Z_k \mid Z_{k-1} = r/n) = \frac{r}{(n+r-1)} = \frac{(r/n)}{(1+r/n-1/n)},$$

and this is increasing in r. Hence

$$\mathbb{E}(Z_k) = \sum_{r=1}^{n} \Pr(Z_{k-1} = r/n)\, \mathbb{E}(Z_k \mid Z_{k-1} = r/n)$$

$$\geqslant \sum_{r/n \geqslant M_{k-1}-\delta} \Pr(Z_{k-1} = r/n)\, \mathbb{E}(Z_k \mid Z_{k-1} = M_{k-1} - \delta), \tag{2·6}$$

so that

$$\mathbb{E}(Z_k) \geqslant \frac{n(1-\epsilon)(1/k-\delta)}{n+n(1/k-\delta)-1} \to \frac{(1-\epsilon)(1/k-\delta)}{1+1/k-\delta} = \frac{(1-\epsilon)(1-k\delta)}{1+k-k\delta} \quad \text{as } n \to \infty, \tag{2·7}$$

which approaches $1/(1+k)$ as $\delta, \epsilon \to 0$. Similarly, splitting the sum in (2·6) into two sums corresponding to $r/n \leqslant M_{k-1} + \delta$ and $r/n > M_{k-1} + \delta$, and taking appropriate bounds yields

$$\mathbb{E}(Z_k) \leqslant \frac{n(1/k+\delta)}{n+n(1/k+\delta)-1} + \epsilon \cdot 1 \to \frac{1+k\delta}{1+k+k\delta} + \epsilon \tag{2·8}$$

which also approaches $1/(1+k)$ as $\delta, \epsilon \to 0$ and (2·7) and (2·8) together yield that $M_k = 1/(1+k)$. We complete the proof by verifying inductively that $\sigma_{Z_k} \to 0$ as $n \to \infty$.

Write $W_k = nZ_k$, $p_i = \Pr(W_k = i)$, $p_{ij} = \Pr(W_k = i \mid W_{k-1} = j)$, $q_j = \Pr(W_{k-1} = j)$, and $\mu_j = \mathbb{E}(W_k \mid W_{k-1} = j)$ for $1 \leqslant i, j \leqslant n$. For the moment let a be arbitrary and consider

$$\sum_{i=1}^{n} p_i (i-a)^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij} q_j \right) (i-a)^2 = \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_{ij} (i-a)^2 \right).$$

Write $i - a$ as $i - \mu_j + \mu_j - a$ and expand to obtain

$$\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_{ij} (i-\mu_j)^2 + 2 \sum_{i=1}^{n} p_{ij} (i-\mu_j)(\mu_j - a) + \sum_{i=1}^{n} p_{ij} (a-\mu_j)^2 \right). \tag{2·9}$$

Now $\sum_{i=1}^{n} p_{ij} (i-\mu_j)^2 = \sigma^2_{Y_{n,j}} = O(n)$ by (2·4). Also $\sum_{i=1}^{n} p_{ij}(i-\mu_j)$ is identically zero, so that the sum of the first two terms in (2·9) is $O(n)$. The remaining term equals

$$\sum_{j=1}^{n} \left(q_j (a-\mu_j)^2 \sum_{i=1}^{n} p_{ij} \right) = \sum_{j=1}^{n} q_j (a-\mu_j)^2. \tag{2·10}$$

By Chebyshev's Inequality, for any positive value of R,

$$\Pr(|W_{k-1} - \mu_{k-1}| \geqslant R\sigma_{W_{k-1}}) \leqslant 1/R^2.$$

Let $\epsilon > 0$ be arbitrary, and take R such that $1/R^2 < \epsilon$. Since $R\sigma_{W_{k-1}} = o(n)$, it follows that there exists intervals I_n of length $o(n)$ such that the probability that W_{k-1} lies outside I_n is less than ϵ. Take $i = i(n)$ to be a fixed member of $I_n \cap X_n$, and put $a = \mathbb{E}(W_k \mid W_{k-1} = i)$. Now recall that

$$\mathbb{E}(W_k|W_{k-1} = r) = \frac{rn}{(r+n-1)} = f(r) \quad \text{say,}$$

$$|f'(r)| = \frac{n(n-1)}{(n+r-1)^2} < 1 \quad \text{for all } r \geqslant 1.$$

Hence $|a - \mathbb{E}(W_k \mid W_{k-1} = j)| \leqslant |i-j| = o(n)$ for all $j \in I_n \cap X_n$. Therefore the summation Σ of (2·10) is such that

$$\Sigma = \sum_{j\in I_n} \Pr(W_{k-1} = j)(a - E(W_k \mid W_{k-1} = j))^2 + \sum_{j\notin I_n} \Pr(W_{k-1} = j)(a - E(W_k \mid W_{k-1} = j))^2$$
$$\leqslant o(n^2) + \epsilon \cdot n^2.$$

Therefore we obtain that

$$\sigma^2_{W_k} \leqslant \sum_{i=1}^{n} p_i(i-a)^2 \leqslant O(n) + o(n^2) + \epsilon \cdot n^2,$$

and hence that $\sigma^2_{Z_k} \leqslant o(1) + \epsilon$, and since ϵ was arbitrary, it follows that $\sigma^2_{Z_k} = o(1)$, whence $\sigma_{Z_k} = o(1)$, as required.

Let $Z_n(\alpha)$ now denote the random variable the value of which is the number of fixed points of a randomly selected $\alpha \in \mathcal{O}_n$. By Proposition 1·5, $Z_n(\alpha)$ also equals the component number of the digraph of α, and also the number of cycle points.

The number of mappings $\alpha \in \mathcal{O}_n$ with $k\alpha = k$ for all $k \in X_n$ is the product of the number of order-preserving mappings from X_{k-1} to X_k and the number of such maps from X_{n-k} to X_{n-k+1}. By the remark preceding the proof of Lemma 2·1 the number of order-preserving maps from a set of order r to a set of order n is $C(r+n-1, n-1)$, whence we conclude that

$$\Pr(k\alpha = k) = \frac{C(2k-2, k-1)\, C(2n-2k, n-k)}{C(2n-1, n-1)}.$$

Hence

$$\begin{aligned}\mathbb{E}(Z_n) &= \sum_{k=1}^{n} \Pr(k\alpha = k) \\ &= C(2n-1, n-1)^{-1} \sum_{k=1}^{n} C(2k-2, k-1)\, C(2n-2k, n-k) \\ &= C(2n-1, n-1)^{-1} \sum_{k=0}^{m} C(2k, k)\, C(2m-2k, m-k),\end{aligned}$$

where $m = n-1$. From Result 1·1 (ii) one now obtains the first statement in the next result.

THEOREM 2·11. $\mathbb{E}(Z_n) = 4^{n-1}/C(2n-1, n-1)$. *Moreover* $(\mathbb{E}(Z_n) - \sqrt{(n\pi)}/2) \to 0$ *as* $n \to \infty$.

Proof. To verify that the mean is of order $\sqrt{(n\pi)}/2$ one readily shows that

$$\frac{2(\mathbb{E}(Z_n))^2}{n} = \frac{(2\cdot 4\cdot 6\ldots(2n-2))^2\cdot 2n}{(1\cdot 3\cdot 5\ldots(2n-1))^2}.$$

This is the product of Wallis which approaches $\pi/2$, from which we infer that $\mathbb{E}(Z_n) \sim \sqrt{(n\pi)}/2$. In the course of the proof of the Wallis limit it is established that

$$\frac{(2\cdot 4\cdot 6\ldots 2n)^2}{(1\cdot 3\cdot 5\ldots(2n-1))^2(2n+1)} \leqslant \frac{\pi}{2} \leqslant \frac{(2\cdot 4\cdot 6\ldots 2n)^2}{(1\cdot 3\cdot 5\ldots(2n-1))^2\, 2n}$$

from which we obtain that

$$\frac{4\mathbb{E}^2(Z_n)}{2n+1} \leqslant \frac{\pi}{2} \leqslant \frac{4\mathbb{E}^2(Z_n)}{2n}.$$

This yields
$$0 \leqslant 4\mathbb{E}^2(Z_n) - n\pi \leqslant 4\mathbb{E}^2(Z_n)/(2n+1).$$

We deduce from above that the ratio on the right approaches $\pi/2$, and thus

$$4\mathbb{E}^2(Z_n) - n\pi = O(1),$$

from which it follows that $\mathbb{E}(Z_n) - \sqrt{(n\pi)}/2 \to 0$.

Remarks. The closest analogue of this result for T_n concerns the random variable $X_n(\alpha)\,(\alpha\in T_n)$, the value of which is the order of the stable range to α (which corresponds to the number of cycle points of the digraph of α). It is shown in [**9**] that

$$(\sqrt{(n\pi/2)} - \mathbb{E}(X_n)) \to \tfrac{1}{3}.$$

The proof of this result is based on Stirling's factorial approximation formula and a result of Ramanujan concerning the 'first half' of the McLaurin series for e^n. By contrast, the component number of $\alpha\in T_n$ has a mean that only increases logarithmically with n. It is shown in [**9**] that if $C_n(\alpha)$ denotes the number of components of a randomly selected $\alpha\in T_n$ then

$$\mathbb{E}(C_n) - \tfrac{1}{2}\log n \to \tfrac{1}{2}(\gamma + \log 2)$$

where γ is the Euler–Mascheroni constant. The first proof of this fact is due to Kruskal[**13**]. From Lemma 2·1 it follows readily that the mean value of $Y_n(\alpha)$ (the rank of a random $\alpha\in\mathcal{O}_n$) satisfies

$$(\mathbb{E}(Y_n) - n/2)\downarrow\tfrac{1}{4}.$$

The corresponding result for $R_n(\alpha)$, the rank of a random $\alpha\in T_n$, is noted in [**9**]:

$$\mathbb{E}(R_n) - n(1-e^{-1}) \to e^{-1}/2.$$

Katz[**12**] considered the problem of finding the probability of 'indecomposability' of a randomly selected $\alpha\in T_n$, by which was meant the probability that α has only one component. He proved that this probability is of order $\sqrt{(\pi/2n)}$. We shall now solve the corresponding problem for $\mathcal{O}_n$. The 'ballot' argument involved also furnishes us with the value of $|\mathscr{C}_n|$.

Recall from the first section the natural one-to-one correspondence between $\alpha\in\mathcal{O}_n$ and a ballot in which candidate A polls n votes while candidate B polls $n-1$ votes. Observe that $i\in X_n$ is a fixed point of $\alpha\in\mathcal{O}_n$ if and only if in the count of the corresponding ballot the ith A appears between the $(i-1)$th and the ith B. In terms of the count, this corresponds to a tie after $2i-2$ votes, with A polling the next vote. Now to say that $\alpha \in \mathcal{O}_n$ is indecomposable means exactly that α has a unique fixed point, which, formulated in terms of the corresponding ballot count, means precisely that A never surrenders the lead once he has taken it (i.e. once A is strictly ahead in the count, he stays strictly ahead).

We consider a more general ballot in which A polls $n+r$ votes while his opponent B polls n votes $(r \geqslant 1)$, and we seek the probability that A never relinquishes the lead once he has gained it. Let Y_1 denote the set of ballots of this type, let Y_2 denote the set of ballots in which A never trails at any point in the count, and let $Y = Y_1 \cup Y_2$. Consider $\alpha \in Y$, and let $(p, 0)$ be the *final* point on the lattice path of α where the count is tied. Let α' be the count of the lattice path which results from reflection of that portion of the path of α from $(0,0)$ to $(p, 0)$ in the x-axis. Then $\alpha' \in Y$ and we observe that the mapping $': Y \to Y$ defines a bijection between Y_1 and Y_2 with $(\alpha')' = \alpha$. In fact, $'$ defines a bijection between $Y_1 \backslash Y_2$ and $Y_2 \backslash Y_1$ as $Y_1 \cap Y_2$ consists precisely of all those counts where A leads throughout, which coincides with the set of fixed points of the mapping $'$.

We next find $|Y_2| \,(= |Y_1|)$ by calculating the order of the complementary set of counts, which consists of all those counts during which A trails B at some point. The number of paths that correspond to counts where A falls behind at some point equals the number of lattice paths from $(0,0)$ to $(2n+r, r)$ which touch or cross the line $y = -1$. By the Reflection Principle (Result 1·3) this equals the number of lattice paths from $(-2, 0)$ to $(2n+r, r)$, which is $C(2n+r, n-1)$; the $n-1$ corresponds to the $n-1$ places where the path dips, leaving $n+r+1$ rises. Hence the probability that A trails B at some time during the count is $C(2n+r, n-1)/C(2n+r, n) = n/(n+r+1)$. Hence the probability of the complementary event that A never trails is $1-n/(n+r+1) = (r+1)/(n+r+1)$, and this equals the probability that A never loses his lead once he has gained it. By taking $r = 1$ and replacing n by $n-1$ we obtain our required result.

THEOREM 2·12. *The probability that $\alpha \in \mathcal{O}_n$ is indecomposable is* $2/(n+1)$.

3. *The semigroup of all decreasing and order-preserving mappings*

We saw in the preceding section that a mapping $\alpha \in \mathcal{O}_n$ is indecomposable if and only if in the corresponding ballot count the winning candidate A never relinquishes his lead once he has gained it, and that this collection of counts is in bijection with the set of counts where A never trails his opponent. We now ask what type of mapping this latter set of counts represents. Candidate A never trails in the count if and only if the ith vote for A appears before the ith vote for B for $i = 1, 2, \ldots, n$; in other words, the number of votes for A preceding the ith for B in the sequence is always greater than or equal to i. In the corresponding mapping, since each $i \in X_n$ is mapped to the number of the next B following the ith A in the sequence, this condition is equivalent to $i\alpha \leqslant i$ for all $i \in X_n$. Therefore the collection of mappings corresponding to the set of counts in which the winning candidate never trails is $\mathcal{C}_n$, the semigroup of order-preserving and decreasing mappings, and $|\mathcal{C}_n|$ equals the cardinality of the set of all indecomposable mappings in $\mathcal{O}_n$. This yields our next result.

THEOREM 3·1. *$|\mathscr{C}_n| = C_n$, the nth Catalan number.*

Proof. From the preceding argument, together with Theorem 2·12 and Result 1·2, we have

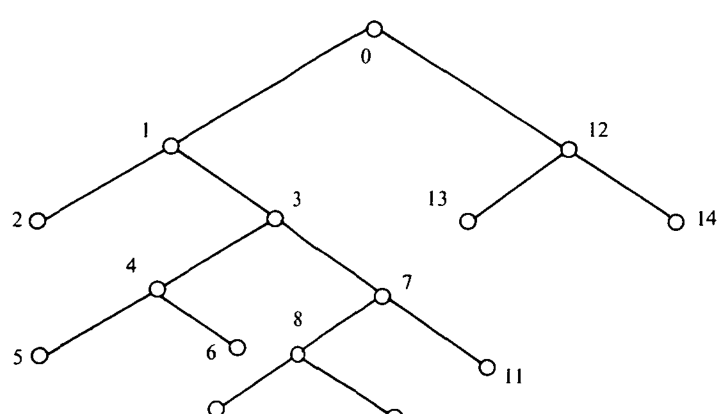
$$|\mathscr{C}_n| = |\{\alpha \in \mathcal{O}_n : \alpha \text{ indecomposable}\}|, = \frac{2}{n+1}\binom{2n-1}{n-1} = \frac{1}{n+1}\binom{2n}{n},$$

which is the nth Catalan number.

Remarks. For each collection of counts of a given ballot there corresponds a dual collection obtained by reversing the count sequence of the ballot. Given a count where A never relinquishes his lead once he has gained it, the dual count has the property that A never relinquishes his eventual winning lead once he has taken it. In our case, since the winning margin for A is one, the characteristic conditions of both types of count are identical, so that the set of indecomposable mappings of $\mathcal{O}_n$ is self-dual under this correspondence. On the other hand, the dual of the condition that A never trails in the count is that A's lead never exceeds his eventually winning margin, which in our case is tantamount to saying that A's lead never exceeds one. This is to say that the number of the next B to follow the ith vote for A in the count is at least i, or in terms of the corresponding mapping α, $i\alpha \geqslant i$ for all $i \in X_n$. Hence the dual to $\mathscr{C}_n$ under this correspondence is the isomorphic semigroup of all increasing and order-preserving mappings on X_n.

There are natural bijections between $\mathscr{C}_n$ and other standard collections of order C_n which thus yield alternative proofs of Theorem 3·1. We give the specific connection between $\mathscr{C}_n$ and the set of all binary trees on n source nodes. In this context it is more natural to take the base set X_n to be $\{0, 1, 2, \ldots, n-1\}$.

By a binary tree we shall mean complete binary tree where every node u is either a source node, meaning that there are exactly two arcs from u leading to its two sons, or that u is an endpoint or leaf. Every node except the uppermost root of the tree is the son of exactly one source node, so that it follows that a binary tree with n source nodes has $2n+1$ nodes in all. We shall also assume that our binary trees are ordered, in the sense that the nodes are consecutively numbered in the order in which they are met while carrying out an anti-clockwise search of the tree, beginning at the root. An example of a binary tree with seven source nodes is given below.



Given a binary tree B on n source nodes (and so with $n+1$ leaf nodes) we associate a mapping $\alpha \in \mathscr{C}_n$ by the rule that $i \mapsto t$, where t leaf nodes precede the ith source node in the ordering of the nodes of B. To see that $\alpha \in \mathscr{O}_n$ observe that our rule certainly defines an order-preserving map on X_n, and since the final source node precedes its two sons in the ordering of the nodes, the maximum possible value of $(n-1)\alpha$ is $n+1-2 = n-1$. We can then prove by induction on n that α also has the decreasing property: remove the two sons of the final source node from B to give a binary tree B' on $n-1$ source nodes, which, by induction, we may assume is associated with a member of $\mathscr{C}_{n-1}$ by the above rule. Indeed, the mapping α' associated with B' is just the restriction of the mapping α to X_{n-1}, and so, by the inductive hypothesis, we have that $\alpha | X_{n-1}$ is decreasing, and we have already observed that $(n-1)\alpha \leqslant n-1$, allowing us to conclude that $\alpha \in \mathscr{C}_n$. The mapping $\alpha \in \mathscr{C}_7$ associated with our binary tree above has its ordered list of images of the members of X_7 the tuple $(0, 0, 1, 1, 3, 3, 6)$.

Conversely, we can associate with any $\alpha \in \mathscr{C}_n$ a binary tree B on n source nodes using the inverse of the foregoing procedure. Suppose inductively that we have associated with $\alpha' = \alpha | X_{n-1}$ a binary tree B' on $n-1$ source nodes in such a way that B' is associated with α' by the procedure of the preceding paragraph. Let k denote $(n-1)\alpha$. The tree B' has n leaf nodes. Take the $(k+1)$th leaf node (with $k+1 \leqslant n$), and replace it by a source node to form the binary tree B. By construction, the mapping associated with B is α, as required.

The number of indecomposable $\alpha \in \mathscr{C}_n$ is easily obtained through our ballot correspondence. In terms of the corresponding count,

(i) $\alpha \in \mathscr{C}_n$ if and only if A never trails B in the count,

(ii) α is indecomposable if and only if A never relinquishes the lead once gained.

The conjunction of this pair of conditions is equivalent to the single constraint that A leads the count throughout. The enumeration of such counts is the classical Bertrand–Whitworth ballot problem, readily solved by the Reflection Principle (see [**1**] or [**10**]), and in general, the probability of such a ballot is $r/(2n+r)$. Again, upon putting $r = 1$ and replacing n by $n-1$ we see that the proportion of such mappings in $\mathscr{C}_n$ is

$$\frac{1}{2n-1}\binom{2n-1}{n-1} = C_{n-1}.$$

This yields

Proposition 3·2. *The proportion of indecomposable mappings in $\mathscr{C}_n$ is*

$$C_{n-1}/C_n = (n+1)/(2(2n-1)) \downarrow \tfrac{1}{4} \quad \textit{as } n \to \infty.$$

It is interesting here that the probability decreases to $\frac{1}{4}$, and not to 0, as is the case with $\mathscr{O}_n$ and T_n. This suggests the existence of a limiting distribution for $Z_n(\alpha)$, where $Z_n(\alpha)$ denotes the component number of a randomly chosen $\alpha \in \mathscr{C}_n$. Our determination of this limiting distribution involves the use of the standard Catalan identity for which we can now offer a new proof.

Proof of Result 1·4. Partition $\mathscr{C}_n$ into n disjoint sets $K_1, K_2, \ldots, K_n$, where $\alpha \in K_k$ if k is the greatest integer such that $k\alpha = k$. Note that $\alpha \in K_1$ if and only if 1 is the unique fixed point of α, which is to say that α is indecomposable. Then

$$|\mathscr{C}_n| = \sum_{k=1}^{n} |K_k|.$$

Now $|K_k| = |\mathscr{C}_{k-1}|\,|\{\alpha\in\mathscr{C}_{n-k+1}: \alpha \text{ indecomposable}\}| = C_{k-1}C_{n-k}$, since, in general, the number of indecomposable mappings in C_n is C_{n-1}. Hence

$$|\mathscr{C}_n| = \sum_{k=1}^{n} C_{k-1}C_{n-k}.$$

We can use this result to show that the expected number of components of a random $\alpha\in\mathscr{C}_n$ is a monotonic function of n increasing to a limit of 3. This contrasts with Theorem 2·11 which shows that the mean component number in $\mathscr{O}_n$ is of order $\sqrt{n}$.

PROPOSITION 3·3. *$\mathbb{E}(Z_n) = 3n/(n+2)$ where $Z_n(\alpha)$ is the component number of a random $\alpha\in\mathscr{C}_n$.*

Proof. We have

$$\begin{aligned}\mathbb{E}(Z_n) &= \sum_{k=1}^{n} \Pr(k\alpha = k) = C_n^{-1}\sum_{k=1}^{n} C_{k-1}C_{n+1-k}\\ &= C_n^{-1}\left(\sum_{k=1}^{n+1} C_{k-1}C_{n+1-k} - C_nC_0\right) = (C_{n+1}-C_n)/C_n\\ &= \frac{2(2n+1)}{n+2} - 1 = \frac{3n}{n+2}\uparrow 3.\end{aligned}$$

Let $N(n,k) = |\{\alpha\in\mathscr{C}_n: \alpha \text{ has exactly } k \text{ fixed points}\}|$.

LEMMA 3·4.

$$N(n,k) = \frac{k}{2n-k}\binom{2n-k}{n}, \quad \textit{for } k = 1,2,\ldots,n.$$

Proof. Using Proposition 3·2 we have

$$\begin{aligned}N(n,1) &= |\{\alpha\in\mathscr{C}_n: \alpha \text{ indecomposable}\}| = C_{n-1}\\ &= (1/n)\,C(2n-2,n-1) = (1/(2n-1)\,C(2n-1,n),\end{aligned}$$

as asserted. Using the argument of the proof of Result 1·4, we focus attention on the final fixed point of a mapping in $\mathscr{C}_n$ and conclude that

$$\begin{aligned}N(n,2) &= \sum_{k=1}^{n-1} N(k,1)N(n-k,1) = \sum_{k=1}^{n-1} C_{k-1}C_{n-k-1},\\ &= \sum_{k=1}^{n-1} C_{k-1}C_{(n-1)-k} = C_{n-1}\end{aligned}$$

by Result 1·4, and $C_{n-1} = (2/(2n-2))\,C(2n-2,N)$, as required.

We next establish the recurrence

$$N(n,k+1) = N(n,k) - N(n-1,k-1) \quad \text{for } k = 1,2,\ldots,n-1. \tag{3·5}$$

The above calculation shows that (3·5) is valid for $k = 1$ (as $N(n,0) = 0$). Take $k\geqslant 2$. By considering the final fixed point t of $\alpha\in\mathscr{C}_n$ with exactly $k+1$ fixed points we see that

$$N(n,k+1) = \sum_{t=k+1}^{n} N(t-1,k)N(n-t+1,1). \tag{3·6}$$

By considering the penultimate fixed point t we also have

$$N(n,k+1) = \sum_{t=k}^{n-1} N(t-1,k-1)N(n-t+1,2). \tag{3·7}$$

But we know that $N(n-t+1,2) = N(n-t+1,1)$, and so, by substituting accordingly in (3·7), we obtain

$$N(n,k+1) = \sum_{t=k}^{n} N(t-1,k-1)N(n-t+1,1) - N(n-1,k-1)N(1,1).$$

Therefore $N(n,k+1) = N(n,k) - N(n-1,k-1)$, as asserted.

It is now a routine matter, using a double induction on n and k, to verify that the solution to our recurrence is given by

$$N(n,k) = (k/(2n-k))\,C(2n-k,n) \quad \text{for } k = 1,2,\ldots,n.$$

THEOREM 3·8. *The probability distribution for the random variable $Z_n(\alpha)$, the component number of a random $\alpha \in \mathscr{C}_n$ is given by*

$$\Pr(Z_n = k) = p(n,k) = \frac{k(n+1)(n-1)(n-2)\ldots(n-k+1)}{2(2n-1)(2n-2)\ldots(2n-k)} \quad \textit{for } k = 1,2,\ldots,n. \tag{3·9}$$

As n increases, Z_n approaches in distribution and in moments the random variable Z, where $p_k = \Pr(Z = k) = k/2^{k+1}$ for $k = 1,2,\ldots$. In particular, the limiting values of the mean and variance are 3 *and* 2 *respectively. Furthermore, for $n > 1$, the probability sequence is bi-modal, with modes $k = 1$ and $k = 2$, and is strictly monotonically decreasing for $k > 2$, features shared with the limiting distribution Z.*

Proof. Except for the statement about moments, this is now immediate: from Theorem 3·1 and Lemma 3·4 we obtain

$$p(n,k) = \frac{N(n,k)}{C_n} = \frac{k}{2n-k}C(2n-k,n)\bigg/\left(\frac{1}{n+1}C(2n,n)\right),$$

which simplifies to yield the expression (3·9). That $\lim_{n\to\infty} p(n,k) = k/2^{k+1}$ follows on dividing each of the last k terms in the numerator and denominator by n, so that Z_n approaches Z in distribution. From (3·9) we deduce the recurrence

$$p(n,k+1) = p(n,k)\cdot\frac{(k+1)(n-k)}{k(2n-k-1)} \quad \text{for } k = 1,2,\ldots,n-1 \tag{3·10}$$

from which the statements on modes and on monotonicity follow. The statement on convergence in moments can be deduced from the following fact.

LEMMA 3·11. *If $n > 2$ then $p(n,k) > p_k$ for $k = 1,2,3$ and $p(n,k) < p_k$ for $k \geqslant 4$.*

Proof. Direct calculation using (3·9) allows one to that the assertions of the Lemma for $k = 1,2,3$ and 4. It remains only to observe that if $p(n,k) < p_k$ for some k, then (3·10) yields

$$p(n,k+1) < p_{k+1}\frac{2(n-k)}{(2n-k-1)},$$

from which we infer that $p(n,k+1) < p_{k+1}$. The result now follows by induction on k. To show convergence in moments, first observe that $\mathbb{E}(Z^t)$ certainly exists for

$t = 1, 2, \ldots$ (indeed the values of $\mathbb{E}(Z^t)$ are integers, and can be calculated recursively). Let t be a positive integer, and let $\epsilon > 0$ be given. Choose $N \geqslant 3$ such that

$$\sum_{k=N+1}^{\infty} k^t p_k < \epsilon/3.$$

Then, for any positive integer n,

$$\begin{aligned}|\mathbb{E}(Z^t) - \mathbb{E}(Z_n^t)| &\leqslant \left|\sum_{k=1}^{N} k^t(p_k - p(n,k))\right| + \sum_{k=N+1}^{\infty} k^t(p_k - p(n,k)) \\ &\leqslant \left|\sum_{k=1}^{N} k^t(p_k - p(n,k))\right| + \sum_{k=N+1}^{\infty} k^t p_k + \sum_{k=N+1}^{\infty} k^t p(n,k).\end{aligned}$$

Since Z_n approaches Z in distribution, it follows that for sufficiently large n the first term is less than $\epsilon/3$; the same is true of the second term, and of the third which is bounded by the second. Hence, for all n sufficiently large, $|\mathbb{E}(Z^t) - \mathbb{E}(Z_n^t)| < \epsilon$, and so Z_n approaches Z in moments also.

Remark. Using the fact that $p(n,k) < p_k$ for all k sufficiently large, it can be shown that $\mathbb{E}(Z_n^t) < \mathbb{E}(Z^t)$ for all positive integers n and t.

The probability distribution for the component number of a random $\alpha \in \mathcal{O}_n$ can be calculated explicitly in a similar fashion. Let us now write $Z_n(\alpha)$ for the component number of a randomly chosen $\alpha \in \mathcal{O}_n$, and let $N(n,k)$ be the cardinality of $\{\alpha \in \mathcal{O}_n : \alpha$ has exactly k components$\}$.

THEOREM 3·12. *Let $p(n,k) = \Pr(Z_n = k)$, where Z_n is the component number of a random $\alpha \in \mathcal{O}_n$. Then*

$$p(n,k) = \frac{2k(n-1)(n-2)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)} \quad \text{for } k = 1, 2, \ldots, n. \tag{3·13}$$

The distribution of Z_n has a unique mode at $[(\frac{1}{2})(\sqrt{(1+2n)}-1)]$, the greatest integer not exceeding $(\frac{1}{2})(\sqrt{(1+2n)}-1)$, unless $1+2n$ is a perfect square, in which case $[(\frac{1}{2})\sqrt{(1+2n)}+1]$ is another mode. The distribution decreases monotonically on either side of the mode(s) with the minimum occurring at $k = n$. Furthermore, $p(n,k) \sim 2k/n$ as $n \to \infty$.

Proof. Assertion (3·13) for $k = 1$ is Theorem 2·12; in this case $N(n,1) = C_n$. In order to establish (3·13) for $k > 1$ it is enough, by Result 1·2, to verify that

$$N(n,k) = \frac{k(2n)!\,(n-1)(n-2)\ldots(n-k+1)}{n!\,(n+k)!} \quad \text{for } k = 1, 2, \ldots, n. \tag{3·14}$$

We do this by proving the analogue of Lemma 3·4.

LEMMA 3·15.

$$N(n,k+1) = N(n+1,k) - 2N(n,k) - N(n,k-1) \quad \textit{for } k = 1, 2, \ldots, n-1.$$

Proof. Let α be a typical member of $\mathcal{O}_n$ with exactly two components, and thus two fixed points, $p < q$, say. Bearing in mind the convexity of the orbits of α (Proposition 1·5), and letting t be the maximum of the orbit of p, we see that

$$N(n,2) = \sum_{t=1}^{n-1} N(t,1)N(n-t,1) = \sum_{t=1}^{n-1} C_t C_{n-t} = \sum_{t=2}^{n} C_{t-1} C_{n+1-t}.$$

Applying Result 1·4 then yields that

$$N(n,2) = C_{n+1} - C_0 C_n - C_n C_0 = C_{n+1} - 2C_n.$$

In other words, $N(n,2) = N(n+1,1) - 2N(n,1)$, in agreement with the statement of our lemma.

Next, for $k \geqslant 2$, by considering the final, and then the penultimate fixed point of a mapping from $\mathcal{O}_n$ with exactly $k+1$ fixed points, we obtain the expressions

$$N(n,k+1) = \sum_{t=k}^{n-1} N(t,k)N(n-t,1), \tag{3·16}$$

$$N(n,k+1) = \sum_{t=k-1}^{n-2} N(t,k-1)N(n-t,2). \tag{3·17}$$

Replacing $N(n-t,2)$ by $N(n-t+1,1) - 2N(n-t,1)$ in (3·17) yields that

$$N(n,k+1) = \sum_{t=k-1}^{n-2} N(t,k-1)N(n-t+1,1) - 2\sum_{t=k-1}^{n-2} N(t,k-1)N(n-t,1).$$

From (3·16) we see that the first term can be written as

$$N(n+1,k) - N(n-1,k-1)N(2,1) - N(n,k-1)N(1,1);$$

while the summation in the second term equals

$$N(n,k) - N(n-1,k-1)N(1,1).$$

Combining these expressions yields

$$N(n,k+1) = N(n+1,k) - 2N(n,k) - N(n,k-1)$$

as required. A routine calculation now shows that the expression for $N(n,k)$ given in (3·14) is correct, and thus (3·13) is also.

From (3·13) comes the recurrence relation

$$p(n,k+1) = p(n,k)\frac{k+1}{k}\frac{(n-k)}{(n+k+1)}. \tag{3·18}$$

From (3·18) it follows that

$$p(n,k+1) > p(n,k) \quad \text{if and only if} \quad k < (\tfrac{1}{2})(\sqrt{(1+2n)} - 1) \tag{3·19}$$

with equality occurring in both sides simultaneously. Using $[x]$ to denote the greatest integer not exceeding a real number x, we see that if $(\frac{1}{2})(\sqrt{(1+2n)}-1)$ is not an integer then Z_n has a unique mode at $[(\frac{1}{2})(\sqrt{(1+2n)}-1)]$, while otherwise Z_n is bi-modal with modes $[(\frac{1}{2})(\sqrt{(1+2n)}\pm 1)]$. That the distribution of Z_n falls away monotonically on either side of the mode(s) follows from (3·19), and that the minimum value of $p(n,k)$ occurs at $k = n$ follows by comparing $p(n,1) = 2/(n+1)$ with $p(n,n) = 1/C(2n-1, n-1)$. Finally, from (3·13),

$$p(n,k) = \frac{2k}{n}\cdot\frac{n(n-1)\dots(n-k+1)}{(n+1)(n+2)\dots(n+k)}$$

and since the bracketed term approaches 1 for large n, it follows that $p(n,k) \sim 2k/n$ as $n \to \infty$.

In [**11**], the cardinality of $E(\mathcal{O}_n)$, the set of idempotents of $\mathcal{O}_n$, was shown to be F_{2n}, the $2n$th Fibonacci number. This was achieved by solving a certain recurrence relation. Determining the order of $E(\mathscr{C}_n)$ is, however, a simple matter, for any $\alpha \in E(\mathscr{C}_n)$ is determined by $\operatorname{im}\alpha$, the image of α. To see this, first recall that $\alpha \in T_n$ is an idempotent if and only if $\operatorname{im}\alpha$ and the set of fixed points of α coincide. If we further suppose that $\alpha \in E(\mathscr{C}_n)$ then, for any $i \in X_n$, $i\alpha \leqslant i$, $i\alpha$ is a fixed point of α, and since α preserves order, $i\alpha$ must be the greatest fixed point less than or equal to i. Hence an idempotent of $\mathscr{C}_n$ is determined by its image and thus $|E(\mathscr{C}_n)| = |\{\operatorname{im}\alpha : \alpha \in \mathscr{C}_n\}|$. Now the subsets of X_n which occur as images of mappings $\alpha \in \mathscr{C}_n$ are exactly the subsets of X_n that contain 1. This yields the first statement in our final result.

THEOREM 3·19. *$|E(\mathscr{C}_n)| = 2^{n-1}$. Furthermore, $R = |E(\mathscr{C}_n)|/|\mathscr{C}_n| \sim \sqrt{(\pi n)}\,(n+1)/2^{n+1}$.*

Proof. We have $R = 2^{n-1}/C_n$, from which one obtains $4^{n+1}R^2/2n(n+1)^2 = W_n$, where W_n is the Wallis product as in the proof of Theorem 2·11. Hence we deduce that $2^{n+1}R/(\sqrt{(n\pi)}\,(n+1)) \to 1$ as $n \to \infty$, as required.

Remark. For an investigation of the number of idempotents of finite full transformation semigroups see [**4**] and [**5**].

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