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The fine structure of 4321 avoiding involutions and 321 avoiding involutions

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Abstract. We study the fine structure of the sets $I(4321)$ (involutions avoiding the pattern 4321) and $I(321)$ (involutions avoiding the pattern 321), connecting the point of view of the substitution decomposition theorems with the one of the associated Motzkin paths. The algebraic generating functions of the simple involutions in $I(4321)$ and $I(321)$ are given, together with other generating functions. The simple involutions in $I(4321)$ and $I(321)$ are characterized through their associated Motzkin paths.

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1 Introduction

The aim of this work is to study the structure of the set $I(4321)$ containing all the involutions in the class of 4321 avoiding permutations, and some of its subsets, in particular $I(321)$. Permutations and in particular involutions with forbidden patterns have been intensively studied through the use of various important techniques. Our focus is on the structure of the sets from the point of view of the *substitution decomposition* properties, particularly of the *involution decomposition*, as given in [1], [4] and [8], and the associated labelled Motzkin paths, as presented in [3].

In Section 2 we briefly enumerate the theorems we use and some definitions. In Section 3 some structure properties of involutions avoiding 4321 are given and related to the associated Motzkin paths, whose properties allow to express the generating function of $I(4321)$ in two variables, leading to the calculation of the algebraic generating functions of the sum and skew decomposable involutions, the simple involutions and their inflations, given in Section 4. In Section 5 the structure of the set $I(321)$ is studied and the analogous algebraic generating functions are calculated.

2 Preliminary notions

For a permutation set S, we denote by S_n the set of the permutations in S of length n, and we refer to $f(x) = \sum |S_n| x^n$ as the *generating function for S.*

An *interval* in the permutation π is a set of contiguous indices $\mathcal{I} = [a, b]$, such that the set of values $\pi(\mathcal{I}) = \{\pi(i) : i \in \mathcal{I}\}$ is also contiguous.

A permutation $\pi \in S_n$ is said to be *simple* if containing only intervals of length 0, 1, n.

A permutation $\pi \in S_n$ *avoids the pattern* $s_k \in S_k$, $k \leq n$, if π does not contain a subsequence order-isomorphic to s_k. The permutations *avoiding the pattern 4321* constitute the class $Av(4321)$, that is any $\pi \in Av(4321)$ contains no descending sequences of length $n \geq 4$.

Given a permutation $\pi \in S_n$, the set $\{1, 2, \ldots, n\}$ can be partitioned into intervals $A_1, \ldots, A_t$ such that $\pi(A_i) = A_i$, $\forall i$. The restrictions of π to the intervals in the finest of these decompositions are called *connected components* of π. A permutation π with a single connected component is called *connected.*

Let $\pi \in S_k$, $\alpha_1, \ldots, \alpha_k \in S$. The *inflation* of π by $\alpha_1, \ldots, \alpha_k$ is the permutation $\pi[\alpha_1, \ldots, \alpha_k]$ obtained by replacing each element s_i of π by a block whose pattern is α_i.
We note that a permutation $\pi \in S$ is connected (or *sum indecomposable*) if and only if π is not an inflation of 12.

An *involution* is a permutation π such that $\pi = \pi^{-1}$. So, the graph of an involution has obviously a symmetry with respect to the line $y = x$.
Let I denote the set of all the involutions, I_n the set of involutions of length n.

An involution π can be decomposed into disjoint cycles

$$\pi = (m_1, M_1)(m_2, M_2) \ldots (m_m, M_m),$$

with $m_i \leq M_i$, and the m_i written in increasing order. We recall that a permutation π has an *excedance* at position j if $\pi(j) > j$, a *deficiency* at position j if $\pi(j) < j$ and a *fixed* point if $\pi(j) = j$.

In the literature there are many bijections between permutations and Motzkin paths provided with some kind of labelling. We use the ideas presented in [3], to which we refer for the bibliography on the subject.

A *labelled Motzkin path* (M, λ) *associated with* $\pi \in I$ is obtained as follows. For every $i = 1, \ldots, n$,

- if i is a fixed point for π, take a horizontal step in the path;
- if i is the first element of a transposition, take an up step in the path;
- if i is the second element af a transposition, take a down step in the path, labelled with h, if i is in the h-th position among integers greater than or equal to i in the cycle decomposition of π.

A Motzkin path is said to be *irreducible* if it does not touch the $x-$axis except for the origin and the final destination. By construction, for each connected component of an involution π we have an irreducible component of its associated labelled Motzkin path. So, an involution π is connected if and only if its associated labelled Motzkin path is irreducible.

In [3] it is shown how associated labelled Motzkin paths with unitary labelling characterize the involutions of $I(4321)$, with a consequence in particular for $I(321)$, while the maximal labelling characterizes $I(3412)$.

Properties regarding *substitution decomposition* and *involution decomposition* are given in [1], [4] and [2]; we enumerate in the following some of those propositions.

PROPOSITION 2.1 *(See [2]) Every permutation, except 1, is the inflation of a unique simple permutation of length at least 2.*

This means that every permutation π determines the unique simple permutation of which π is an inflation.

PROPOSITION 2.2 *(See [2]) If* $\pi = \sigma[\alpha_1, \ldots, \alpha_m]$ *where* σ *is simple of length* $m \geq 4$, *then the* α_i *are unique.*

As for the case of the simple permutations 12 and 21 a unique decomposition can still be given, as in the following.

PROPOSITION 2.3 *(See [2]) If* π *is an inflation of 12, then there is a unique* α_1, *not an inflation of 12, such that* $\pi = 12[\alpha_1, \alpha_2]$ *for some* α_2, *which is itself unique. The same holds with 12 replaced by 21, and* α_1 *not an inflation of 21.*

Specifically for the involutions we recall the following propositions.

PROPOSITION 2.4 *A permutation* $\pi = 12[\alpha_1, \alpha_2]$ *is an involution if and only if* α_1 *and* α_2 *are involutions.*

PROPOSITION 2.5 *Let be* $\pi = \sigma[\alpha_1, \ldots, \alpha_m]$ *inflation of a simple permutation* $\sigma \neq 21$. *Then if* π *is an involution, also* σ *is an involution, and the following equalities hold:* $\alpha_i = \alpha^{-1}_{\sigma^{-1}(i)} = \alpha^{-1}_{\sigma(i)}$, *for* $i = 1, \ldots, m$.

The equalities say that every transposition in σ must be inflated with a couple of substitutions one the inverse of the other.

PROPOSITION 2.6 *The involutions that are inflation of 21 are precisely those of the form:*

$21[\alpha_1, \alpha_2]$, *where neither* α_1 *nor* α_2 *are inflations of* 21, *and* $\alpha_1 = \alpha_2^{-1}$;

$321[\alpha_1, \alpha_2, \alpha_3]$, *where neither* α_1 *nor* α_3 *are inflations of* 21, $\alpha_1 = \alpha_3^{-1}$, *and* α_2 *is an involution.*

3 Properties of involutions avoiding 4321

We denote $I(4321)$ the set of the involutions avoiding the pattern 4321, and $I(4321)_n$ the subset of the involutions of length n; analogously $I(321)$ and $I(321)_n$ for the involutions avoiding the pattern 321. Obviously one has $I(4321) \supset I(321)$.
We note that an involution in $I(321)$ is the merge of two increasing sequences of integers, the one of its maxima (or excedances) and the one of its minima (or deficiencies), while an involution in $I(4321)$,

containing 321, is obtained by interlacing three ascending sequences of integers: the excedances, the deficiencies and the fixed points.

The main result of this section is the characterization of the simple involutions in $I(4321)$ through their associated Motzkin paths, which is obtained in Proposition 3.7. We give in the following some properties of involutions in $I(4321)$.

PROPOSITION 3.1 *If* $\pi \in I(4321)$, $\pi = 12[\alpha_1, \alpha_2]$, *then* α_1 and α_2 *are in* $I(4321)$.

PROPOSITION 3.2 *Let* $\pi \in I(4321)$ *be an involution, which is an inflation of 21. Then* $\pi = 21[\alpha_1, \alpha_2]$, *where* $\alpha_1 = \alpha_2 = 1\,2\ldots n$, *or* $\pi = 321[\alpha_1, \alpha_2, \alpha_3]$, *where* $\alpha_1 = \alpha_3 = 1\,2\ldots n$, *and* $\alpha_2 = 1\,2\ldots m$.

Proof. It is clear to see that otherwise π would contain a descending sequence of length 4. □

The following properties of the simple involutions are easily obtained.

PROPOSITION 3.3 *A simple involution has no consecutive fixed points.*

PROPOSITION 3.4 *Let* σ *be a simple involution. Then if* σ *has a pair of excedances consecutive in the permutation, the corresponding deficiencies cannot be consecutive in* σ.

The same holds with excedance replaced by deficiency.

Proof. Let $\sigma = (m_1, M_1)(m_2, M_2)\cdots(m_m, M_m)$; if M_h and M_{h+1} are consecutive in the permutation, $m_{h+1} = m_h + 1$, hence the thesis. □

PROPOSITION 3.5 *Let* $\sigma \in I(4321)_n$, $n > 3$, σ *simple. An inflation* π *of* σ *is again in* $I(4321)$ *if and only if* π *is obtained by inflating any number of transpositions and fixed points through* $\alpha_i = \alpha^{-1}_{\sigma^{-1}(i)} = \alpha^{-1}_{\sigma(i)} = 123\ldots m$.

Proof. By inflating a transposition of σ or a fixed point through a substitution presenting an inversion, we would obtain an inflation presenting a descending sequence of length 4, so not belonging to $Av(4321)$. □

The associated Motzkin paths suggest that the involutions in $I(4321)$ which are inflation of 21 can be obtained by expanding the involutions in $I(321)$ which are inflation of 21, through any number of fixed points at the maximal height, see Figure 1.

For sake of simplicity the unitary labelling will be omitted in the figures. The involution corresponding to a given Motzkin path has as fixed points the horizontal steps and commutes every up step with the first successive down step.

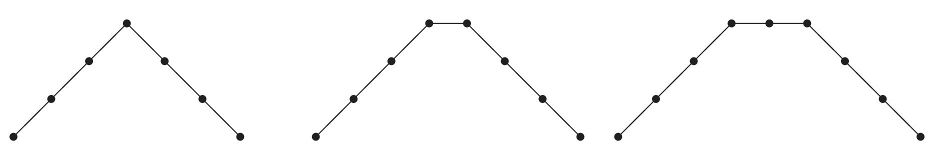

Figure 1: $456123 \in I_6(321)$, $5674123 \in I_7(4321) \setminus I_7(321)$, $\tau \in I(4321) \setminus I(321)$

Through the use of associated labelled Motzkin paths and the characterization contained in the next Theorem, we obtain the following properties of simple involutions in $I(4321)$.

THEOREM 3.6 *(See [3], Theorem 3.) Let π_n be an involution with (M, λ) as the associated labelled Motzkin path of length n. Then π_n avoids 4321 if and only if $\lambda = \nu$ (where ν is the unitary labelling).*

As a consequence, it is shown that *π_n avoids 321 if and only if λ is the unitary labelling and all horizontal steps in M are at height 0* (see [3], Proposition 4).

PROPOSITION 3.7 *Let $\pi_n \in I(4321)_n$, having (M, ν) as the associated labelled Motzkin path of length n (with ν the unitary labelling). Then π_n is simple if and only if all the following three properties hold:*

i) (M, ν) is an irreducible Motzkin path;

ii) there are no consecutive horizontal steps;

iii) let $\{U_1, \dots, U_s\}$ and $\{D_1, \dots, D_s\}$ be the sequences of the up and of the down steps in (M, ν). If two up steps U_i and U_{i+1} are consecutive up steps in (M, ν), then the corresponding D_i and D_{i+1} are never consecutive down steps in (M, ν).

Proof. Let $\pi_n \in I(4321)_n$ be simple, so connected: then the Motzkin path is irreducible, with no adjacent fixed points.

Moreover (M, ν) is such that by construction, the excedances of π_n correspond to the up steps, the deficiencies to the down steps. Hence, by Proposition 3.4, if U_i, U_{i+1} are consecutive in (M, ν), D_i, D_{i+1} cannot be consecutive.

Conversely, if (M, ν) satisfies i), ii) and iii), the involution $\pi_n \in I(4321)_n$ is connected, so not an inflation of 12. The involution π_n can be neither inflation of 21 nor inflation of another simple involution because of Proposition 3.2 and Proposition 3.5, so it is simple, as claimed. □

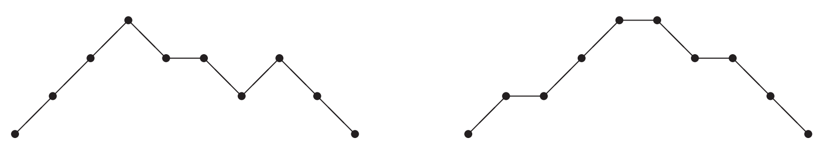

Figure 2: Motzkin paths related to 4 6 8 1 5 2 9 3 7 and to 6 2 8 9 5 1 7 3 4

The involution $\sigma =$ 4 6 8 1 5 2 9 3 7 $\in I(4321)$, whose Motzkin path is illustrated in Fig.2a, is simple, because the conditions i), ii), iii) are fulfilled. On the contrary $\pi =$ 6 2 8 9 5 1 7 3 4 $\in I(4321)$ of Fig.2b is not simple, because the pair of consecutive up steps 8, 9 corresponds to the pair of consecutive down steps 3, 4 .

From Theorem 3.6 and Proposition 3.7 one derives a nice graphical criterion for deciding if an irreducible Motzkin path represents a simple involution avoiding 4321. Moreover, a feasible procedure to obtain new simple involutions from a given one is outlined in the following proposition.

PROPOSITION 3.8 *The two following properties hold.*

i) Starting from a simple involution $\pi_{2n} \in I(321)_{2n}$, a simple involution $\sigma_{2n+1} \in I(4321)_{2n+1}$ can be obtained by adjoining a horizontal step to the Dyck path (D, ν) representing π_{2n}, in any position different from the first and the last point.

ii) Starting from a Dyck path (D,ν) associated with an irreducible not simple involution $\pi \in I(321)$, one obtains a Motzkin path associated with a simple involution $\sigma \in I(4321)$ by inserting in D all the horizontal steps necessary to break the consecutiveness of corresponding up and down steps.

Proof. For *i)*, the thesis easily follows remembering that any simple involution, being irreducible, contains no horizontal steps at level zero in the associated Motzkin path, so the simple involutions of $I(4321)$ generate horizontal steps only at levels different from zero. Adjoining a horizontal step to the Dyck path (D,ν) associated to $\pi_{2n} \in I(321)_{2n}$ corresponds to adjoining a fixed point, so generating a simple involution in $I(4321)$, as claimed. While for *ii)*, the thesis immediately follows from Theorem 3.6 and Proposition 3.7. □

For instance, starting from $\pi_6 = 351624$ with associated Dyck path that we can describe as usual in the form $UUDUDD$, if we want to insert a fixed point in the fourth position, we obtain the Motzkin path $UUDHUDD$, and the required involution is 3614725.

In [3], Theorem 6, (i), it is shown that the set $I_n(4321)$ is closed under *reverse-complement*; also, if the involution π corresponds to the labelled Motzkin path (M,ν), ν the unitary labelling, the involution π^{rc} corresponds to the path obtained by reflecting M over the line $x/2$, ν unitary (see also [3], Proposition 2). Then we obtain immediately:

THEOREM 3.9 *The reverse-complement bijection preserves the fine structure of $I(4321)$.*

Let, for instance, σ and π be as in Fig.2: then $\sigma^{rc} = 371859246$, $\pi^{rc} = 673951284$, as illustrated in the following figure.

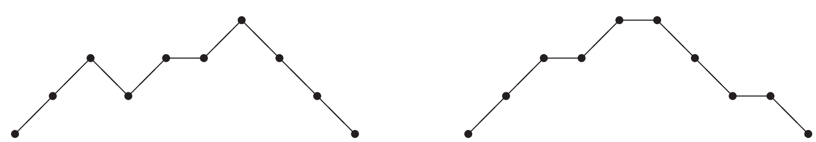

Figure 3: $\sigma^{rc} =$ 371859246 and $\pi^{rc} =$ 673951284

4 Generating functions of subsets of I(4321)

Given a set of involutions, we denote by f, α, β, γ, δ the following generating functions:
f denotes the generating function of the whole set;
α the generating function of the involutions which are inflation of 12;
β the generating function of the involutions which are inflation of 21;
γ the generating function of simple involutions different from 1, 12 e 21;
δ the generating function of the involutions which are inflation of simple involutions of length $n > 2$.

It is well known that the generating function f of the whole set $I(4321)$ is

$$f = -1 + \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},$$

whose expansion's coefficients are the Motzkin numbers $1, 2, 4, 9, 21, 51, 127, \ldots$.

Proposition 4.1 *The generating function β of the involutions in $I(4321)$ which are inflations of 21 is*

$$\beta = \left(\frac{x^2}{1-x^2}\right)\left(\frac{1}{1-x}\right).$$

Proof. In fact, the thesis easily follows from the description of these involutions given in Proposition 2.2. □

On the basis of the substitution theorems for involutions, of Proposition 3.1 and Proposition 4.1, it is possible to write the following relations for the generating functions:

$$\begin{cases} f = x + \alpha + \beta + \gamma + \delta = -1 + \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2} \\ \beta = \left(\frac{x^2}{1-x^2}\right)\left(\frac{1}{1-x}\right) \\ \alpha = (x + \beta + \gamma + \delta)(x + \alpha + \beta + \gamma + \delta) \end{cases} . \quad (1)$$

From (1) we obtain

$$\begin{aligned} \gamma + \delta &= 1/4\left(2 - \frac{2}{(-1+x^2)^2} - \frac{3}{-1+x} - 2x - \frac{1}{1+x} - 2\sqrt{1-2x-3x^2}\right) \\ &= (f+1)x^2 - \beta, \end{aligned}$$

whose expansion is $2x^5 + 6x^6 + 18x^7 + 47x^8 + 123x^9 + 318x^{10} + o(x^{11})$.

The function $\gamma + \delta$ counts the simple involutions in $I(4321)$ and their inflations. It can obviously be obtained directly, from the following considerations.

The function $f = -1 + \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2} = x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + \ldots$ enumerates all the involutions; by adjoining an up step and a down step respectively at the beginning and at the end of the Motzkin path one obtains a path associated to an irreducible involution; all the irreducible involutions are the inflations of 21, the simple ones and their inflations. Hence the claimed property.

By Theorem 3.6, involutions in $I_n(4321)$ with k fixed points are in bijection with Motzkin paths with k horizontal steps; so the expression of the function f in two variables can be used, enumerating the fixed points of the involutions (see [5]; [3], Proposition 13, and [10], sequence A097610). Precisely:

$$f(x,y) = \frac{1 - xy - \sqrt{1 - 2xy + x^2y^2 - 4x^2}}{2x^2},$$

whose expansion is $1 + yx + (1+y^2)x^2 + (3y + y^3)x^3 + (2 + 6y^2 + y^4)x^4 + \ldots$, where the coefficient $c_{k,n}$ of $c_{k,n}y^kx^n$ indicates the number of involutions of length n having k fixed points. (Putting $y = 1$ one obtains the total number of involutions of length n).

The function β is immediately written in the form

$$\beta(x,y) = \frac{x^2}{(1-x^2)(1-xy)},$$

where the variables have the same interpretation, so giving

$$(\gamma + \delta)(x,y) = \frac{1 - xy - \sqrt{1 - 2xy + x^2y^2 - 4x^2}}{2} - \frac{x^2}{(1-x^2)(1-xy)},$$

whose expansion is $2yx^5 + (1+5y^2)x^6 + (9y+9y^3)x^7 + \ldots$.

Through the substitution y/x for y, one obtains

$$(\gamma+\delta)(x,y) = 1/2\left(1 - y - \sqrt{1-2y+y^2-4x^2}\right) - \frac{x^2}{(1-x^2)(1-y)},$$

whose expansion is $x^6 + (2x^4+9x^6)y + (5x^4+29x^6)y^2 + (9x^4+69x^6)y^3 + \ldots$, where the coefficient $c_{n,k}$ of $c_{n,k}x^n y^k$ now indicates the number of involutions of length $n+k$ having k fixed points and $n/2$ transpositions.

Using the expression of $(\gamma+\delta)(x,y)$ in the last form, one can easily obtain the function γ enumerating the simple involutions of length greater than 2, through Proposition 2.5. In fact, we observe that if we had the generating function $\gamma(x,y)$ of the simple involutions in $I(4321)$, with x^2 and y counting respectively transpositions and fixed points, we could inflate x^2 through the substitution $\frac{x^2}{1-x^2}$ and y through $\frac{y}{1-y}$ in order to obtain $(\gamma+\delta)(x,y)$, which counts the simple involutions and their inflations. Having the function $(\gamma+\delta)(x,y)$, we can express $\gamma(x,y)$ through the inverse substitutions $\frac{x^2}{1+x^2}$ and $\frac{y}{1+y}$. We obtain in fact the following.

THEOREM 4.2 *The generating function $\gamma(x,y)$ of the simple involutions in $Av(4321)$ is*

$$\gamma(x,y) = 1/2\left(\frac{1}{1+y} - 2x^2(1+y) - \sqrt{-4 + \frac{4}{(1+x^2)} + \frac{1}{(1+y)^2}}\right),$$

whose expansion is

$$\begin{aligned} &(x^6+x^8+3x^{10}+6x^{12}+\ldots) + (2x^4+5x^6+13x^8+\ldots)y + (3x^4+14x^6+54x^8+\ldots)y^2 \\ + \quad &(x^4+18x^6+\ldots)y^3 + (10x^6+145x^8+\ldots)y^4 + \ldots. \end{aligned}$$

By substituting x for y in $\gamma(x,y)$, we obtain the generating function in one variable:

$$\gamma(x) = 1/2\left(\frac{1}{1+x} - 2x^2(1+x) - \sqrt{-4 + \frac{4}{(1+x^2)} + \frac{1}{(1+x)^2}}\right),$$

whose expansion is $2x^5 + 4x^6 + 6x^7 + 15x^8 + 31x^9 + 67x^{10} + 155x^{11} + 343x^{12} + 787x^{13} + 1829x^{14} + \ldots$.

The sequence of coefficients is not in Sloane [10].

In [3], Theorem 6 and Corollary 7, the Motzkin paths associated with I(4321, 132) and I(4321, 312) are characterized and enumerative results are given as a consequence. Many important enumerative results about restricted involutions are to be found in [6], [7] and the literature therein cited; we only want to show here as another example how for $I(4321, 132)$, $I(4321,\ 213)$ and $I(4321, 312)$ the generating functions can be reobtained through the substitution decomposition theorems.

PROPOSITION 4.3 *i) The generating functions of subsets respectively of $I(4321,132)$ and $I(4321,213)$ satisfy the following equations:*

$$\begin{cases} f = x + \alpha + \beta \\ \beta = \left(\frac{x^2}{1-x^2}\right)\left(\frac{1}{1-x}\right) \\ \alpha = (x + \beta)\frac{x}{1-x} \end{cases} ; \tag{2}$$

$$\begin{cases} f = x\, +\, \alpha + \beta \\ \beta = \left(\frac{x^2}{1-x^2}\right)\left(\frac{1}{1-x}\right) \\ \alpha = xf \end{cases} \quad . \tag{3}$$

From (2) and (3) one derives the same generating function f*:*

$$f = -\frac{x - x^3 + x^4}{(-1+x)^3(1+x)} \;=\; x + 2x^2 + 3x^3 + 5x^4 + \ldots,$$

and the sequence $1, 2, 3, 5, 7, 10, 13, 17, ..., 1 + \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor, ...$ *as in the cited papers.*

ii) The generating functions of subsets of $I(4321,\, 312)$ *satisfy the following equations:*

$$\begin{cases} f = x\, +\, \alpha + \beta \\ \beta = x^2\, +\, x^3 \\ \alpha = (x\, + \beta)(f) \end{cases} \quad . \tag{4}$$

From (4) we obtain

$$f = \frac{-x - x^2 - x^3}{-1 + x + x^2 + x^3},$$

whose expansion gives the coefficients $0, 1, 2, 4, 7, 13, 24, 44, 81...,$ *(Tribonacci numbers), as in the cited papers.*

In [12] we give more details for Proposition 4.3 and a list of the simple involutions in $I(4321)$ for some n.

5 The fine structure of I(321)

As for the involutions avoiding 321, the following properties are easily derived.

Proposition 5.1 *If* $\pi \in I(321)$*,* $\pi \,=\, 12[\alpha_1,\, \alpha_2]$*, then* α_1 and α_2 *are in I(321).*

Proposition 5.2 *If* $\pi \in I(321)$ *is inflation of* 21*, then*

$$\pi \;=\; 21[1\,2\,\ldots\,m,\, 1\,2\,\ldots\,m] \;=\; (m+1)\,(m+2)\ldots\,(2m)\,1\,2\ldots\,m.$$

As a consequence, we have for the generating function β *of involutions inflation of 21*

$$\beta = \frac{x^2}{1 - x^2}.$$

Proposition 5.3 *If* $\sigma \in I(321)_n$*,* $n > 2$*, is a simple involution, then* σ *has no fixed points, and* n *is even. If* π *is an inflation of a simple involution* $\sigma \in I(321)_n$*,* $n > 2$*, then the length of* π *is even.*

Proof. A simple involution $\sigma \in I(321)$ has no fixed points, because of the symmetry of the two ascending sequences in respect of $y = x$. So n is even.

As for an inflation of a simple involution, it must have even length because always obtained through couples of substitutions one the inverse of the other, by Proposition 2.5. □

As an immediate consequence of Theorem 3.6, one derives

PROPOSITION 5.4 *The set $I(321)$ has infinitely many simple involutions.*

It is well known that the generating function f of $I(321)$ is

$$f = \frac{1 - 4x^2 - \sqrt{1 - 4x^2}}{2(-x + 2x^2)},$$

whose expansion gives the central binomial coefficients, (see [9] and [10], A001405):

It is then possible to write the equations (4) describing the fine structure of $I(321)$, where the symbols have the meaning already given in Section 4:

$$\begin{cases} f = x + \alpha + \beta + \gamma + \delta = \frac{1-4x^2-\sqrt{1-4x^2}}{2(-x+2x^2)} \\ \beta = \frac{x^2}{1-x^2} \\ \alpha = (x + \beta + \gamma + \delta)(x + \alpha + \beta + \gamma + \delta) \end{cases} . \tag{5}$$

The system (5) allows to express

$$\alpha = \frac{1 + x - 4x^2 - 4x^3 - \sqrt{1 + 2x - 7x^2 - 12x^3 + 16x^4 + 16x^5 - 16x^6}}{2(-x + 2x^2)}$$

whose expansion gives the coefficients 1, 3, 5, 10, 18, 35, 65, 126, 238, 462, 882, 1716, 3300, 6435, 12441, 24310, 47190, 92378, 179894, ... (see [10], A107232), and

$$\zeta = \gamma + \delta = \frac{1 - 3x^2 - \sqrt{1 - 6x^2 + 9x^4 - 4x^6}}{2(1 - x^2)}$$

whose expansion is $x^6 + 4x^8 + 13x^{10} + 41x^{12} + 131x^14 + \dots$, where the coefficients $1, 4, 13, 41, 131, 428, 1429, \dots$ are Catalan -1 numbers, (see [10] , A001453).

Using the same procedure as in Section 4, now for one variable only, in order to calculate the generating function γ of the simple involutions in $I(321)$ we note that $\zeta(x^2) = \gamma\left(\frac{x^2}{1-x^2}\right)$ and conversely $\gamma(x^2) = \zeta\left(\frac{x^2}{1+x^2}\right)$.

Hence one obtains

THEOREM 5.5 *The algebraic generating function γ of the simple involutions avoiding 321 is*

$$\gamma = \frac{1 - x^2 - 2x^4 - \sqrt{1 - 2x^2 - 3x^4}}{2(1 + x^2)}.$$

The expansion of γ gives the coefficients

$$1, 0, 1, 0, 3, 0, 6, 0, 15, 0, 36, 0, 91, 0, 232, 0, 603, 0, 1585, \dots.$$

The non zero coefficients are Riordan numbers, see [10] , A005043.

For the simple involutions of length $2n$ avoiding 321, we have a characterization in terms of Dyck paths of length $2n$, and another one in terms of Motzkin paths of length $n-1$. The first one is given through the analogous of Proposition 3.7, with an analogous proof.

PROPOSITION 5.6 *(See [3], Proposition 4.) Let π_n be an involution with (M,λ) as the associated labelled Motzkin path of length n. Then π_n avoids 321 if and only if $\lambda = \nu$ (where ν is the unitary labelling) and all horizontal steps in M are at height 0.*

PROPOSITION 5.7 *Let $\sigma_n \in I(321)_n$ with (M,ν) as the associated labelled Motzkin path of length n (whith ν unitary). Then σ_n is simple if and only if both the following properties hold:*

i) (M,ν) is an irreducible Dyck path;

ii) let $\{U_1,\ldots,U_m\}$ and $\{D_1,\ldots,D_m\}$ be the sequences of the up and of the down steps in (M,ν). If two up steps U_i and U_{i+1} are consecutive up steps in (M,ν), then the corresponding D_i and D_{i+1} are never consecutive down steps in (M,ν).

Proof. Let $\sigma_n \in I(321)_n$ be simple, so with no fixed points: then by construction (M,ν) has no horizontal steps, therefore being a Dyck path, irreducible because a simple involution is connected.

Moreover (M,ν) is such that, always by construction, the maxima of σ correspond to the up steps, the minima to the down steps. Hence, through Proposition 3.7, if U_i, U_{i+1} are consecutive in (M,ν), D_i, D_{i+1} cannot be consecutive.

Conversely, if (M,ν) satisfies i) and ii), the involution $\sigma \in I(321)_n$ is connected, with no fixed points, so again through Proposition 3.7 we derive that σ_n is simple, as requested. □

Remark. From Propositions 5.6 and 5.7 one derives a nice graphical criterion for deciding if an irreducible Dyck path represents a simple involution avoiding 321, and a feasible procedure to obtain new simple involutions from a given one.

A different graphical representation can be obtained through the interpretation of the Riordan numbers presented in [10], given by Emeric Deutsch (2003): *the coefficients* $1,0,1,1,3,6,15,36,91,232,$ $603,1585,\ldots$ *enumerate the Motzkin paths of length n with no horizontal steps at level* 0. (We call them conventionally *short* Motzkin paths). In [11] we define a bijection between the simple involutions of $I(321)_{2n+2}$ and the *short* Motzkin paths of length n, which gives a procedure to calculate all the simple involutions of given length.

Moreover, given $\sigma_{2n+2} \in I(321)_{2n+2}$, it is possible to decide which simple involutions are contained in σ_{2n+2} as patterns.

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